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DUALITY-REFLECTION FORMULAS OF MULTIPLE POLYLOGARITHMS AND THEIR ℓ-ADIC GALOIS ANALOGUES

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Abstract. In this paper, we derive formulas of complex and ℓ-adic multiple polylogarithms, which have two aspects: a duality in terms of indexes and a reflection in terms of variables. We provide an algebraic proof of these formulas by using algebraic relations between associators arising from the S_3-symmetry of the projective line minus three points.

1. Introduction and main results

The purpose of the present paper is to derive a series of functional equations that generalizes Oi-Ueno's reflection formulas between complex multiple polylogarithms at z and $1-z$. This specializes to the duality formula for multiple zeta values when $z \to 1$. We also show the ℓ-adic Galois analog of these equations by tracing the same argument in a parallel way to the complex case.

For a multi-index $\mathbf{k} = (k_1 \dots, k_d) \in \mathbb{N}^d$ and a topological path $\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, z\right)$ from the standard tangential base point $\overrightarrow{01}$ to a $\mathbb{C}$-rational base point z, the complex multiple polylogarithm $Li_{\mathbf{k}}(z;\gamma)$ is defined as an iterated integral along γ (see §2.1 for details). As is well known, $Li_{\mathbf{k}}(z;\gamma)$ coincides with a certain signed coefficient of the KZ solution

$$G_0(X,Y)(z;\gamma) \in \mathbb{C}\langle\langle X, Y\rangle\rangle.$$

The multiple zeta value $\zeta(\mathbf{k})$ appears as its special value at the tangential base point $\overrightarrow{10}$ with the straight path $\delta \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, \overrightarrow{10}\right)$ along the unit interval $(0,1) \subset \mathbb{P}^1(\mathbb{R})\backslash\{0,1,\infty\}$. Our main result of the complex case is then as follows.

Theorem 1.1 (The duality-reflection formula of complex multiple polylogarithms)**.** *Given a (possibly, tangential base) point z of $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$ and a path $\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, z\right)$, define the path γ' associated to γ by*

$$\gamma' := \delta \cdot \phi(\gamma) \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, 1-z\right), \tag{1.1}$$

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where $\phi \in \mathrm{Aut}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}\right)$ *is given by* $\phi(t) = 1 - t$ *and paths are composed from left to right. For any* $n, m \in \mathbb{Z}_{\geq 2}$*, the following holds:*

(1.2)
$$\sum_{j=0}^{m-1} \frac{(-\log(z;\gamma))^j}{j!} Li_{\underbrace{1,\ldots,1}_{n-2\ times},m-j}(z;\gamma) + \sum_{j=0}^{n-2} \frac{(-\log(1-z;\gamma'))^j}{j!} Li_{\underbrace{1,\ldots,1}_{m-2\ times},n-j}(1-z;\gamma') = \zeta(\underbrace{1,\ldots,1}_{n-2\ times},m),$$

where $\log(z;\gamma) := \int_{\delta^{-1}\cdot\gamma} \frac{dt}{t}$ *is the logarithm function with respect to* γ*.*

This functional equation has two aspects: a duality $n \leftrightarrow m$ with respect to indexes and a reflection $z \leftrightarrow 1 - z$ with respect to variables. We derive the functional equation from an algebraic relation (chain rule)

$$G_0(X,Y)(z;\gamma) = G_0(Y,X)(1-z;\gamma') \cdot G_0(X,Y)(\overrightarrow{10};\delta),$$

where $G_0(X,Y)(\overrightarrow{10};\delta)$ is the so-called Drinfeld associator.

We also deal with the ℓ-adic Galois case for any prime number ℓ. Let K be a subfield of $\mathbb{C}$ and $G_K := \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of K with respect to its algebraic closure $\overline{K}$. For a K-rational base point z of $\mathbb{P}^1\backslash\{0,1,\infty\}$, consider each topological path $\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right)$ as a pro-ℓ étale path $\gamma \in \pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_{\overline{K}}\backslash\{0,1,\infty\};\overrightarrow{01},z\right)$ by the comparison map. Then, for $\sigma \in G_K$, the ℓ-adic Galois multiple polylogarithm $Li^{\ell}_{\mathbf{k}}(z;\gamma,\sigma)$ is defined as a certain signed coefficient of the ℓ-adic Galois associator

$$\mathfrak{f}^{z,\gamma}_{\sigma}(X,Y) \in \mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$$

