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ON $G(A)_{\mathbb{Q}}$ OF RINGS OF FINITE REPRESENTATION TYPE

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Abstract. Let $(A, \mathfrak{m})$ be an excellent Henselian Cohen-Macaulay local ring of finite representation type. If the AR-quiver of A is known then by a result of Auslander and Reiten one can explicity compute $G(A)$ the Grothendieck group of finitely generated A-modules. If the AR-quiver is not known then in this paper we give estimates of $G(A)_{\mathbb{Q}} = G(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ when $k = A/\mathfrak{m}$ is perfect. As an application we prove that if A is an excellent equi-characteristic Henselian Gornstein local ring of positive even dimension with char $A/\mathfrak{m} \neq 2, 3, 5$ (and $A/\mathfrak{m}$ perfect) then $G(A)_{\mathbb{Q}} \cong \mathbb{Q}$.

1. introduction

Let $(A, \mathfrak{m})$ be a Henselian Noetherian local ring. Then it is well-known that the category of finitely generated A-modules satisfy the Krull-Schmidt property, i.e., every finitely generated A-module is uniquely a direct sum of indecomposable A-modules (with local endomorphism rings). Now assume that A is Cohen-Macaulay. Then we say A is of finite (Cohen-Macaulay) representation type if A has only finitely many indecomposable maximal Cohen-Macaulay A-modules upto isomorphism. To study (not necessarily commutative) Artin algebra's Auslander and Reiten introduced the theory of almost-split sequences. These are now called AR-sequences. Later Auslander and Reiten extended the theory of AR-sequences to the case of commutative Henselian isolated singularities. Good references for this topic are [10] and [7]. Let $\operatorname{CM}(A)$ denote the full subcategory of maximal Cohen-Macaulay (= MCM) A-modules.

Remark 1.1. *Note we can define Grothendieck group of any extension closed subcategory $\mathcal{S}$ of* $\operatorname{mod}(A)$ *the category of all finitely generated A-modules, we denote it by $G(\mathcal{S})$. By* [10, 13.2] *the natural map $G(\operatorname{CM}(A)) \to G(\operatorname{mod}(A))$ is an isomorphism. Throughout this section we work with $G(\operatorname{CM}(A))$ and by abuse of notation denote it by $G(A)$.*

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Let A be a Henselian Cohen-Macaulay local of finite representation type. Set $\mathcal{I}_A$ to be the set of all indecomposable MCM A-modules upto isomorphism. If W is a subset of $\mathcal{I}_A$ then let $\operatorname{add}(W)$ be the set consisting of finite direct sums of elements of W. Also let $\mathcal{AR}(A)$ denote the set of all AR-sequences in A upto isomorphism. Let $F(A)$ be the free abelian group generated on $\operatorname{add}(\mathcal{I}_A)$. Let $\mathcal{AR}_0(A)$ be the subgroup of $F(A)$ generated by

$$\{X_1 - X_2 + X_3 \mid \text{ there is a sequence } 0 \to X_1 \to X_2 \to X_3 \to 0 \text{ in } \mathcal{AR}(A)\}.$$

By a result due to Auslander-Reiten [2, 2.2] (also see [10, 13.7]) we have $G(A) = F(A)/\mathcal{AR}_0(A)$.

Computing AR-sequences is usually a tedious task and usually we assume A is equi-characteristic, complete with algebraically closed residue field. The main objective of this paper is to give estimates of rank of $G(A)$ when the residue field is perfect but not necessarily algebraically closed.

In our introduction let us assume $(A, \mathfrak{m})$ is an excellent Henselian Cohen-Macaulay local of finite representation type and containing a field isomorphic to $k = A/\mathfrak{m}$ which we also denote by k. (In our proofs we will deal with a more general case). We assume k is perfect. Let $\overline{k}$ be the algebraic closure of k. Let

$$\mathcal{C}_k = \{E \mid E \text{ is a finite extension of } \ k, \text{ and } E \subseteq \overline{k}\}.$$

