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DIRAC PAIRS ON JACOBI ALGEBROIDS

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Abstract. We define Dirac pairs on Jacobi algebroids, which is a generalization of Dirac pairs on Lie algebroids introduced by Kosmann-Schwarzbach. We show the relationship between Dirac pairs on Lie and on Jacobi algebroids, and that Dirac pairs on Jacobi algebroids characterize several compatible structures on Jacobi algebroids.

1. Introduction

Poisson and symplectic structures on smooth manifolds have wide application in the theory of integrable systems on smooth manifolds, especially even dimensional manifolds. These structures are generalized on Lie algebroids. One of further generalizations of Poisson structures on Lie algebroids is Dirac structures, which are defined on Lie bialgebroids in general [7]. Dirac structures on a Lie algebroid A are defined by using the Lie bialgebroid canonically determined for A. In terms of applications in the theory of integrable systems, compatible two structures, for example, $P\Omega$- and ΩN-structures [12], are often used. The notion dealing with these compatible structures in a unified way is a Dirac pair, which was introduced by Kosmann-Schwarzbach [4].

On the other hand, contact structures can be defined on odd dimensional manifolds, and Jacobi structures are generalizations of contact structures. Moreover Jacobi structures are generalized as structures on Jacobi algebroids. As a generalization of both Jacobi structures on Jacobi algebroids and Dirac structures on Lie bialgebroids, we can define Dirac structures on Jacobi bialgebroids [14]. As in the case of Lie algebroid, Dirac structures on Jacobi algebroids can also be defined naturally. In addition, we can define several compatible structures on Jacobi algebroids, for example, $J\Omega$- and ΩN-structures. In this paper, we define Dirac pairs on Jacobi bialgebroids and prove that $J\Omega$- and ΩN-structures can be characterized by Dirac pairs. Furthermore, we investigate relationships between Dirac pairs on Lie and Jacobi bialgebroids.

This paper is divided into four sections. In Section 2, we recall several definitions, properties and examples of Lie and Jacobi algebroids, relations, Dirac structures on Lie and Jacobi bialgebroids, and Dirac pairs on Lie bialgebroids. Here Jacobi algebroids (resp. bialgebroids) are generalizations of

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Lie algebroids (resp. bialgebroids). In Section 3, we define Dirac pairs on Jacobi bialgebroids. A Dirac pair is a pair (L, L') of two Dirac structures such that the induced relation $\mathrm{N}_{L,L'}$ is Nijenhuis. In Theorem 3.2, we show that $(\overline{\mathrm{graph}\, \pi_1^\sharp}, \overline{\mathrm{graph}\, \pi_2^\sharp})$, $(\overline{\mathrm{graph}\, \pi_1^\sharp}, \mathrm{graph}\, \omega_2^\flat)$ and $(\mathrm{graph}\, \omega_1^\flat, \mathrm{graph}\, \omega_2^\flat)$ are Dirac pairs on a Jacobi bialgebroid $((A, \phi_0), (A^*, X_0))$ over M if and only if $(\overline{\mathrm{graph}\, \tilde{\pi}_1^\sharp}, \overline{\mathrm{graph}\, \tilde{\pi}_2^\sharp})$, $(\overline{\mathrm{graph}\, \tilde{\pi}_1^\sharp}, \mathrm{graph}\, \tilde{\omega}_2^\flat)$ and $(\mathrm{graph}\, \tilde{\omega}_1^\flat, \mathrm{graph}\, \tilde{\omega}_2^\flat)$ are Dirac pairs on the induced Lie bialgebroid $(\tilde{A}_{\bar{\phi}_0}, \tilde{A}^*_{\hat{X}_0})$ over $M \times \mathbb{R}$, respectively. Here π_i in $\Gamma(\Lambda^2 A)$ and ω_i in $\Gamma(\Lambda^2 A^*)$ $(i = 1, 2)$ are elements satisfying the Maurer-Cartan type equation, and we set $\tilde{\pi}_i := e^{-t}\pi_i$ in $\Gamma(\Lambda^2 \tilde{A})$ and $\tilde{\omega}_i := e^t \omega_i$ in $\Gamma(\Lambda^2 \tilde{A}^*)$, where t is the standard coordinate in $\mathbb{R}$. Since this theorem means that the condition to be a Dirac pair is preserved between $((A, \phi_0), (A^*, X_0))$ and $(\tilde{A}_{\bar{\phi}_0}, \tilde{A}^*_{\hat{X}_0})$, it is important. This is the main theorem in this paper. In Section 4, we consider Jacobi pairs and ϕ_0-presymplectic pairs defined by using Dirac pairs on Jacobi algebroids. We show the relationship between Jacobi (resp. ϕ_0-presymplectic) pairs and Poisson (resp. presymplectic) pairs, and prove that there exists a one-to-one correspondence between the non-degenerate Jacobi pairs and the ϕ_0-symplectic pairs on Jacobi algebroids. Moreover, we introduce JΩ- and ΩN-structures on Jacobi algebroids. These structures are defined as generalizations of PΩ- and ΩN-structures on Lie algebroids [12]. In addition to these, there are also PN (or Poisson-Nijenhuis) structures on Lie algebroids [12], [5]. However there exists already a generalization of PN structures on Jacobi algebroids called Jacobi-Nijenhuis structures [13], [2]. We show the relationship between JΩ-(resp. ΩN-)structures on Jacobi algebroids and PΩ-(resp. ΩN-)structures on Lie algebroids, and prove that JΩ- and ΩN-structures can be characterized by Dirac pairs on Jacobi algebroids.

2. Preliminaries

2.1. **Lie and Jacobi algebroids.** A *Lie algebroid* over a manifold M is a vector bundle $A \to M$ equipped with a Lie bracket $[\cdot,\cdot]_A$ on $\Gamma(A)$ and a bundle map $\rho_A : A \to TM$ over M, called the *anchor*, satisfying the following condition: for any X, Y in $\Gamma(A)$ and f in $C^\infty(M)$,

$$[X, fY]_A = f[X, Y]_A + (\rho_A(X)f)Y.$$

Let $A := (A, [\cdot,\cdot]_A, \rho_A)$ be a Lie algebroid over M. The *Schouten bracket* on $\Gamma(\Lambda^* A)$ is defined similarly to the Schouten bracket $[\cdot,\cdot]$ on $\mathfrak{X}^*(M)$. That is, the Schouten bracket $[\cdot,\cdot]_A : \Gamma(\Lambda^k A) \times \Gamma(\Lambda^l A) \to \Gamma(\Lambda^{k+l-1} A)$ is defined as the unique extension of the Lie bracket $[\cdot,\cdot]_A$ on $\Gamma(A)$ such that

$$[f, g]_A = 0;$$

$[X, f]_A = \rho_A(X)f$;

$[X, Y]_A$ is the Lie bracket on $\Gamma(A)$;

$[D_1, D_2 \wedge D_3]_A = [D_1, D_2]_A \wedge D_3 + (-1)^{(a_1+1)a_2} D_2 \wedge [D_1, D_3]_A$;

$[D_1, D_2]_A = -(-1)^{(a_1-1)(a_2-1)}[D_2, D_1]_A$

for any f, g in $C^\infty(M)$, X, Y in $\Gamma(A)$ and D_i in $\Gamma(\Lambda^{a_i} A)$. The *differential* of the Lie algebroid A is an operator $d_A : \Gamma(\Lambda^k A^*) \to \Gamma(\Lambda^{k+1} A^*)$ defined by for any ω in $\Gamma(\Lambda^k A^*)$ and $X_0, \dots, X_k$ in $\Gamma(A)$,

$$(d_A\omega)(X_0, \dots, X_k) = \sum_{i=0}^{k} (-1)^i \rho_A(X_i)(\omega(X_0, \dots, \hat{X}_i, \dots, X_k)) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j]_A, X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k).$$

For any X in $\Gamma(A)$, the *Lie derivative* $\mathcal{L}^A_X : \Gamma(\Lambda^k A^*) \to \Gamma(\Lambda^k A^*)$ is defined by the *Cartan formula* $\mathcal{L}^A_X := d_A \iota_X + \iota_X d_A$ and $\mathcal{L}^A_X$ is extended on $\Gamma(\Lambda^* A)$ in the same way as the usual Lie derivative $\mathcal{L}_X$. Then it follows that $\mathcal{L}^A_X D = [X, D]_A$ for any D in $\Gamma(\Lambda^* A)$.

