

AN IMPROVEMENT OF THE INTEGRABILITY OF THE STATE SPACE OF THE Φ_3^4 -PROCESS AND THE SUPPORT OF THE Φ_3^4 -MEASURE CONSTRUCTED BY THE LIMIT OF STATIONARY PROCESSES OF APPROXIMATING STOCHASTIC QUANTIZATION EQUATIONS

SEIICHIRO KUSUOKA

ABSTRACT. This is a remark paper for the Φ_3^4 -measure and the associated flow on the torus which are constructed in [1] by the limit of the stationary processes of the stochastic quantization equations of approximation measures. We improve the integrability of the state space of the Φ_3^4 -process and the support of the Φ_3^4 -measure. For the improvement, we improve the estimates of the Hölder continuity in time of the solutions to approximation equations. In the present paper, we only discuss the estimates different from those in [1].

1. INTRODUCTION

Recently by the new theories such as regularity structure [9] and paracontrolled calculus [8], singular nonlinear stochastic partial differential equations became solvable via renormalization. In particular, the singular stochastic partial differential equations associated to stochastic quantization of the Φ_3^4 -measure are solved (see [1], [5], [6], [7], [10], [11], [12], [13], [14] and [15]). Moreover, the Φ_3^4 -measure is able to be constructed from the stochastic quantization equations (see [1], [3], [4], [7] and [13]). For the detail of the history and background of the Φ_3^4 -measure and stochastic quantization, see the introduction of [1].

In [1], we considered the probability measures which approximate the Φ_3^4 -measure, and the stochastic quantization equations associated to them, and provided the stationary solutions to the approximating stochastic quantization equations. By proving the tightness of the stationary solutions we obtain the Φ_3^4 -process as a limit. Moreover, we constructed the Φ_3^4 -measure as a limit of the marginal distributions. Here, note that the approximation sequence of the marginal distributions is an approximation of the (formally defined) Φ_3^4 -measure. The most remarkable advantage of considering the stationary solutions is that we are able to construct the time-global limit

Mathematics Subject Classification. Primary 60H17; Secondary 81S20, 81T08, 60H15, 35Q40, 35R60, 35K58.

Key words and phrases. stochastic quantization, quantum field theory, singular SPDE.

This research was partially supported by JSPS KAKENHI Grant Numbers 17K14204 and 21H00988.

process and the Φ_3^4 -measure directly. This is a difference between [1] and the earlier result [13]. We remark that there is another delicate difference between [1] and [13]. In [1], we first prepare the probability measures $\{\mu_N\}$ approximating the (formally defined) Φ_3^4 -measure, and consider the stochastic partial differential equations associated to the stochastic quantization of $\{\mu_N\}$. On the other hand, in [13], they first consider the stochastic quantization equation associated to the (formally defined) Φ_3^4 -measure and show the existence of the global solution to the stochastic quantization equation by approximation. So, between the arguments of [1] and [13] there is a difference on the order of the two operations: approximation and stochastic quantization. This makes a delicate difference in the concerned stochastic partial differential equations. Indeed, approximation operators appear in the stochastic quantization equation in the case of [1] (see Eq. (4.1) in [1]). Because of the difference, we only have an energy functional with square and fourth-power integrals in [1], while the p th-power integrability of energy functionals is obtained for all $p \in [1, \infty)$ in [13]. Hence, we have some restriction on the integrability of the function spaces in the argument of [1].

In the present paper, we improve the integrability of the state space of the Φ_3^4 -process and the support of the Φ_3^4 -measure obtained by [1]. We will show the tightness of the approximating processes in smaller Besov spaces by improving the estimates of the Hölder continuity in time (see Proposition 3.4) and the estimate uniform in time (see Proposition 3.5). They enable us to improve the main estimate in [1] (see Theorem 3.6) and by using the estimate and the Besov embedding theorem we obtain the better integrability of the state space $B_{12/5}^{-1/2-\varepsilon}$ for the limit process and the support $B_\infty^{-1/2-\varepsilon}$ of our Φ_3^4 -measure (see Theorem 3.7). We remark that in the setting of [13], which is different from our setting as mentioned above, much more integrability for the state space of the Φ_3^4 -process is obtained. On the other hand, the supports of the Φ_3^4 -measures obtained here and obtained in [13] are the same.

We also remark that the state space of the Φ_3^4 -process and the support of the Φ_3^4 -measure obtained in the present paper are different. Note that null sets of the Φ_3^4 -measure can be ignored in the support of the measure, but cannot in the state space of the Φ_3^4 -process. Only polar sets can be ignored in the state space of the Φ_3^4 -process. Moreover, generally polar sets of processes are smaller than null sets of the invariant measures. Hence, such a difference naturally appears in the main theorem (see Theorem 3.7).

The organization of the present paper is as follows. In Section 2 we recall the notation and setting of [1]. In Section 3 we consider the improvement of the integrability. To do it, we give some estimates better than those in

[1]. We only discuss the different parts of the argument in [1] and show the main theorem (Theorem 3.7).

