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POSITIVE SOLUTIONS TO A NONLINEAR THREE-POINT BOUNDARY VALUE PROBLEM WITH SINGULARITY

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Abstract. In this paper, we discuss the existence and uniqueness of positive solutions to a singular boundary value problem of fractional differential equations with three-point integral boundary conditions. The nonlinear term f possesses singularity and also depends on the first-order derivative u'. Our approach is based on Leray-Schauder fixed point theorem and Banach contraction principle. Examples are presented to confirm the application of the main results.

1. Introduction

In this paper, we discuss the existence and uniqueness of positive solutions to the following singular nonlinear fractional differential equation:

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t), u'(t)) = 0, \ \ t \in (0,1), \\ \\ u(0) = u'(0) = 0, \ \beta u(1) + D^{r}u(\eta) = \displaystyle\int_0^1 u(s)ds, \end{cases} \tag{1.1}$$

where $2 < \alpha \leq 3,\ 0 < r < 1,\ \beta, \eta \in (0,1)$, D^{α} and D^{r} are standard Riemann-Louville fractional derivatives, $f \in C\left((0,1] \times I\!R^{+} \times I\!R^{+},\ I\!R^{+}\right)$ and $u \in C^{1}\left([0,1],\ I\!R^{+}\right)$.

Many authors have dealt with singular boundary value problems(BVPs for short) of fractional differential equation in recent times, see [1], [4], [5], [6], [9], [13], [15], [17], [18], [19], [20], [22], [23] and the references cited therein.

The author in [11], by using the Leray-Schauder continuation principle in a cone, obtained the existence of positive solutions to the following singular boundary value problem of nonlinear fractional differential equation:

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t), u'(t)) = 0, \ \ t \in (0,1), \\ \\ u(0) = u(1) = 0, \end{cases} \tag{1.2}$$

where $0 < \alpha \leq 2$, D^{α} is the Riemann-Louville fractional derivative and $f : (0,1] \times [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ is singular with respect to the time variable.

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In [10], the authors discussed the existence and uniqueness of positive solutions to the following three-point boundary value problem:

$$\begin{cases} D^{\alpha}u(t)+f(t,u(t),u'(t))=0, \quad 0<\alpha\leq 2, \\ u(0)=0, \;\; D^{p}u(1)=\delta D^{p}u(\eta), \quad 0<p<1, \end{cases} \tag{1.3}$$

where $0<\delta<p<1$, $0<\eta\leq 1$, D^{α} and D^{p} are the standard Caputo fractional derivatives.

Guezane-Lakoud[8] investigated the existence of positive solutions to the following initial value problem of fractional order:

$$\begin{cases} D^{q}u(t)=f(t,u(t),u'(t)), \quad 0<t<1, \\ u(0)=u'(0)=u''(0)=0, \end{cases} \tag{1.4}$$

where $2<q\leq 3$, D^{q} is the standard Riemann-Louville fractional derivative and $f:[0,1]\times I\!R\times I\!R\longrightarrow I\!R$ is a given continuous function.

Moreover, Z. Bai[3] discussed the existence and uniqueness of positive solutions to the following three-point boundary value problem:

$$\begin{cases} D^{\alpha}u(t)+f(t,u(t))=0, \quad 0<t<1, \\ u(0)=0, \;\; \beta u(\eta)=u(1), \quad 0<\alpha\leq 2, \end{cases} \tag{1.5}$$

where $0<\eta\leq 1$, D^{α} is the standard Riemann-Louville fractional derivative and the function f is continuous on $[0,1]\times[0,\infty)$.

Inspired by the works in [11], [10], [8] and [3], the aim of this paper is to establish the existence and uniqueness of positive solutions to the BVP(1.1). Here, the boundary condition is of integral type involving the fractional derivative $D^{r}u(t)$ of the unknown function. The nonlinear term f possesses singularity at $t=0$, i.e, $\lim\limits_{t\to 0^{+}} f(t,u,u')=+\infty$. This paper improves the works of the authors in [3], [8], [10] and [11]. In the papers [3] and [10], the issues of singularity and integral boundary conditions were not considered while the authors in [8] and [11] did not treat integral boundary conditions. To the best of our knowledge, no work has been done on the existence and uniqueness of positive solutions to the singular BVP(1.1) in the literature. Our approach is based on the application of Leray-Schauder fixed-point theorem in a cone and Banach contraction principle.

Throughout this work, we assume the following conditions hold:

- $\mathbf{C_1}$. $f:(0,1]\times[0,\infty]\times[0,\infty)\longrightarrow[0,\infty)$ is continuous.
- $\mathbf{C_2}$. There exists a constant $q\in(0,1)$ such that $t^{q}f(t,u,p)$ is continuous on $[0,1]\times[0,\infty)\times[0,\infty)$, $\quad p=u'(t)\in C\left([0,1],I\!R^{+}\right)$, $u\in C^{1}\left([0,1],\ I\!R^{+}\right)$.

In the rest of the paper, we recall some basic definitions and some known results in Section 2. The existence and uniqueness results are established in Section 3. Finally, we present two examples in Section 4 to demonstrate the practicability of the main results.

2. Preliminary Results

In this section, we recall some basic definitions and results. Further, we obtain the expression of the kernel $G(t,s)$ associated with the BVP(1.1).

Definition 2.1(see [2], [3]) - The Riemann-Liouville fractional integral of order $\alpha > 0$ for a given continuous function $f : (0,\infty) \longrightarrow I\!R$ is defined to be

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

provided the right side is pointwise defined on $(0,\infty)$.

Definition 2.2(see [2], [3]) - The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a given continuous function $f : (0,\infty) \longrightarrow I\!R$ is defined to be

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds,$$

$n-1 < \alpha \leq n$, provided the right side is pointwise defined on $(0,\infty)$, where $n = [\alpha]+1$ and $[\alpha]$ is the integer part of the number α.

Remark 2.3(see [2], [10]) - If $\alpha > 0$ and $u \in C[0,1] \cap L^1[0,1]$, then the following relation

$$D_{0^+}^{\alpha} I_{0^+}^{\alpha} u(t) = u(t)$$

holds almost everywhere on $[0,1]$ and it is valid at any point $t \in [0,1]$.