For each E in $\mathcal{C}_k$ set $A^E = A \otimes_k E$. We have an obvious directed system of rings $\{A^E\}_{E\in\mathcal{C}_k}$. Set $T = \lim_{E\in\mathcal{C}_k} A^E = A \otimes_k \overline{k}$. Then A^E, T are excellent Henselian Cohen-Macaulay local of finite representation type. So $\widehat{T}$ the completion of T is Cohen-Macaulay of finite representation type. It is not difficult to show $G(T) \cong G(\widehat{T})$. For $k \subseteq F \subseteq E$, where $E, F \in \mathcal{C}_k$, we have an obvious map $\eta^E_F \colon G(A^F) \to G(A^E)$ given by $M \to M \otimes_{A^F} A^E$. It is clear that we have a direct system of abelian groups $\{G(A^E)\}_{E\in\mathcal{C}_k}$. So we have an abelian group $\lim_{E\in\mathcal{C}_k} G(A^E)$ and natural maps $\eta_E \colon G(A^E) \to \lim_{E\in\mathcal{C}_k} G(A^E)$. Let $E \in \mathcal{C}_k$. As T is a flat A^E-algebra we have an obvious map $\xi_E \colon G(A^E) \to G(T)$ given by $M \to M \otimes_{A^E} T$. The maps ξ_E are compatible with η^E_F whenever $k \subseteq F \subseteq E$. So we have a natural map

$$\xi \colon \lim_{E\in\mathcal{C}_k} G(A^E) \to G(T).$$

Our main result is

Theorem 1.2. *ξ is an isomorphism.*

Theorem 1.2 does not give us any estimates on $G(A)$. If H is an abelian group then we set $H_{\mathbb{Q}} = H \otimes_{\mathbb{Z}} \mathbb{Q}$ and if $f \colon H \to L$ is a homomorphism of abelian groups then we set $f_{\mathbb{Q}}$ to be map $H_{\mathbb{Q}} \to L_{\mathbb{Q}}$ induced by f.

It is well-known that direct limits commutes with tensor products, see [8, Theorem A1, p. 270]. So we have an isomorphism

$$\xi_{\mathbb{Q}} \colon \lim_{E \in \mathcal{C}_k} G(A^E)_{\mathbb{Q}} \to G(T)_{\mathbb{Q}}.$$

Our next result is essentially an observation.

Proposition 1.3. *Let $F \in \mathcal{C}_k$ be any. Then the map*

$$(\eta_F)_{\mathbb{Q}} \colon G(A^F)_{\mathbb{Q}} \to \lim_{E \in \mathcal{C}_k} G(A^E)_{\mathbb{Q}}$$

is an injection. In particular we have an injection from $G(A)_{\mathbb{Q}}$ to $G(T)_{\mathbb{Q}}$.

Complete equi-characteristic Gorenstein local rings of finite representation type (with $\operatorname{char} A/\mathfrak{m} \neq 2, 3, 5$ and $A/\mathfrak{m}$ algebraically closed) are precisely the ADE-singularities, see [7, 9.8]. Furthermore in this case their AR-quiver is known and so their Grothendieck groups have been computed, see [10, 13.10]. As an easy consequence to our results we show that

Corollary 1.4. *Let $(A, \mathfrak{m})$ be an excellent equi-characteristic Henselian Gorenstein local ring of finite representation type. Assume $k = A/\mathfrak{m}$ is perfect* $\operatorname{char} k \neq 2, 3, 5$. *Then*

(1) *If* $\dim A$ *is positive and even then* $G(A)_{\mathbb{Q}} \cong \mathbb{Q}$.
(2) *If* $\dim A$ *is odd then* $\dim_{\mathbb{Q}} G(A)_{\mathbb{Q}} \leq 3$.

In fact in (1) we have $G(A)_{\mathbb{Q}} \cong G(\widehat{T})_{\mathbb{Q}}$. In section five we give an example which shows that in (2); $G(A)_{\mathbb{Q}}$ can be a proper subspace of $G(\widehat{T})_{\mathbb{Q}}$. The same example shows that we cannot in general compare torsion of $G(A)$ with torsion of $G(\widehat{T})$.