2. PREPARATION

In this section we recall the notation and setting of [1]. Let Λ be the three-dimensional torus given by $(\mathbb{R}/(2\pi\mathbb{Z}))^3$. Let L^p and $W^{s,p}$ be the p th-order integrable function space and the Sobolev space respectively, with respect to the Lebesgue measure on Λ , for $s \in \mathbb{R}$ and $p \in [1, \infty]$. Denote by $\langle \cdot, \cdot \rangle$ the inner product on $L^2(\Lambda; \mathbb{C})$. Let $\{e_k; k \in \mathbb{Z}^3\}$ be the Fourier basis on $L^2(\Lambda; \mathbb{C})$ and $k^2 := \sum_{j=1}^3 k_j^2$ for $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$.

To define approximation operators on $\mathcal{D}'(\Lambda)$ (the space of distributions on Λ), let $\psi^{(1)}$ be a nonincreasing C^∞ -function on $[0, \infty)$ such that $\psi^{(1)}(r) = 1$ for $r \in [0, 1]$ and $\psi^{(1)}(r) = 0$ for $r \in [2, \infty)$, and let $\psi^{(2)}$ be a nonincreasing function on $[0, \infty)$ such that $\psi^{(2)}(r) = 1$ for $r \in [0, 2]$ and $\psi^{(2)}(r) = 0$ for $r \in [4, \infty)$. We remark that $\psi^{(2)}$ is not necessary continuous. For $N \in \mathbb{N}$, $i = 1, 2$ and $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$, denote $\psi^{(i)}(2^{-N}|k_1|)\psi^{(i)}(2^{-N}|k_2|)\psi^{(i)}(2^{-N}|k_3|)$ by $\psi_N^{(i), \otimes 3}(k)$, and define $P_N^{(i)}$ by the mapping from $\mathcal{D}'(\Lambda)$ to $C^\infty(\Lambda)$ given by

$$P_N^{(i)} f := \sum_{k \in \mathbb{Z}^3} \psi_N^{(i), \otimes 3}(k) \langle f, e_k \rangle e_k.$$

Let μ_0 be the centered Gaussian measure on $\mathcal{D}'(\Lambda)$ with the covariance operator $[2(-\Delta + m_0^2)]^{-1}$ where Δ is the Laplacian on Λ and $m_0 > 0$, and let

$$C_1^{(N)} := \frac{1}{2(2\pi)^3} \sum_{k \in \mathbb{Z}^3} \frac{\left(\psi_N^{(1), \otimes 3}(k)\right)^2}{k^2 + m_0^2},$$

$$C_2^{(N)} := \frac{1}{2(2\pi)^6} \sum_{l_1, l_2 \in \mathbb{Z}^3} \frac{\left(\psi_N^{(1), \otimes 3}(l_1)\right)^2 \left(\psi_N^{(1), \otimes 3}(l_2)\right)^2 \left(\psi_N^{(1), \otimes 3}(l_1 + l_2)\right)^2}{(l_1^2 + m_0^2)(l_2^2 + m_0^2)(l_1^2 + l_2^2 + (l_1 + l_2)^2 + 3m_0^2)}.$$

The constants $C_1^{(N)}$ and $C_2^{(N)}$ are renormalization constants, and satisfy $\lim_{N \rightarrow \infty} C_1^{(N)} = \lim_{N \rightarrow \infty} C_2^{(N)} = \infty$. Let $\lambda_0 \in (0, \infty)$ and $\lambda \in (0, \lambda_0]$ be fixed. Define a function U_N on $\mathcal{D}'(\Lambda)$ by

$$U_N(\phi) = \int_{\Lambda} \left\{ \frac{\lambda}{4} (P_N^{(1)} \phi)(x)^4 - \frac{3\lambda}{2} \left(C_1^{(N)} - 3\lambda C_2^{(N)} \right) (P_N^{(1)} \phi)(x)^2 \right\} dx,$$

and consider the probability measure μ_N on $\mathcal{D}'(\Lambda)$ given by

$$\mu_N(d\phi) = A_N^{-1} \exp(-U_N(\phi)) \mu_0(d\phi)$$

where A_N is the normalizing constant. We remark that $\{\mu_N\}$ is an approximation sequence for the Φ_3^4 -measure which will be constructed below as a stationary probability measure of the flow associated with the stochastic quantization equation.

Letting $\dot{W}_t(x)$ be a Gaussian white noise with parameter $(t, x) \in (-\infty, \infty) \times \Lambda$, we consider the stochastic partial differential equation on Λ

$$(2.1) \quad \begin{cases} \partial_t \tilde{X}_t^N(x) = \dot{W}_t(x) - (-\Delta + m_0^2) \tilde{X}_t^N(x) \\ \quad - \lambda P_N^{(1)} \left\{ (P_N^{(1)} \tilde{X}_t^N)^3(x) \right. \\ \quad \left. - 3 \left(C_1^{(N)} - 3\lambda C_2^{(N)} \right) P_N^{(1)} \tilde{X}_t^N(x) \right\}, \\ \tilde{X}_0^N(x) = \xi_N(x) \end{cases}$$

where ξ_N is an initial value which has μ_N as its law and is independent of \dot{W}_t . Then, \tilde{X}^N is a stationary process (see Theorem 4.1 of [1]). Supplementary we prepare Z_t defined by the solution to the stochastic partial differential equation on Λ :

$$(2.2) \quad \begin{cases} \partial_t Z_t(x) = \dot{W}_t(x) - (-\Delta + m_0^2) Z_t(x), & (t, x) \in (-\infty, \infty) \times \Lambda, \\ Z_0(x) = \zeta(x), & x \in \Lambda \end{cases}$$

where ζ is a random variable which has μ_0 as its law and is independent of $\{\dot{W}_t; t \in [0, \infty)\}$. We choose a pair of the initial values (ξ_N, ζ) so that the paired process (\tilde{X}^N, Z) is a stationary process. For the existence of such a pair, see Section 4 of [1].