Lemma 2.4(see [2]) - *Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L^1(0,1)$, then the fractional differential equation $D^{\alpha}u(t) = 0$ has $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, for $c_i \in I\!R$, $i = 1,2,\ldots,n$, as a unique solution, where n is the smallest integer greater than or equal to α.*

Lemma 2.5(see [2], [3]) - *Assume that $u \in C(0,1) \cap L^1(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1(0,1)$. Then*

$$\left\{ \begin{array}{l} I^{\alpha} D^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \\ \\ \qquad \text{for } c_i \in I\!R, \quad i = 1,2,\ldots,n, \quad n \geq \alpha. \end{array} \right. \tag{2.1}$$

Lemma 2.6(see [12]) - *Assume that $h(t) \in L^1[0,1]$ and α, ν are two constants such that $\alpha > 1 \geq \nu \geq 0$. Then*

$$D_{0^+}^{\nu} \int_0^t (t-s)^{\alpha-1} h(s)ds = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\nu)} \int_0^t (t-s)^{\alpha-\nu-1} h(s)ds.$$

Lemma 2.7 - *Let $2 < \alpha \leq 3, \quad 0 < r < 1$ and $\alpha\sigma > 1$. Assume that $\sigma, \ \delta_0, \ \delta_1, \ \delta_2$ are positive real numbers. If $\phi_o \in L^1[0,1]$ is a given function, then the unique solution of the BVP*

$$\begin{cases} D^\alpha u(t) + \phi_o(t) = 0, \quad 0 < t < 1, \\ \\ u(0) = u'(0) = 0, \ \ \beta u(1) + D^r u(\eta) = \displaystyle\int_0^1 u(s)ds, \end{cases} \tag{2.2}$$

is given by

$$u(t) = \int_0^1 G(t,s)\phi_o(s)ds, \tag{2.3}$$

where

$$G(t,s) = \begin{cases} \dfrac{\delta_2 t^{\alpha-1}(\eta - s)^{\alpha-r-1} + \delta_1[t(1-s)]^{\alpha-1} - \delta_0(t-s)^{\alpha-1}}{\sigma(\alpha\sigma - 1)\Gamma(\alpha)}, \\ \hfill s \leq t, \ s \leq \eta, \\ \\ \dfrac{\delta_1[t(1-s)]^{\alpha-1} - \delta_0(t-s)^{\alpha-1}}{\sigma(\alpha\sigma - 1)\Gamma(\alpha)}, \qquad 0 < \eta \leq s \leq t \leq 1, \\ \\ \dfrac{\delta_1[t(1-s)]^{\alpha-1} + \delta_2 t^{\alpha-1}(\eta - s)^{\alpha-r-1}}{\sigma(\alpha\sigma - 1)\Gamma(\alpha)}, \ 0 \leq t \leq s \leq \eta < 1, \\ \\ \dfrac{\delta_1[t(1-s)]^{\alpha-1}}{\sigma(\alpha\sigma - 1)\Gamma(\alpha)}, \qquad 0 \leq t \leq s \leq 1, \ \eta \leq s. \end{cases} \tag{2.4}$$

Proof. By Lemma 2.5, the BVP(2.2) can be reduced to an equivalent integral equation

$$\begin{aligned} u(t) &= -I^\alpha \phi_o(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\phi_o(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}. \end{aligned}$$

Using boundary condition $u(0) = u'(0) = 0$ with $\alpha \leq 3$, we have $c_2 = c_3 = 0$.

$$\implies u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\phi_o(s)ds + c_1 t^{\alpha-1}.$$

$$u(1) = -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}\phi_o(s)ds + c_1. \tag{2.5}$$

$$\beta u(1) = -\frac{\beta}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}\phi_o(s)ds + \beta c_1.$$

By Lemma 2.6, we obtain

$$D^r u(t) = -\frac{1}{\Gamma(\alpha - r)} \int_0^t (t-s)^{\alpha-r-1}\phi_o(s)ds + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-r)} t^{\alpha-r-1}.$$

$$D^r u(\eta) = -\frac{1}{\Gamma(\alpha - r)} \int_0^\eta (\eta-s)^{\alpha-r-1}\phi_o(s)ds + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-r)} \eta^{\alpha-r-1}.$$

Using boundary condition $\beta u(1) + D^r u(\eta) = \displaystyle\int_0^1 u(s)ds$ and setting $\left(\beta + \frac{\Gamma(\alpha)}{\Gamma(\alpha-r)}\eta^{\alpha-r-1}\right) = \sigma > 0$, we have

$$c_1 = \frac{1}{\sigma}\int_0^1 u(s)ds + \frac{\beta}{\sigma\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_o(s)ds + \frac{1}{\sigma\Gamma(\alpha-r)}\int_0^\eta (\eta-s)^{\alpha-r-1}\phi_o(s)ds. \tag{2.6}$$

Substituting (2.6) into (2.5) gives

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_o(s)ds + \frac{t^{\alpha-1}}{\sigma}\int_0^1 u(s)ds + \frac{\beta t^{\alpha-1}}{\sigma\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_o(s)ds + \frac{t^{\alpha-1}}{\sigma\Gamma(\alpha-r)}\int_0^\eta (\eta-s)^{\alpha-r-1}\phi_o(s)ds. \tag{2.7}$$

Integrating both sides of (2.7) with respect to t from 0 to 1 gives

$$\int_0^1 u(t)dt = -\frac{1}{\Gamma(\alpha)}\int_0^1\int_0^t (t-s)^{\alpha-1}\phi_o(s)dsdt + \frac{1}{\alpha\sigma}\int_0^1 u(s)ds + \frac{\beta}{\alpha\sigma\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_o(s)ds + \frac{1}{\alpha\sigma\Gamma(\alpha-r)}\int_0^\eta (\eta-s)^{\alpha-r-1}\phi_o(s)ds.$$

Setting $a_0 = \displaystyle\int_0^1 u(t)dt$ and then solving for a_0 gives

$$a_0 = -\frac{\alpha\sigma}{(\alpha\sigma-1)\Gamma(\alpha)}\int_0^1\int_0^t (t-s)^{\alpha-1}\phi_o(s)dsdt + \frac{\beta}{(\alpha\sigma-1)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_o(s)ds + \frac{1}{(\alpha\sigma-1)\Gamma(\alpha-r)}\int_0^\eta (\eta-s)^{\alpha-r-1}\phi_o(s)ds. \tag{2.8}$$

