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TRAVELING FRONT SOLUTIONS FOR PERTURBED REACTION-DIFFUSION EQUATIONS

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Abstract. Traveling front solutions have been studied for reaction-diffusion equations with various kinds of nonlinear terms. One of the interesting subjects is the existence and non-existence of them. In this paper, we prove that, if a traveling front solution exists for a reaction-diffusion equation with a nonlinear term, it also exists for a reaction-diffusion equation with a perturbed nonlinear term. In other words, a traveling front is robust under perturbation on a nonlinear term.

1. Introduction

In this paper we study a reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u), \quad x \in \mathbb{R},\ t > 0, \tag{1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{2}$$

where u_0 is a given bounded and uniformly continuous function from $\mathbb{R}$ to $\mathbb{R}$. Now f is of class C^1 in an open interval including $[0,1]$ and satisfies $f(0)=0$, $f(1)=0$ and

$$f'(1) < 0. \tag{3}$$

Equation (1) with such a nonlinear term f appears in many models, and it has often a traveling front solution. See [1, 2, 7, 8, 21, 16, 20] for a general theory of traveling front solutions. Equation (1) is called bistable or multistable if we assume $f'(0) < 0$ in addition. If $f(u) = -u(u-a)(u-1)$ for $a \in (0,1)$, (1) is called the Nagumo equation or the Allen–Cahn equation. See [15, 1, 2, 5, 7, 19, 6, 18, 20] for traveling fronts of (1) for bistable or multistable nonlinear terms. Traveling fronts of (1) for the Fisher–KPP equations have been studied. A typical nonlinear term is $f(u) = u(1-u)$. See [9, 12, 14, 4, 21] for traveling fronts of (1) for the Fisher–KPP equations. For traveling fronts of (1) for combustion models, see [10, 11, 3, 17] for instance. For traveling fronts of (1) for degenerate monostable nonlinear terms, see [13, 22, 23].

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If $U \in C^2(\mathbb{R})$ and $c \in \mathbb{R}$ satisfy

$$\begin{cases} U''(y) + cU'(y) + f(U(y)) = 0, & y \in \mathbb{R}, \\ U(-\infty) = 1, \quad U(\infty) = 0, \end{cases} \tag{4}$$

$u(x,t) = U(x - ct)$ becomes a traveling front solution to (1). We call (4) the profile equation of (c, U), if it exists. In this case we necessarily have

$$U'(y) < 0, \qquad y \in \mathbb{R}$$

by using [7, Lemma 2.1]. Assume that f_0 is of class C^1 in an open interval including $[0,1]$ with $f_0(0) = 0$, $f_0(1) = 0$ and

$$f_0'(1) < 0, \tag{5}$$

and assume that there exist $U_0 \in C^2(\mathbb{R})$ and $c_0 \in \mathbb{R}$ that satisfy

$$\begin{cases} U_0''(y) + c_0U_0'(y) + f_0(U_0(y)) = 0, & y \in \mathbb{R}, \\ U_0(-\infty) = 1, \quad U_0(\infty) = 0. \end{cases} \tag{6}$$

Then we necessarily have

$$U_0'(y) < 0, \qquad y \in \mathbb{R}. \tag{7}$$

Assume that $f - f_0 \in C_0^1(0,1]$. Here $C_0^1(0,1]$ is the set of functions in $C^1(0,1]$ whose supports lie in $(0,1]$. The following is the main assertion in this paper.

Theorem 1. *Assume that there exists (c_0, U_0) that satisfies (6). Assume that $f - f_0 \in C_0^1(0,1]$ and let $\|f - f_0\|_{C^1[0,1]}$ be small enough. Then there exists (c, U) that satisfies (4). If $\|f - f_0\|_{C^1[0,1]}$ goes to zero, c converges to c_0 and $\|U - U_0\|_{C^2(\mathbb{R})}$ goes to zero.*

We write the proof of Theorem 1 in Section 2. See Figure 1 in Section 2 for an idea of the proof. Theorem 1 asserts that a traveling front is robust under perturbation on a nonlinear term by assuming (5). If we assume $f_0'(0) < 0$ in addition, Theorem 1 shows that traveling fronts for bistable or multistable nonlinear terms are robust under perturbations. See Corollary 10 in Section 3 for this argument.

For the robustness of traveling fronts, one can see [7, 8, 1, 2, 19] for instance. However, the existence of (c, U) to (4) is an open problem as far as the authors know if one assumes the existence of (c_0, U_0) to (6) without assuming (5) and just assumes that $\|f - f_0\|_{C^1[0,1]}$ is small enough. Theorem 1 might be a new step to attack this general robustness problem of traveling fronts.

2. Proof of Theorem 1

In view of (4), we search (c, U) that satisfies

$$\begin{gathered} \frac{\mathrm{d}}{\mathrm{d}y}\begin{pmatrix} U \\ U' \end{pmatrix} = \begin{pmatrix} U' \\ -cU' - f(U) \end{pmatrix}, \qquad y \in \mathbb{R}, \\ U'(y) < 0, \qquad y \in \mathbb{R}, \\ U(-\infty) = 1, \quad U(\infty) = 0. \end{gathered} \tag{8}$$

Equations (4) and (8) are equivalent. Using (6), we have (c_0, U_0) that satisfies

$$\begin{gathered} \frac{\mathrm{d}}{\mathrm{d}y}\begin{pmatrix} U_0 \\ U_0' \end{pmatrix} = \begin{pmatrix} U_0' \\ -c_0U_0' - f_0(U_0) \end{pmatrix}, \qquad y \in \mathbb{R}, \\ U_0'(y) < 0, \qquad y \in \mathbb{R}, \\ U_0(-\infty) = 1, \quad U_0(\infty) = 0. \end{gathered} \tag{9}$$