Example 1. (i) For any manifold M, the tangent bundle $(TM, [\cdot,\cdot], \mathrm{id}_{TM})$ is a Lie algebroid over M, where $[\cdot,\cdot]$ is the usual Lie bracket on the vector fields $\mathfrak{X}(M)$.

(ii) For any vector bundle A over M, we set $[\cdot,\cdot]_A := 0$ and $\rho_A := 0$. Then $A_0 := (A, [\cdot,\cdot]_A, \rho_A)$ is a Lie algebroid. We call $([\cdot,\cdot]_A, \rho_A)$ the *trivial Lie algebroid structure* on A.

Example 2 ([3]). Let A be a vector bundle over a manifold M and set $A \oplus \mathbb{R} := A \oplus (M \times \mathbb{R})$. Then the sections $\Gamma(\Lambda^k(A \oplus \mathbb{R}))$ and $\Gamma(\Lambda^k(A \oplus \mathbb{R})^*)$ can be identified with $\Gamma(\Lambda^k A) \oplus \Gamma(\Lambda^{k-1} A)$ and $\Gamma(\Lambda^k A^*) \oplus \Gamma(\Lambda^{k-1} A^*)$, respectively. Now, assume that $A = (A, [\cdot,\cdot]_A, \rho_A)$ is a Lie algebroid over M. Then $(A \oplus \mathbb{R}, [\cdot,\cdot]_{A\oplus\mathbb{R}}, \rho_A \circ \mathrm{pr}_1)$ is also a Lie algebroid over M, where the bracket $[\cdot,\cdot]_{A\oplus\mathbb{R}}$ is defined by

$$[(X, f), (Y, g)]_{A\oplus\mathbb{R}} := ([X, Y]_A, \rho_A(X)g - \rho_A(Y)f)$$

and the map $\mathrm{pr}_1 : A \oplus \mathbb{R} \to A$ is the canonical projection to the first factor.

Next, we define Jabobi algebroids. A pair (A, ϕ_0) is a *Jacobi algebroid* over M if $A = (A, [\cdot,\cdot]_A, \rho_A)$ is a Lie algebroid over M and ϕ_0 in $\Gamma(A^*)$ is d_A-closed, that is, $d_A\phi_0 = 0$.

Example 3. For a Lie algebroid $A \oplus \mathbb{R}$ in Example 2, we set $\phi_0 := (0, 1)$ in $\Gamma(A^* \oplus \mathbb{R}) = \Gamma(A^*) \oplus C^\infty(M)$. Then $(A \oplus \mathbb{R}, \phi_0)$ is a Jacobi algebroid.

Example 4. For any Lie algebroid A over M, we set $\phi_0 := 0$. Then (A, ϕ_0) is a Jacobi algebroid. We call ϕ_0 the *trivial Jacobi algebroid structure* on A. Therefore any Lie algebroid is a Jacobi algebroid.

For a Jacobi algebroid (A, ϕ_0), there is the ϕ_0-*Schouten bracket* $[\cdot,\cdot]_{A,\phi_0}$ on $\Gamma(\Lambda^* A)$ given by

$$[D_1, D_2]_{A,\phi_0} := [D_1, D_2]_A + (a_1 - 1)D_1 \wedge \iota_{\phi_0} D_2 - (-1)^{a_1+1}(a_2 - 1)\iota_{\phi_0} D_1 \wedge D_2$$

for any D_i in $\Gamma(\Lambda^{a_i} A)$, where $[\cdot,\cdot]_A$ is the Schouten bracket of the Lie algebroid A. The ϕ_0-*differential* d_{A,ϕ_0} and the ϕ_0-*Lie derivative* $\mathcal{L}^{A,\phi_0}_X$ are defined by

$$d_{A,\phi_0}\omega := d_A\omega + \phi_0 \wedge \omega, \quad \mathcal{L}^{A,\phi_0}_X := \iota_X \circ d_{A,\phi_0} + d_{A,\phi_0} \circ \iota_X$$

for any ω in $\Gamma(\Lambda^* A^*)$ and X in $\Gamma(A)$. For any π in $\Gamma(\Lambda^2 A)$, ξ and η in $\Gamma(A^*)$, it follows that

(2.1)
$$\frac{1}{2}[\pi, \pi]_{A,\phi_0}(\xi, \eta, \cdot) = [\pi^\sharp\xi, \pi^\sharp\eta]_A - \pi^\sharp\left(\mathcal{L}^{A,\phi_0}_{\pi^\sharp\xi}\eta - \mathcal{L}^{A,\phi_0}_{\pi^\sharp\eta}\xi - d_{A,\phi_0}\langle\pi^\sharp\xi, \eta\rangle\right),$$

where a bundle map $\pi^\sharp : A^* \to A$ over M is defined by $\langle\pi^\sharp\xi, \eta\rangle := \pi(\xi, \eta)$.

We call a d_{A,ϕ_0}-closed 2-cosection ω, i.e., $d_{A,\phi_0}\omega = 0$, a ϕ_0-*presymplectic structure* on (A, ϕ_0). A ϕ_0-presymplectic structure ω is called a ϕ_0-*symplectic structure* if ω is non-degenerate.

Example 5. We consider $A := TM \oplus \mathbb{R}$ and $\phi_0 := (0, 1)$ in $\Omega^1(M) \oplus C^\infty(M)$. Then any ω in $\Omega^2(M) \oplus \Omega^1(M)$ can be written as $\omega = (\alpha, \beta)$ $(\alpha \in \Omega^2(M), \beta \in \Omega^1(M))$. Then $\omega = (\alpha, \beta)$ is $(0, 1)$-presymplectic on $(TM \oplus \mathbb{R}, (0, 1))$ if and only if $\alpha = d\beta$. Moreover setting $\dim M = 2n + 1$, we see that a $(0, 1)$-presymplectic strucutre $\omega = (d\beta, \beta)$ is non-degenerate if and only if $\beta \wedge (d\beta)^n \neq 0$, that is, β is a *contact structure* on M. Therefore a $(0, 1)$-symplectic structure on $(TM \oplus \mathbb{R}, (0, 1))$ is just a contact structure on M.

As a generalization of Poisson structures on Lie algebroids, we define Jacobi structures on Jacobi algebroids. That is, a *Jacobi structure* on a Jacobi algebroid (A, ϕ_0) is a 2-section π in $\Gamma(\Lambda^2 A)$ satisfying the condition

$$[\pi, \pi]_{A,\phi_0} = 0. \tag{2.2}$$

For any Lie algebroid A equipped with the trivial Jacobi algebroid structure 0, it follows that $[\cdot,\cdot]_{A,0} = [\cdot,\cdot]_A$. Hence Jacobi structures on $(A, 0)$ are just Poisson structures on A.

It is well known that there exists a one-to-one correspondence between ϕ_0-symplectic structures on (A, ϕ_0) and non-degenerate Jacobi structures on (A, ϕ_0). In fact, for a non-degenerate Jacobi structure π on (A, ϕ_0), a

2-cosection ω_π characterized by $\omega_\pi^\flat = -(\pi^\sharp)^{-1}$ is ϕ_0-symplectic on (A, ϕ_0), where for any 2-cosection ω, a bundle map $\omega^\flat : A \to A^*$ over M is defined by $\langle \omega^\flat X, Y\rangle := \omega(X, Y)$.