(see §2.2 for details). This ℓ-adic multiple polylogarithm is an ℓ-adic étale avatar of $Li_{\mathbf{k}}(z;\gamma)$ introduced by Wojtkowiak. The ℓ-adic Galois multiple zeta value (or called the ℓ-adic multiple Soulé element) $\boldsymbol{\zeta}^{\ell}_{\mathbf{k}}(\sigma)$ is defined as its special value $Li^{\ell}_{\mathbf{k}}\left(\overrightarrow{10};\delta,\sigma\right)$. Our another main result is then as follows.

Theorem 1.2 (The duality-reflection formula of ℓ-adic Galois multiple polylogarithms)**.** *Given a* K*-rational (possibly, tangential base) point* z *of* $\mathbb{P}^1\backslash\{0,1,\infty\}$ *and* $\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right)$*, define the path* γ' *associated to* γ *as in (1.1). For any* $\sigma \in G_K$*, the following holds:*

(1.3)
$$\sum_{j=0}^{m-1} \frac{(\rho_{z,\gamma}(\sigma))^j}{j!} Li^{\ell}_{\underbrace{1,\ldots,1}_{n-2\ times},m-j}(z;\gamma,\sigma) + \sum_{j=0}^{n-2} \frac{(\rho_{1-z,\gamma'}(\sigma))^j}{j!} Li^{\ell}_{\underbrace{1,\ldots,1}_{m-2\ times},n-j}(1-z;\gamma',\sigma) = \boldsymbol{\zeta}^{\ell}_{\underbrace{1,\ldots,1}_{n-2\ times},m}(\sigma),$$

where $\rho_{z,\gamma} : G_K \to \mathbb{Z}_\ell$ is the Kummer 1-cocycle defined by $\sigma(z^{1/\ell^k}) = \zeta_{\ell^k}^{\rho_{z,\gamma}(\sigma)} z^{1/\ell^k}$ with respect to the ℓ-th power roots $\{z^{1/\ell^k}\}_k$ determined by γ.

By reinterpreting the proof of the complex functional equation (1.2) after replacing $G_0(X,Y)(z;\gamma)$ by $\mathfrak{f}_\sigma^{z,\gamma}(X,Y)$, we derive the ℓ-adic functional equation (1.3) from the following chain rule between ℓ-adic Galois associators

$$\mathfrak{f}_\sigma^{z,\gamma}(X,Y) = \mathfrak{f}_\sigma^{1-z,\gamma'}(Y,X) \cdot \mathfrak{f}_\sigma^{\overrightarrow{10},\delta}(X,Y)$$

along the path composition (1.1) (cf. [N21, p.701, the key identity $(*)$]).

Remark 1.3. The formula (1.2) is a generalization of the following functional equation (1.4) due to Oi and Ueno in [Oi09],[OU13]. The formula (1.3) is a generalization of the following functional equation (1.5) due to Nakamura in [NS22],[N21].

(1.4)
$$\sum_{j=0}^{m-1} \frac{(-\log(z;\gamma))^j}{j!} Li_{m-j}(z;\gamma) + Li_{\underbrace{1,\ldots,1}_{m-2 \text{ times}},2}(1-z;\gamma') = \zeta(m),$$

(1.5)
$$\sum_{j=0}^{m-1} \frac{(\rho_{z,\gamma}(\sigma))^j}{j!} Li^{\ell}_{m-j}(z;\gamma,\sigma) + Li^{\ell}_{\underbrace{1,\ldots,1}_{m-2 \text{ times}},2}(1-z;\gamma',\sigma) = \boldsymbol{\zeta}^{\ell}_m(\sigma) \quad (\sigma \in G_K).$$

Remark 1.4. By setting $z = \overrightarrow{10}$ (i.e. $z \to 1$ along the real interval) in (1.2) and (1.3), we obtain the well-known duality formula of multiple zeta values and its ℓ-adic Galois analog.