We study the following ordinary differential equation

$$\begin{cases} p'(z) = -c - \dfrac{f(z)}{p(z)}, & 0 < z < 1, \\ p(z) < 0, & 0 < z < 1, \\ p(0) = 0, \quad p(1) = 0. \end{cases} \tag{10}$$

We write the solution of (10) as $p(z; c, f)$ if it exists. There exists a solution (c, U) to (8) if and only if $p(z; c, f)$ exists. Indeed, if (c, U) satisfies (8), we define p by $p(U(y)) = U'(y)$ for $y \in \mathbb{R}$, and have (10). If $p(z; c, f)$ satisfies (10), we define

$$y = \int_a^U \frac{\mathrm{d}z}{p(z)}, \qquad 0 < z < 1, \tag{11}$$

and have (8). Here a is an arbitrarily given number. Similarly, there exists a solution (c_0, U_0) to (9) if and only if $p(z; c_0, f_0)$ exists. By the standing assumption, we have $p(z; c_0, f_0)$ that satisfies

$$\begin{cases} p_z(z; c_0, f_0) = -c_0 - \dfrac{f_0(z)}{p(z; c_0, f_0)}, & 0 < z < 1, \\ p(z; c_0, f_0) < 0, & 0 < z < 1, \\ p(0; c_0, f_0) = 0, \quad p(1; c_0, f_0) = 0. \end{cases} \tag{12}$$

Now we choose $\alpha_0 \in (0, 1)$ such that we have

$$f_0(u) > 0 \qquad \text{if} \quad u \in [\alpha_0, 1).$$

Also we choose $\alpha \in (0, 1)$ such that we have

$$f(u) > 0 \qquad \text{if} \quad u \in [\alpha, 1).$$

Now we can have $|\alpha - \alpha_0| \to 0$ as $\|f - f_0\|_{C^1[0,1]} \to 0$. We set

$$\alpha_* = \frac{1+\alpha_0}{2}. \tag{13}$$

It suffices to assume that $\|f - f_0\|_{C^1[0,1]}$ is small enough and we always have

$$\alpha < \alpha_*.$$

Now we use the following assertion.

Lemma 2 ([20])**.** *For every $s \in \mathbb{R}$ there exists $p_+(z; s, f)$ defined for $z \in [\alpha, 1]$, such that one has*

$$(p_+)_z\,(z; s, f) = -s - \frac{f(z)}{p_+(z; s, f)}, \qquad z \in (\alpha, 1), \tag{14}$$

$$p_+(z; s, f) < 0, \qquad z \in [\alpha, 1), \tag{15}$$

$$p_+(1; s, f) = 0, \tag{16}$$

$$(p_+)_z\,(1; s, f) = \frac{-s + \sqrt{s^2 - 4f'(1)}}{2} > 0. \tag{17}$$

If $s_1 < s_2$, one has

$$p_+(z; s_1, f) < p_+(z; s_2, f), \qquad z \in [\alpha, 1).$$

Proof. This assertion follows from [20, Theorem 1.1] and its proof. □

Since $f - f_0 \in C_0^1(0, 1]$, we can choose $z_* \in (0, 1)$ with

$$f(z) = f_0(z) \qquad \text{if} \quad 0 \le z \le z_*. \tag{18}$$

Let $s \in \mathbb{R}$ be arbitrarily given and let $p_+(z; s, f)$ be given by Lemma 2. We choose $M \ge 1$ large enough such that we have

$$|s| + \frac{\|f\|_{C[0,1]}}{M} \le M. \tag{19}$$

In Lemma 2, $p_+(z; s, f)$ is defined only on $[\alpha, 1]$. We extend $p_+(z; s, f)$ for all possible z, say $z \in (\zeta_0(s, f), 1)$. Then we have

$$\zeta_0(s, f) \le \alpha < \alpha_*.$$

Since f is defined in an open interval including $[0,1]$, $\zeta_0(s,f)$ can be a negative value. Now we have

$$(20)\qquad \begin{aligned} &(p_+)_z\,(z;s,f) = -s - \frac{f(z)}{p_+(z;s,f)}, \qquad z \in (\zeta_0(s,f),1),\\ &p_+(z;s,f) < 0, \qquad z \in (\zeta_0(s,f),1),\\ &p_+(1;s,f) = 0,\\ &(p_+)_z\,(1;s,f) = \frac{-s+\sqrt{s^2-4f'(1)}}{2} > 0. \end{aligned}$$

Now we assert the following lemma.

Lemma 3. *Let $s \in \mathbb{R}$ be arbitrarily given and let $M \geq 1$ satisfy (19). Let $p_+(z;s,f)$ be given by Lemma 2 and one extends $p_+(z;s,f)$ for all possible z, say $z \in (\zeta_0(s,f),1)$. Then one has*

$$(21)\qquad 0 < -p_+(z;s,f) < 2M, \qquad \zeta_0(s,f) < z < 1.$$

One has

$$p_+(0;s,f) < 0, \quad \zeta_0(s,f) < 0,$$

or one has

$$(22)\qquad \zeta_0(s,f) \in [0,\alpha), \quad p_+(\zeta_0(s,f);s,f) = 0.$$

Proof. Assume that there exists $\eta_0 \in (0,1)$ with

$$-p_+(\eta_0;s,f) \geq 2M.$$

Then we can define $\eta_1 \in (\eta_0, 1]$ by

$$\eta_1 = \sup\{\eta \in (\eta_0,1) \mid -p_+(z;s,f) \geq M \quad \text{for all } z \in [\eta_0,\eta]\}.$$

Using $p_+(1;s,f) = 0$, we have $0 < \eta_0 < \eta_1 < 1$. Using (19) and (20), we obtain

$$\begin{aligned} &-p_+(\eta_1;s,f)\\ =&-p_+(\eta_0;s,f) - \int_0^1 (p_+)_z\,(\theta\eta_1 + (1-\theta)\eta_0;s,f)\,\mathrm{d}\theta\,(\eta_1-\eta_0)\\ \geq& 2M - M(\eta_1-\eta_0) > M. \end{aligned}$$

This contradicts the definition of η_1. Now we obtain (21).