Let $p_A : A \to M$ be a vector bundle over M. We set $\tilde{A} := A \times \mathbb{R}$. Then $p_{\tilde{A}} : \tilde{A} \to M \times \mathbb{R}$, $p_{\tilde{A}}(X, t) := (p_A(X), t)$, is a vector bundle over $M \times \mathbb{R}$. The sections $\Gamma(\tilde{A})$ can be identified with the set of time-dependent sections of A. We assume that (A, ϕ_0) is a Jacobi algebroid. Under the above identification, we can define two Lie algebroid structures $([\cdot,\cdot]_A^{\hat{\phi}_0}, \hat{\rho}_A^{\phi_0})$ and $([\cdot,\cdot]_A^{\bar{\phi}_0}, \bar{\rho}_A^{\phi_0})$ on $\tilde{A}$, where for any $\tilde{X}$ and $\tilde{Y}$ in $\Gamma(\tilde{A})$,

(2.3)
$$[\tilde{X}, \tilde{Y}]_A^{\hat{\phi}_0} := e^{-t}\left([\tilde{X}, \tilde{Y}]_A + \langle \phi_0, \tilde{X}\rangle \left(\frac{\partial \tilde{Y}}{\partial t} - \tilde{Y}\right) - \langle \phi_0, \tilde{Y}\rangle \left(\frac{\partial \tilde{X}}{\partial t} - \tilde{X}\right)\right),$$

$$\hat{\rho}_A^{\phi_0}(\tilde{X}) := e^{-t}\left(\rho_A(\tilde{X}) + \langle \phi_0, \tilde{X}\rangle \frac{\partial}{\partial t}\right), \tag{2.4}$$

$$[\tilde{X}, \tilde{Y}]_A^{\bar{\phi}_0} := [\tilde{X}, \tilde{Y}]_A + \langle \phi_0, \tilde{X}\rangle \frac{\partial \tilde{Y}}{\partial t} - \langle \phi_0, \tilde{Y}\rangle \frac{\partial \tilde{X}}{\partial t}, \tag{2.5}$$

$$\bar{\rho}_A^{\phi_0}(\tilde{X}) := \rho_A(\tilde{X}) + \langle \phi_0, \tilde{X}\rangle \frac{\partial}{\partial t}. \tag{2.6}$$

The definition and properties of Jacobi bialgebroids are the followings. Jacobi bialgebroids are important to define Dirac structures in Subsection 2.2.

Definition 1 ([3])**.** Let $A = (A, [\cdot,\cdot]_A, \rho_A)$ be a Lie algebroid over M, A^* the dual vector bundle of A with a Lie algebroid structure $([\cdot,\cdot]_{A^*}, \rho_{A^*})$, ϕ_0 and X_0 a Jacobi algebroid structures on A and on $A^* = (A^*, [\cdot,\cdot]_{A^*}, \rho_{A^*})$, respectively. Then a pair $((A, \phi_0), (A^*, X_0))$ is a *Jacobi bialgebroid* over M if for any X, Y in $\Gamma(A)$ and P in $\Gamma(\Lambda^k A)$,

$$d_{A^*,X_0}[X, Y]_A = [d_{A^*,X_0}X, Y]_{A,\phi_0} + [X, d_{A^*,X_0}Y]_{A,\phi_0},$$
$$\mathcal{L}_{X_0}^{A,\phi_0}P + \mathcal{L}_{\phi_0}^{A^*,X_0}P = 0,$$

where d_{A^*,X_0} is the X_0-differential and $\mathcal{L}_{\phi_0}^{A^*,X_0}$ is the X_0-Lie derivative of (A^*, X_0) with respect to ϕ_0.

Example 6 (Lie bialgebroids [10])*.* Let A and A^* be vector bundles in duality equipped with Lie algebroid structures and the trivial Jacobi algebroid structures 0. Then a pair $((A, 0), (A^*, 0))$ is a Jacobi bialgebroid if and only if a pair (A, A^*) is a Lie bialgebroid.

Example 7 ([3])*.* For any Jacobi algebroid (A, ϕ_0) and its dual bundle $(A_0^*, 0)$ equipped with the trivial Lie and Jacobi algebroid structure, a pair $((A, \phi_0), (A_0^*, 0))$ is a Jacobi bialgebroid.

Proposition 2.1 is the relation between a Jacobi and Lie bialgebroid.

Proposition 2.1 ([3]). A pair $((A, \phi_0), (A^*, X_0))$ is a Jacobi bialgebroid over M if and only if a pair $\left(\tilde{A}^{-}_{\phi_0}, \tilde{A}^{*\wedge}_{X_0}\right) = ((\tilde{A}, \overline{[\cdot,\cdot]}^{\phi_0}_A, \bar{\rho}^{\phi_0}_A), (\tilde{A}^*, \widehat{[\cdot,\cdot]}^{X_0}_A, \hat{\rho}^{X_0}_A))$ is a Lie bialgebroid over $M \times \mathbb{R}$.

Proposition 2.2 follows immediately from Proposition 2.1.

Proposition 2.2 ([3]). If $((A, \phi_0), (A^*, X_0))$ is a Jacobi bialgebroid, then so is $((A^*, X_0), (A, \phi_0))$.

2.2. **Dirac structures on Jacobi algebroids.** To define Dirac structures on a Jacobi bialgebroid $((A, \phi_0), (A^*, X_0))$, we introduce the following pairings $(\cdot,\cdot)_\pm$ and bracket $[\![\cdot,\cdot]\!]$ on the Whitney sum $A \oplus A^*$:

$$
\begin{aligned}
(X+\xi, Y+\eta)_\pm &:= \frac{1}{2}\left(\langle \xi, Y\rangle \pm \langle \eta, X\rangle\right);\\
[\![X+\xi, Y+\eta]\!] &:= ([X,Y]_{A,\phi_0} + \mathcal{L}^{A^*,X_0}_{\xi} Y - \mathcal{L}^{A^*,X_0}_{\eta} X - d_{A^*,X_0}(X+\xi, Y+\eta)_-)\\
&\qquad + ([\xi,\eta]_{A^*,X_0} + \mathcal{L}^{A,\phi_0}_{X}\eta - \mathcal{L}^{A,\phi_0}_{Y}\xi + d_{A,\phi_0}(X+\xi, Y+\eta)_-);
\end{aligned}
$$

We notice that the pairings $(\cdot,\cdot)'_\pm$ and the bracket $[\![\cdot,\cdot]\!]'$ defined as above on $A^* \oplus A$ for a Jacobi bialgebroid $((A^*, X_0), (A, \phi_0))$ satisfy

$$
(\cdot,\cdot)'_\pm = \pm(\cdot,\cdot)_\pm, \quad [\![\cdot,\cdot]\!]' = [\![\cdot,\cdot]\!]. \tag{2.7}
$$

Definition 2 ([14]). Let $((A, \phi_0), (A^*, X_0))$ be a Jacobi bialgebroid over M. A subbundle L of $A \oplus A^*$ is a *Dirac structure* on $((A, \phi_0), (A^*, X_0))$ if it is maximally isotropic under the pairing $(\cdot,\cdot)_+$ and $\Gamma(L)$ is closed under the bracket $[\![\cdot,\cdot]\!]$. By (2.7), the Dirac structures on $((A, \phi_0), (A^*, X_0))$ and on $((A^*, X_0), (A, \phi_0))$ coincide. For a Jacobi algebroid (A, ϕ_0), we call a Dirac structure on a Jacobi bialgebroid $((A, \phi_0), (A^*_0, 0))$ in Example 7 a *Dirac structure on* (A, ϕ_0).

Let $((A, \phi_0), (A^*, X_0))$ be a Jacobi bialgebroid over M, π and ω elements in $\Gamma(\Lambda^2 A)$ and in $\Gamma(\Lambda^2 A^*)$, respectively. We set

$$
\begin{aligned}
\operatorname{graph}\pi^\sharp &:= \{\pi^\sharp\xi + \xi \mid \xi \in A^*\} \subset A \oplus A^*,\\
\operatorname{graph}\omega^\flat &:= \{X + \omega X \mid X \in A\} \subset A \oplus A^*.
\end{aligned}
$$

Theorem 2.3 ([14]). *With the above notations,* $\operatorname{graph}\pi^\sharp$ *(resp.* $\operatorname{graph}\omega^\flat$*), the graph of a bundle map* $\pi^\sharp$ *(resp.* $\omega^\flat$*), is a Dirac structure on* $((A, \phi_0), (A^*, X_0))$ *if and only if* π *(resp.* ω*) satisfies the Maurer-Cartan type equation:*

$$
d_{A^*,X_0}\pi + \frac{1}{2}[\pi,\pi]_{A,\phi_0} = 0 \quad \left(\textit{resp. } d_{A,\phi_0}\omega + \frac{1}{2}[\omega,\omega]_{A^*,X_0} = 0\right). \tag{2.8}
$$

Remark. A Dirac structure on a Jacobi bialgebroid $((A, 0), (A^*, 0))$ in Example 6 is called a *Dirac structure on a Lie bialgebroid* (A, A^*) and a Dirac structure on a Lie bialgebroid (A, A_0^*) is called a *Dirac structure on a Lie algebroid* A. Then (2.8) coincides with the Maurer-Cartan type equation for a Lie bialgebroid (A, A^*) introduced in [7].