$$\zeta(\underbrace{1,\ldots,1}_{m-2 \text{ times}},n) = \zeta(\underbrace{1,\ldots,1}_{n-2 \text{ times}},m), \tag{1.6}$$

$$\boldsymbol{\zeta}^{\ell}_{\underbrace{1,\ldots,1}_{m-2 \text{ times}},n}(\sigma) = \boldsymbol{\zeta}^{\ell}_{\underbrace{1,\ldots,1}_{n-2 \text{ times}},m}(\sigma) \quad (\sigma \in G_K). \tag{1.7}$$

Remark 1.5. In [F04], Furusho constructed the theory of the p-adic KZ equation and studied the p-adic multiple polylogarithm, which is a p-adic crystalline avatar of $Li_{\mathbf{k}}(z;\gamma)$. Using the results in [F04], it is possible to obtain a p-adic analog of (1.2) in the same way as in the proof of (1.2).

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2. Preliminaries

In this section, we review the basic properties of complex multiple polylogarithms and ℓ-adic Galois multiple polylogarithms in preparation for proving the main theorems (1.2) and (1.3).

For a (possibly, tangential base) point z on $\mathbb{P}^1\backslash\{0,1,\infty\}$, we shall write

$$\pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right)$$

for the set of homotopy classes of piece-wise smooth topological paths on $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$ from the tangential base point $\overrightarrow{01}$ to z, and

$$\pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\},\overrightarrow{01}\right):=\pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},\overrightarrow{01}\right)$$

for the topological fundamental group of $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$ at the base point $\overrightarrow{01}$ with respect to the path composition $\gamma_1\cdot\gamma_2:=\gamma_1\gamma_2$, i.e. paths are composed from left to right. Let

$$l_0,l_1\in\pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\},\overrightarrow{01}\right)$$

be smooth loops circling counterclockwise around 0, 1 respectively, as FIGURE 1 shows. In FIGURE 1, the dashed line represents $\mathbb{P}^1(\mathbb{R})\backslash\{0,1,\infty\}$ and the upper half-plane is located above. Then, $\{l_0,l_1\}$ is a free generating system of $\pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\},\overrightarrow{01}\right)$. Fix a smooth path

$$\gamma\in\pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right). \tag{2.1}$$

Moreover, we denote by

$$\delta\in\pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},\overrightarrow{10}\right)$$

the straight path on $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ along the real interval as FIGURE 1 shows. Let $\phi\in\mathrm{Aut}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}\right)$ be the automorphism of $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$ defined by $\phi(t)=1-t$. Then, we shall define

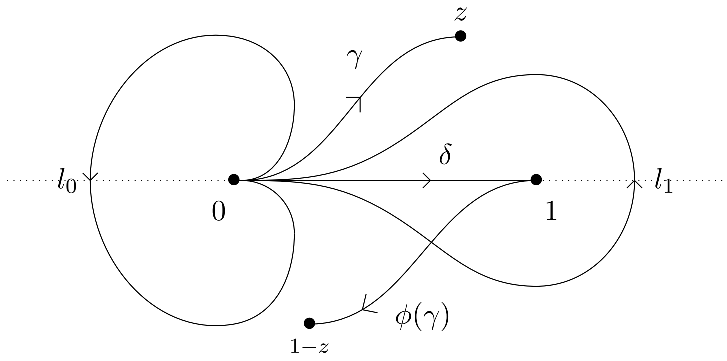
$$\gamma':=\delta\cdot\phi(\gamma)\in\pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},1-z\right). \tag{2.2}$$

2.1. Complex multiple polylogarithms.

Let z be a $\mathbb{C}$-rational (possibly, tangential base) point on $\mathbb{P}^1\backslash\{0,1,\infty\}$. For a pair $\mathbf{k}=(k_1\dots,k_d)\in\mathbb{N}^d$ and a fixed path $\gamma(=\gamma_z)\in\pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right)$, we shall define the complex logarithm

$$\log(z;\gamma):=\int_{\delta^{-1}\cdot\gamma}\frac{dt}{t} \tag{2.3}$$

FIGURE 1. Topological paths on $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$



and the complex multiple polylogarithm $Li_{\mathbf{k}}(z;\gamma)$ as the iterated integral along γ below:

$$Li_{\mathbf{k}}(z;\gamma) := \begin{cases} \int_\gamma \frac{1}{t} Li_{k_1,\cdots,k_d-1}(t;\gamma_t)\,dt & (k_d \neq 1), \\ \int_\gamma \frac{1}{1-t} Li_{k_1,\cdots,k_{d-1}}(t;\gamma_t)\,dt & (k_d = 1), \end{cases} \tag{2.4}$$

$$Li_1(z;\gamma) := -\log(1-z;\gamma') = \int_\gamma \frac{dt}{1-t}, \tag{2.5}$$

which can be analytically continued to the pointed universal covering space of $\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}, \overrightarrow{01}\right)$. In particular, we define the multiple zeta value

$$\zeta(\mathbf{k}) := Li_{\mathbf{k}}\left(\overrightarrow{10};\delta\right) \in \mathbb{R}. \tag{2.6}$$