Substituting (2.8) into (2.7) gives

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_o(s)ds - \frac{1}{(\alpha\sigma-1)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_o(s)ds$$

$$+\frac{\beta}{\sigma(\alpha\sigma-1)\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}\phi_o(s)ds$$

$$+\frac{1}{\sigma(\alpha\sigma-1)\Gamma(\alpha-r)}\int_0^\eta t^{\alpha-1}(\eta-s)^{\alpha-r-1}\phi_o(s)ds$$

$$+\frac{\beta}{\sigma\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}\phi_o(s)ds+\frac{1}{\sigma\Gamma(\alpha-r)}\int_0^\eta t^{\alpha-1}(\eta-s)^{\alpha-r-1}\phi_o(s)ds.$$

If we set $\dfrac{\Gamma(\alpha)}{\Gamma(\alpha-r)}=c_0,\ \ \alpha\sigma^2=\delta_0,\ \ \alpha\beta\sigma=\delta_1,\ \ \alpha\sigma c_0=\delta_2$ and then simplifying, we have

$$u(t)=-\frac{\delta_0}{\sigma(\alpha\sigma-1)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_o(s)ds \tag{2.9}$$

$$+\frac{\delta_1}{\sigma(\alpha\sigma-1)\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}\phi_o(s)ds$$

$$+\frac{\delta_2}{\sigma(\alpha\sigma-1)\Gamma(\alpha)}\int_0^\eta t^{\alpha-1}(\eta-s)^{\alpha-r-1}\phi_o(s)ds.$$

$$\Longrightarrow u(t)=\int_0^1 G(t,s)\phi_o(s)ds.$$

For $t\le\eta$, we have

$$u(t)=-\frac{\delta_0}{\sigma(\alpha\sigma-1)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_o(s)ds$$

$$+\frac{\delta_1}{\sigma(\alpha\sigma-1)\Gamma(\alpha)}\left(\int_0^t+\int_t^\eta+\int_\eta^1\right)t^{\alpha-1}(1-s)^{\alpha-1}\phi_o(s)ds$$

$$+\frac{\delta_2}{\sigma(\alpha\sigma-1)\Gamma(\alpha)}\left(\int_0^t+\int_t^\eta\right)t^{\alpha-1}(\eta-s)^{\alpha-r-1}\phi_o(s)ds.$$

$$=\frac{1}{\sigma(\alpha\sigma-1)\Gamma(\alpha)}\int_0^t\left[\delta_2 t^{\alpha-1}(\eta-s)^{\alpha-r-1}+\delta_1 t^{\alpha-1}(1-s)^{\alpha-1}-\delta_0(t-s)^{\alpha-1}\right]\phi_o(s)ds$$

$$+\frac{1}{\sigma(\alpha\sigma-1)\Gamma(\alpha)}\int_t^\eta\left[\delta_1 t^{\alpha-1}(1-s)^{\alpha-1}+\delta_2 t^{\alpha-1}(\eta-s)^{\alpha-r-1}\right]\phi_o(s)ds$$

$$+\frac{1}{\sigma(\alpha\sigma-1)\Gamma(\alpha)}\int_\eta^1 \delta_1 t^{\alpha-1}(1-s)^{\alpha-1}\phi_o(s)ds.$$

$\therefore\ u(t)=\displaystyle\int_0^1 G(t,s)\phi_o(s)ds.$

Similarly, for $t\ge\eta$, we have

$$u(t) = \frac{1}{\sigma(\alpha\sigma-1)\Gamma(\alpha)} \int_0^\eta \left[\delta_2 t^{\alpha-1}(\eta-s)^{\alpha-r-1} + \delta_1 t^{\alpha-1}(1-s)^{\alpha-1} \right.$$

$$\left. -\delta_0(t-s)^{\alpha-1}\right] \phi_o(s)ds$$

$$+ \frac{1}{\sigma(\alpha\sigma-1)\Gamma(\alpha)} \int_\eta^t \left[\delta_1 t^{\alpha-1}(1-s)^{\alpha-1} - \delta_0(t-s)^{\alpha-1}\right] \phi_o(s)ds$$

$$+ \frac{1}{\sigma(\alpha\sigma-1)\Gamma(\alpha)} \int_t^1 \delta_1 t^{\alpha-1}(1-s)^{\alpha-1}\phi_o(s)ds.$$

Hence, $u(t) = \displaystyle\int_0^1 G(t,s)\phi_o(s)ds$, where $G(t,s)$ is defined by (2.4). □

By Lemma 2.7, the solution $u(t)$ of the BVP(1.1) is represented by

$$u(t) = \int_0^1 G(t,s)f(s,u,p)ds, \quad p = u'(t). \tag{2.10}$$

Let $\mathcal{B}^* = \{u(t) \in C[0,1] : u'(t) \in C[0,1]\}$ be a Banach space equipped with the norm

$$\|u\| = \max_{0\le t\le 1}|u(t)| + \max_{0\le t\le 1}|u'(t)| \quad \text{(see [16])}$$

and $\mathcal{K}_o \subset \mathcal{B}^*$ be a cone defined by

$$\mathcal{K}_o = \left\{u \in \mathcal{B}^* : u(t) \ge 0, \ \ |u'(t)| \ge 0\right\}.$$

Define an integral operator $\mathcal{A} : \mathcal{K}_o \longrightarrow \mathcal{B}^*$ by

$$\mathcal{A}u(t) = \int_0^1 G(t,s)f(s,u,p)ds, \ \ u \in \mathcal{K}_o. \tag{2.11}$$