Remark 1.5. *Let $(R, \mathfrak{n})$ be an equicharacteristic excellent Cohen-Macaulay local ring, not necessarily Henselian. Suppose R has finite representation type (i.e., R has only finitely many indecomposable MCM modules upto isomorphism). Then R is an isolated singularity,* [4, Corollary 2]. *If R is also a homomorphic image of an excellent regular local ring then by* [5, Theorem 1.5] *the natural map $G(R) \to G(\widehat{R})$ is injective. By* [7, Corollary 10.11] *we get that $\widehat{R}$ has finite representation type and obviously it is Henselian. So our results are applicable.*

For example if R is Gorenstein and $\dim R$ *is positive and even then by Corollary 1.4 it follows that $G(\widehat{R})_{\mathbb{Q}} \cong \mathbb{Q}$. Also as R is a domain we have an obvious map $G(R) \to \mathbb{Z}$ which is surjective. It follows that $G(R)_{\mathbb{Q}} \cong \mathbb{Q}$*

We now describe in brief the contents of this paper. In section two we discuss some preliminaries on AR-sequences that we need. In section three we describe our construction. In section four we prove Theorem 1.2, Proposition 1.3 and Corollary 1.4. Finally in section five we discuss an example which shows that torsion does not behave well with our construction.

Convention: Throughout this paper all rings are commutative Noetherian and all modules (unless stated otherwise) are finitely generated.

2. Some preliminaries on Auslander-Reiten sequences

In this section we discuss some preliminaries on Auslander-Reiten (AR) sequences that we need. The reference for this section is [10, Chapter 2]. Let $(A, \mathfrak{m})$ be a Henselian Cohen-Macaulay local ring.

2.1. Let $M \in \mathrm{CM}(A)$ be indecomposable. We define a set of short exact sequences in $\mathrm{CM}(A)$ as follows:

$$\mathcal{S}(M) = \{s\colon 0 \to N_s \to E_s \to M \to 0 \mid N_s \text{ is indecomposable and } s \text{ is non-split}\}.$$

If M is non-free then $\mathcal{S}(M)$ is non-empty, [10, 2.2]. Define a partial order $>$ on $\mathcal{S}(M)$ as follows. Let $s, t \in \mathcal{S}(M)$. Then we say $s > t$ if there is $f \in \mathrm{Hom}_A(N_s, N_t)$ such that $\mathrm{Ext}^1_A(M, f)(s) = t$. This is equivalent to the existence of a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_s & \longrightarrow & E_s & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow & & \downarrow j & & \\ 0 & \longrightarrow & N_t & \longrightarrow & E_t & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where j is the identity map. We write $s \sim t$ if f is an isomorphism.

2.2. we have the following properties of $>$ on $\mathcal{S}(M)$

(1) If $s > t$ and $t > l$ then $s > l$; (obvious).
(2) If $s > t$ and $t > s$ then $s \sim t$; see [10, 2.4].
(3) If $s, t \in \mathcal{S}(M)$ then there exists $u \in \mathcal{S}(M)$ such that $s > u$ and $t > u$; see [10, 2.6].

By 2.2 it follows that if there is a minimal element in $\mathcal{S}(M)$ then it is a minimal element in $\mathcal{S}(M)$ (upto isomorphism).

Definition 2.3. An AR-sequence ending at M is the unique minimal element of $\mathcal{S}(M)$ (if it exists).

For a more concrete description of AR-sequences see [10, 2.9].

2.4. The following two results are basic. The first is [10, 3.4].

Theorem 2.5. *Let $(A, \mathfrak{m})$ be a Henselian Cohen-Macaulay local ring. Let $M \in \mathrm{CM}(A)$ be non-free and indecomposable. The following two conditions are equivalent:*

(i) *M is locally free on the punctured spectrum of A.*
(ii) *There exists an AR-sequence ending at M.*

The second result is [1] (when A is complete), [10, 4.22] (when A has a canonical module) and [4, Corollary 2] (in general).

Theorem 2.6. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring. If A is of finite representation type then A is an isolated singularity.*

3. A construction

In this section we describe a construction that is essential to us. This was constructed in [9].

3.1. Let $(A, \mathfrak{m})$ be a Henselian local ring with perfect residue field k. Let $\overline{k}$ be the algebraic closure of k. Let

$$\mathcal{C}_k = \{E \mid E \text{ is a finite extension of } \ k, \text{ and } E \subseteq \overline{k}\}.$$

Order $\mathcal{C}_k$ with the inclusion as partial order. Note that $\mathcal{C}_k$ is a directed set, for if $E, F \in \mathcal{C}_k$ then the composite field $EF \in \mathcal{C}_k$ and clearly $EF \supseteq E$ and $EF \supseteq F$. We prove