If $\zeta_0(s,f) < 0$, we have $p_+(0;s,f) < 0$. It suffices to prove (22) by assuming $\zeta_0(s,f) \geq 0$. Then necessarily we have $\zeta_0(s,f) \in [0,\alpha)$. Assume that (22) does not hold true. Then we have

$$\beta = \limsup_{z \to \zeta_0(s,f)} (-p_+(z;s,f)) \in (0,2M].$$

Using (20), we obtain

$$(p_+)_z\,(\zeta_0(s,f);s,f) = -s + \frac{f(0)}{\beta}.$$

Since the right-hand side is bounded, it is bounded on a neighborhood of $(\zeta_0(s,f),-\beta)$ and we can extend $p_+(z;s,f)$ for $z \in (\zeta_0(s,f)-\delta, \zeta_0(s,f))$ with some $\delta > 0$ that is small enough. This contradicts the definition of $\zeta_0(s,f)$. Thus we obtain (22) and complete the proof. ☐

Now we have

$$\begin{aligned} &\zeta_0(c_0,f_0) = 0, \\ &p_+(z;c_0,f_0) = p(z;c_0,f_0), \qquad 0<z<1. \end{aligned} \tag{23}$$

Now we assert the following proposition.

Proposition 4. *Let $s \in \mathbb{R}$ be arbitrarily given. Then one has*

$$\begin{aligned} &p_+(z;s,f) - p_+(z;c_0,f_0) \\ =&\int_z^1 \left(s - c_0 + \frac{f(z') - f_0(z')}{p_+(z';c_0,f_0)} \right) \exp\left(-\int_z^{z'} \frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta \right) \mathrm{d}z' \end{aligned}$$

for $\zeta_0(s,f) < z < 1$.

Proof. We put

$$w(z) = p_+(z;s,f) - p_+(z;c_0,f_0)$$

and have

$$w'(z) = -s + c_0 - \frac{f(z)}{p_+(z;s,f)} + \frac{f_0(z)}{p_+(z;c_0,f_0)}$$

for $\zeta_0(s,f) < z < 1$. Now we have

$$-\frac{f(z)}{p_+(z;s,f)} + \frac{f_0(z)}{p_+(z;c_0,f_0)} = \frac{-f(z)p_+(z;c_0,f_0) + f_0(z)p_+(z;s,f)}{p_+(z;s,f)p_+(z;c_0,f_0)}$$

and

$$\begin{aligned} &-f(z)p_+(z;c_0,f_0) + f_0(z)p_+(z;s,f) \\ =&-f(z)\left(p_+(z;c_0,f_0) - p_+(z;s,f)\right) - f(z)p_+(z;s,f) + f_0(z)p_+(z;s,f) \\ =&f(z)w(z) - \left(f(z) - f_0(z)\right)p_+(z;s,f). \end{aligned}$$

Then we obtain

$$w'(z) - \frac{f(z)}{p_+(z;s,f)p_+(z;c_0,f_0)}w(z) = -s + c_0 - \frac{f(z) - f_0(z)}{p_+(z;c_0,f_0)}$$

for $\zeta_0(s,f) < z < 1$. Then we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}z}\left(w(z)\exp\left(\int_z^1 \frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right)\right)\\
&=\left(w'(z)-\frac{f(z)}{p_+(z;s,f)p_+(z;c_0,f_0)}w(z)\right)\\
&\qquad\qquad\times\exp\left(\int_z^1 \frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right)\\
&=\left(-s+c_0-\frac{f(z)-f_0(z)}{p_+(z;c_0,f_0)}\right)\exp\left(\int_z^1 \frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right).
\end{aligned}$$

Let $\theta' \in (z,1)$ be arbitrarily given. Integrating the both sides of the equality stated above over (z,θ'), we have

$$\begin{aligned}
&-w(z)\exp\left(\int_z^1 \frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right)\\
&+w(\theta')\exp\left(\int_{\theta'}^1 \frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right)\\
=&-\int_z^{\theta'}\left(s-c_0+\frac{f(z')-f_0(z')}{p_+(z';c_0,f_0)}\right)\exp\left(\int_{z'}^1 \frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right)\mathrm{d}z'
\end{aligned}$$

for $\zeta_0(s,f) < z < \theta'$. Now we find

$$\begin{aligned}
w(z) = {}& w(\theta')\exp\left(-\int_z^{\theta'} \frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right)\\
&+\int_z^{\theta'}\left(s-c_0+\frac{f(z')-f_0(z')}{p_+(z';c_0,f_0)}\right)\\
&\qquad\times\exp\left(-\int_z^{z'} \frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right)\mathrm{d}z'
\end{aligned}\tag{24}$$

for $\zeta_0(s,f) < z < \theta'$. Using

$$\begin{aligned}
&f(\zeta) > 0 \qquad \text{if} \quad \zeta \in (\alpha_*,1),\\
&p_+(\zeta;s,f) < 0, \quad p_+(\zeta;c_0,f_0) < 0, \qquad \zeta_0(s,f) < \zeta < 1,
\end{aligned}$$