The complex multiple polylogarithm $Li_{\mathbf{k}}(z;\gamma)$ is closely related to the KZ (Knizhnik-Zamolodchikov) equation. The KZ equation on $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$ is the differential equation

$$\frac{d}{dz}G(X,Y)(z) = \left(\frac{X}{z} + \frac{Y}{z-1}\right)G(X,Y)(z)$$

where $G(X,Y)(z)$ is an analytic (i.e. each of whose coefficient is analytic) function with values in the non-commutative formal power series algebra $\mathbb{C}\langle\langle X,Y\rangle\rangle$. There exists a unique solution $G_0(X,Y)(z;\gamma) \in \mathbb{C}\langle\langle X,Y\rangle\rangle$ attached to $\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right)$ characterized by the asymptotic behavior

$$G_0(X,Y)(z;\gamma) \approx \sum_{m=0}^{\infty} \frac{1}{m!}(X\cdot\log(z;\gamma))^m \ (z\to 0). \tag{2.7}$$

Moreover, we define the Drinfeld associator

$$\Phi(X,Y) := G_0(X,Y)\left(\overrightarrow{10};\delta\right) \in \mathbb{C}\langle\langle X,Y\rangle\rangle.$$

Then the following chain rule holds (cf. [D90],[F04],[F14]):

$$G_0(X,Y)(z;\gamma) = G_0(Y,X)(1-z;\gamma')\cdot\Phi(X,Y). \tag{2.8}$$

This algebraic relation reflects the path composition (2.2). Let M be the non-commutative free monoid generated by the non-commuting indeterminates X, Y. Since $G_0(X,Y)(z;\gamma)$ is group-like in $\mathbb{C}\langle\langle X,Y\rangle\rangle$, the expansion of $G_0(X,Y)(z;\gamma)$ looks like

$$G_0(X,Y)(z;\gamma) = 1 + \sum_{w\in \mathrm{M}\setminus\{1\}} \mathtt{Coeff}_w(G_0(X,Y)(z;\gamma))\cdot w \tag{2.9}$$

where $\{\mathtt{Coeff}_w(G_0(X,Y)(z;\gamma))\}_{w\in\mathrm{M}}$ is a family of complex numbers. For $w(\mathbf{k}) := X^{k_d-1}Y\cdots X^{k_1-1}Y \in \mathrm{M}$ and $j\in\mathbb{N}$, the following equalities hold (cf. [F04],[F14],[LM96]):

$$\mathtt{Coeff}_{w(\mathbf{k})}(G_0(X,Y)(z;\gamma)) = (-1)^d\cdot Li_{\mathbf{k}}(z;\gamma), \tag{2.10}$$

$$\mathtt{Coeff}_{X^j}(G_0(X,Y)(z;\gamma)) = \frac{\log^j(z;\gamma)}{j!}, \tag{2.11}$$

$$\mathtt{Coeff}_{X^j}(G_0(X,Y)(1-z;\gamma')) = \frac{\log^j(1-z;\gamma')}{j!}. \tag{2.12}$$

2.2. **ℓ-adic Galois multiple polylogarithms.** Let ℓ be a prime number and K a subfield of $\mathbb{C}$ with the algebraic closure $\overline{K}\subset\mathbb{C}$. Suppose that z is a K-rational (possibly, tangential base) point on $\mathbb{P}^1\setminus\{0,1,\infty\}$. Then the ℓ-adic Galois (multiple) polylogarithm introdecud by Zdzisław Wojtkowiak in his series of papers [W0]-[W3] is defined as follows.