Theorem 3.2. *[[9, 4.2]](with hypotheses as in 3.1) There exists a direct system of local rings $\{(A^E, \mathfrak{m}^E) \mid E \in \mathcal{C}_k\}$ such that*

(1) *A^E is a finite flat extension with $\mathfrak{m}A^E = \mathfrak{m}^E$. Furthermore $A^E/\mathfrak{m}^E \cong E$ over k.*
(2) *A^E is Henselian.*
(3) *For any $F, E \in \mathcal{C}_k$ with $F \subseteq E$ the maps in the direct system $\theta_F^E \colon A^F \to A^E$ is flat and local with $\mathfrak{m}^F A^E = \mathfrak{m}^E$.*

The ring $T = \lim_{E \in \mathcal{C}_k} A^E$ will have nice properties.

3.3. Construction-C.1: For every $E \in \mathcal{C}_k$ we construct a ring A^E as follows. As k is perfect, E is a separable extension of k. So by primitive element theorem $E = k(\alpha_E)$ for some $\alpha_E \in E$. Let

$$p_E(X) = p_{E,\alpha_E}(X) = \operatorname{Irr}(\alpha_E, k),$$

be the unique monic minimal polynomial of α_E over k. Let $f_E(X) = f_{E,\alpha_E}(X)$ be a monic polynomial in $A[X]$ such that $\overline{f_E(X)} = p_E(X)$. Set

$$A^E = \frac{A[X]}{(f_E(X))}.$$

Our construction of course depends on choice of α_E and the choice of $f_E(X)$. We will simply fix one choice of α_E and $f_E(X)$.

Remark 3.4. *If A contains a field isomorphic to k then we can choose $A^E = A \otimes_k E$. However note that in general, even if A contains a field, it need not contain a field isomorphic to k.*

3.5. Construction-C.2: Let $k \subseteq F \subseteq E$ be a tower of fields. In [9, 4.5] we constructed a ring homomorphism $\theta_F^E \colon A^F \to A^E$ such that the following holds:

Proposition 3.6. [9, 4.6] *(with hypotheses as in 3.5)*

(i) *θ_F^E is a homomorphism of A-algebra's.*
(ii) *θ_F^E is a local map and $\mathfrak{m}^F A^E = \mathfrak{m}^E$.*
(iii) *A^E is a flat A^F-module (via θ_F^E).*
(iv) *If $k \subseteq F \subseteq E \subseteq L$ is a tower of fields then we have a commutative diagram*

$$\begin{array}{ccc} A^F & & \\ {\scriptstyle \theta_F^E}\downarrow & \searrow {\scriptstyle \theta_F^L} & \\ A^E & \xrightarrow[\theta_E^L]{} & A^L \end{array}$$

3.7. Construction-C.3: By 3.6 we have a directed system of rings $\{A^E\}_{E \in \mathcal{C}_k}$. Set

$$T = \lim_{E \in \mathcal{C}_k} A^E,$$

and let $\theta_E \colon E \to T$ be the maps such that for any $F \subseteq E$ in $\mathcal{C}_k$ we have $\theta_E \circ \theta_F^E = \theta_F$. For $F \in \mathcal{C}_k$ set

$$\mathcal{C}_F = \{E \mid E \text{ is a finite extension of } F\}.$$

Then clearly $\mathcal{C}_F$ is cofinal in $\mathcal{C}_k$. Thus we have

$$T = \lim_{E \in \mathcal{C}_F} A^E.$$

We have the following properties of T.

Theorem 3.8. *[See* [9, 4.8]*] (with hypotheses as in 3.7)*

(i) *T is a Noetherian ring.*
(ii) *T is a flat A-module.*
(iii) *T is a flat A^F-module for any $F \in \mathcal{C}_k$.*
(iv) *The map θ_E is injective for any $E \in \mathcal{C}_k$.*
(v) *By (iv) we may write $T = \bigcup_{E \in \mathcal{C}_k} A^E$. Set $\mathfrak{m}^T = \bigcup_{E \in \mathcal{C}_k} \mathfrak{m}^E$. Then $\mathfrak{m}^T$ is the unique maximal ideal of T.*
(vi) *$\mathfrak{m}T = \mathfrak{m}^T$.*
(vii) *$T/\mathfrak{m}^T \cong \overline{k}$.*
(viii) *T is a Henselian ring.*

The following result is definitely known to experts. We give a proof for the convenience of the reader. For the definition of etale map see [7, 10.2].

