A SUBCLASS OF STRONGLY CLOSE-TO-CONVEX FUNCTIONS ASSOCIATED WITH JANOWSKI FUNCTION

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Abstract. The aim of this paper is to introduce a new subclass of strongly close-to-convex functions by subordinating to Janowski function. Certain properties such as coefficient estimates, distortion theorem, argument theorem, inclusion relations and radius of convexity are established for this class. The results obtained here will generalize various earlier known results.

1. Introduction

Let $\mathcal{A}$ be the class of functions f which are analytic in the open unit disc $E = \{z : | z | < 1\}$ and have expansion of the form

$$(1.1) \qquad f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Further, the class of functions $f \in \mathcal{A}$ which are univalent in E, is denoted by $\mathcal{S}$.

For two analytic functions f and g in E, f is said to be subordinate to g (symbolically $f \prec g$) if there exists a Schwarzian function $w(z) = \sum_{n=1}^{\infty} c_n z^n$ which satisfies the conditions $w(0) = 0$ and $|w(z)| \leq 1$, such that $f(z) = g(w(z))$. Further, if g is univalent in E, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

For $0 \leq \alpha < 1$, the classes of starlike functions and convex functions of order α are denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ respectively and defined as

$$\mathcal{S}^*(\alpha) = \left\{ f : f \in \mathcal{A}, Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in E \right\}$$

and

$$\mathcal{K}(\alpha) = \left\{ f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{f'(z)}\right) > \alpha, z \in E \right\}.$$

It is obvious that $f \in \mathcal{K}(\alpha)$ if and only if $zf' \in \mathcal{S}^*(\alpha)$. Particularly $\mathcal{S}^*(0) \equiv \mathcal{S}^*$, which is the class of starlike functions and $\mathcal{K}(0) \equiv \mathcal{K}$, the class of convex functions. For $\alpha = \frac{1}{2}$, $\mathcal{S}^*(\frac{1}{2})$ is the class of starlike functions of

order $\frac{1}{2}$.

The concept of close-to-convex functions was introduced by Kaplan [7]. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}$ of close-to-convex functions if there exists a function $g \in \mathcal{S}^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0 (z \in E).$$

A function $f \in \mathcal{A}$ is said to be starlike with respect to symmetric points if it satisfies the following condition:

$$Re\left(\frac{2zf'(z)}{f(z) - f(-z)}\right) > 0.$$

The class of starlike functions with respect to symmetric points is denoted by $\mathcal{S}_s^*$ and was introduced by Sakaguchi [13]. It is obvious that the functions in the class $\mathcal{S}_s^*$ are close-to-convex, as $\dfrac{f(z) - f(-z)}{2}$ is a starlike function [3] in E.

Following the idea of the class $\mathcal{S}_s^*$, Gao and Zhou [4] studied the class $\mathcal{K}_S$ given by:

$$\mathcal{K}_s = \left\{f : f \in \mathcal{A}, Re\left(\frac{-z^2 f'(z)}{g(z)g(-z)}\right) > 0, g \in \mathcal{S}^*\left(\frac{1}{2}\right), z \in E\right\},$$

where $\mathcal{S}^*\left(\frac{1}{2}\right)$ is the class of starlike functions of order $\frac{1}{2}$.

Further, Kowalczyk and Les-Bomba [8] extended the class $\mathcal{K}_S$ by introducing the class $\mathcal{K}_S(\gamma), (0 \leq \gamma < 1)$, which is mentioned below:

$$\mathcal{K}_s(\gamma) = \left\{f : f \in \mathcal{A}, Re\left(\frac{-z^2 f'(z)}{g(z)g(-z)}\right) > \gamma, g \in \mathcal{S}^*\left(\frac{1}{2}\right), z \in E\right\}.$$

For $\gamma = 0$, the class $\mathcal{K}_S(\gamma)$ reduces to the class $\mathcal{K}_S$.

Later on, Prajapat [10] established the class $\chi_t(\gamma)(|t| \leq 1, t \neq 0, 0 \leq \gamma < 1)$, which consists of the functions $f \in \mathcal{A}$ such that

$$Re\left[\frac{tz^2 f'(z)}{g(z)g(tz)}\right] > \gamma,$$

where $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$.
In particular $\chi_{-1}(\gamma) \equiv \mathcal{K}_S(\gamma)$ and $\chi_{-1}(0) \equiv \mathcal{K}_S$.

For $-1 \leq B < A \leq 1$, Janowski [6] introduced the class $\mathcal{P}(A, B)$, the subclass of $\mathcal{A}$ which consists of functions of the form $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ such that $p(z) \prec \dfrac{1 + Az}{1 + Bz}$. This class played very important role in defining various subclasses of analytic functios.