Example 8. For any Jacobi algebroid (A, ϕ_0), the Maurer-Cartan type equations for (A, ϕ_0) are

$$[\pi, \pi]_{A,\phi_0} = 0, \quad d_{A,\phi_0}\omega = 0.$$

The former means that π in $\Gamma(\Lambda^2 A)$ is a Jacobi structure on (A, ϕ_0) and the latter means that ω in $\Gamma(\Lambda^2 A^*)$ is a ϕ_0-presymplectic structure on (A, ϕ_0).

2.3. **Relations.** For any vector bundles U and V over a manifold M, we call a subset of a direct product of the sections $\Gamma(U)$ and $\Gamma(V)$ a *relation*. Let R be a subset of $U \times V$. Then the relation $\underline{R}$ induced by R is defined by

$$\underline{R} := \{(X, Y) \in \Gamma(U) \times \Gamma(V) \,|\, \forall p \in M, (X_p, Y_p) \in R\}.$$

We also call R a relation. We notice that $\underline{R} = \Gamma(R)$ if $R \subset U \times V$ is a vector bundle over M. For any bundle map $\phi : U \to V$, we get $\underline{\mathrm{graph}\,\phi} = \mathrm{graph}\,\underline{\phi}$. Here $\underline{\phi} : \Gamma(U) \to \Gamma(V)$ is the map induced by ϕ. In the rest of this paper, we shall omit underline and denote the induced relation and map by the same symbols.

Let U, V and W be vector bundles over a manifold M. The *composition* $R' * R$ of relations $R \subset \Gamma(U) \times \Gamma(V)$ and $R' \subset \Gamma(V) \times \Gamma(W)$, the *inverse* $\overline{R}$ and the *dual* R^* of a relation $R \subset \Gamma(U) \times \Gamma(V)$ are defined by

$$\begin{aligned} R' * R &:= \{(u, w) \in \Gamma(U) \times \Gamma(W) \,|\, \exists v \in \Gamma(V), (u, v) \in R \text{ and } (v, w) \in R'\}, \\ \overline{R} &:= \{(v, u) \in \Gamma(V) \times \Gamma(U) \,|\, (u, v) \in R\}, \\ R^* &:= \{(\beta, \alpha) \in \Gamma(V^*) \times \Gamma(U^*) \,|\, \forall (u, v) \in R, \langle \alpha, u\rangle = \langle \beta, v\rangle\}. \end{aligned}$$

Moreover for relations $R \subset \Gamma(U) \times \Gamma(V)$ and $R' \subset \Gamma(V) \times \Gamma(W)$, we set

$$R' \diamond R := \{(u, v, w) \in \Gamma(U) \times \Gamma(V) \times \Gamma(W) \,|\, (u, v) \in R, (v, w) \in R'\}.$$

We notice that $\overline{R' * R} = \overline{R} * \overline{R'}$ and that $\overline{R^*} = \overline{R}^*$. Let $\phi : U \to V$ and $\phi' : V \to W$ be bundle maps. Then we obtain $\mathrm{graph}\,\phi' * \mathrm{graph}\,\phi = \mathrm{graph}(\phi' \circ \phi)$. It is clear that $\overline{\mathrm{graph}\,\phi} = \mathrm{graph}(\phi^{-1})$ if ϕ is invertible.

We define the Nijenhuis torsion of relations in Lie algebroids.

Definition 3 ([4])**.** Let $(A, [\cdot, \cdot]_A, \rho_A)$ be a Lie algebroid over M. Then the *Nijenhuis torsion* of a relation $R \subset \Gamma(A) \times \Gamma(A)$ (or $A \times A$) is a map $\mathcal{T}_R : R \times R \times (R^* \diamond R^*) \to C^\infty(M)$ defined by

$$\mathcal{T}_R((X_1, Y_1), (X_2, Y_2), (\alpha, \beta, \gamma))$$

$$:= \langle \alpha, [Y_1, Y_2]_A \rangle - \langle \beta, [Y_1, X_2]_A + [X_1, Y_2]_A \rangle + \langle \gamma, [X_1, X_2]_A \rangle$$

for all $(X_1, Y_1), (X_2, Y_2)$ in R and (α, β, γ) in $R^* \diamond R^*$. A relation R is *Nijenhuis* if $\mathcal{T}_R$ vanishes.

It follows easily that $\mathcal{T}_R = \mathcal{T}_{\overline{R}}$. A section N in $\Gamma(A^* \otimes A)$ is a *Nijenhuis structure* on A, that is, N satisfies that

$$\mathcal{T}_N(X, Y) := [NX, NY]_A - N[NX, Y]_A - N[X, NY]_A + N^2[X, Y]_A$$

vanishes for any X and Y in $\Gamma(A)$, if and only if graph N is a Nijenhuis relation on A.

2.4. **Dirac pairs on Lie bialgebroids.** For any relations L and $L' \subset A \times A^*$, where A and A^* are vector bundles in duality over M, we set $\mathrm{N}_{L,L'} := \overline{L} * L'$. Then we get $\mathrm{N}_{L,L'} = \overline{\mathrm{N}_{L',L}}$.

Definition 4 ([4])**.** Let (A, A^*) be a Lie bialgebroid over M, L and L' Dirac structures on (A, A^*) (see Remark 2.2). Then (L, L') is a *Dirac pair on* (A, A^*) if $N_{L,L'}$ is a Nijenhuis relation. A Dirac pair (L, L') on (A, A_0^*) is called a *Dirac pair on* A.

Since $\mathcal{T}_{\mathrm{N}_{L,L'}} = \mathcal{T}_{\overline{\mathrm{N}_{L',L}}} = \mathcal{T}_{\mathrm{N}_{L',L}}$, if (L, L') is a Dirac pair, then so is (L', L).

Let A be a Lie algebroid over M. Then a 2-section on A is Poisson if and only if its graph is a Dirac structure on A, i.e., $[\pi, \pi]_A = 0$. A pair (π_1, π_2) of two Poisson structures on A is a *Poisson pair* if a pair $(\overline{\mathrm{graph}\, \pi_1^\sharp}, \overline{\mathrm{graph}\, \pi_2^\sharp})$ is a Dirac pair on A. A Poisson pair (π_1, π_2) is *non-degenerate* if both π_1 and π_2 are non-degenerate. If two Poisson structures π_1 and π_2 on A are compatible, i.e., $\pi_1 + \pi_2$ is also Poisson, then (π_1, π_2) is a Poisson pair by Theorem 2.3 in [4]. Conversely, if a Poisson pair (π_1, π_2) satisfies

$$A^* = (\pi_1^\sharp)^{-1}(\mathrm{Im}\, \pi_2^\sharp) \cap (\pi_2^\sharp)^{-1}(\mathrm{Im}\, \pi_1^\sharp), \tag{2.9}$$

then two Poisson structures π_1 and π_2 on A are compatible. In particular, since a non-degenerate Poisson pair (π_1, π_2) satisfies (2.9), the two Poisson structures π_1 and π_2 on A are compatible.

A 2-cosection on A is presymplectic, i.e., it is d_A-closed, if and only if its graph is a Dirac structure on A. A pair (ω, ω') of two presymplectic structures on A is a *presymplectic pair* if a pair $(\mathrm{graph}\, \omega^\flat, \mathrm{graph}\, \omega'^\flat)$ is a Dirac pair. A presymplectic pair (ω, ω') is *symplectic pair* if both ω and ω' are symplectic. The following proposition for Poisson and presymplectic pairs holds.

Proposition 2.4 ([4])*.* Symplectic pairs are in one-to-one correspondence with non-degenerate Poisson pairs.

At the end of this subsection, we describe PΩ- and ΩN-structures on a Lie algebroid.

Definition 5 ([4]). Let A be a Lie algebroid over M, π a 2-section on A and ω a 2-cosection on A. Then a pair (π, ω) is a *PΩ-structure* on A if π is Poisson and both ω and ω' are d_A-closed, where ω' is a 2-cosection characterized by $\omega'^\flat = \omega^\flat \circ \pi^\sharp \circ \omega^\flat$.

Definition 6 ([4]). Let A be a Lie algebroid over M, ω a 2-cosection on A and N a $(1,1)$-tensor field on A. Then a pair (ω, N) is an *ΩN-structure* on A if $\omega^\flat \circ N = N^* \circ \omega^\flat$, N is Nijenhuis and both ω and ω_N are d_A-closed, where ω_N is a 2-cosection characterized by $\omega_N^\flat = \omega^\flat \circ N$. We can also define a *weak* *ΩN-structure* on A by replacing $\mathcal{T}_N = 0$ with $\omega^\flat(\mathcal{T}_N(X, Y)) = 0$ for any X and Y in $\Gamma(A)$.