We shall write

$$\pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_{\overline{K}}\setminus\{0,1,\infty\};\overrightarrow{01},z\right)$$

for the pro-ℓ-finite set of pro-ℓ étale paths on $\mathbb{P}^1_{\overline{K}}\setminus\{0,1,\infty\}$ from the K-rational tangential base point $\overrightarrow{01}$ to z, and

$$\pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_{\overline{K}}\setminus\{0,1,\infty\},\overrightarrow{01}\right) := \pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_{\overline{K}}\setminus\{0,1,\infty\};\overrightarrow{01},\overrightarrow{01}\right)$$

for the pro-ℓ étale fundamental group of $\mathbb{P}^1_{\overline{K}}\setminus\{0,1,\infty\}$ with the base point $\overrightarrow{01}$. By the canonical comparison map

$$\pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\};\overrightarrow{01},*\right) \to \pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_{\overline{K}}\setminus\{0,1,\infty\};\overrightarrow{01},*\right) \tag{2.13}$$

for $* \in \left\{\overrightarrow{01}, z, 1-z\right\}$, regard topological paths $l_0, l_1, \gamma, \gamma'$ on $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$ as pro-ℓ étale paths on $\mathbb{P}^1_{\overline{K}}\backslash\{0,1,\infty\}$. Then $\pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_{\overline{K}}\backslash\{0,1,\infty\}, \overrightarrow{01}\right)$ is the free pro-ℓ group of rank 2 with topologically generating system $\{l_0, l_1\}$.

We focus on the natural action of the absolute Galois group

$$G_K := \mathrm{Gal}(\overline{K}/K)$$

on $\pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_{\overline{K}}\backslash\{0,1,\infty\}; \overrightarrow{01}, z\right)$ (cf. [NW99, (1.1)]). Since z is K-rational, this Galois action is well-defined. For each $\sigma \in G_K$, we define a pro-ℓ étale loop

$$\mathfrak{f}_\sigma^{z,\gamma} := \gamma \cdot \sigma(\gamma)^{-1} \in \pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_{\overline{K}}\backslash\{0,1,\infty\}, \overrightarrow{01}\right). \tag{2.14}$$

Consider the multiplicative Magnus embedding into the algebra of non-commutative formal power series

$$E : \pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_{\overline{K}}\backslash\{0,1,\infty\}, \overrightarrow{01}\right) \hookrightarrow \mathbb{Q}_\ell\langle\langle X, Y\rangle\rangle$$

defined by $E(l_0) = \exp(X) := \sum_{n=0}^{\infty} \frac{1}{n!}X^n$, $E(l_1) = \exp(Y)$. We get a formal power series

$$\mathfrak{f}_\sigma^{z,\gamma}(X,Y) := E(\mathfrak{f}_\sigma^{z,\gamma}) \in \mathbb{Q}_\ell\langle\langle X, Y\rangle\rangle \tag{2.15}$$

called the ℓ-adic Galois associator associated to γ. If $z = \overrightarrow{10}$ and $\gamma = \delta$, it is called the ℓ-adic Ihara associator in [F07, Definition 2.32]. By (2.2) and (2.14), the following algebraic relation (chain rule) holds:

$$\mathfrak{f}_\sigma^{z,\gamma}(X,Y) = \mathfrak{f}_\sigma^{1-z,\gamma'}(Y,X) \cdot \mathfrak{f}_\sigma^{\overrightarrow{10},\delta}(X,Y). \tag{2.16}$$

The power series $\mathfrak{f}_\sigma^{z,\gamma}(X,Y)$ is an ℓ-adic Galois analog of the KZ fundamental solution $G_0(X,Y)(z;\gamma)$ in (2.7), and the relation (2.16) is an ℓ-adic Galois analog of the chain rule (2.8) of KZ fundamental solutions. Since $\mathfrak{f}_\sigma^{z,\gamma}(X,Y)$ is group-like in $\mathbb{Q}_\ell\langle\langle X, Y\rangle\rangle$, the expansion of $\mathfrak{f}_\sigma^{z,\gamma}(X,Y)$ looks like

$$\mathfrak{f}_\sigma^{z,\gamma}(X,Y) = 1 + \sum_{w \in \mathrm{M}\backslash\{1\}} \mathtt{Coeff}_w\left(\mathfrak{f}_\sigma^{z,\gamma}(X,Y)\right) \cdot w, \tag{2.17}$$