Lemma 3.9. *If A is excellent then*

(1) *A^E is excellent for all $E \in \mathcal{C}_k$.*
(2) *In the directed system $\{A^E\}_{E\in\mathcal{C}_k}$ each map $A^F \to A^E$ (when $F \subseteq E$) is etale.*
(3) *$T = \lim_{E\in\mathcal{C}_k} A^E$ is excellent.*

Proof. (1) We have $A^E = A[X]/(f_E(X))$. So A^E is excellent.
(2) This follows from 3.2.
(3) This follows from [3, 5.3]. □

The significance of T is that certain crucial properties descend to a finite extension E of k, see [9, 4.9].

Lemma 3.10. *(with hypotheses as above)*

(1) *Let M be a T-module. Then there exists $E \in \mathcal{C}_k$ and an A^E-module N such that $M = N \otimes_{A^E} T$.*
(2) *Let N_1, N_2 be A^E-modules for some $E \in \mathcal{C}_k$. Suppose there is a T-linear map $f\colon N_1 \otimes_{A^E} T \to N_2 \otimes_{A^E} T$. Then there exists $K \in \mathcal{C}_k$ with $K \supseteq E$ and an A^K-linear map $g\colon N_1 \otimes_{A^E} A^K \to N_2 \otimes_{A^E} A^K$ such that $f = g \otimes T$. Furthermore if f is an isomorphism then so is g.*

We now relate finite representation property of our construction.

Lemma 3.11. *Assume A is Cohen-Macaulay, excellent and of finite representation type. Then*

(1) *A^E is Cohen-Macaulay of finite representation type for each $E \in \mathcal{C}_k$.*
(2) *$T = \lim_{E\in\mathcal{C}_k} A^E$ is Cohen-Macaulay of finite representation type.*
(3) *$\widehat{T}$, the $\mathfrak{m}^T$ completion of T, is Cohen-Macaulay of finite representation type.*
(4) *If A is Gorenstein then A^E is Gorenstein for each $E \in \mathcal{C}_k$. Furthermore T, $\widehat{T}$ are Gorenstein.*

Proof. We first note that as $\mathfrak{m}A^E = \mathfrak{m}^E$. So A^E is Cohen-Macaulay, see [8, Corollary, p. 181]. Furthermore if A is Gorenstein then so is A^E, see [8, 23.4]. Similarly as $\mathfrak{m}T = \mathfrak{m}^T$ we get T is Cohen-Macaulay (and is Gorenstein if A is). So $\widehat{T}$ is also Cohen-Macaulay (and is Gorenstein if T is).

For (1), (2) recall that in the directed system $\{A^E\}_{E\in\mathcal{C}_k}$ each map $A^F \to A^E$ (when $F \subseteq E$) is etale (3.9(2). The result follows from [7, 10.8].

For (3) use 3.9 and [7, 10.10]. □

The following results on comparing AR-sequences is crucial for us.

Lemma 3.12. *Let the setup be as in Lemma 3.11. Let M^T be a non-free indecomposable MCM T-module and let $s^T\colon 0 \to N^T \to L^T \to M^T \to 0$ be an AR-sequence ending at M^T. By 3.10 there exists $E \in \mathcal{C}_k$ and MCM A^E-modules M^E, N^E, L^E such that*

(1) $M^E \otimes_{A^E} T = M^T$, $N^E \otimes_{A^E} T = N^T$ *and* $L^E \otimes_{A^E} T = L^T$.

(2) *A short exact sequence,* $s^E\colon 0 \to N^E \to L^E \to M^E \to 0$, *of A^E-modules such that* $s^E \otimes_{A^E} T = s^T$.

Then

(a) s^E *is the AR-sequence in* A^E *ending at* M^E.

(b) *If* $E \subseteq F$ *then* $s^F = s^E \otimes_{A^E} A^F$ *is the AR-sequence in* A^F *ending at* $M^F = M^E \otimes_{A^E} A^F$.

Proof. (a) As N^T, M^T are indecomposable we get N^E, M^E are indecomposable. Let β be an AR-sequence in A^E ending at M^E. Then $s^E > \beta$. So $s^T = s^E \otimes T > \beta \otimes T$. But s^T is the AR-sequence ending at M^T. So $\beta \otimes T > s^T$. Therefore $s^E \otimes T \sim \beta \otimes T$ (see 2.2(2)). As T is a faithfully flat A^E-algebra we get that $s^E \sim \beta$. The result follows.