These structures are characterized in terms of Dirac pairs on Lie algebroids.

Proposition 2.5 ([4]). Let A be a Lie algebroid over M, π a Poisson structure on A, ω a presymplectic structure on A and N a $(1,1)$-tensor field on A.

(i) If a pair (π, ω) is a PΩ-structure on A, then a pair $(\overline{\operatorname{graph} \pi^\sharp}, \operatorname{graph} \omega^\flat)$ is a Dirac pair on A. Conversely, if $(\overline{\operatorname{graph} \pi^\sharp}, \operatorname{graph} \omega^\flat)$ is a Dirac pair on A, and if π is nondegenerate, then a pair (π, ω) is a PΩ-structure on A;

(ii) If a pair (ω, N) is an ΩN-structure on A, and if $\mathrm{N}^*_{L,L'} = \mathrm{N}^+_{(\omega,N)}$, where $L := \operatorname{graph} \omega^\flat$, $L' := \operatorname{graph} \omega_N^\flat$ and $\mathrm{N}^+_{(\omega,N)} := \{(\omega^\flat X, \omega_N^\flat X) \mid X \in A\} \subset \mathrm{N}^*_{L,L'}$, then a pair (L, L') is a Dirac pair on A. Conversely, if $(\operatorname{graph} \omega^\flat, \operatorname{graph} \omega_N^\flat)$ is a Dirac pair on A, then a pair (ω, N) is a weak ΩN-structure on A.

3. Dirac pairs on Jacobi bialgebroids

In this section, we generalize Dirac pairs on a Lie bialgebroid and introduce Dirac pairs on a Jacobi bialgebroid. We prove that similar properties for Dirac pairs on a Lie bialgebroid also hold for them on a Jacobi bialgebroid.

We start with the definition of Dirac pairs on Jacobi bialgebroids. This is defined as is the case on Lie bialgebroids.

Definition 7. Let $((A, \phi_0), (A^*, X_0))$ be a Jacobi bialgebroid over M, L and L' Dirac structures on $((A, \phi_0), (A^*, X_0))$ (see Definition 2). Then (L, L') is a *Dirac pair on* $((A, \phi_0), (A^*, X_0))$ if $\mathrm{N}_{L,L'}$ is a Nijenhuis relation. If (L, L') are a Dirac pair on $((A, \phi_0), (A_0^*, 0))$, then (L, L') is called *Dirac pair on* (A, ϕ_0).

Since $\mathcal{T}_{\mathrm{N}_{L,L'}} = \mathcal{T}_{\overline{\mathrm{N}_{L',L}}} = \mathcal{T}_{\mathrm{N}_{L',L}}$, if (L, L') is a Dirac pair, then so is (L', L).

We obtain the following property of Nijenhuis relations on Jacobi bialgebroids. This is a generalization of Theorem 2.3 in [4].

Lemma 3.1. *Let (A, ϕ_0) be a Jacobi algebroid. For any π, π' in $\Gamma(\Lambda^2 A)$, the Nijenhuis torsion of $\mathrm{N}_{L,L'}$, where $L := \overline{\operatorname{graph} \pi^\sharp}$ and $L := \overline{\operatorname{graph} \pi'^\sharp}$, satisfies the following:*

$$\begin{aligned}&\mathcal{T}_{\mathrm{N}_{L,L'}}((\pi'^\sharp\xi_1, \pi^\sharp\xi_1), (\pi'^\sharp\xi_2, \pi^\sharp\xi_2), (\xi, \xi', \xi''))\\ &\qquad = [\pi, \pi]_{A,\phi_0}(\xi_1, \xi_2, \xi) + [\pi', \pi']_{A,\phi_0}(\xi_1, \xi_2, \xi'') - 2[\pi, \pi']_{A,\phi_0}(\xi_1, \xi_2, \xi').\end{aligned}$$

By using (2.1), Lemma 3.1 can be shown exactly as Theorem 2.3 in [4].

The following theorem extends the correspondence between Jacobi and Lie bialgebroids in Proposition 2.1 to that between Dirac pairs on Jacobi and Lie bialgebroids. By this theorem, we see that it will be possible to use the theory of Dirac pairs on Lie bialgebroids in the study of Dirac pairs on Jacobi bialgebroids. This is one of the main theorems in this paper.

Theorem 3.2. *Let $((A, \phi_0), (A^*, X_0))$ be a Jacobi bialgebroid over M, π_i a 2-section on A satisfying the Maurer-Cartan type equation and ω_i a 2-cosection on A satisfying the Maurer-Cartan type equation ($i = 1, 2$). Let $\left(\tilde{A}^-_{\phi_0}, \tilde{A}^{*\wedge}_{X_0}\right)$ be the induced Lie bialgebroid over $M \times \mathbb{R}$ (see Proposition 2.1). We set $\tilde{\pi}_i := e^{-t}\pi_i$ in $\Gamma(\Lambda^2 \tilde{A})$ and $\tilde{\omega}_i := e^t\omega_i$ in $\Gamma(\Lambda^2 \tilde{A}^*)$, where t is the standard coordinate in $\mathbb{R}$. Then:*

(i) $(\overline{\operatorname{graph} \tilde{\pi}_1^\sharp}, \overline{\operatorname{graph} \tilde{\pi}_2^\sharp})$ *is a Dirac pair on* $\left(\tilde{A}^-_{\phi_0}, \tilde{A}^{*\wedge}_{X_0}\right)$ *if and only if* $(\overline{\operatorname{graph} \pi_1^\sharp}, \overline{\operatorname{graph} \pi_2^\sharp})$ *is a Dirac pair on* $((A, \phi_0), (A^*, X_0))$*;*

(ii) $(\overline{\operatorname{graph} \tilde{\pi}_1^\sharp}, \operatorname{graph} \tilde{\omega}_2^\flat)$ *is a Dirac pair on* $\left(\tilde{A}^-_{\phi_0}, \tilde{A}^{*\wedge}_{X_0}\right)$ *if and only if* $(\overline{\operatorname{graph} \pi_1^\sharp}, \operatorname{graph} \omega_2^\flat)$ *is a Dirac pair on* $((A, \phi_0), (A^*, X_0))$*;*

(iii) $(\operatorname{graph} \tilde{\omega}_1^\flat, \operatorname{graph} \tilde{\omega}_2^\flat)$ *is a Dirac pair on* $\left(\tilde{A}^-_{\phi_0}, \tilde{A}^{*\wedge}_{X_0}\right)$ *if and only if* $(\operatorname{graph} \omega_1^\flat, \operatorname{graph} \omega_2^\flat)$ *is a Dirac pair on* $((A, \phi_0), (A^*, X_0))$.

In order to prove Theorem 3.2, we need the following lemmas.

Lemma 3.3. *Let $((A, \phi_0), (A^*, X_0))$ be a Jacobi bialgebroid over M, π a 2-section on A and ω a 2-cosection on A. Then* $\operatorname{graph} \tilde{\pi}^\sharp$ *(resp.* $\operatorname{graph} \tilde{\omega}^\flat$*) is a Dirac structure on* $\left(\tilde{A}^-_{\phi_0}, \tilde{A}^{*\wedge}_{X_0}\right)$ *if and only if* $\operatorname{graph} \pi^\sharp$ *(resp.* $\operatorname{graph} \omega^\flat$*) is a Dirac structure on* $((A, \phi_0), (A^*, X_0))$.