Using the concept of subordination, Singh et al. [14] introduced the class $\chi_t(A, B)(|t| \leq 1, t \neq 0)$ that consists of the functions $f \in \mathcal{A}$ which satisfy the condition

$$\frac{tz^2 f'(z)}{g(z)g(tz)} \prec \frac{1 + Az}{1 + Bz}, -1 \leq B < A \leq 1, z \in E,$$

where $g \in \mathcal{S}^* \left(\frac{1}{2}\right)$.

The following observations are obvious:

(i) $\chi_t(1 - 2\gamma, -1) \equiv \chi_t(\gamma)$.

(ii) $\chi_{-1}(1 - 2\gamma, -1) \equiv \mathcal{K}_S(\gamma)$.

(iii) $\chi_{-1}(1, -1) \equiv \mathcal{K}_S$.

Raina et al. [11] defined the class of strongly close-to-convex functions of order β, as below:

$$\mathcal{C}'_\beta = \left\{ f : f \in \mathcal{A}, \left| arg \left\{ \frac{zf'(z)}{g(z)} \right\} \right| < \frac{\beta\pi}{2}, g \in \mathcal{K}, 0 < \beta \leq 1, z \in E \right\},$$

or equivalently

$$\mathcal{C}'_\beta = \left\{ f : f \in \mathcal{A}, \frac{zf'(z)}{g(z)} \prec \left(\frac{1 + z}{1 - z} \right)^\beta, g \in \mathcal{K}, 0 < \beta \leq 1, z \in E \right\}.$$

In particular, $\mathcal{C}'_1 \equiv \mathcal{C}'$, the subclass of close-to-convex functions studied by Abdel-Gawad and Thomas [1].

Getting motivated by the above mentioned work, now we define a new subclass of strongly close-to-convex functions as follows:

Let $\chi_t(A, B; \beta)$, $(|t| \leq 1, t \neq 0, 0 < \beta \leq 1)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the conditions,

$$\frac{tz^2 f'(z)}{g(z)g(tz)} \prec \left(\frac{1 + Az}{1 + Bz} \right)^\beta, -1 \leq B < A \leq 1, z \in E,$$

where $g \in \mathcal{S}^* \left(\frac{1}{2}\right)$.

We have the following observations:

(i) $\chi_t(A, B; 1) \equiv \chi_t(A, B)$.

(ii) $\chi_t(1 - 2\gamma, -1; 1) \equiv \chi_t(\gamma)$.
(iii) $\chi_{-1}(1 - 2\gamma, -1; 1) \equiv \mathcal{K}_s(\gamma)$.
(iv) $\chi_{-1}(1, -1; 1) \equiv \mathcal{K}_s$.

As $f \in \chi_t(A, B; \beta)$, therefore by the definition of subordination, there exists a Scwarz function w such that

$$(1.2) \qquad \frac{tz^2 f'(z)}{g(z)g(tz)} = \left(\frac{1 + Aw(z)}{1 + Bw(z)}\right)^{\beta}.$$

In this paper, we establish the coefficient estimates, inclusion relation, distortion theorem, argument theorem and radius of convexity for the functions in the class $\chi_t(A, B; \beta)$. Some earlier known results follow as special cases.

Throughout the paper, to avoid repetition, we lay down once for all that $-1 \leq B < A \leq 1, 0 < |t| \leq 1, t \neq 0, 0 < \beta \leq 1, z \in E$.

2. PROPERTIES OF THE CLASS $\chi_t(A, B; \beta)$

2.1. **Coefficient Estimates.** To prove the results in this subsection, we must require the following lemmas:

Lemma 2.1. [12] *Let,*

$$(2.1) \qquad \left(\frac{1 + Aw(z)}{1 + Bw(z)}\right)^{\beta} = (P(z))^{\beta} = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

then

$$|p_n| \leq \beta(A - B), n \geq 1.$$

Lemma 2.2. [15] *Let $g \in S^* \left(\frac{1}{2}\right)$, then for*

$$(2.2) \qquad G(z) = \frac{g(z)g(tz)}{tz} = z + \sum_{n=2}^{\infty} d_n z^n \in S^*,$$

we have, $|d_n| \leq n$.