we have

$$\lim_{\theta'\to 1} w(\theta')\exp\left(-\int_z^{\theta'} \frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right) = 0$$

and

$$\lim_{\theta'\to 1}\int_z^{\theta'}\left(s-c_0+\frac{f(z')-f_0(z')}{p_+(z';c_0,f_0)}\right)\times\exp\left(-\int_z^{z'}\frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right)\mathrm{d}z'$$
$$=\int_z^{1}\left(s-c_0+\frac{f(z')-f_0(z')}{p_+(z';c_0,f_0)}\right)\times\exp\left(-\int_z^{z'}\frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right)\mathrm{d}z'$$

for $\zeta_0(s,f)<z<1$. Passing to the limit of $\theta'\to 1$ in (24), we obtain

$$w(z)=\int_z^{1}\left(s-c_0+\frac{f(z')-f_0(z')}{p_+(z';c_0,f_0)}\right)\exp\left(-\int_z^{z'}\frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right)\mathrm{d}z'$$

for $\zeta_0(s,f)<z<1$. This completes the proof. □

Now we take $\varepsilon_0\in(0,1-\alpha_*)$ small enough such that we have

$$(p_+)_z\,(z;c_0,f_0)>\frac{1}{2}\,(p_+)_z\,(1;c_0,f_0)>0\qquad\text{if}\quad z\in(1-\varepsilon_0,1).\tag{25}$$

We show that $|p_+(\alpha_*;s,f)-p_+(\alpha_*;c_0,f_0)|$ converges to 0 as $|s-c_0|+\|f-f_0\|_{C^1[0,1]}$ goes to 0 in the following lemma.

Lemma 5. *Let $\alpha_*\in(0,1)$ be as in (13) and let $\varepsilon_0\in(0,1-\alpha_*)$ satisfy (25). Then one has*

$$\sup_{z\in[\alpha_*,1]}|p_+(z;s,f)-p_+(z;c_0,f_0)|$$
$$\le(1-\alpha_*)|s-c_0|+\frac{(1-\varepsilon_0-\alpha_*)\|f-f_0\|_{C[0,1]}}{\min_{z'\in[\alpha_*,1-\varepsilon_0]}\left(-p_+(z';c_0,f_0)\right)}+\frac{\varepsilon_0\|f-f_0\|_{C^1[0,1]}}{\min_{\zeta'\in[1-\varepsilon_0,1]}|(p_+)_z(\zeta';c_0,f_0)|}.$$

Proof. We have

$$f(z)>0\qquad\text{if}\quad z\in[\alpha_*,1),$$
$$p_+(z;s,f)<0\qquad\text{if}\quad z\in[\alpha_*,1),$$
$$p_+(z;c_0,f_0)<0\qquad\text{if}\quad z\in(0,1).$$

Then, using Proposition 4, we have

$$\max_{z\in[\alpha_*,1]} |p_+(z;s,f)-p_+(z;c_0,f_0)| \le \int_{\alpha_*}^1 \left(|s-c_0| + \left|\frac{f(z')-f_0(z')}{p_+(z';c_0,f_0)}\right|\right) \mathrm{d}z'.$$

Now we find

$$\begin{aligned}(26)\qquad &\int_{\alpha_*}^1 \left(|s-c_0| + \left|\frac{f(z')-f_0(z')}{p_+(z';c_0,f_0)}\right|\right) \mathrm{d}z' \\ &\le (1-\alpha_*)\,|s-c_0| + \int_{\alpha_*}^1 \left|\frac{f(z')-f_0(z')}{p_+(z';c_0,f_0)}\right| \mathrm{d}z'.\end{aligned}$$

If $z' \in (\alpha_*, 1-\varepsilon_0]$, we have

$$\left|\frac{f(z')-f_0(z')}{p_+(z';c_0,f_0)}\right| \le \frac{\|f-f_0\|_{C[0,1]}}{\min_{z'\in[\alpha_*,1-\varepsilon_0]}(-p_+(z';c_0,f_0))}$$

and thus

$$\int_{\alpha_*}^{1-\varepsilon_0} \left|\frac{f(z')-f_0(z')}{p_+(z';c_0,f_0)}\right| \mathrm{d}z' \le \frac{(1-\varepsilon_0-\alpha_*)\|f-f_0\|_{C[0,1]}}{\min_{z'\in[\alpha_*,1-\varepsilon_0]}(-p_+(z';c_0,f_0))}.$$

If $z' \in (1-\varepsilon_0, 1)$, we have

$$\frac{f(z')-f_0(z')}{p_+(z';c_0,f_0)} = \frac{f'(\zeta')-f_0'(\zeta')}{(p_+)_z(\zeta';c_0,f_0)}$$

for some $\zeta' \in (z', 1)$. Thus, if $z' \in (1-\varepsilon_0, 1)$, we find

$$\left|\frac{f(z')-f_0(z')}{p_+(z';c_0,f_0)}\right| \le \frac{\|f-f_0\|_{C^1[0,1]}}{\min_{\zeta'\in[1-\varepsilon_0,1]}|(p_+)_z(\zeta';c_0,f_0)|}$$

and

$$\int_{1-\varepsilon_0}^{1} \left|\frac{f(z')-f_0(z')}{p_+(z';c_0,f_0)}\right| \mathrm{d}z' \le \frac{\varepsilon_0\|f-f_0\|_{C^1[0,1]}}{\min_{\zeta'\in[1-\varepsilon_0,1]}|(p_+)_z(\zeta';c_0,f_0)|}.$$