Proof. Let $\hat{d}^{X_0}_{A^*}$ be the differential of $\tilde{A}^{*\wedge}_{\phi_0}$. By long calculations, we see that

$$[\tilde{\pi}, \tilde{\pi}]^{-\phi_0}_{A} = e^{-2t}[\pi, \pi]_{A,\phi_0}, \tag{3.1}$$

$$\hat{d}_{A^*}^{X_0}\tilde{\pi} = e^{-2t} d_{A^*,X_0}\pi \tag{3.2}$$

hold on $\Gamma(\tilde{A}^*)$. Therefore we obtain

$$\hat{d}_{A^*}^{X_0}\tilde{\pi} + \frac{1}{2}[\tilde{\pi},\tilde{\pi}]_A^{\bar{\phi}_0} = e^{-2t}\left(d_{A^*,X_0}\pi + \frac{1}{2}[\pi,\pi]_{A,\phi_0}\right) \tag{3.3}$$

on $\Gamma(\tilde{A}^*)$. Since $\Gamma(\tilde{A}^*)$ can be regarded as the set of curves in $\Gamma(A^*)$, $\tilde{\pi}$ satisfies the Maurer-Cartan type equation for $\left(\tilde{A}^-_{\phi_0}, \tilde{A}^{*\wedge}_{X_0}\right)$ if and only if π satisfies the Maurer-Cartan type equation for $((A,\phi_0),(A^*,X_0))$. We can show the same for ω. □

Lemma 3.4. *In the notation of Theorem 3.2, any* $(1,1)$*-tensor field* N *on* $\tilde{A}^-_{\phi_0}$ *independent of* t *can be regarded as a* $(1,1)$*-tensor field on* A *in general. Then* N *is Nijenhuis on* $\tilde{A}^-_{\phi_0}$ *if and only if* N *is Nijenhuis on* A*.*

Proof. We set the Nijenhuis torsion of N on $\tilde{A}^-_{\phi_0}$ and on A by $\mathcal{T}_N^{\tilde{A}^-_{\phi_0}}$ and $\mathcal{T}_N^A$ respectively. By a straightforward calculation, we have for any $\tilde{X}$ and $\tilde{Y}$ in $\Gamma(\tilde{A})$,

$$\mathcal{T}_N^{\tilde{A}^-_{\phi_0}}(\tilde{X},\tilde{Y}) = \mathcal{T}_N^A(\tilde{X},\tilde{Y}). \tag{3.4}$$

Since $\tilde{X}$ and $\tilde{Y}$ in $\Gamma(\tilde{A})$ can be regarded as curves in $\Gamma(A)$, $\mathcal{T}_N^{\tilde{A}^-_{\phi_0}} = 0$ is equivalent with $\mathcal{T}_N^A = 0$. □

Proof of Theorem 3.2. We prove (i). We set $L_i := \overline{\operatorname{graph}\pi_i^\sharp}$ and $\tilde{L}_i := \overline{\operatorname{graph}\tilde{\pi}_i^\sharp}$, $i=1,2$. By Lemma 3.1 and the equation (3.1), we compute that $\mathcal{T}_{\mathrm{N}_{\tilde{L}_1,\tilde{L}_2}} = e^{-2t}\mathcal{T}_{\mathrm{N}_{L_1,L_2}}$ on $\mathrm{N}_{\tilde{L}_1,\tilde{L}_2} \times \mathrm{N}_{\tilde{L}_1,\tilde{L}_2} \times \left(\mathrm{N}^*_{\tilde{L}_1,\tilde{L}_2} \diamond\ \mathrm{N}^*_{\tilde{L}_1,\tilde{L}_2}\right)$. Since any element in $\tilde{L}_i$ can be regarded as a curve in L_i, it is clear that any element in $\mathrm{N}_{\tilde{L}_1,\tilde{L}_2}$ and in $\mathrm{N}^*_{\tilde{L}_1,\tilde{L}_2}\diamond\mathrm{N}^*_{\tilde{L}_1,\tilde{L}_2}$ can also be regarded as a curve in N_{L_1,L_2} and in $\mathrm{N}^*_{L_1,L_2}\diamond\mathrm{N}^*_{L_1,L_2}$, respectively. Therefore the condition $\mathcal{T}_{\mathrm{N}_{\tilde{L}_1,\tilde{L}_2}} = 0$ is equivalent to the condition $\mathcal{T}_{\mathrm{N}_{L_1,L_2}} = 0$. This means (i).

Next, we have $\tilde{\pi}_1^\sharp \circ \tilde{\omega}_2^\flat = \pi_1^\sharp \circ \omega_2^\flat$, so that the $(1,1)$-tensor field $\tilde{\pi}_1^\sharp \circ \tilde{\omega}_2^\flat$ is independent of t. By the definition, $(\overline{\operatorname{graph}\tilde{\pi}_1^\sharp}, \operatorname{graph}\tilde{\omega}_2^\flat)$ is a Dirac pair on $\left(\tilde{A}^-_{\phi_0}, \tilde{A}^{*\wedge}_{X_0}\right)$ if and only if $\mathrm{N}_{\tilde{L}_1,\tilde{L}'_2}$ is a Nijenhuis relation on $\tilde{A}^-_{\phi_0}$, where we set $\tilde{L}_1 := \overline{\operatorname{graph}\tilde{\pi}_1^\sharp}$ and $\tilde{L}'_2 := \operatorname{graph}\tilde{\omega}_2^\flat$. This condition is equivalent with the condition $\pi_1^\sharp \circ \omega_2^\flat$ is a Nijenhuis structure on $\tilde{A}^-_{\phi_0}$ since $\mathrm{N}_{\tilde{L}_1,\tilde{L}'_2} = \operatorname{graph}(\pi_1^\sharp \circ \omega_2^\flat)$. Then by Lemma 3.4, this is equivalent with that $\pi_1^\sharp \circ \omega_2^\flat$ is a Nijenhuis structure on A. Similarly, $(\overline{\operatorname{graph}\pi_1^\sharp}, \operatorname{graph}\omega_2^\flat)$ is a Dirac pair on $((A,\phi_0),(A^*,X_0))$ if and only if $\pi_1^\sharp \circ \omega_2^\flat$ is a Nijenhuis structure on A. Therefore we obtain (ii).

Finally we prove (iii). We set $L_i' := \operatorname{graph}\omega_i^\flat$ and $\tilde{L}_i' := \operatorname{graph}\tilde{\omega}_i^\flat$. A pair $(\tilde{X},\tilde{Y})$ belongs to $\mathrm{N}_{\tilde{L}_1',\tilde{L}_2'}$ if and only if $\tilde{\omega}_1^\flat\tilde{X} = \tilde{\omega}_2^\flat\tilde{Y}$ holds by the definition. By differentiating both sides of $\tilde{\omega}_1^\flat\tilde{X} = \tilde{\omega}_2^\flat\tilde{Y}$ with respect to t, we obtain $\tilde{\omega}_1^\flat\frac{\partial\tilde{X}}{\partial t} = \tilde{\omega}_2^\flat\frac{\partial\tilde{Y}}{\partial t}$ since $\tilde{\omega}_i = e^t\omega_i$ and $\tilde{\omega}_1^\flat\tilde{X} = \tilde{\omega}_2^\flat\tilde{Y}$. This means that a pair $\left(\frac{\partial\tilde{X}}{\partial t},\frac{\partial\tilde{Y}}{\partial t}\right)$ belongs to $\mathrm{N}_{\tilde{L}_1',\tilde{L}_2'}$. Therefore by using this fact and the definition (2.5) of $\bar{[\cdot,\cdot]}_A^{\phi_0}$, it follows that $\mathcal{T}_{\mathrm{N}_{\tilde{L}_1',\tilde{L}_2'}} = \mathcal{T}_{\mathrm{N}_{L_1',L_2'}}$ on $\mathrm{N}_{\tilde{L}_1',\tilde{L}_2'}\times\mathrm{N}_{\tilde{L}_1',\tilde{L}_2'}\times\left(\mathrm{N}^*_{\tilde{L}_1',\tilde{L}_2'}\diamond\ \mathrm{N}^*_{\tilde{L}_1',\tilde{L}_2'}\right)$. Similarly to (i), we see that $\mathcal{T}_{\mathrm{N}_{\tilde{L}_1',\tilde{L}_2'}} = 0$ and $\mathcal{T}_{\mathrm{N}_{L_1',L_2'}} = 0$ are equivalent. □

Remark. It follows immediately that $(\operatorname{graph}\tilde{\omega}_1^\flat, \overline{\operatorname{graph}\tilde{\pi}_2^\sharp})$ is a Dirac pair on $\left(\tilde{A}^-_{\phi_0},\tilde{A}^{*\wedge}_{X_0}\right)$ if and only if $(\operatorname{graph}\omega_1^\flat,\ \overline{\operatorname{graph}\pi_2^\sharp})$ is a Dirac pair on $((A,\phi_0),(A^*,X_0))$ by (iii) in Theorem 3.2 and the fact that if a pair (L,L') is a Dirac pair on $\left(\tilde{A}^-_{\phi_0},\tilde{A}^{*\wedge}_{X_0}\right)$ or $((A,\phi_0),(A^*,X_0))$, so is (L',L).

4. Dirac pairs on Jacobi algebroids

In this section, we consider a Dirac pair (L,L') on a Jacobi algebroid (A,ϕ_0), i.e., a pair consisting of two Dirac structures L and L' on (A,ϕ_0) such that $\mathrm{N}_{L,L'}$ is a Nijenhuis relation on (A,ϕ_0).