Next we prepare notation of Besov spaces and paraproducts. Let χ and φ be functions in $C^\infty([0, \infty); [0, 1])$ such that the supports of χ and φ are included by $[0, 4/3)$ and $[3/4, 8/3]$ respectively, and that

$$\chi(r) + \sum_{j=0}^{\infty} \varphi(2^{-j}r) = 1, \quad r \in [0, \infty).$$

Then, it is easy to see that

$$\begin{aligned} \varphi(2^{-j}r)\varphi(2^{-k}r) &= 0, \quad r \in [0, \infty), \quad j, k \in \mathbb{N} \cup \{0\} \text{ such that } |j - k| \geq 2, \\ \chi(r)\varphi(2^{-j}r) &= 0, \quad r \in [0, \infty), \quad j \in \mathbb{N}. \end{aligned}$$

Let $\mathcal{S}(\mathbb{R}^3)$ and $\mathcal{S}'(\mathbb{R}^3)$ be the Schwartz space and the space of tempered distributions on \mathbb{R}^3 , respectively. For $f \in \mathcal{D}'(\Lambda)$, we can define the periodic extension $\tilde{f} \in \mathcal{S}'(\mathbb{R}^3)$. By this extension, we define the (Littlewood-Paley)

nonhomogeneous dyadic blocks $\{\Delta_j; j \in \mathbb{N} \cup \{-1, 0\}\}$ by setting

$$\begin{aligned}\Delta_{-1}f(x) &= \left[\mathcal{F}^{-1} \left(\chi(|\cdot|) \mathcal{F}f \right) \right] (x), & x \in \Lambda, \\ \Delta_j f(x) &= \left[\mathcal{F}^{-1} \left(\varphi(2^{-j}|\cdot|) \mathcal{F}f \right) \right] (x), & x \in \Lambda, j \in \mathbb{N} \cup \{0\},\end{aligned}$$

where \mathcal{F} and \mathcal{F}^{-1} are the Fourier transform and inverse Fourier transform operators on \mathbb{R}^3 . We remark that

$$\Delta_{-1}f = \sum_{k \in \mathbb{Z}^3} \chi(|k|) \langle f, e_k \rangle e_k, \quad \Delta_j f = \sum_{k \in \mathbb{Z}^3} \varphi(2^{-j}|k|) \langle f, e_k \rangle e_k$$

hold for $f \in \mathcal{D}'(\Lambda)$ and $j \in \mathbb{N} \cup \{0\}$. We define the Besov norm $\|\cdot\|_{B_{p,r}^s}$ and the Besov space $B_{p,r}^s$ on Λ with $s \in \mathbb{R}$ and $p, r \in [1, \infty]$ by

$$\|f\|_{B_{p,r}^s} := \begin{cases} \left(\sum_{j=-1}^{\infty} 2^{jsr} \|\Delta_j f\|_{L^p}^r \right)^{1/r}, & r \in [1, \infty), \\ \sup_{j \in \mathbb{N} \cup \{-1, 0\}} 2^{js} \|\Delta_j f\|_{L^p}, & r = \infty, \end{cases}$$

$$B_{p,r}^s := \{f \in \mathcal{D}'(\Lambda); \|f\|_{B_{p,r}^s} < \infty\}.$$

For simplicity of notation, we denote $B_{p,\infty}^s$ by B_p^s for $s \in \mathbb{R}$ and $p \in [1, \infty]$. Let

$$S_j f := \sum_{k=-1}^{j-1} \Delta_k f, \quad j \in \mathbb{N} \cup \{0\}.$$

For simplicity of notation, let $\Delta_{-2}f := 0$ and $S_{-1}f := 0$. We define

$$\begin{aligned}f \otimes g &:= \sum_{j=0}^{\infty} (S_j f) \Delta_{j+1} g, & f \circledast g &:= g \otimes f, \\ f \ominus g &:= \sum_{j=-1}^{\infty} \Delta_j f (\Delta_{j-1} g + \Delta_j g + \Delta_{j+1} g).\end{aligned}$$

By the definitions of $\{\Delta_j\}$, $\{S_j\}$, \otimes , \ominus , and \circledast , we have

$$fg = f \otimes g + f \ominus g + f \circledast g.$$

Let $f \otimes\otimes g := f \otimes g + f \ominus g$ and $f \circledast\otimes g := f \circledast g + f \ominus g$. For the properties of Besov spaces and paraproducts, see Section 2 in [1] or [2]. We also remark that $P_N^{(1)}$ is a bounded operator on B_p^s for $p \in (1, \infty)$ and $s \in \mathbb{R}$, and moreover, sufficiently good for commutator estimates with paraproducts (see Section 2 of [1]).