Then we obtain

$$\begin{aligned}&\int_{\alpha_*}^1 \left|\frac{f(z')-f_0(z')}{p_+(z';c_0,f_0)}\right| \mathrm{d}z' \\ &\le \frac{(1-\varepsilon_0-\alpha_*)\|f-f_0\|_{C[0,1]}}{\min_{z'\in[\alpha_*,1-\varepsilon_0]}(-p_+(z';c_0,f_0))} + \frac{\varepsilon_0\|f-f_0\|_{C^1[0,1]}}{\min_{\zeta'\in[1-\varepsilon_0,1]}|(p_+)_z(\zeta';c_0,f_0)|}.\end{aligned}$$

Combining this inequality and (26), we complete the proof. □

Lemma 5 asserts that $|p_+(z;s,f)-p_+(z;c_0,f_0)|$ converges to 0 on an interval $[\alpha_*,1]$ as $|s-c_0|+\|f-f_0\|_{C^1[0,1]}$ goes to 0. Does this convergence hold true for every compact interval in $(0,1]$? To answer this question, we assert the following lemma.

Lemma 6. *Let $s \in \mathbb{R}$. Let $z_* \in (0,1)$ satisfy (18) and let $z_1 \in (0, z_*)$ be arbitrarily given. As $|s-c_0| + \|f-f_0\|_{C^1[0,1]}$ goes to zero, $\zeta_0(s,f)$ converges to zero and*

$$\sup_{z\in[z_1,1]} |p_+(z;s,f) - p_+(z;c_0,f_0)|$$

converges to zero.

Proof. We will prove $\zeta_0(s,f) < z_1$ if $|s-c_0| + \|f-f_0\|_{C^1[0,1]}$ is small enough. Let (c_0, U_0) satisfy (9). There exists $-\infty < y_0 < y_1 < \infty$ such that we have

$$U_0(y_0) = \alpha_*, \quad U_0(y_1) = \frac{z_1}{2}.$$

For $s \in \mathbb{R}$, let $V = V(y)$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}y}\begin{pmatrix} V \\ V' \end{pmatrix} = \begin{pmatrix} V' \\ -sV' - f(V) \end{pmatrix}, \qquad y \in \mathbb{R} \tag{27}$$

with

$$V(y_0) = \alpha_*, \quad V'(y_0) = p_+(\alpha_*; s, f).$$

Now we define

$$w(y) = \begin{pmatrix} w_1(y) \\ w_2(y) \end{pmatrix} = \begin{pmatrix} V(y) - U_0(y) \\ V'(y) - U_0'(y) \end{pmatrix}, \qquad y \in \mathbb{R}.$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}y}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_2 \\ -sV' + c_0U_0' - f(V) + f_0(U_0) \end{pmatrix}, \qquad y \in \mathbb{R}.$$

Now we have

$$f(V) - f(U_0) = [f(\theta V + (1-\theta)U_0)]_{\theta=0}^{\theta=1} = \int_0^1 f'(\theta V + (1-\theta)U_0)\,\mathrm{d}\theta\,(V - U_0)$$

for $y \in \mathbb{R}$. Now we define

$$h(y) = \int_0^1 f'(\theta V(y) + (1-\theta)U_0(y))\,\mathrm{d}\theta, \qquad y \in \mathbb{R},$$

$$A(y) = \begin{pmatrix} 0 & -1 \\ h(y) & s \end{pmatrix}, \qquad y \in \mathbb{R},$$

$$g(y) = -\begin{pmatrix} 0 \\ (s-c_0)U_0'(y) + f(U_0(y)) - f_0(U_0(y)) \end{pmatrix}, \qquad y \in \mathbb{R}.$$

Now we have

$$\sup_{y\in\mathbb{R}} |A(y)| \le \sqrt{1 + s^2 + \|f\|^2_{C^1[0,1]}}.$$

Here

$$|A| = \sup_{x_1^2+x_2^2=1} \left| A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right|$$

for a 2×2 real matrix A. Then, we obtain

$$w'(y) + A(y)w(y) = g(y), \qquad y \in \mathbb{R}$$

and

$$w(y) = w(y_0)\exp\left(-\int_{y_0}^{y} A(y')\mathrm{d}y'\right) + \int_{y_0}^{y} \exp\left(-\int_{y'}^{y} A(y'')\,\mathrm{d}y''\right) g(y')\,\mathrm{d}y'$$

for $y \in \mathbb{R}$. Now we have

$$\sup_{y\in\mathbb{R}} |g(y)| \le |s - c_0| \max_{\eta\in\mathbb{R}} \left|U_0'(\eta)\right| + \|f - f_0\|_{C[0,1]}.$$

Thus, as $|s - c_0| + \|f - f_0\|_{C[0,1]}$ goes to zero,

$$\max_{y\in[y_0,y_1]} |w(y)|$$

converges to zero. Taking $|s - c_0| + \|f - f_0\|_{C[0,1]}$ small enough, we have

$$|w(y_1)| < \frac{z_1}{4},$$

$$\max_{y\in[y_0,y_1]} |w(y)| < \frac{1}{2} \min_{y\in[y_0,y_1]} \left(-U_0'(y)\right).$$

We define $p(\,\cdot\,; s, f)$ by

$$p(V(y); s, f) = V'(y), \qquad y_0 \le y < y_1.$$

Then we have

$$V(y_1) < \frac{z_1}{2} + \frac{z_1}{4} = \frac{3}{4} z_1$$

and

$$p_z(z; s, f) = -s - \frac{f(z)}{p(z; s, f)}, \qquad \frac{3}{4} z_1 < z \le \alpha_*,$$

$$p(z; s, f) < 0, \qquad \frac{3}{4} z_1 < z \le \alpha_*,$$

$$p(\alpha_*; s, f) = p_+(\alpha_*; s, f) < 0.$$

This $p(z; s, f)$ is an extension of $p_+(z; s, f)$ given by Lemma 2. Thus we obtain $\zeta_0(s, f) < z_1$. Combining Lemma 5 and the argument stated above, we have

$$\sup_{z\in[z_1,1]} |p_+(z; s, f) - p_+(z; c_0, f_0)| \to 0$$

as $|s - c_0| + \|f - f_0\|_{C^1[0,1]}$ goes to zero. This completes the proof. □

Lemma 2 asserts that $p_+(z; s, f)$ is strictly monotone increasing in s on $[\alpha_*, 1)$. In the following lemma, we assert that $p_+(z; s, f)$ is strictly monotone increasing in s on the whole interval $(0, 1)$.