4.1. Jacobi and ϕ_0-presymplectic pairs on Jacobi algebroids.

In this subsection, we investigate Jacobi and ϕ_0-presymplectic pairs, which are defined by using Dirac pairs on (A,ϕ_0). We show that these pairs have properties similar to Poisson and presymplectic pairs on a Lie algebroid. In addition, we show the relationship between Jacobi and Poisson pairs.

By Example 8, for any Jacobi algebroid (A,ϕ_0), a 2-section π in $\Gamma(\Lambda^2A)$ is a Jacobi structure on (A,ϕ_0) if and only if $\operatorname{graph}\pi^\sharp$ is a Dirac structure on (A,ϕ_0). Similarly, a 2-cosection ω in $\Gamma(\Lambda^2A^*)$ is a ϕ_0-presymplectic structure on (A,ϕ_0) if and only if $\operatorname{graph}\omega^\flat$ is a Dirac structure on (A,ϕ_0). We define Jacobi and ϕ_0-presymplectic pairs as analogy of Poisson and presymplectic pairs on a Lie algebroid.

Definition 8. Let (A,ϕ_0) be a Jacobi algebroid, π_i a Jacobi structure on (A,ϕ_0) and ω_i a ϕ_0-presymplectic structure on (A,ϕ_0) $(i=1,2)$.

(i) A pair (π_1, π_2) is a *Jacobi pair* if a pair $(\overline{\operatorname{graph} \pi_1^\sharp}, \overline{\operatorname{graph} \pi_2^\sharp})$ is a Dirac pair on (A, ϕ_0). A Jacobi pair (π_1, π_2) is *non-degenerate* if both π_1 and π_2 are non-degenerate;
(ii) A pair (ω_1, ω_2) is a ϕ_0*-presymplectic pair* if a pair $(\operatorname{graph} \omega_1^\flat, \operatorname{graph} \omega_2^\flat)$ is a Dirac pair on (A, ϕ_0). A ϕ_0-presymplectic pair consisting of two ϕ_0-symplectic structures is called a ϕ_0*-symplectic pair.*

It follows immediately from Lemma 3.1 and Definition 8 that a pair (π_1, π_2) consisting of compatible Jacobi structures is a Jacobi pair. Conversely, if a Jacobi pair (π_1, π_2) satisfies (2.9), then π_1 and π_2 are compatible. In particular, since a non-degenerate Jacobi pair (π_1, π_2) satisfies (2.9), the two Jacobi structures π_1 and π_2 are compatible. It is well known that compatible two Jacobi structures themselves are induced by Jacobi-Nijenhuis structures [13], [2] and so on.

The following proposition is a relationship between ϕ_0-symplectic pairs and non-degenerate Jacobi pairs. The proof is similar to Proposition 2.4 (See [4]).

Proposition 4.1. There exists a one-to-one correspondence between ϕ_0-symplectic pairs and non-degenerate Jacobi pairs.

The following example is analogy of Example 3.5 in [4].

Example 9. Let $M := T^*\mathbb{R}^2 \times \mathbb{R}$ and β be the canonical contact form on M. In canonical coordinates (x_1, x_2, y_1, y_2, z) on M, we can write $\beta = -\sum_i y_i dx_i + dz$. We define Ω, ω_H, ω_E and ω_P by

$$\begin{aligned}
\Omega &:= (d\beta, \beta);\\
\omega_{\mathrm{H}} &:= (d\beta_{\mathrm{H}}, \beta_{\mathrm{H}}),\ \beta_{\mathrm{H}} := -y_1 dx_1 + y_2 dx_2 + dz;\\
\omega_{\mathrm{E}} &:= (d\beta_{\mathrm{E}}, \beta_{\mathrm{E}}),\ \beta_{\mathrm{E}} := -y_2 dx_1 + y_1 dx_2 + dz;\\
\omega_{\mathrm{P}} &:= (d\beta_{\mathrm{P}}, \beta_{\mathrm{P}}),\ \beta_{\mathrm{P}} := -y_2 dx_1 + dz.
\end{aligned}$$

Then (Ω, ω_H) and (Ω, ω_E) are a $(0,1)$-symplectic pairs on $(TM \oplus \mathbb{R}, (0,1))$ and (Ω, ω_P) is a $(0,1)$-presymplectic pair on $(TM \oplus \mathbb{R}, (0,1))$. 2-forms $d\beta, d\beta_{\mathrm{H}}, d\beta_{\mathrm{E}}$ and $d\beta_{\mathrm{P}}$ on $T^*\mathbb{R}^2$ coincide with presymplectic structures in Example 3.5 in [4].

Now, we show two relationships between Jacobi and Poisson pairs.

If π is Poisson on a Lie algebroid A over M, then $(\pi, 0)$ is Jacobi on a Jacobi algebroid $(A \oplus \mathbb{R}, (0,1))$ over M. It is well-known that compatible Poisson structures π_1 and π_2 on a Lie algebroid A induce compatible Jacobi structures $(\pi_1, 0)$ and $(\pi_2, 0)$ on a Jacobi algebroid $(A \oplus \mathbb{R}, (0,1))$. The following theorem is a generalization of this relation.

Theorem 4.2. *Let (π_1, π_2) be a pair of 2-sections on a Lie algebroid A over M. Then (π_1, π_2) is a Poisson pair on A if and only if $((\pi_1, 0), (\pi_2, 0))$ is a Jacobi pair on a Jacobi algebroid $(A \oplus \mathbb{R}, (0, 1))$ over M.*

Proof. It follows immediately that (ξ, f) belongs to $((\pi_1, 0)^\sharp)^{-1}(\mathrm{Im}\,(\pi_2, 0)^\sharp) \cap ((\pi_2, 0)^\sharp)^{-1}(\mathrm{Im}\,(\pi_1, 0)^\sharp)$ if and only if ξ belongs to $(\pi_1^\sharp)^{-1}(\mathrm{Im}\,\pi_2^\sharp) \cap (\pi_2^\sharp)^{-1}(\mathrm{Im}\,\pi_1^\sharp)$. For any (ξ_i, f_i) in $\Gamma(A^*) \oplus C^\infty(M)$ and (ξ, f) in $((\pi_1, 0)^\sharp)^{-1}(\mathrm{Im}\,(\pi_2, 0)^\sharp) \cap ((\pi_2, 0)^\sharp)^{-1}(\mathrm{Im}\,(\pi_1, 0)^\sharp)$, it follows that

$$[(\pi_1, 0), (\pi_2, 0)]_{A \oplus \mathbb{R}, (0,1)}((\xi_1, f_1), (\xi_2, f_2), (\xi, f)) = [\pi_1, \pi_2]_A(\xi_1, \xi_2, \xi),$$

so that the consequence holds by Lemma 3.1. □

The other relation between Jacobi and Poisson pairs is the following theorem.

Theorem 4.3. *Let (π_1, π_2) be a pair of 2-sections on a Jacobi algebroid (A, ϕ_0) over M. Then (π_1, π_2) is a Jacobi pair on (A, ϕ_0) if and only if $(\tilde{\pi}_1, \tilde{\pi}_2)$ is a Poisson pair on a Lie algebroid $\tilde{A}^-_{\phi_0}$ over $M \times \mathbb{R}$, where $\tilde{\pi}_i := e^{-t}\pi_i$ in $\Gamma(\tilde{A})$.*

Proof. By Lemma 3.3, a 2-section π on A is a Jacobi structure on (A, ϕ_0) if and only if a 2-section $\tilde{\pi}$ on $\tilde{A}$ is a Poisson structure on $\tilde{A}^-_{\phi_0}$. By the definitions of Jacobi and Poisson pairs and Theorem 3.2, a pair (π_1, π_2) is a Jacobi pair on (A, ϕ_0) if and only if a pair $(\tilde{\pi}_1, \tilde{\pi}_2)$ is a Poisson pair on $\tilde{A}_{\bar{\phi}_0}$. □

4.2. **JΩ- and ΩN-structures.** In this subsection, we define JΩ- and ΩN-structures on Jacobi algebroids, and show a relationship between JΩ- (resp. ΩN-) structures on Jacobi algebroids and PΩ- (resp. ΩN-) structures on Lie algebroids. By using the relationship, we show that JΩ- and ΩN-structures on Jacobi algebroids can be characterized in terms of Dirac pairs.