Theorem 2.3. *If $f \in \chi_t(A, B; \beta)$, then*

$$(2.3) \qquad |a_n| \leq 1 + \frac{\beta(n - 1)(A - B)}{2}.$$

Proof. As $f \in \chi_t(A, B; \beta)$, therefore (1.2) can be expressed as

$$(2.4) \qquad \frac{zf'(z)}{G(z)} = (P(z))^{\beta}.$$

Using (1.1),(2.1) and (2.2) in (2.4), it yields

$$(2.5) \qquad 1 + \sum_{n=2}^{\infty} n a_n z^{n-1} = \left(1 + \sum_{n=2}^{\infty} d_n z^{n-1} \right) \left(1 + \sum_{n=1}^{\infty} p_n z^n \right).$$

Comparing the coefficients of z^{n-1} on both sides of (2.5), we have

$$(2.6) \qquad n a_n = d_n + d_{n-1} p_1 + d_{n-2} p_2 + ... + d_2 p_{n-2} + p_{n-1}.$$

Using Lemma 2.1 and Lemma 2.2, it gives

$$(2.7) \qquad n|a_n| \le n + \beta(A - B)[(n - 1) + (n - 2) + ... + 2 + 1].$$

Hence from (2.7), (2.3) can be easily obtained.

$\square$

Substituting for $\beta = 1$, Theorem 2.3 gives the following result due to Singh et al. [14].

Corollary 2.1. *If $f \in \chi_t(A, B)$, then*

$$|a_n| \le 1 + \frac{(n - 1)(A - B)}{2}.$$

On putting $A = 1 - 2\gamma$, $B = -1$ and $\beta = 1$ in Theorem 2.3, the following result due to Prajapat [10] is obvious:

Corollary 2.2. *If $f \in \chi_t(\gamma)$, then*

$$|a_n| \le 1 + (n - 1)(1 - \gamma).$$

2.2. **Inclusion Relation.** The following lemma is useful in the proof of the result in this subsection:

Lemma 2.4. [11] *Let $-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1$, then*

$$\left(\frac{1 + A_1 z}{1 + B_1 z} \right)^{\beta} \prec \left(\frac{1 + A_2 z}{1 + B_2 z} \right)^{\beta}.$$

Theorem 2.5. *If $-1 \le B_2 = B_1 < A_1 \le A_2 \le 1$, then*

$$\chi_t(A_1, B_1; \beta) \subset \chi_t(A_2, B_2; \beta).$$

Proof. For $f \in \chi_t(A_1, B_1; \beta)$, we have

$$\frac{t z^2 f'(z)}{g(z) g(tz)} \prec \left(\frac{1 + A_1 z}{1 + B_1 z} \right)^{\beta}.$$

As $-1 \leq B_2 = B_1 < A_1 \leq A_2 \leq 1$, therefore by Lemma 2.4, we obtain

$$\frac{tz^2 f'(z)}{g(z)g(tz)} \prec \left(\frac{1 + A_2 z}{1 + B_2 z}\right)^{\beta},$$

which proves that $f \in \chi_t(A_2, B_2; \beta)$.

$\square$

2.3. Distortion Theorem.

Theorem 2.6. *If $f \in \chi_t(A, B; \beta)$, then for $|z| = r, 0 < r < 1$, we have*

$$(2.8) \qquad \left(\frac{1 - Ar}{1 - Br}\right)^{\beta} \cdot \frac{1}{(1 + r)^2} \leq |f'(z)| \leq \left(\frac{1 + Ar}{1 + Br}\right)^{\beta} \cdot \frac{1}{(1 - r)^2}$$

and

$$(2.9) \qquad \int_0^r \left(\frac{1 - At}{1 - Bt}\right)^{\beta} \cdot \frac{1}{(1 + t)^2} dt \leq |f(z)| \leq \int_0^r \left(\frac{1 + At}{1 + Bt}\right)^{\beta} \cdot \frac{1}{(1 - t)^2} dt.$$

Proof. (2.4) can be expressed as

$$(2.10) \qquad\qquad |f'(z)| = \frac{|G(z)|}{|z|}(P(z))^{\beta}.$$

Mehrok [9] proved that

$$\frac{1 - Ar}{1 - Br} \leq |P(z)| \leq \frac{1 + Ar}{1 + Br},$$

which implies

$$(2.11) \qquad\qquad \left(\frac{1 - Ar}{1 - Br}\right)^{\beta} \leq |P(z)|^{\beta} \leq \left(\frac{1 + Ar}{1 + Br}\right)^{\beta}.$$

By Lemma 2.2, G is a starlike function and so due to Mehrok [9], we have

$$(2.12) \qquad\qquad \frac{r}{(1 + r)^2} \leq |G(z)| \leq \frac{r}{(1 - r)^2}.$$

(2.10) together with (2.11) and (2.12) yields (2.8). On integrating (2.8) from 0 to r, (2.9) follows.