Lemma 7. *Let* $-\infty < s_1 < s_2 < \infty$ *be arbitrarily given. Let* $z_{\mathrm{init}} \in (0,1)$ *be arbitrarily given. Assume that* $p_+(z_{\mathrm{init}}; s_1, f)$ *and* $p_+(z_{\mathrm{init}}; s_2, f)$ *exist and satisfy*

$$p_+(z_{\mathrm{init}}; s_1, f) < p_+(z_{\mathrm{init}}; s_2, f) < 0.$$

Then one has

$$\zeta_0(s_1, f) \le \zeta_0(s_2, f) < z_{\mathrm{init}}$$

and

$$p_+(z; s_1, f) < p_+(z; s_2, f) < 0 \qquad \text{for all} \quad z \in (\zeta_0(s_2, f), z_{\mathrm{init}}].$$

Proof. We put

$$q(z) = p_+(z; s_2, f) - p_+(z; s_1, f), \qquad \max\{\zeta_0(s_2, f), \zeta_0(s_1, f)\} \le z \le z_{\mathrm{init}}.$$

Then we have

$$q'(z) = -(s_2 - s_1) - \frac{f(z)}{p_+(z; s_2, f)} + \frac{f(z)}{p_+(z; s_1, f)}, \qquad \max\{\zeta_0(s_2, f), \zeta_0(s_1, f)\} < z < z_{\mathrm{init}},$$

$$q(z_{\mathrm{init}}) > 0.$$

Consequently we get

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(q(z)\exp\left(-\int_z^{z_{\mathrm{init}}} \frac{f(\zeta)}{p_+(\zeta; s_1, f)p_+(\zeta; s_2, f)}\,\mathrm{d}\zeta\right)\right) = -\,(s_2 - s_1)\exp\left(-\int_z^{z_{\mathrm{init}}} \frac{f(\zeta)}{p_+(\zeta; s_1, f)p_+(\zeta; s_2, f)}\,\mathrm{d}\zeta\right) < 0$$

for

$$\max\{\zeta_0(s_2, f), \zeta_0(s_1, f)\} < z < z_{\mathrm{init}}.$$

Then we find

$$q(z)\exp\left(-\int_z^{z_{\mathrm{init}}} \frac{f(\zeta)}{p_+(\zeta; s_1, f)p_+(\zeta; s_2, f)}\,\mathrm{d}\zeta\right) > 0, \qquad \max\{\zeta_0(s_2, f), \zeta_0(s_1, f)\} < z < z_{\mathrm{init}}.$$

Thus we obtain

$$q(z) > 0, \qquad \max\{\zeta_0(s_2, f), \zeta_0(s_1, f)\} < z < z_{\mathrm{init}}.$$

Then, using $q(z_{\mathrm{init}}) > 0$, we obtain

$$q(z) = p_+(z; s_2, f) - p_+(z; s_1, f) > 0, \quad \max\{\zeta_0(s_2, f), \zeta_0(s_1, f)\} < z < z_{\mathrm{init}}.$$

Now we obtain $\zeta_0(s_1, f) \le \zeta_0(s_2, f)$. This completes the proof. □

Let $\delta_0 \in (0,1)$ be arbitrarily given. We have $\zeta_0(c_0+\delta_0, f_0) \in [0,1)$ with

$$p_+(\zeta_0(c_0+\delta_0, f_0); c_0+\delta_0, f_0) = 0,$$
$$p_+(z; c_0-\delta_0, f_0) < p_+(z; c_0, f_0) < p_+(z; c_0+\delta_0, f_0) < 0, \quad z \in (\zeta_0(c_0+\delta_0, f_0), 1),$$
$$p_+(z; c_0-\delta_0, f_0) < 0, \qquad z \in (0,1).$$

Taking $\delta_0 \in (0,1)$ small enough and applying Lemma 6, we have

$$0 \le \zeta_0(c_0+\delta_0, f_0) < z_*.$$

Taking $\delta_0 \in (0,1)$ smaller if necessary and taking $\|f - f_0\|_{C^1[0,1]}$ small enough, we also have

$$0 \le \zeta_0(c_0+\delta_0, f) < z_* \tag{28}$$

by Lemma 6.