where $\{\mathtt{Coeff}_w(\mathfrak{f}_\sigma^{z,\gamma}(X,Y))\}_{w\in\mathrm{M}}$ is a family of ℓ-adic numbers. For $\mathbf{k} = (k_1 \ldots, k_d) \in \mathbb{N}^d$ and $w(\mathbf{k}) := X^{k_d-1}Y \cdots X^{k_1-1}Y$, we shall define the ℓ-adic Galois multiple polylogarithm and the ℓ-adic Galois multiple zeta value

$$Li_{\mathbf{k}}^{\ell}(z;\gamma,\sigma) := (-1)^d \cdot \mathtt{Coeff}_{w(\mathbf{k})}(\mathfrak{f}_\sigma^{z,\gamma}(X,Y)), \tag{2.18}$$

$$\boldsymbol{\zeta}_{\mathbf{k}}^{\ell}(\sigma) := Li_{\mathbf{k}}^{\ell}\left(\overrightarrow{10};\delta,\sigma\right). \tag{2.19}$$

As $\boldsymbol{\zeta}_{\mathbf{k}}^{\ell}(\sigma)$ is called the ℓ-adic multiple Soulé element in [F07, Definition 2.32], $\boldsymbol{\zeta}_k^{\ell}(\sigma)$ is closely related to the Soulé character (cf. [F07, Examples 2.33], [NW99, REMARK 2]).

Let $\rho_{z,\gamma}$ (resp. $\rho_{1-z,\gamma'}$) : $G_K \to \mathbb{Z}_\ell$ be the Kummer 1-cocycle of $\{z^{1/\ell^k}\}_k$ (resp. $\{(1-z)^{1/\ell^k}\}_k$) determined by γ (resp. γ'). For $j \in \mathbb{N}$, the following holds (cf. [NW99],[NW20],[NS22]):

$$\mathtt{Coeff}_{X^j}(\mathfrak{f}_\sigma^{z,\gamma}(X,Y)) = \frac{(-\rho_{z,\gamma}(\sigma))^j}{j!}, \tag{2.20}$$

$$\mathtt{Coeff}_{X^j}(\mathfrak{f}_\sigma^{1-z,\gamma'}(X,Y)) = \frac{(-\rho_{1-z,\gamma'}(\sigma))^j}{j!}. \tag{2.21}$$

The ℓ-adic Galois multiple polylogarithm is similar to the complex multiple polylogarithm as the TABLE 1 shows.

TABLE 1. Analogy between ℓ-adic Galois side and complex side

ℓ-adic Galois side	complex side
z : K-ratinal base point on $\mathbb{P}^1\backslash\{0,1,\infty\}$	z : $\mathbb{C}$-rational base point on $\mathbb{P}^1\backslash\{0,1,\infty\}$
$\mathfrak{f}_\sigma^{z,\gamma}(X,Y) \in \mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$ $(\sigma \in G_K)$	$G_0(X,Y)(z;\gamma) \in \mathbb{C}\langle\langle X,Y\rangle\rangle$
$\mathfrak{f}_\sigma^{\overrightarrow{10},\delta}(X,Y) \in \mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$	$\Phi(X,Y) = G_0(X,Y)\left(\overrightarrow{10};\delta\right) \in \mathbb{C}\langle\langle X,Y\rangle\rangle$
$\mathfrak{f}_\sigma^{z,\gamma}(X,Y) = \mathfrak{f}_\sigma^{1-z,\gamma'}(Y,X) \cdot \mathfrak{f}_\sigma^{\overrightarrow{10},\delta}(X,Y)$	$G_0(X,Y)(z;\gamma) = G_0(Y,X)(1-z;\gamma') \cdot \Phi(X,Y)$
$Li_{\mathbf{k}}^\ell(z;\gamma,\sigma) \in \mathbb{Q}_\ell$	$Li_{\mathbf{k}}(z;\gamma) \in \mathbb{C}$
$\boldsymbol{\zeta}_{\mathbf{k}}^\ell : G_K \to \mathbb{Q}_\ell$	$\zeta(\mathbf{k}) \in \mathbb{R}$
$Li_1^\ell(z;\gamma,\sigma) = \rho_{1-z,\gamma'}(\sigma)$	$Li_1(z;\gamma) = -\log(1-z,\gamma')$

3. Proof of main results

In this section, we prove Theorem 1.1 and Theorem 1.2. We fix a topological path $\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right)$. All other symbols are the same as in the previous sections.