Now we prepare notation of the polynomials of Ornstein-Uhlenbeck processes as follows.

$$\begin{aligned}
\mathcal{Z}_t^{(1,N)} &:= P_N^{(1)} Z_t, \\
\mathcal{Z}_t^{(2,N)} &:= (P_N^{(1)} Z_t)^2 - C_1^{(N)}, \\
\mathcal{Z}_t^{(3,N)} &:= (P_N^{(1)} Z_t)^3 - 3C_1^{(N)} P_N^{(1)} Z_t, \\
\mathcal{Z}_t^{(0,2,N)} &:= \int_{-\infty}^t e^{(t-s)(\Delta - m_0^2)} P_N^{(1)} \mathcal{Z}_s^{(2,N)} ds, \\
\mathcal{Z}_t^{(0,3,N)} &:= \int_{-\infty}^t e^{(t-s)(\Delta - m_0^2)} P_N^{(1)} \mathcal{Z}_s^{(3,N)} ds, \\
\mathcal{Z}_t^{(2,2,N)} &:= \mathcal{Z}_t^{(2,N)} \ominus P_N^{(1)} \mathcal{Z}_t^{(0,2,N)} - C_2^{(N)}, \\
\mathcal{Z}_t^{(2,3,N)} &:= \mathcal{Z}_t^{(2,N)} \ominus P_N^{(1)} \mathcal{Z}_t^{(0,3,N)} - 3C_2^{(N)} \mathcal{Z}_t^{(1,N)},
\end{aligned}$$

for $t \in (-\infty, \infty)$ and $N \in \mathbb{N}$. Denote $P_N^{(2)} \tilde{X}^N$ by X^N . To show the tightness of the laws of $\{X^N\}$, by using these notations we transform (2.1) for a better equation. In the present paper, we omit the detail of the transformation and just write the result of the transformation. Consider the following:

$$\begin{aligned}
X_t^{N,(2)} &:= P_N^{(2)} \left(\tilde{X}_t^N - Z_t \right) + \lambda \mathcal{Z}_t^{(0,3,N)}, \\
X_t^{N,(2),<} &:= -3\lambda \int_0^t e^{(t-s)(\Delta - m_0^2)} P_N^{(1)} \\
&\quad \left[\left(P_N^{(1)} X_s^{N,(2)} - \lambda P_N^{(1)} \mathcal{Z}_s^{(0,3,N)} \right) \ominus \mathcal{Z}_s^{(2,N)} \right] ds, \\
X_t^{N,(2),\geq} &:= X_t^{N,(2)} - X_t^{N,(2),<}.
\end{aligned}$$

Note that $(X_0^{N,(2),<}, X_0^{N,(2),\geq}) = (0, X_0^{N,(2)}) = (0, P_N^{(2)} (\xi_N - \zeta) + \lambda \mathcal{Z}_0^{(0,3,N)})$. Let

$$\begin{aligned}
\Psi_t^{(1)}(w) &:= \int_0^t e^{(t-s)(\Delta - m_0^2)} (P_N^{(1)})^2 \left[\left(w_s - \lambda P_N^{(1)} \mathcal{Z}_s^{(0,3,N)} \right) \ominus \mathcal{Z}_s^{(2,N)} \right] ds \\
&\quad - \left(w_t - \lambda P_N^{(1)} \mathcal{Z}_t^{(0,3,N)} \right) \ominus \int_0^t e^{(t-s)(\Delta - m_0^2)} (P_N^{(1)})^2 \mathcal{Z}_s^{(2,N)} ds, \\
\Psi_t^{(2)}(w) &:= \left[\left(w_t - \lambda P_N^{(1)} \mathcal{Z}_t^{(0,3,N)} \right) \ominus \int_0^t e^{(t-s)(\Delta - m_0^2)} (P_N^{(1)})^2 \mathcal{Z}_s^{(2,N)} ds \right] \\
&\quad \ominus \mathcal{Z}_t^{(2,N)} \\
&\quad - \left(w_t - \lambda P_N^{(1)} \mathcal{Z}_t^{(0,3,N)} \right) \\
&\quad \times \left[\int_0^t e^{(t-s)(\Delta - m_0^2)} (P_N^{(1)})^2 \mathcal{Z}_s^{(2,N)} ds \ominus \mathcal{Z}_t^{(2,N)} \right],
\end{aligned}$$