(b) Note

$$s^F \otimes_{A^F} T = (s^E \otimes_{A^E} A^F) \otimes_{A^F} T \cong s^T.$$

The result follows from (a). □

4. Proof of our main result 1.2

In this section we prove our main result. We require several preparatory results to prove it. Throughout this section $(A, \mathfrak{m})$ is an excellent Cohen-Macaulay local ring of finite representation type with $k = A/\mathfrak{m}$ perfect. Fix an algebraic closure $\overline{k}$ of k. Let

$$\mathcal{C}_k = \{E \mid E \text{ is a finite extension of } \ k, \text{ and } E \subseteq \overline{k}\}.$$

For $E \in \mathcal{C}_k$ let A^E be as in 3.5. If $k \subseteq F \subseteq E$ let $\theta^E_F\colon A^F \to A^E$ be as in 3.5. As discussed above $\{A^E \mid E \in \mathcal{C}_k\}$ forms a direct system of rings. As before set $T = \lim_{E\in\mathcal{C}_k} A^E$. By 3.11 we get that A^E has finite representation type for each $E \in \mathcal{C}_k$. Furthermore T and $\widehat{T}$ also have finite representation type.

4.1. Construction-K.1: Let $k \subseteq F \subseteq E$. As A^E is a flat A^F-algebra we have an obvious map $\eta^E_F\colon G(A^F) \to G(A^E)$ given by $M \to M \otimes_{A^F} A^E$. After tensoring with $\mathbb{Q}$ denote this map by $(\eta^E_F)_{\mathbb{Q}}$. It is clear that we have a direct system of abelian groups $\{G(A^E)\}_{E\in\mathcal{C}_k}$. So we have an abelian group $\lim_{E\in\mathcal{C}_k} G(A^E)$ and natural maps $\eta_E\colon G(A^E) \to \lim_{E\in\mathcal{C}_k} G(A^E)$.

Next we show

Lemma 4.2. *Let $k \subseteq F \subseteq E$. Then the map* $(\eta^E_F)_{\mathbb{Q}}\colon G(A^F)_{\mathbb{Q}} \to G(A^E)_{\mathbb{Q}}$ *is an inclusion of $\mathbb{Q}$-vector spaces.*

Proof. We note that via $\theta_F^E \colon A^F \to A^E$ we get that A^E is a finite free A^F-module, say of rank r. It follows that any MCM A^E-module is also an MCM A^F-module. So we have the obvious map $\phi \colon G(A^E) \to G(A^F)$.

Set $\delta = (\phi \otimes \mathbb{Q}) \circ (\eta_F^E)_{\mathbb{Q}}$. Let M be a MCM A^F-module. Then note that $\delta([M]) = r[M]$. So δ is an isomorphism. In particular $(\eta_F^E)_{\mathbb{Q}}$ is an inclusion. □

As an immediate consequence we get Proposition 1.3, which we restate for the convenience of the reader.

Corollary 4.3. *Let $F \in \mathcal{C}_k$ be any. The map*

$$(\eta_F)_{\mathbb{Q}} \colon G(A^F)_{\mathbb{Q}} \to \lim_{E \in \mathcal{C}_k} G(A^E)_{\mathbb{Q}}$$

is injective.

Proof. See Chapter III, Exercise 19 in [6]. □

4.4. Construction-K.2: Let $E \in \mathcal{C}_k$. As T is a flat A^F-algebra we have an obvious map $\xi_E \colon G(A^E) \to G(T)$ given by $M \to M \otimes_{A^E} T$. The maps ξ_E are compatiable with η_F^E whenever $k \subseteq F \subseteq E$. So we have a natural map

$$\xi \colon \lim_{E \in \mathcal{C}_k} G(A^E) \to G(T).$$

We restate Theorem 1.2 for the convenience of the reader.

Theorem 4.5. *ξ is an isomorphism.*

The proof of Theorem 4.5 requires a few preliminaries.