For convenience, we set

$$N = \sigma(\alpha\sigma-1)\Gamma(\alpha), \quad R_1 = \frac{(\alpha-1)\mathcal{M}_o\delta_1}{N}\frac{\Gamma(\alpha)\Gamma(1-q)}{\Gamma(\alpha-q+1)},$$

$$R_2 = \frac{(\alpha-1)\mathcal{M}_o\delta_2}{N}\eta^{(\alpha-r-q)}\frac{\Gamma(\alpha-r)\Gamma(1-q)}{\Gamma(\alpha-r-q+1)},$$

$$R_3 = \frac{(\alpha-1)\mathcal{M}_o\delta_0}{N}\frac{\Gamma(\alpha-1)\Gamma(1-q)}{\Gamma(\alpha-q)}.$$

Lemma 2.8 - *Let* $2 < \alpha \leq 3, \quad 0 < q < 1, \quad \dfrac{\Gamma(\alpha)}{\Gamma(\alpha - r)} = c_0, \quad \alpha\sigma^2 = \delta_0,$ $\alpha\beta\sigma = \delta_1$ *and* $\alpha\sigma c_0 = \delta_2$. *Then*

$$(i) \qquad \max_{0\leq t\leq 1} |\int_0^1 G(t,s)s^{-q}ds| = \frac{1}{N}| \left[-\delta_0 \frac{\Gamma(\alpha)\Gamma(1-q)}{\Gamma(\alpha-q+1)} + \delta_1 \frac{\Gamma(\alpha)\Gamma(1-q)}{\Gamma(\alpha-q+1)} \right.$$

$$\left. + \, \delta_2 \eta^{(\alpha-r-q)} \frac{\Gamma(\alpha-r)\Gamma(1-q)}{\Gamma(\alpha-r-q+1)} \right] |.$$

$$(ii) \qquad \max_{0\leq t\leq 1} |\int_0^1 \frac{\partial}{\partial t} G(t,s)s^{-q}ds| = \frac{(\alpha-1)}{N}| \left[-\delta_0 \frac{\Gamma(\alpha-1)\Gamma(1-q)}{\Gamma(\alpha-q)} \right.$$

$$+ \, \delta_1 \frac{\Gamma(\alpha)\Gamma(1-q)}{\Gamma(\alpha-q+1)}$$

$$\left. + \, \delta_2 \eta^{(\alpha-r-q)} \frac{\Gamma(\alpha-r)\Gamma(1-q)}{\Gamma(\alpha-r-q+1)} \right] |.$$

Proof. (i) By equation (2.9), we have

$$\begin{aligned}
\int_0^1 G(t,s)s^{-q}ds &= -\frac{\delta_0}{N}\int_0^t (t-s)^{\alpha-1}s^{-q}ds + \frac{\delta_1}{N}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}s^{-q}ds \\
&\quad + \frac{\delta_2}{N}\int_0^\eta t^{\alpha-1}(\eta-s)^{\alpha-r-1}s^{-q}ds. \\
&= -\frac{\delta_0}{N}t^{\alpha-q}B(\alpha,\ 1-q) + \frac{\delta_1}{N}t^{\alpha-1}B(\alpha,\ 1-q) \\
&\quad + \frac{\delta_2}{N}t^{\alpha-1}\eta^{\alpha-r-q}B(\alpha-r,\ 1-q).
\end{aligned}$$

$$\therefore \qquad \max_{0\leq t\leq 1} |\int_0^1 G(t,s)s^{-q}ds| = \frac{1}{N}| \left[-\delta_0 \frac{\Gamma(\alpha)\Gamma(1-q)}{\Gamma(\alpha-q+1)} + \delta_1 \frac{\Gamma(\alpha)\Gamma(1-q)}{\Gamma(\alpha-q+1)} \right.$$

$$\left. + \, \delta_2 \eta^{(\alpha-r-q)} \frac{\Gamma(\alpha-r)\Gamma(1-q)}{\Gamma(\alpha-r-q+1)} \right] |.$$

To prove (ii), we have

$$\begin{aligned}\int_0^1 \frac{\partial}{\partial t}G(t,s)s^{-q}ds &= -\frac{\delta_0}{N}\int_0^t \frac{\partial}{\partial t}(t-s)^{\alpha-1}s^{-q}ds \\ &\quad + \frac{\delta_1}{N}\int_0^1 \frac{\partial}{\partial t}t^{\alpha-1}(1-s)^{\alpha-1}s^{-q}ds \\ &\quad + \frac{\delta_2}{N}\int_0^\eta \frac{\partial}{\partial t}t^{\alpha-1}(\eta-s)^{\alpha-r-1}s^{-q}ds. \\ &= -\frac{(\alpha-1)\delta_0}{N}t^{\alpha-q-1}B(\alpha-1,\ 1-q) \\ &\quad + \frac{(\alpha-1)\delta_1}{N}t^{\alpha-2}B(\alpha,\ 1-q) \\ &\quad + \frac{(\alpha-1)\delta_2}{N}t^{\alpha-2}\eta^{\alpha-r-q}B(\alpha-r,\ 1-q).\end{aligned}$$

$$\begin{aligned}\therefore \quad \max_{0\le t\le 1}\Big|\int_0^1 \frac{\partial}{\partial t}G(t,s)s^{-q}ds\Big| = \frac{(\alpha-1)}{N}\Big|\Big[&-\delta_0\frac{\Gamma(\alpha-1)\Gamma(1-q)}{\Gamma(\alpha-q)} \\ &+\delta_1\frac{\Gamma(\alpha)\Gamma(1-q)}{\Gamma(\alpha-q+1)} \\ &+\delta_2\eta^{(\alpha-r-q)}\frac{\Gamma(\alpha-r)\Gamma(1-q)}{\Gamma(\alpha-r-q+1)}\Big]\Big|.\end{aligned}$$

This completes the proof. □

Lemma 2.9(see [15]) - *Let* $2<\alpha\le 3,\quad 0<q<1,\quad F:(0,1]\longrightarrow \mathbb{R}$ *is continuous and* $\lim_{t\to 0^+} F(t)=+\infty$. *Suppose that* $t^qF(t)$ *is a continuous function on* $[0,1]$. *Then the function*

$$H(t)=\int_0^1 G(t,s)F(s)ds$$

is continuous on $[0,1]$, *where* $G(t,s)$ *is defined by (2.4).*

Lemma 2.10 - *Let* $2<\alpha\le 3$ *and* $0<q<1$. *Assume that conditions* $C_1,\ C_2$ *are satisfied. Then the operator* $\mathcal{A}:\mathcal{K}_o\longrightarrow\mathcal{K}_o$ *is completely continuous.*

Proof. Obviously, the operator $\mathcal{A}:\mathcal{K}_o\longrightarrow\mathcal{K}_o$ is continuous in view of the fact that $f(t,.)$ and $G(t,s)$ are continuous and nonnegative.