$$\begin{aligned}
 \Phi_t^{(1)}(w) &:= -3 \left(\mathcal{Z}_t^{(1,N)} - \lambda P_N^{(1)} \mathcal{Z}_t^{(0,3,N)} \right) \otimes w_t^2 \\
 &\quad + 3\lambda \left[\left(2\mathcal{Z}_t^{(1,N)} - \lambda P_N^{(1)} \mathcal{Z}_t^{(0,3,N)} \right) P_N^{(1)} \mathcal{Z}_t^{(0,3,N)} \right] \otimes w_t, \\
 \Phi_t^{(2)}(w) &:= -3 \left(w_t - \lambda P_N^{(1)} \mathcal{Z}_t^{(0,3,N)} \right) \otimes \mathcal{Z}_t^{(2,N)} + 3\lambda \mathcal{Z}_t^{(2,3,N)} \\
 &\quad + 9\lambda \left(w_t - \lambda P_N^{(1)} \mathcal{Z}_t^{(0,3,N)} \right) \\
 &\quad \times \left(\mathcal{Z}_t^{(2,2,N)} - \mathcal{Z}_t^{(2,N)} \ominus \int_{-\infty}^0 e^{(t-s)(\Delta - m_0^2)} \left(P_N^{(1)} \right)^2 \mathcal{Z}_s^{(2,N)} ds \right) \\
 &\quad - \lambda^2 \left(3\mathcal{Z}_t^{(1,N)} - \lambda P_N^{(1)} \mathcal{Z}_t^{(0,3,N)} \right) \left(P_N^{(1)} \mathcal{Z}_t^{(0,3,N)} \right)^2, \\
 \Phi_t^{(3)}(w) &:= -3 \left(\mathcal{Z}_t^{(1,N)} - \lambda P_N^{(1)} \mathcal{Z}_t^{(0,3,N)} \right) \otimes w_t^2 \\
 &\quad + 3\lambda \left[\left(2\mathcal{Z}_t^{(1,N)} - \lambda P_N^{(1)} \mathcal{Z}_t^{(0,3,N)} \right) P_N^{(1)} \mathcal{Z}_t^{(0,3,N)} \right] \otimes w_t.
 \end{aligned}$$

Then, in view of the argument in Section 4 of [1], the pair $(X_t^{N,(2),<}, X_t^{N,(2),\geq})$ satisfies the coupled partial differential equation:

$$(2.3) \quad \left\{ \begin{aligned}
 &(\partial_t - \Delta + m_0^2) X_t^{N,(2),<} \\
 &= -3\lambda P_N^{(1)} \left[\left(P_N^{(1)} X_t^{N,(2),<} + P_N^{(1)} X_t^{N,(2),\geq} - \lambda P_N^{(1)} \mathcal{Z}_t^{(0,3,N)} \right) \right. \\
 &\quad \left. \otimes \mathcal{Z}_t^{(2,N)} \right], \\
 &(\partial_t - \Delta + m_0^2) X_t^{N,(2),\geq} \\
 &= -\lambda P_N^{(1)} \left[\left(P_N^{(1)} X_t^{N,(2),<} + P_N^{(1)} X_t^{N,(2),\geq} \right)^3 \right] \\
 &\quad + \lambda P_N^{(1)} \Phi_t^{(1)} \left(P_N^{(1)} X_t^{N,(2),<} + P_N^{(1)} X_t^{N,(2),\geq} \right) \\
 &\quad + \lambda P_N^{(1)} \Phi_t^{(2)} \left(P_N^{(1)} X_t^{N,(2),<} + P_N^{(1)} X_t^{N,(2),\geq} \right) \\
 &\quad + \lambda P_N^{(1)} \Phi_t^{(3)} \left(P_N^{(1)} X_t^{N,(2),<} + P_N^{(1)} X_t^{N,(2),\geq} \right) \\
 &\quad - 3\lambda P_N^{(1)} \left[\left(P_N^{(1)} X_t^{N,(2),\geq} \right) \ominus \mathcal{Z}_t^{(2,N)} \right] \\
 &\quad + 9\lambda^2 P_N^{(1)} \left[\Psi_t^{(1)} \left(P_N^{(1)} X_t^{N,(2),<} + P_N^{(1)} X_t^{N,(2),\geq} \right) \ominus \mathcal{Z}_t^{(2,N)} \right] \\
 &\quad + 9\lambda^2 P_N^{(1)} \Psi_t^{(2)} \left(P_N^{(1)} X_t^{N,(2),<} + P_N^{(1)} X_t^{N,(2),\geq} \right).
 \end{aligned} \right.$$

By showing the tightness of the laws of $X_t^{N,(2)} = X_t^{N,(2),\geq} + X_t^{N,(2),<}$, we will obtain the tightness of the laws of X_t^N .

Remark. Some typos in [1] are corrected in (2.3). Precisely, in [1], $P_N^{(1)}$ of $P_N^{(1)} \mathcal{Z}_t^{(0,3,N)}$ is dropped in the equation corresponding to (2.3) and also in the coefficients $\Psi^{(i)}$ and $\Phi^{(i)}$.