We start with the definitions of JΩ- and ΩN-structures on a Jacobi algebroid.

Definition 9. Let (A, ϕ_0) be a Jacobi algebroid over M, π a 2-section on A, ω a 2-cosection on A and N a $(1, 1)$-tensor field on A. In the definitions of PΩ- and ΩN-structures on a Lie algebroid (Definition 5 and 6), by replacing the conditions "Poisson" and "d_A-closed" with "Jacobi" and "d_{A,ϕ_0}-closed", respectively, we obtain the definitions of a J*Ω-structure* (π, ω) and an *(weak)* ΩN-*structure* (ω, N) on a Jacobi algebroid.

It is clear that the definitions of JΩ- and (weak) ΩN-structures on a Jacobi algebroid (A, ϕ_0) coincide with the definitions of PΩ- and (weak) ΩN-structures on a Lie algebroid A when $\phi_0 = 0$.

First, the following proposition means that there is a one-to-one correspondence between JΩ-structures on a Jacobi algebroid (A,ϕ_0) and PΩ-structures on a Lie algebroid $\tilde{A}^-_{\phi_0}$.

Proposition 4.4. Let (A,ϕ_0) be a Jacobi algebroid over M. Then a pair (π,ω) is a JΩ-structure on (A,ϕ_0) if and only if a pair $(\tilde{\pi},\tilde{\omega})$ is a PΩ-structure on $\tilde{A}^-_{\phi_0}$, where $\tilde{\pi}=e^{-t}\pi, \tilde{\omega}=e^t\omega$.

Proof. By Lemma 3.3, a 2-section π on A is a Jacobi structure on (A,ϕ_0) if and only if a 2-section $\tilde{\pi}$ on $\tilde{A}$ is a Poisson structure on $\tilde{A}^-_{\phi_0}$, and a 2-cosection ω on A is a ϕ_0-presymplectic structure on (A,ϕ_0) if and only if a 2-cosection $\tilde{\omega}$ on $\tilde{A}$ is a presymplectic structure on $\tilde{A}^-_{\phi_0}$. Setting $\widetilde{(\omega')}:=e^t\omega'$ and $(\tilde{\omega})':=\tilde{\omega}^\flat\circ\tilde{\pi}^\sharp\circ\tilde{\omega}^\flat$, we obtain $\widetilde{(\omega')}=(\tilde{\omega})'$ since $\tilde{\pi}^\sharp\circ\tilde{\omega}^\flat=\pi^\sharp\circ\omega^\flat$. Therefore, since $\bar{d}_A^{\phi_0}(\tilde{\omega})'=\bar{d}_A^{\phi_0}\widetilde{(\omega')}=e^t d_{A,\phi_0}\omega'$ by Lemma 3.3, it follows that ω' is d_{A,ϕ_0}-closed if and only if $(\tilde{\omega})'$ is $\bar{d}_A^{\phi_0}$-closed. □

Proposition 4.5. Let (A,ϕ_0) be a Jacobi algebroid over M. Then a pair (ω,N) is an ΩN- (resp. a weak ΩN-)structure on (A,ϕ_0) if and only if a pair $(\tilde{\omega},N)$ is an ΩN- (resp. a weak ΩN-)structure on $\tilde{A}^-_{\phi_0}$, where $\tilde{\omega}=e^t\omega$ and a $(1,1)$-tensor field N on A is regarded as a $(1,1)$-tensor field independent of t on $\tilde{A}^-_{\phi_0}$.

Proof. By Lemma 3.3, a 2-cosection ω on A is a ϕ_0-presymplectic structure on (A,ϕ_0) if and only if a 2-cosection $\tilde{\omega}$ on $\tilde{A}$ is a presymplectic structure on $\tilde{A}^-_{\phi_0}$. We have $\tilde{\omega}^\flat\circ N=e^t\omega^\flat\circ N$ and $N^*\circ\tilde{\omega}^\flat=e^tN^*\circ\omega^\flat$, so that the commutativity of $\tilde{\omega}$ and N is equivalent with that of ω and N. Since $\widetilde{(\omega_N)}=(\tilde{\omega})_N$ holds, where $\widetilde{(\omega_N)}:=e^t\omega_N$, ω_N is d_{A,ϕ_0}-closed if and only if $(\tilde{\omega})_N$ is $\bar{d}_A^{\phi_0}$-closed by Lemma 3.3. Finally, by Lemma 3.4 (resp. the equation (3.4)), the consequence holds. □

The following theorem is characterizations of JΩ- and (weak) ΩN-structures on a Jacobi algebroid (A,ϕ_0) by Dirac pairs, and a generalization of Proposition 2.5.

Theorem 4.6. *Let (A,ϕ_0) be a Jacobi algebroid over M, π a Jacobi structure on (A,ϕ_0), ω a ϕ_0-presymplectic structure on (A,ϕ_0) and N a $(1,1)$-tensor field on A. Then*

(i) *If a pair (π,ω) is a JΩ-structure on (A,ϕ_0), then a pair $(\overline{\operatorname{graph}\pi^\sharp}, \operatorname{graph}\omega^\flat)$ is a Dirac pair on (A,ϕ_0). Conversely, if $(\overline{\operatorname{graph}\pi^\sharp}, \operatorname{graph}\omega^\flat)$ is a Dirac pair on (A,ϕ_0), and if π is non-degenerate, then a pair (π,ω) is a JΩ-structure on (A,ϕ_0).*

(ii) *If a pair* (ω, N) *is an* ΩN*-structure on* (A, ϕ_0)*, and if* $\mathrm{N}^*_{L,L'} = \mathrm{N}^+_{(\omega,N)}$*, where* $L := \mathrm{graph}\,\omega^\flat$*,* $L' := \mathrm{graph}\,\omega^\flat_N$ *and* $\mathrm{N}^+_{(\omega,N)} := \{(\omega^\flat X, \omega^\flat_N X) \mid X \in A\} \subset \mathrm{N}^*_{L,L'}$*, then a pair* (L, L') *is a Dirac pair on* (A, ϕ_0)*. Conversely, if* $(\mathrm{graph}\,\omega^\flat, \mathrm{graph}\,\omega^\flat_N)$ *is a Dirac pair on* (A, ϕ_0)*, then a pair* (ω, N) *is a weak* ΩN*-structure on* (A, ϕ_0)*.*

Proof. (i) holds by Proposition 4.4, (i) in Proposition 2.5 and (ii) in Theorem 3.2. Next, prove (ii). We set $\tilde{L} := \mathrm{graph}\,\tilde{\omega}^\flat$, $\tilde{L}' := \mathrm{graph}\,\tilde{\omega}^\flat_N$. We notice that relations $\mathrm{N}^+_{(\tilde{\omega},N)}, \mathrm{N}_{\tilde{L},\tilde{L}'}$ and $\mathrm{N}^*_{\tilde{L},\tilde{L}'}$ can be regarded as the sets of all curves in $\mathrm{N}^+_{(\omega,N)}, \mathrm{N}_{L,L'}$ and $\mathrm{N}^*_{L,L'}$, respectively. Then we obtain (ii) by Proposition 4.5, (ii) in Proposition 2.5 and (iii) in Theorem 3.2. □

Remark. Theorem 4.6 can also be proved directly by long calculations. However, as above, we can prove it more easily by using Theorem 3.2, Proposition 4.4, Proposition 4.5 and the theory of Dirac pairs on Lie algebroids.

Example 10. In Example 9, we denote the opposite of the non-degenerate Jacobi structure corresponding with a $(0, 1)$-symplectic structure Ω on $(TM \oplus \mathbb{R}, (0, 1))$ by Π, i.e., Π is a 2-vector field characterized by $\Pi^\sharp = (\Omega^\flat)^{-1}$. Then it follows from Theorem 4.6 that three pairs $(\Pi, \omega_{\mathrm{H}})$, $(\Pi, \omega_{\mathrm{E}})$ and $(\Pi, \omega_{\mathrm{P}})$ are JΩ-structures on $(TM \oplus \mathbb{R}, (0, 1))$ and that three pairs (Ω, N_{H}), (Ω, N_{E}) and (Ω, N_{P}) are ΩN-structures on $(TM \oplus \mathbb{R}, (0, 1))$, where $N_{\mathrm{X}} := \Pi^\sharp \circ \omega^\flat_{\mathrm{X}}$ for $\mathrm{X} = \mathrm{H}, \mathrm{E}, \mathrm{P}$.

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