For $u\in\mathcal{K}_o,\ \mathcal{A}u\ge 0$ and $\mathcal{A}u(t)\in\mathcal{K}_o$. Also, for $t,s\in[0,1]$ and by the expression of $G(t,s)$ in (2.9), we have

$$|\frac{\partial}{\partial t}G(t,s)s^{-q}| = \left|\frac{(\alpha-1)}{N}t^{\alpha-2}\left[\delta_1(1-s)^{\alpha-1}+\delta_2(\eta-s)^{\alpha-r-1} - \delta_0(1-\frac{s}{t})^{\alpha-2}\right]s^{-q}\right| \geq 0.$$

Thus, $|(\mathcal{A}u)'(t)| = \int_0^1 |\frac{\partial}{\partial t}G(t,s)s^{-q}| \cdot s^q f(s,u,p)ds \geq 0$, which implies that $|(\mathcal{A}u)'(t)| \in \mathcal{K}_o$ and hence $\mathcal{A}(\mathcal{K}_o) \subset \mathcal{K}_o$.
Let Ω_o be a bounded set. Then there exists a constant $R > 0$ such that $\|u\| \leq R$, for all $u \in \Omega_o$.
Define $\mathcal{M}_o = \max_{0\leq t\leq 1} |t^q f(t,u,p)|$, $\mathcal{L}_1 = |\max_{0\leq t\leq 1}\int_0^1 G(t,s)s^{-q}ds|$,
$\mathcal{L}_2 = |\max_{0\leq t\leq 1}\int_0^1 \frac{\partial}{\partial t}G(t,s)s^{-q}ds|$, $R = \mathcal{M}_o(\mathcal{L}_1+\mathcal{L}_2) < \infty$ and $p \in [0,\infty)$.
Then for all $u \in \Omega_o$, we have

$$\begin{aligned} |\mathcal{A}u(t)| &= |\int_0^1 G(t,s)s^{-q}\cdot s^q f(s,u,p)ds| \\ &\leq \mathcal{M}_o|\max_{0\leq t\leq 1}\int_0^1 G(t,s)s^{-q}ds| \\ &\leq \mathcal{M}_o\mathcal{L}_1. \\ |(\mathcal{A}u)'(t)| &= |\int_0^1 \frac{\partial}{\partial t}G(t,s)s^{-q}\cdot s^q f(s,u,p)ds| \\ &\leq \mathcal{M}_o|\max_{0\leq t\leq 1}\int_0^1 \frac{\partial}{\partial t}G(t,s)s^{-q}ds| \\ &\leq \mathcal{M}_o\mathcal{L}_2. \end{aligned}$$

In view of the definition of norm $\|u\| = \max_{0\leq t\leq 1}|u(t)| + \max_{0\leq t\leq 1}|u'(t)|$, we have

$$\|\mathcal{A}u\| \leq \mathcal{M}_o(\mathcal{L}_1+\mathcal{L}_2) = R.$$

Hence the set $\mathcal{A}(\Omega_o)$ is bounded. Next, we show that $\mathcal{A}(\Omega_o)$ is equicontinuous: Since $G(t,s)$ is continuous on $[0,1]\times[0,1]$, it is uniformly continuous as well. Let $u \in \Omega_o \subset \mathcal{K}_o$ and $t_1,\ t_2 \in [0,1]$ with $t_1 < t_2$. Then for any fixed $s \in [0,1]$ and any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that whenever $|t_2 - t_1| < \delta$, we have
$|G(t_2,s) - G(t_1,s)| \leq \dfrac{(1-q)\varepsilon}{2\mathcal{M}_o}$ and $|\mathcal{A}u(t_2) - \mathcal{A}u(t_1)| \leq \varepsilon$.

Let $\delta = \min\left\{\frac{\varepsilon}{2\mathrm{R_o}},\ \left(\frac{\varepsilon}{4\alpha\mathrm{R_o}}\right)^{\frac{1}{\alpha-1}},\ \left(\frac{\varepsilon}{2^{(\alpha+1)}\mathrm{R_o}}\right)^{\frac{1}{\alpha-1}}\right\}$, $\mathrm{R_o} = \mathrm{R_1}+\mathrm{R_2}+\mathrm{R_3}$.

$$
\begin{aligned}
|\mathcal{A}u(t_2) - \mathcal{A}u(t_1)| &= \left| \int_0^1 [G(t_2,s) - G(t_1,s)] s^{-q} ds \cdot s^q f(s,u,p) \right| \\
&\le \int_0^1 |[G(t_2,s) - G(t_1,s)]| s^{-q} ds \cdot \max_{0 \le t \le 1} |s^q f(s,u,p)| \\
&\le \frac{(1-q)\varepsilon}{2\mathcal{M}_o} \int_0^1 s^{-q} ds \cdot \mathcal{M}_o \\
&\le \frac{(1-q)\varepsilon}{2\mathcal{M}_o} \cdot \frac{1}{(1-q)} \cdot \mathcal{M}_o = \frac{\varepsilon}{2}.
\end{aligned}
$$

$$
\begin{aligned}
|(\mathcal{A}u)'(t_2) - (\mathcal{A}u)'(t_1)| &= \left| \int_0^1 \frac{\partial}{\partial t} [G(t_2,s) - G(t_1,s)] s^{-q} ds \cdot s^q f(s,u,p) \right| \\
&\le \mathcal{M}_o \left| \int_0^1 \frac{\partial}{\partial t} [G(t_2,s) - G(t_1,s)] s^{-q} ds \right|
\end{aligned}
$$