For estimates we prepare the following. For $\eta \in [0, 1)$, $\gamma \in (0, 1/4)$ and $\varepsilon \in (0, 1]$ define $\mathfrak{X}_{\lambda, \eta, \gamma}^N(t)$ and $\mathfrak{Y}_\varepsilon^N(t)$ by

$$\begin{aligned} \mathfrak{X}_{\lambda, \eta, \gamma}^N(t) &:= \int_0^t \left(\left\| \nabla X_s^{N, (2), \geq} \right\|_{L^2}^2 + \left\| X_s^{N, (2)} \right\|_{L^2}^2 + \lambda \left\| P_N^{(1)} X_s^{N, (2)} \right\|_{L^4}^4 \right) ds \\ &\quad + \sup_{s', t' \in [0, t]; s' < t'} \frac{(s')^\eta \left\| X_{t'}^{N, (2)} - X_{s'}^{N, (2)} \right\|_{L^{4/3}}}{(t' - s')^\gamma}, \\ \mathfrak{Y}_\varepsilon^N(t) &:= \int_0^t \left\| X_s^{N, (2), <} \right\|_{B_4^{1-\varepsilon}}^3 ds + \int_0^t \left\| X_s^{N, (2), \geq} \right\|_{B_{4/3}^{1+\varepsilon}} ds. \end{aligned}$$

To simplify the notation, we denote by Q a positive polynomial built with the following quantities

$$(2.4) \quad \begin{aligned} &\sup_{t \in [0, T]} \left\| \mathcal{Z}_t^{(1, N)} \right\|_{B_\infty^{-(1+\varepsilon)/2}}, \quad \sup_{t \in [0, T]} \left\| P_N^{(2)} \mathcal{Z}_t \right\|_{B_\infty^{-(1+\varepsilon)/2}}, \\ &\sup_{t \in [0, T]} \left\| \mathcal{Z}_t^{(2, N)} \right\|_{B_\infty^{-1-\varepsilon/24}}, \quad \sup_{t \in [0, T]} \left\| \mathcal{Z}_t^{(2, 2, N)} \right\|_{B_\infty^{-\varepsilon/4}}, \\ &\sup_{t \in [0, T]} \left\| \mathcal{Z}_t^{(0, 2, N)} \right\|_{B_\infty^{1-\varepsilon/2}}, \quad \sup_{t \in [0, T]} \left\| \mathcal{Z}_t^{(0, 3, N)} \right\|_{B_\infty^{1/2-\varepsilon/4}}, \\ &\sup_{t \in [0, T]} \left\| \mathcal{Z}_t^{(2, 3, N)} \right\|_{B_\infty^{-(1+\varepsilon)/2}}, \quad \sup_{t \in [0, T]} \left\| \mathcal{Z}_t^{(1, N)} \left(P_N^{(1)} \mathcal{Z}_t^{(0, 3, N)} \right) \right\|_{B_\infty^{-(1+\varepsilon)/2}}, \\ &\sup_{t \in [0, T]} \left\| \mathcal{Z}_t^{(1, N)} \left(P_N^{(1)} \mathcal{Z}_t^{(0, 3, N)} \right)^2 \right\|_{B_\infty^{-(1+\varepsilon)/2}}, \\ &\quad \text{and} \quad \sup_{s, t \in [0, T]; s < t} \frac{\left\| \mathcal{Z}_t^{(0, 3, N)} - \mathcal{Z}_s^{(0, 3, N)} \right\|_{L^\infty}}{(t - s)^\gamma}, \end{aligned}$$

with coefficients depending on λ_0 , ε , η , γ and T , and we also denote by C a positive constant depending on λ_0 , ε , η , γ and T . We remark that Q and C can be different from line to line. A constant depending on an extra parameter δ is denoted by C_δ . As in Section 3 of [1], we have the square integrability of those in (2.4) with respect to the probability measure. In view of this fact and hypercontractivity of Gaussian random variables, any polynomial consists of the elements in (2.4) are integrable with respect to the probability measure, i.e. $E[Q] \leq C$.

3. IMPROVEMENT OF INTEGRABILITY

Let $\alpha \in [0, 1/2)$ and choose $\varepsilon \in (0, 1/16]$, $\gamma \in (0, 1/8)$ and $\eta \in (1/2, 1)$ such that $4\varepsilon < \gamma$, $\eta > \alpha + 2\gamma$ and $2\alpha + 4\gamma + \varepsilon < 1$. In the present paper, we only see the difference from [1] and omit the argument of the parts which are the same as those in [1].

We prepare some lemmas for estimates of the terms in (2.3), which are different versions of estimates in [1].

Lemma 3.1. *Let $p \in [1, 2]$, $s, t \in [0, T]$ and $\delta \in (0, 1]$,*

$$\begin{aligned} & \int_s^t (t-u)^{-\alpha/2-\gamma} \left\| \Psi_u^{(1)}(P_N^{(1)} X^{N,(2)}) \ominus \mathcal{Z}_u^{(2,N)} \right\|_{B_p^\varepsilon} du \\ & \leq \delta \sup_{u,v \in [0,t]; u < v} \frac{u^\eta \left\| P_N^{(1)} X_v^{N,(2)} - P_N^{(1)} X_u^{N,(2)} \right\|_{L^p}}{(v-u)^\gamma} \\ & \quad + C \int_0^t \left(\left\| X_u^{N,(2)} \right\|_{B_2^{15/16}}^2 + \left\| P_N^{(1)} X_u^{N,(2)} \right\|_{L^4}^4 \right)^{7/8} du + C_\delta Q. \end{aligned}$$