$\square$

On putting $\beta = 1$ in Theorem 2.6, the following result due to Singh et al. [14] is obvious:

Corollary 2.3. *If $f \in \chi_t(A, B)$, then*

$$\frac{1 - Ar}{(1 - Br)(1 + r)^2} \leq |f'(z)| \leq \frac{1 + Ar}{(1 + Br)(1 - r)^2}$$

and

$$\int_0^r \frac{1 - At}{(1 - Bt)(1 + t)^2} dt \leq |f(z)| \leq \int_0^r \frac{1 + At}{(1 + Bt)(1 - t)^2} dt.$$

Taking $A = 1 - 2\gamma, B = -1, \beta = 1$, Theorem 2.6 gives the following result due to Prajapat [10]:

Corollary 2.4. *If $f \in \chi_t(\gamma)$, then*

$$\frac{1 - (1 - 2\gamma)r}{(1 + r)^3} \leq |f'(z)| \leq \frac{1 + (1 - 2\gamma)r}{(1 - r)^3}$$

and

$$\int_0^r \frac{1 - (1 - 2\gamma)t}{(1 + t)^3} dt \leq |f(z)| \leq \int_0^r \frac{1 + (1 - 2\gamma)t}{(1 - t)^3} dt.$$

2.4. **Argument Theorem.**

Theorem 2.7. *If $f \in \chi_t(A, B; \beta)$, then for $|z| = r, 0 < r < 1$, we have*

$$\left| arg f'(z) \right| \leq \beta sin^{-1} \left(\frac{(A - B)r}{1 - ABr^2} \right) + 2sin^{-1}r.$$

Proof. (2.4) can be represented as

$$f'(z) = \frac{G(z)}{z} (P(z))^\beta,$$

which implies

$$(2.13) \qquad \left| arg f'(z) \right| \leq \beta |arg P(z)| + \left| arg \frac{G(z)}{z} \right|.$$

But G is a starlike function and so using the result due to Mehrok [9], we have

$$(2.14) \qquad \left| arg \frac{G(z)}{z} \right| \leq 2sin^{-1}r.$$

Mehrok [9] established that,

$$(2.15) \qquad |arg P(z)| \leq sin^{-1} \left(\frac{(A - B)r}{1 - ABr^2} \right).$$

Using (2.14) and (2.15) in (2.13), the proof is obvious.

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On putting $\beta = 1$ in Theorem 2.7, the following result due to Singh et al. [14] is obvious:

Corollary 2.5. *If $f \in \chi_t(A, B)$, then*

$$|arg f'(z)| \leq sin^{-1}\left(\frac{(A-B)r}{1-ABr^2}\right) + 2sin^{-1}r.$$

2.5. **Radius of Convexity.**

Lemma 2.8. [2] *If $P(z) = \dfrac{1 + Aw(z)}{1 + Bw(z)}, -1 \leq B < A \leq 1$, then for $|z| = r < 1$, we have*

$$Re\frac{zP'(z)}{P(z)} \geq \begin{cases} -\frac{(A-B)r}{(1-Ar)(1-Br)}, & \text{if } R_1 \leq R_2, \\ 2\frac{\sqrt{(1-B)(1-A)(1+Ar^2)(1+Br^2)}}{(A-B)(1-r^2)} \\ \quad -\frac{(1-ABr^2)}{(A-B)(1-r^2)} + \frac{(A+B)}{(A-B)}, & \text{if } R_1 \geq R_2, \end{cases}$$

where $R_1 = \sqrt{\dfrac{(1-A)(1+Ar^2)}{(1-B)(1+Br^2)}}$ and $R_2 = \dfrac{1-Ar}{1-Br}$.

Theorem 2.9. *Let $f \in \chi_t(A, B; \beta)$, then*

$$Re\frac{(zf'(z))'}{f'(z)} \geq \begin{cases} \dfrac{1-r}{1+r} - \beta\dfrac{(A-B)}{(1-Ar)(1-Br)}, & \text{if } R_1 \leq R_2, \\ \dfrac{1-r}{1+r} + \dfrac{(A+B)}{(A-B)} \\ \quad +2\dfrac{\sqrt{(1-B)(1-A)(1+Ar^2)(1+Br^2)}}{(A-B)(1-r^2)} \\ \quad -2\dfrac{(1-ABr^2)}{(A-B)(1-r^2)}, & \text{if } R_1 \geq R_2, \end{cases}$$

where R_1 and R_2 are defined in Lemma 2.8.

Proof. As $f \in \chi_t(A, B; \beta)$, we have

$$zf'(z) = G(z)(P(z))^{\beta}.$$

Differentiating it logarithmically, we get

(2.16) $$\frac{(zf'(z))'}{f'(z)} = \frac{zG'(z)}{G(z)} + \beta\frac{zP'(z)}{P(z)}.$$

But $G \in \mathcal{S}^*$, so due to [9], we have

(2.17) $$Re\left(\frac{zG'(z)}{G(z)}\right) \geq \frac{1-r}{1+r}.$$

Using (2.17) and Lemma 2.8 in (2.16), the proof of Theorem 2.9 is obvious. $\square$

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