$$
\begin{aligned}
\le\ & \left| \frac{(\alpha-1)\mathcal{M}_o\delta_1}{N} \int_0^1 {t_2}^{\alpha-2} - {t_1}^{\alpha-2} (1-s)^{\alpha-1} s^{-q} ds \right| \\
&+ \left| \frac{(\alpha-1)\mathcal{M}_o\delta_2}{N} \int_0^\eta {t_2}^{\alpha-2} - {t_1}^{\alpha-2} (\eta-s)^{\alpha-r-1} s^{-q} ds \right| \\
&+ \left| \frac{(\alpha-1)\mathcal{M}_o\delta_0}{N} \int_0^{t_1} [(t_2-s)^{\alpha-2} - (t_1-s)^{\alpha-2}] s^{-q} ds \right| \\
&- \left| \frac{(\alpha-1)\mathcal{M}_o\delta_0}{N} \int_{t_1}^{t_2} (t_2-s)^{\alpha-2} s^{-q} ds \right| \\
\le\ & \left| \frac{(\alpha-1)\mathcal{M}_o\delta_1}{N} \int_0^1 {t_2}^{\alpha-2} - {t_1}^{\alpha-2} (1-s)^{\alpha-1} s^{-q} ds \right| \\
&+ \left| \frac{(\alpha-1)\mathcal{M}_o\delta_2}{N} \int_0^\eta {t_2}^{\alpha-2} - {t_1}^{\alpha-2} (\eta-s)^{\alpha-r-1} s^{-q} ds \right| \\
&+ \left| \frac{(\alpha-1)\mathcal{M}_o\delta_0}{N} \int_0^{t_1} [(t_2-s)^{\alpha-2} - (t_1-s)^{\alpha-2}] s^{-q} ds \right|
\end{aligned}
$$

$$
\begin{aligned}
|(\mathcal{A}u)'(t_2) - (\mathcal{A}u)'(t_1)| \le \frac{(\alpha-1)\mathcal{M}_o\delta_1}{N} ({t_2}^{\alpha-2} - {t_1}^{\alpha-2}) B(1-q,\ \alpha) \\
+ \frac{(\alpha-1)\mathcal{M}_o\delta_2}{N} ({t_2}^{\alpha-2} - {t_1}^{\alpha-2}) \eta^{\alpha-r-q} B(1-q,\ \alpha-r)
\end{aligned}
$$

$$+\frac{(\alpha-1)\mathcal{M}_o\delta_0}{N}({t_2}^{\alpha-q-1}-{t_1}^{\alpha-q-1})B(1-q,\ \alpha-1)$$
$$=(R_1+R_2)({t_2}^{\alpha-2}-{t_1}^{\alpha-2})+R_3({t_2}^{\alpha-q-1}-{t_1}^{\alpha-q-1})$$

$\therefore |(\mathcal{A}u)'(t_2)-(\mathcal{A}u)'(t_1)| \leq (R_1+R_2)({t_2}^{\alpha-2}-{t_1}^{\alpha-2})+R_3({t_2}^{\alpha-q-1}-{t_1}^{\alpha-q-1})$.

To continue the proof, we consider the following cases.

Case 1: For $t_1=0,\ \ t_2<\delta$,

$$\begin{aligned}
|(\mathcal{A}u)'(t_2)-(\mathcal{A}u)'(t_1)| &\leq (R_1+R_2)({t_2}^{\alpha-2}-{t_1}^{\alpha-2}) \\
&\qquad + R_3({t_2}^{\alpha-q-1}-{t_1}^{\alpha-q-1}) \\
&< (R_1+R_2){t_2}^{\alpha-2}+R_3{t_2}^{\alpha-q-1} \\
&< (R_1+R_2)\delta+R_3\delta \\
&= (R_1+R_2+R_3)\delta \\
&= \delta R_o=\frac{\varepsilon}{2}.
\end{aligned}$$

Case 2: For $\delta\leq t_1<t_2<1$, with the application of mean value theorem(see [3]), we have

$$\begin{aligned}
|(\mathcal{A}u)'(t_2)-(\mathcal{A}u)'(t_1)| &\leq (R_1+R_2)({t_2}^{\alpha-2}-{t_1}^{\alpha-2}) \\
&\qquad + R_3({t_2}^{\alpha-q-1}-{t_1}^{\alpha-q-1}) \\
&\leq (R_1+R_2)(\alpha-2)\delta^{\alpha-1}+R_3(\alpha-q-1)\delta^{\alpha-1} \\
&< (R_1+R_2)2\alpha\delta^{\alpha-1}+R_3\cdot 2\alpha\delta^{\alpha-1} \\
&= (R_1+R_2+R_3)2\alpha\delta^{\alpha-1} \\
&= 2\alpha\delta^{\alpha-1}R_o=\frac{\varepsilon}{2}.
\end{aligned}$$

Case 3: For $0\leq t_1<\delta,\ \ t_2<2\delta$ with $\max\left\{(2\delta)^{\alpha-2},\ \ (2\delta)^{\alpha-q-1}\right\}\leq 2^\alpha\delta^{\alpha-1}$, we have

$$\begin{aligned}
|(\mathcal{A}u)'(t_2)-(\mathcal{A}u)'(t_1)| &\leq (R_1+R_2)({t_2}^{\alpha-2}-{t_1}^{\alpha-2}) \\
&\qquad + R_3({t_2}^{\alpha-q-1}-{t_1}^{\alpha-q-1}) \\
&< (R_1+R_2){t_2}^{\alpha-2}+R_3{t_2}^{\alpha-q-1} \\
&< (R_1+R_2)(2\delta)^{\alpha-2}+R_3(2\delta)^{\alpha-q-1} \\
&< (R_1+R_2)2^\alpha\delta^{\alpha-1}+R_3\cdot 2^\alpha\delta^{\alpha-1} \\
&= (R_1+R_2+R_3)2^\alpha\delta^{\alpha-1} \\
&= 2^\alpha\delta^{\alpha-1}R_o=\frac{\varepsilon}{2}.
\end{aligned}$$

In either case, with the definition of norm $\|\cdot\|$, we obtain

$$|\mathcal{A}u(t_2)-\mathcal{A}u(t_1)|\leq\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

which shows that $\mathcal{A}(\Omega_o)$ is equicontinuous. By Arzela-Ascoli theorem, we conclude that $\overline{\mathcal{A}(\Omega_o)}$ is relatively compact and hence the operator $\mathcal{A} : \mathcal{K}_o \longrightarrow \mathcal{K}_o$ is completely continuous. □

The next lemma, which is also given in [7] and [21], is crucial in proving our existence results.