Proof. Applying Lemmas 4.3 and 2.3 in [1] and Hölder's inequality, we have

$$\begin{aligned} & \int_s^t (t-u)^{-\alpha/2-\gamma} \left\| \Psi_u^{(1)}(P_N^{(1)} X^{N,(2)}) \ominus \mathcal{Z}_u^{(2,N)} \right\|_{B_p^\varepsilon} du \\ & \leq Q \int_s^t (t-u)^{-\alpha/2-\gamma} \left(\int_0^u (u-v)^{-21/32} \left\| P_N^{(1)} X_v^{N,(2)} \right\|_{B_p^{15/16}} dv \right) du \\ & \quad + Q \int_s^t (t-u)^{-\alpha/2-\gamma} \left(\sup_{r \in [0,u]} \frac{r^\eta \left\| P_N^{(1)} X_u^{N,(2)} - P_N^{(1)} X_r^{N,(2)} \right\|_{L^p}}{(u-r)^\gamma} \right)^{1/2} \\ & \quad \times \left(\left\| P_N^{(1)} X_u^{N,(2)} \right\|_{L^p}^{1/2} \right. \\ & \quad \quad \left. + \int_0^u v^{-\eta/2} (u-v)^{\gamma/2-1-3\varepsilon/2} \left\| P_N^{(1)} X_v^{N,(2)} \right\|_{L^p}^{1/2} dv \right) du \\ & + Q \\ & \leq Q \int_s^t (t-u)^{-\alpha/2-\gamma} \left(\int_0^u (u-v)^{-21/32} \left\| P_N^{(1)} X_v^{N,(2)} \right\|_{B_p^{15/16}} dv \right) du \\ & \quad + Q \left(\sup_{r,u \in [0,t]; r < u} \frac{r^\eta \left\| P_N^{(1)} X_u^{N,(2)} - P_N^{(1)} X_r^{N,(2)} \right\|_{L^p}}{(u-r)^\gamma} \right)^{1/2} \\ & \quad \times \left[\int_s^t (t-u)^{-\alpha/2-\gamma} \left\| P_N^{(1)} X_u^{N,(2)} \right\|_{L^p}^{1/2} du \right. \end{aligned}$$

$$\begin{aligned}
& + \int_s^t (t-u)^{-\alpha/2-\gamma} \left(\int_0^u v^{-\eta/2} (u-v)^{\gamma/2-1-3\varepsilon/2} \right. \\
& \qquad \qquad \qquad \left. \times \left\| P_N^{(1)} X_v^{N,(2)} \right\|_{L^p}^{1/2} dv \right) du \Big] \\
& + Q \\
\leq & Q \int_0^t (t-u)^{-\alpha/2-\gamma+11/32} \left\| P_N^{(1)} X_v^{N,(2)} \right\|_{B_p^{15/16}} dv \\
& + Q \left(\sup_{r,u \in [0,t]; r < u} \frac{r^\eta \left\| P_N^{(1)} X_u^{N,(2)} - P_N^{(1)} X_r^{N,(2)} \right\|_{L^p}}{(u-r)^\gamma} \right)^{1/2} \\
& \times \left[\int_s^t (t-u)^{-\alpha/2-\gamma} \left\| P_N^{(1)} X_u^{N,(2)} \right\|_{L^p}^{1/2} du \right. \\
& \qquad \qquad \qquad \left. + \int_0^t v^{-\eta/2} (t-v)^{-(\alpha+\gamma+3\varepsilon)/2} \left\| P_N^{(1)} X_v^{N,(2)} \right\|_{L^p}^{1/2} dv \right] \\
& + Q \\
\leq & \int_0^t \left\| P_N^{(1)} X_v^{N,(2)} \right\|_{B_p^{15/16}}^{7/4} dv \\
& + Q \left(\sup_{r,u \in [0,t]; r < u} \frac{r^\eta \left\| P_N^{(1)} X_u^{N,(2)} - P_N^{(1)} X_r^{N,(2)} \right\|_{L^p}}{(u-r)^\gamma} \right)^{1/2} \\
& \times \left[\left(\int_s^t (t-u)^{-\alpha-2\gamma} du \right)^{1/2} \left(\int_s^t \left\| P_N^{(1)} X_u^{N,(2)} \right\|_{L^p} du \right)^{1/2} \right. \\
& \qquad \qquad \qquad \left. + \left(\int_0^t v^{-\eta} (t-v)^{-(\alpha+\gamma+3\varepsilon)} dv \right)^{1/2} \left(\int_0^t \left\| P_N^{(1)} X_v^{N,(2)} \right\|_{L^p} dv \right)^{1/2} \right] \\
& + Q.
\end{aligned}$$