Proof of Theorem 1.1, Theorem 1.2. Let $n,m \in \mathbb{Z}_{\geq 2}$. The following computations are inspired by a remark given in the Appendix of Furusho's lecture note [F14, A.24] and an insight about the ℓ-adic Oi-Ueno's equation in Nakamura's Oberwolfach Report [N21].

First, we show Theorem 1.1. Since $G_0(X,Y)(z;\gamma)$ is a group-like element in $\mathbb{C}\langle\langle X,Y\rangle\rangle$, the shuffle relation holds for $\{\mathtt{Coeff}_w(G_0(X,Y)(z;\gamma))\}_{w\in\mathrm{M}}$ (cf. [Ree58]), i.e. for $w,w' \in \mathrm{M}$,

$$\mathtt{Coeff}_{w\sqcup w'} = \mathtt{Coeff}_w \cdot \mathtt{Coeff}_{w'}. \tag{3.1}$$

By the definition of the shuffle product,

(3.2)

$$X^j \sqcup X^{m-j-1}Y^{n-1} = X(X^{j-1} \sqcup X^{m-j-1}Y^{n-1}) + X(X^j \sqcup X^{m-j-2}Y^{n-1}).$$

For $w, w' \in \mathrm{M}$, we set

$$\mathtt{Coeff}_{w+w'} := \mathtt{Coeff}_w + \mathtt{Coeff}_{w'}.$$

Then, we obtain

$$\begin{aligned}
&\sum_{j=0}^{m-1} \frac{(-\log(z;\gamma))^j}{j!} Li_{\underbrace{1,\ldots,1}_{n-2 \text{ times}},m-j}(z;\gamma) \\
=&\sum_{j=0}^{m-1} (-1)^{n+j-1} \cdot \mathtt{Coeff}_{X^j}\Big(G_0(Y,X)(z;\gamma)\Big) \cdot \mathtt{Coeff}_{X^{m-j-1}Y^{n-1}}\Big(G_0(X,Y)(z;\gamma)\Big) \\
&\quad (\text{by } (2.10), (2.11)) \\
=&\sum_{j=0}^{m-1} (-1)^{n+j-1} \cdot \mathtt{Coeff}_{X^j \sqcup X^{m-j-1}Y^{n-1}}\Big(G_0(X,Y)(z;\gamma)\Big) \quad (\text{by } (3.1)) \\
=&(-1)^{n+m-2} \cdot \mathtt{Coeff}_{Y(X^{m-1} \sqcup Y^{n-2})}\Big(G_0(X,Y)(z;\gamma)\Big) \quad (\text{by } (3.2)).
\end{aligned}$$

Using (2.6), (2.10), (2.12), $\log\left(\overrightarrow{10};\delta\right) = 0$, (3.1) and (3.2), we have the following equalities by making the same computations as above:

$$\begin{aligned}
&\sum_{j=0}^{n-2} \frac{(-\log(1-z;\gamma'))^j}{j!} Li_{\underbrace{1,\ldots,1}_{m-2 \text{ times}},n-j}(1-z;\gamma') \\
&= (-1)^{n+m-3} \cdot \mathtt{Coeff}_{X(Y^{m-1} \sqcup X^{n-2})}\Big(G_0(X,Y)(1-z;\gamma')\Big),
\end{aligned}$$

and

$$\begin{aligned}
\zeta(\underbrace{1,\ldots,1}_{n-2 \text{ times}},m) &= \left(Li_{\underbrace{1,\ldots,1}_{n-2 \text{ times}},m}\left(\overrightarrow{10};\delta\right) + \sum_{j=1}^{m-1} \frac{\left(-\log\left(\overrightarrow{10};\delta\right)\right)^j}{j!} Li_{\underbrace{1,\ldots,1}_{n-2 \text{ times}},m-j}\left(\overrightarrow{10};\delta\right) \right) \\
&=(-1)^{n+m-2} \cdot \mathtt{Coeff}_{Y(X^{m-1} \sqcup Y^{n-2})}\Big(G_0(Y,X)\left(\overrightarrow{10};\delta\right)\Big).
\end{aligned}$$