Lemma 2.11(Leray-Schauder) - *Let Ω be the convex subset of a Banach space X, $0 \in \Omega$, $\Phi : \Omega \longrightarrow \Omega$ be completely continuous operator. Then, either*
(i) Φ has at least one fixed point in Ω, or
(ii) the set $\{x \in \Omega : x = \lambda\Phi x,\ \ 0 < \lambda < 1\}$ is unbounded.

3. Main Results

In this section, we establish the existence and uniqueness of positive solutions to the BVP(1.1).

Theorem 3.1 - *Let $2 < \alpha \leq 3$ and $0 < q < 1$. Assume that conditions C_1, C_2 are satisfied. Then the BVP(1.1) has at least one positive solution.*

Proof. By Lemma 2.10, we have that $\mathcal{A} : \mathcal{K}_o \longrightarrow \mathcal{K}_o$ is completely continuous. We only show that the set $\Psi = \{u \in \Omega_o : u = \lambda\mathcal{A}u,\ \ 0 < \lambda < 1\}$ is bounded. Let $u \in \Omega_o$. Then we have $u = \lambda\mathcal{A}u$ and

$$\begin{aligned}
|u(t)| &= |\lambda \int_0^1 G(t,s)s^{-q} \cdot s^q f(s,u,p)ds| \\
&\leq |\lambda \int_0^1 G(t,s)s^{-q}ds||s^q f(s,u,p)| \\
&\leq \mathcal{M}_o \,|\max_{0\leq t\leq 1} \int_0^1 G(t,s)s^{-q}ds| \\
&\leq \mathcal{M}_o\mathcal{L}_1.
\end{aligned}$$

By the proof of Lemma 2.10, we obtain

$$\begin{aligned}
|(\mathcal{A}u)'(t)| &= |\lambda \int_0^1 \frac{\partial}{\partial t}G(t,s)s^{-q} \cdot s^q f(s,u,p)ds| \\
&\leq \mathcal{M}_o \,|\max_{0\leq t\leq 1} \int_0^1 \frac{\partial}{\partial t}G(t,s)s^{-q}ds| \\
&\leq \mathcal{M}_o\mathcal{L}_2.
\end{aligned}$$

Thus, by the definition of norm $\|u\| = \max_{0\leq t\leq 1}|u(t)| + \max_{0\leq t\leq 1}|u'(t)|$, we have $\|u(t)\| \leq \mathcal{M}_o(\mathcal{L}_1 + \mathcal{L}_2) = R < \infty$.

Hence, the set Ψ is bounded and independent of λ. By Lemma 2.11, the operator $\mathcal{A}$ has a fixed point in $\mathcal{K}_o$ which is the positive solution for the BVP(1.1). □

Lemma 3.2 - *Let $2 < \alpha \leq 3,\ 0 < q < 1$ and $\mathcal{L} = (\mathcal{L}_1 + \mathcal{L}_2) > 0$. Assume that conditions $C_1,\ C_2$ are satisfied. Suppose there exists a constant $K > 0$ such that*

$$|t^q f(t,u,p) - t^q f(t,v,\bar{p})| \leq K(|u-v| + |p-\bar{p}|), \quad \bar{p} = v'(t),$$

for all $t \in [0,1]$ and $u,\ v,\ p,\ \bar{p} \in [0,\infty)$. Then the BVP(1.1) has a unique solution provided $K\mathcal{L} < 1$.

Proof. By equation (2.11), we have

$$\begin{aligned}
|\mathcal{A}u(t) - \mathcal{A}v(t)| &= \left| \int_0^1 G(t,s)s^{-q}s^q f(s,u,p)ds \right. \\
&\quad \left. - \int_0^1 G(t,s)s^{-q}s^q f(s,v,\bar{p})ds \right| \\
&\leq \left| \max_{0\leq t\leq 1} \int_0^1 G(t,s)s^{-q}ds \right| |s^q f(s,u,p) - s^q f(s,v,\bar{p})| \\
&\leq K\mathcal{L}_1(|u-v| + |p-\bar{p}|) \\
&\leq K\mathcal{L}_1\|u-v\|. \qquad (3.1)
\end{aligned}$$

$$\begin{aligned}
|(\mathcal{A}u)'(t) - (\mathcal{A}v)'(t)| &= \left| \int_0^1 \frac{\partial}{\partial t} G(t,s)s^{-q}s^q f(s,u,p)ds \right. \\
&\quad \left. - \int_0^1 \frac{\partial}{\partial t} G(t,s)s^{-q}s^q f(s,v,\bar{p})ds \right| \\
&\leq \left| \max_{0\leq t\leq 1} \int_0^1 \frac{\partial}{\partial t} G(t,s)s^{-q}ds \right| \\
&\qquad \times |s^q f(s,u,p) - s^q f(s,v,\bar{p})| \\
&\leq K\mathcal{L}_2(|u-v| + |p-\bar{p}|) \\
&\leq K\mathcal{L}_2\|u-v\|. \qquad (3.2)
\end{aligned}$$

Therefore, by the definition of norm $\|\cdot\|$ with equations (3.1) and (3.2), we obtain

$$\begin{aligned}
|\mathcal{A}u(t) - \mathcal{A}v(t)| &\leq K\mathcal{L}_1\|u-v\| + K\mathcal{L}_2\|u-v\| \\
&\leq K(\mathcal{L}_1 + \mathcal{L}_2)\|u-v\| \\
&= K\mathcal{L}\|u-v\|.
\end{aligned}$$

Hence, by Banach contraction principle, the BVP(1.1) has a unique solution provided $K\mathcal{L} < 1$. $\square$

4. Illustrative Examples

1. Consider the nonlinear boundary value problem:

$$\left\{\begin{array}{l} D^{\frac{5}{2}}u(t) + \dfrac{45 + 8t - 20t^2e^{-u} + |u'|}{\sqrt{t}} = 0, \quad t \in (0,1), \\ \\ u(0) = u'(0) = 0, \quad \frac{3}{4}u(1) + D^{\frac{1}{2}}u(\frac{1}{4}) = \displaystyle\int_0^1 u(s)ds. \end{array}\right. \tag{4.1}$$

Here, $\alpha = \frac{5}{2}$, $\beta = \frac{3}{4}$, $r = \frac{1}{2}$, $\eta = \frac{1}{4}$ and

$$f(t,u,u') = \frac{45 + 8t - 20t^2e^{-u} + |u'(t)|}{\sqrt{t}}.$$

Clearly, $f(t,u,|u'|)$ is continuous on $(0,1] \times I\!R^+ \times I\!R^+$, $\lim\limits_{t\to 0^+} f(t,\cdot) = +\infty$, $t^{\frac{1}{2}}f(t,u,|u'|)$ is continuous on $[0,1] \times I\!R^+ \times I\!R^+$ with $q = \frac{1}{2}$.