4.6. Construction-K.3: We know that T is of finite representation type. Let $\mathcal{I}_T = \{M_1, \ldots, M_s\}$. By 3.10 we can choose $F \in \mathcal{C}_k$ and indecomposable MCM A^F-modules $M_1^F, \ldots, M_s^F$ with $M_i = M_i^F \otimes_{A^F} T$ for $i = 1, \ldots, s$. Set

$$\mathcal{I}'_{A^F} = \{M_1^F, \ldots, M_s^F\}.$$

Let $\mathcal{I}_{A^F}$ be the set of all indecomposable MCM A^F-modules.

Remark 4.7. $\mathcal{I}'_{A^F} = \mathcal{I}_{A^F}$
To see this first set $V = M_1^F \oplus \cdots \oplus M_s^F$. Note that if $U \in \mathcal{I}_{A^F}$ then $U \otimes_{A^F} T \in \operatorname{add}_T(V \otimes_{A^F} T)$. The extension $A^F \to T$ is flat. By [7, 2.18] *we get $U \in \operatorname{add}_{A^F} V$. By Krull-Schmidt theorem it follows that $U = M_i^F$ for some i. The result follows.*

By 3.12 we may further assume (after possibly taking a finite extension of F) that $\mathcal{AR}(A^F) \otimes_{A^F} T = \mathcal{AR}(T)$.

Consider the set

$$\mathcal{C}_F = \{E \mid E \text{ is a finite extension of } F, \text{ and } E \subseteq \overline{k}\}.$$

Then $\mathcal{C}_F$ is co-final in $\mathcal{C}_k$. Also for $E \in \mathcal{C}_F$ we may choose $\mathcal{I}_{A^E}$ with $\mathcal{I}_{A^E} = \mathcal{I}_{A^F} \otimes_{A^F} A^E$, the set of isomorphism classes of indecompsable MCM A^E-modules. Also note $\mathcal{AR}_{A_E} = \mathcal{AR}_{A_F} \otimes_{A^F} A^E$, by 3.12 .

We now give

Proof of Theorem 4.5. By a result of Auslander Reiten [2, 2.2] it follows that the natural maps $G(A^E) \to G(A^L)$ and the map $G(A^E) \to G(T)$ are isomorphisms. So the map

$$\xi' \colon \lim_{E \in \mathcal{C}_F} G(A^E) \to G(T), \quad \text{is an isomorphism.}$$

As $\mathcal{C}_F$ is co-final in $\mathcal{C}_k$ we get

$$\lim_{E \in \mathcal{C}_F} G(A^E) = \lim_{E \in \mathcal{C}_k} G(A^E) \quad \text{and } \xi' = \xi.$$

So ξ is an isomorphism. □

We now give

Proof of 1.4. By 4.2 and 4.5 we have an injection $G(A)_\mathbb{Q} \to G(T)_\mathbb{Q}$. Also $G(T) \cong G(\widehat{T})$. By [7, 10.17] $\widehat{T}$ is an ADE-singularity. The Grothendieck groups of ADE-singularities have been computed, see [10, 13.10].

(2) We have

$$\dim_\mathbb{Q} G(A)_\mathbb{Q} \leq \dim_\mathbb{Q} G(\widehat{T})_\mathbb{Q} \leq 3.$$

The result follows.

(1) We have

$$\dim_\mathbb{Q} G(A)_\mathbb{Q} \leq \dim_\mathbb{Q} G(\widehat{T})_\mathbb{Q} = 1.$$

Also as A is an isolated singularity of dimension ≥ 2 we get that A is a domain. So we have an obvious surjective map $\operatorname{rank} \colon G(Q) \to \mathbb{Z}$ which maps M to $\operatorname{rank}(M)$. It follows that $\dim_\mathbb{Q} G(A)_\mathbb{Q} \geq 1$. The result follows. □

5. An example

We now give an example which proves two things:

(1) If $\dim A$ is odd then $G(A)_\mathbb{Q}$ can be proper subspace of $G(\widehat{T})_\mathbb{Q}$.

(2) In general we cannot compare torsion subgroups of $G(A)$ and $G(\widehat{T})$

The example is $A = \mathbb{R}[[x,y]]/(x^2+y^2)$. Note $\widehat{T} = \mathbb{C}[[x,y]]/(x^2+y^2)$ is the A_1-singularity. By [10, p. 134] we get that $G(A) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. While $G(\widehat{T}) = \mathbb{Z}^2$, see [10, 13.10]. This proves both our assertions.

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