Combining these equalities and the following equality

$$\begin{aligned}
&\mathtt{Coeff}_{Y(X^{m-1} \sqcup Y^{n-2})}\Big(G_0(X,Y)(z;\gamma)\Big) \\
&= \mathtt{Coeff}_{Y(X^{m-1} \sqcup Y^{n-2})}\Big(G_0(Y,X)(1-z;\gamma')\Big) \\
&\qquad + \mathtt{Coeff}_{Y(X^{m-1} \sqcup Y^{n-2})}\Big(G_0(X,Y)\left(\overrightarrow{10};\delta\right)\Big) \quad (\text{by } (2.8)) \\
&= \mathtt{Coeff}_{X(Y^{m-1} \sqcup X^{n-2})}\Big(G_0(X,Y)(1-z;\gamma')\Big)
\end{aligned}$$

$$+\,\mathtt{Coeff}_{Y(X^{m-1}\sqcup Y^{n-2})}\Big(G_0(X,Y)\left(\overrightarrow{10};\delta\right)\Big),$$

we get the desired equation (1.2). This completes the proof of Theorem 1.1.

Next, we show Theorem 1.2. Let $\sigma \in G_K$. Since $\mathfrak{f}_\sigma^{z,\gamma}(X,Y)$ is group-like in $\mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$, the shuffle relation holds for $\{\mathtt{Coeff}_w(\mathfrak{f}_\sigma^{z,\gamma}(X,Y))\}_{w\in\mathrm{M}}$. Using (2.18), (2.19), (2.20), (2.21), (3.1) and (3.2), we obtain the following equalities by making the same computations as above:

$$\begin{aligned}
&\sum_{j=0}^{m-1}\frac{(\rho_{z,\gamma}(\sigma))^j}{j!}Li^{\ell}_{\underbrace{1,\dots,1}_{n-2\ \text{times}},m-j}(z;\gamma,\sigma)\\
&= (-1)^{n+m-2}\cdot \mathtt{Coeff}_{Y(X^{m-1}\sqcup Y^{n-2})}\Big(\mathfrak{f}_\sigma^{z,\gamma}(X,Y)\Big),\\
&\sum_{j=0}^{n-2}\frac{(\rho_{1-z,\gamma'}(\sigma))^j}{j!}Li^{\ell}_{\underbrace{1,\dots,1}_{m-2\ \text{times}},n-j}(1-z;\gamma',\sigma)\\
&= (-1)^{n+m-3}\cdot \mathtt{Coeff}_{X(Y^{m-1}\sqcup X^{n-2})}\Big(\mathfrak{f}_\sigma^{1-z,\gamma'}(X,Y)\Big),\\
&\boldsymbol{\zeta}^{\ell}_{\underbrace{1,\dots,1}_{n-2\ \text{times}},m}(\sigma) = (-1)^{n+m-2}\cdot \mathtt{Coeff}_{Y(X^{m-1}\sqcup Y^{n-2})}\left(\mathfrak{f}_\sigma^{\overrightarrow{10},\delta}(X,Y))\right).
\end{aligned}$$

Combining these equalities and the following equality

$$\begin{aligned}
&\mathtt{Coeff}_{Y(X^{m-1}\sqcup Y^{n-2})}\Big(\mathfrak{f}_\sigma^{z,\gamma}(X,Y)\Big)\\
&= \mathtt{Coeff}_{Y(X^{m-1}\sqcup Y^{n-2})}\Big(\mathfrak{f}_\sigma^{1-z,\gamma'}(Y,X)\Big) + \mathtt{Coeff}_{Y(X^{m-1}\sqcup Y^{n-2})}\left(\mathfrak{f}_\sigma^{\overrightarrow{10},\delta}(X,Y)\right)\\
&\qquad \text{(by (2.16))}\\
&= \mathtt{Coeff}_{X(Y^{m-1}\sqcup X^{n-2})}\Big(\mathfrak{f}_\sigma^{1-z,\gamma'}(X,Y)\Big) + \mathtt{Coeff}_{Y(X^{m-1}\sqcup Y^{n-2})}\left(\mathfrak{f}_\sigma^{\overrightarrow{10},\delta}(X,Y)\right),
\end{aligned}$$

we get the equation (1.3). This completes the proof of Theorem 1.2. □

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