By simple calculation, we have $c_o = \dfrac{\Gamma(\alpha)}{\Gamma(\alpha - r)} = \dfrac{3\sqrt{\pi}}{4} = 1.329340388$,

$$\sigma = \left(\beta + \frac{\Gamma(\alpha)}{\Gamma(\alpha - r)}\eta^{\alpha - r - 1}\right) = \frac{3}{4} + \frac{3\sqrt{\pi}}{4}\left(\frac{1}{4}\right) = 1.082335097.$$

$$\delta_0 = \alpha\sigma^2 = 2.928623155, \quad \delta_1 = \alpha\beta\sigma = 2.029378307,$$

$$\alpha\sigma - 1 = 1.705837743, \quad \delta_2 = \alpha\sigma c_o = 3.596979394,$$

$$\begin{aligned} N = \sigma(\alpha\sigma - 1)\Gamma(\alpha) &= (1.082335097)(1.705837743)(1.329340388) \\ &= 2.454345285. \end{aligned}$$

$$\mathcal{M}_o = \max_{0\le t\le 1} |t^q f(t,u,u')| = \max_{0\le t\le 1} |45 + 8t - 20t^2e^{-u} + |u'|| \le 54.63368722,$$

for $(t,u,|u'|) \in [0,1] \times [0,4] \times [0,2]$.

$$\begin{aligned} \mathcal{L}_1 &= \left|\max_{0\le t\le 1}\int_0^1 G(t,s)s^{-q}ds\right| \\ &= \frac{1}{N}\left|-\delta_0 B\left(\frac{5}{2}, \frac{1}{2}\right) + \delta_1 B\left(\frac{5}{2}, \frac{1}{2}\right) + \delta_2\eta^{\frac{3}{2}}B\left(2, \frac{1}{2}\right)\right| \\ &= \frac{1}{N}\left|-\delta_0\frac{3\pi}{8} + \delta_1\frac{3\pi}{8} + \delta_2\left(\frac{1}{4}\right)^{\frac{3}{2}}\left(\frac{4}{3}\right)\right| \\ &= \frac{|-0.459901313\,|}{2.454345285} = 0.187382482. \end{aligned}$$

$$
\begin{aligned}
\mathcal{L}_2 &= \left|\max_{0\le t\le 1}\int_0^1 \frac{\partial}{\partial t}G(t,s)s^{-q}ds\right| \\
&= \frac{\alpha-1}{N}\left|-\delta_0 B\left(\frac{3}{2},\,\frac{1}{2}\right)+\delta_1 B\left(\frac{5}{2},\,\frac{1}{2}\right)+\delta_2\eta^{\frac{3}{2}}B\left(2,\,\frac{1}{2}\right)\right| \\
&= \frac{\alpha-1}{N}\left|-\delta_0\frac{\pi}{2}+\delta_1\frac{3\pi}{8}+\delta_2\left(\frac{1}{4}\right)^{\frac{3}{2}}\left(\frac{4}{3}\right)\right| \\
&= \frac{\alpha-1}{N}|-1.609968936| \\
&= \frac{1.5(1.609968936)}{2.454345285}=0.983950147.
\end{aligned}
$$

$$
\begin{aligned}
\text{Also,}\quad \|\mathrm{u(t)}\| \le \mathcal{M}_{\mathrm{o}}(\mathcal{L}_1+\mathcal{L}_2) &= 54.63368722(1.171332629) \\
&= 63.99422048 = R.
\end{aligned}
$$

This implies the set $\Psi=\{u\in\Omega_o : u=\lambda\mathcal{A}u,\ \ 0<\lambda<1\}$ is bounded. Thus, by Lemma 2.11 and Theorem 3.1, the BVP(4.1) has at least one positive solution. □

2. Consider the nonlinear boundary value problem:

$$
\begin{cases}
D^{\frac{5}{2}}u(t)+\dfrac{45+8t+0.25e^t u+0.25e^t|u'|}{\sqrt{t}\cdot e^t(1+t)^2}=0, \quad t\in(0,1), \\[2ex]
u(0)=u'(0)=0, \quad \frac{3}{4}u(1)+D^{\frac{1}{2}}u(\frac{1}{4})=\displaystyle\int_0^1 u(s)ds.
\end{cases} \tag{4.2}
$$

Clearly, conditions C_1, C_2 are satisfied with $q=\frac{1}{2}$.
For $t\in[0,1]$ and $u,\ v,\ u',\ v'\in[0,\infty)$,

$$
\begin{aligned}
|t^{\frac{1}{2}}f(t,u,u')-t^{\frac{1}{2}}f(t,v,v')| &= \left|\frac{0.25e^t u+0.25e^t u'-0.25e^t v-0.25e^t v'}{e^t(1+t)^2}\right| \\
&\le \frac{0.25}{(1+t)^2}(|u-v|+|u'-v'|) \\
&\le \frac{0.25}{(1+t)}(|u-v|+|u'-v'|) \\
&\le 0.125\|u-v\|,
\end{aligned}
$$

with $K=0.125$. By simple calculation, we obtain

$$\mathcal{L}=(\mathcal{L}_1+\mathcal{L}_2)=0.187382482+0.983950147=1.171332629.$$

$$\therefore\ \ K\mathcal{L}=0.125(1.171332629)=0.146416578<1.$$

Thus, by Theorem 3.2, the BVP(4.2) has a unique solution.

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