Now we have

$$p_+(z_*; c_0-\delta_0, f_0) < p_+(z_*; c_0, f_0) < p_+(z_*; c_0+\delta_0, f_0) < 0.$$

Taking $\|f - f_0\|_{C^1[0,1]}$ small enough and applying Lemma 6, we have

$$p_+(z_*; c_0-\delta_0, f) < p_+(z_*; c_0, f_0) < p_+(z_*; c_0+\delta_0, f) < 0.$$

Recalling (18) and applying Lemma 7, we obtain

$$\begin{aligned} &p_+(z; c_0-\delta_0, f) < p_+(z; c_0, f_0), \qquad z \in (0, z_*], \\ &p_+(z; c_0-\delta_0, f) < p_+(z; c_0, f_0) < p_+(z; c_0+\delta_0, f) < 0, \\ &\qquad\qquad z \in (\zeta_0(c_0+\delta_0, f), z_*] \end{aligned} \tag{29}$$

and

$$p_+(\zeta_0(c_0+\delta_0, f); c_0-\delta_0, f) < p_+(\zeta_0(c_0+\delta_0, f); c_0, f_0) < p_+(\zeta_0(c_0+\delta_0, f); c_0+\delta_0, f) = 0.$$

Using (29) and $p_+(0; c_0, f_0) = 0$, we have

$$\zeta_0(c_0-\delta_0) \le 0$$

and

$$(p_+)_z(z; c_0-\delta_0, f) = -(c_0-\delta_0) - \frac{f(z)}{p_+(z; c_0-\delta_0, f)}, \qquad 0 < z < 1, \tag{30}$$

$$p_+(z; c_0-\delta_0, f) < 0, \qquad 0 < z < 1, \tag{31}$$

$$p_+(1; c_0-\delta_0, f) = 0. \tag{32}$$

To prove Theorem 1 we have $\zeta = p_+(z; c_0+\delta_0, f)$ in the (z, ζ) plane in Figure 1. We study $\zeta = p_+(z; c_0-\delta_0, f)$ in the following lemma and will show the existence of $\zeta = p_+(z;c,f)$ with $p_+(0;c,f) = 0$ for some $c \in [c_0 - \delta_0, c_0 + \delta_0]$.

Lemma 8. *Assume* $|s - c_0| \le 1$ *and*

$$\|f - f_0\|_{C^1[0,1]} \le 1. \tag{33}$$

Take $M \ge 1$ large enough such that one has (19) for all $s \in [c_0 - 1, c_0 + 1]$ and for all f with (33). Assume that $|s-c_0| + \|f - f_0\|_{C^1[0,1]}$ is small enough such that one has (28). Then there exists $\gamma \in [0, 2M]$ such that one has

$$\gamma = \lim_{z\to 0} \left(-p_+(z; c_0 - \delta_0, f)\right).$$

Proof. We define $W = W(y)$ by

$$\frac{\mathrm{d}}{\mathrm{d}y}\begin{pmatrix} W \\ W' \end{pmatrix} = \begin{pmatrix} W' \\ -(c_0 - \delta_0)W' - f(W) \end{pmatrix}, \qquad y \in \mathbb{R},$$
$$W(0) = \alpha_*, \quad W'(0) = p_+(\alpha_*; c_0 - \delta_0, f) < 0.$$

Now we have

$$W'(y) = p_+(W(y); c_0 - \delta_0, f), \qquad 0 \le y < \infty.$$

Using (29), $p_+(0; c_0, f_0) = 0$ and Lemma 3, we have one of the following (i) or (ii).

(i) One has

$$W'(y) < 0, \qquad y \in [0, \infty)$$

and

$$\lim_{y\to\infty} \begin{pmatrix} W(y) \\ W'(y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(ii) There exists $y_0 \in (0, \infty)$ such that one has

$$W(y_0) = 0, \quad W'(y_0) < 0.$$

In Case (i), we can extend $p_+(z; c_0 - \delta_0, f)$ by

$$p_+(W(y); c_0 - \delta_0, f) = W'(y), \qquad y \in [0, \infty)$$

and obtain

$$\gamma = \lim_{z\to 0} \left(-p_+(z; c_0 - \delta_0, f)\right) = 0.$$

In Case (ii), we can extend $p_+(z; c_0 - \delta_0, f)$ by

$$p_+(W(y); c_0 - \delta_0, f) = W'(y), \qquad y \in [0, y_0)$$

and obtain

$$\gamma = \lim_{z\to 0} \left(-p_+(z; c_0 - \delta_0, f)\right) = -W'(y_0) \in (0, 2M].$$

This completes the proof. ∎

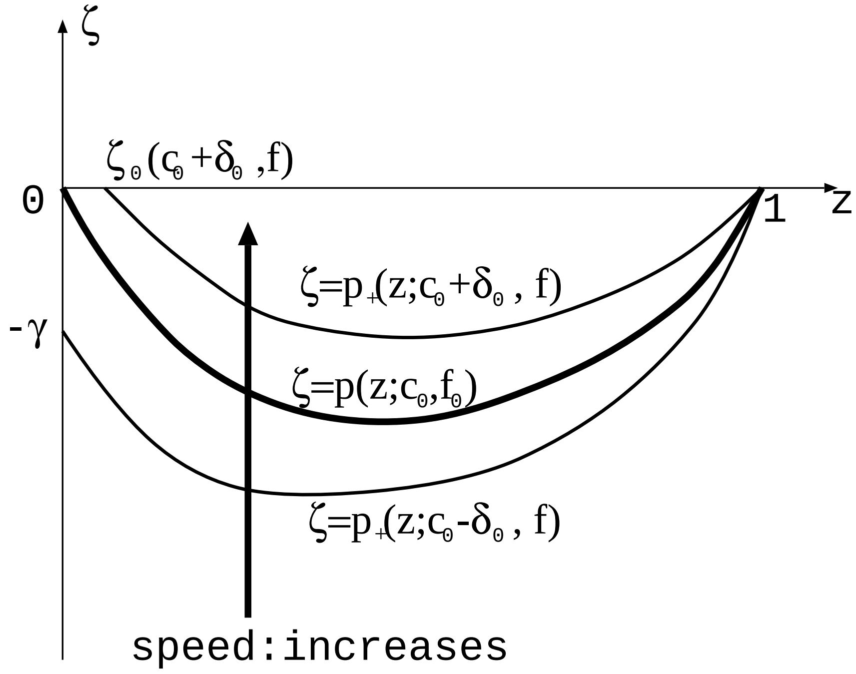

FIGURE 1. Search $c \in [c_0 - \delta_0, c_0 + \delta_0]$ with $p_+(0; c, f) = 0$.

Now we are ready to prove the main theorem.

Proof of Theorem 1. By the assumption we have (28). By the definition of $\zeta_0(c_0 + \delta_0, f) \in [0, z_*)$, we have

$$p_+(\zeta_0(c_0 + \delta_0, f); c_0 + \delta_0, f) = 0.$$
$$p_+(z; c_0 + \delta_0, f) < 0, \qquad \zeta_0(c_0 + \delta_0, f) < z < 1.$$

By Lemma 8, we have

$$\lim_{z \to 0} p_+(z; c_0 - \delta_0, f) = -\gamma \in (-\infty, 0].$$

Recalling (18) and applying Lemma 7, we obtain $c \in [c_0 - \delta_0, c_0 + \delta_0]$ with

$$\lim_{z \to 0} p_+(z; c, f) = 0,$$
$$p_+(z; c, f) < 0, \qquad 0 < z < 1.$$

See Figure 1. Thus $p_+(z; c, f)$ satisfies (10). Defining U by (11), we find that (c, U) satisfies the profile equation (4). As $\|f - f_0\|_{C^1[0,1]}$ goes to zero, we can take $\delta_0 \in (0, 1)$ arbitrarily small. Then c converges to c_0. From (11) and Lemma 6, $\|U - U_0\|_{C(\mathbb{R})}$ converges to zero as $\|f - f_0\|_{C^1[0,1]}$ goes to zero. By

$$U'(y) = p_+(U(y); s, f), \qquad y \in \mathbb{R}$$

and Lemma 6, $\|U - U_0\|_{C^1(\mathbb{R})}$ converges to zero. Then $\|U - U_0\|_{C^2(\mathbb{R})}$ converges to zero as $\|f - f_0\|_{C^1[0,1]}$ goes to zero. This completes the proof. □

3. Auxiliary results

In this section, we assume

$$f_0'(0) < 0 \tag{34}$$

instead of (5). We assume that f_0 is of class C^1 in an open interval including $[0,1]$ with $f_0(0) = 0$, $f_0(1) = 0$ and (34), and assume that there exist $U_0 \in C^2(\mathbb{R})$ and $c_0 \in \mathbb{R}$ that satisfy (6). We define

$$g_0(u) = -f_0(1-u)$$

in an open interval including $[0,1]$. Then we have

$$g_0(0) = 0, \quad g_0(1) = 0, \quad g_0'(1) < 0.$$

Defining

$$\begin{aligned} &s_0 = -c_0, \\ &V_0(y) = 1 - U_0(-y), \qquad y \in \mathbb{R}, \end{aligned}$$

we have

$$\begin{aligned} &V_0''(y) + s_0 V_0'(y) + g_0(V_0(y)) = 0, \qquad y \in \mathbb{R}, \\ &V_0'(y) < 0, \qquad y \in \mathbb{R}, \\ &V_0(-\infty) = 1, \quad V_0(\infty) = 0. \end{aligned}$$

Let $C_0^1[0,1)$ be the set of functions in $C^1[0,1)$ whose supports lie in $[0,1)$.

Corollary 9. *Let f_0 be of class C^1 in an open interval including $[0,1]$ with*

$$f_0(0) = 0, \quad f_0(1) = 0, \quad f_0'(0) < 0.$$

Assume that there exists (c_0, U_0) that satisfies (6). Assume that $f - f_0 \in C_0^1[0,1)$ and let $\|f - f_0\|_{C^1[0,1]}$ be small enough. Then there exists (c, U) that satisfies (4). If $\|f - f_0\|_{C^1[0,1]}$ goes to zero, c converges to c_0 and $\|U - U_0\|_{C^2(\mathbb{R})}$ goes to zero.

Proof. Combining Theorem 1 and the argument stated above, we have this corollary. □

Now we consider the existence of a traveling front to (1) for a perturbed bistable or multistable nonlinear term f.

Corollary 10. *Let f_0 be of class C^1 in an open interval including $[0,1]$ with $f_0(0) = 0$, $f_0(1) = 0$, $f_0'(0) < 0$ and $f_0'(1) < 0$. Assume that there exists (c_0, U_0) that satisfies (6). Assume that $f - f_0 \in C^1[0,1]$ and let $\|f - f_0\|_{C^1[0,1]}$ be small enough. Then there exists (c, U) that satisfies (4). If $\|f - f_0\|_{C^1[0,1]}$ goes to zero, c converges to c_0 and $\|U - U_0\|_{C^2(\mathbb{R})}$ goes to zero.*

Proof. We have

$$f(u) - f_0(u) = h_-(u) + h_+(u),$$

in an open interval including $[0,1]$ with $h_+ \in C_0^1(0,1]$ and $h_- \in C_0^1[0,1)$. As $\|f - f_0\|_{C^1[0,1]}$ goes to zero, we can take $h_+ \in C_0^1(0,1]$ and $h_- \in C_0^1[0,1)$ such that $\|h_+\|_{C^1[0,1]}$ and $\|h_-\|_{C^1[0,1]}$ go to zero. First we apply Theorem 1 to $f_0(u) + h_+(u)$ and we obtain a solution to (4) for $f_0(u) + h_+(u)$. Then, we apply Corollary 9 to $f_0(u) + h_+(u) + h_-(u)$ and we obtain a solution to (4) for $f(u) = f_0(u) + h_+(u) + h_-(u)$. This completes the proof. □

Corollary 10 asserts that a traveling front to (1) for a perturbed bistable or multistable nonlinear term is robust under perturbation in $C^1[0,1]$.

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