Hence, noting that $2\alpha + 4\gamma + \varepsilon < 1$ and that

$$\begin{aligned}
& \left(\int_0^t v^{-\eta} (t-v)^{-(\alpha+\gamma+3\varepsilon)} dv \right) \left(\int_0^t \left\| P_N^{(1)} X_v^{N,(2)} \right\|_{L^p} dv \right) \\
& \leq C t^{1-\eta-(\alpha+\gamma+3\varepsilon)} \cdot t^{1/2} \left(\int_0^t \left\| P_N^{(1)} X_v^{N,(2)} \right\|_{L^p}^2 dv \right)^{1/2} \\
& \leq C \left(\int_0^t \left\| P_N^{(1)} X_v^{N,(2)} \right\|_{L^p}^2 dv \right)^{1/2},
\end{aligned}$$

we obtain

$$\begin{aligned}
 & \int_s^t (t-u)^{-\alpha/2-\gamma} \left\| \Psi_u^{(1)}(P_N^{(1)} X^{N,(2)}) \ominus \mathcal{Z}_u^{(2,N)} \right\|_{B_p^\varepsilon} du \\
 & \leq \int_0^t \left\| P_N^{(1)} X_v^{N,(2)} \right\|_{B_p^{15/16}}^{7/4} dv \\
 & \quad + Q \left(\sup_{r,u \in [0,t]; r < u} \frac{r^\eta \left\| P_N^{(1)} X_u^{N,(2)} - P_N^{(1)} X_r^{N,(2)} \right\|_{L^p}}{(u-r)^\gamma} \right)^{1/2} \\
 & \quad \quad \quad \times \left(\int_0^t \left\| P_N^{(1)} X_v^{N,(2)} \right\|_{L^p}^2 dv \right)^{1/2} \\
 & \leq \int_0^t \left\| P_N^{(1)} X_v^{N,(2)} \right\|_{B_p^{15/16}}^{7/4} dv \\
 & \quad + \delta \sup_{r,u \in [0,t]; r < u} \frac{r^\eta \left\| P_N^{(1)} X_u^{N,(2)} - P_N^{(1)} X_r^{N,(2)} \right\|_{L^p}}{(u-r)^\gamma} \\
 & \quad + \left(\int_0^t \left\| P_N^{(1)} X_v^{N,(2)} \right\|_{L^p}^2 dv \right)^{7/4} + C_\delta Q.
 \end{aligned}$$

This inequality and Hölder's inequality yield the assertion. \square

Lemmas 3.2 and 3.3 below improve Lemmas 4.8 and 4.9 of [1], respectively. These improvements enable us to find the smaller state space and support of the Φ_3^4 -process and the Φ_3^4 -measure, respectively.

Lemma 3.2. *For $p \in [1, 2]$, $t \in [0, T]$ and $\delta \in (0, 1]$,*

$$\left\| \Phi_t^{(3)}(P_N^{(1)} X^{N,(2)}) \right\|_{B_p^{-(1+\varepsilon)/2}} \leq \delta \left\| P_N^{(1)} X_t^{N,(2)} \right\|_{L^4}^4 + \delta^{-1} Q.$$

Proof. Estimates of the para-products (see Proposition 2.1 (ii) in [1]) imply

$$\begin{aligned}
 \left\| \Phi_t^{(3)}(P_N^{(1)} X^{N,(2)}) \right\|_{B_p^{-(1+\varepsilon)/2}} & \leq Q \left(\left\| (P_N^{(1)} X_t^{N,(2)})^2 \right\|_{L^p} + \left\| P_N^{(1)} X_t^{N,(2)} \right\|_{L^p} \right) \\
 & \leq Q \left\| P_N^{(1)} X_t^{N,(2)} \right\|_{L^4}^2 + Q \\
 & \leq \delta \left\| P_N^{(1)} X_t^{N,(2)} \right\|_{L^4}^4 + \delta^{-1} Q.
 \end{aligned}$$

Thus, we have the inequality. \square

Lemma 3.3. *For $p \in [1, 2]$, $t \in [0, T]$ and $\delta \in (0, 1]$,*

$$\left\| (P_N^{(1)} X_t^{N,(2),\geq}) \ominus \mathcal{Z}_t^{(2,N)} \right\|_{B_p^{\varepsilon/8}}$$

$$\begin{aligned} &\leq \delta \left(\left\| \nabla X_t^{N,(2),\geq} \right\|_{L^2}^2 + \left\| P_N^{(1)} X_t^{N,(2)} \right\|_{L^4}^4 \right) \\ &\quad + \delta \left\| X_t^{N,(2),<} \right\|_{L^p}^2 + \delta \left\| X_t^{N,(2),\geq} \right\|_{B_p^{1+\varepsilon}} + \delta^{-2} Q. \end{aligned}$$

Proof. An estimates of the resonance term (see Proposition 2.1 (iv) in [1]) implies

$$(3.1) \quad \begin{aligned} &\left\| (P_N^{(1)} X_t^{N,(2),\geq}) \ominus \mathcal{Z}_t^{(2,N)} \right\|_{B_p^{\varepsilon/8}} \\ &\leq C \left\| \mathcal{Z}_t^{(2,N)} \right\|_{B_\infty^{-1-\varepsilon/8}} \left\| P_N^{(1)} X_t^{N,(2),\geq} \right\|_{B_p^{1+\varepsilon/4}}. \end{aligned}$$

By the interpolation inequality of Besov spaces (see Proposition 2.1 (vii) in [1]) we have

$$(3.2) \quad \begin{aligned} &\left\| P_N^{(1)} X_t^{N,(2),\geq} \right\|_{B_p^{1+\varepsilon/4}} \\ &\leq \left\| P_N^{(1)} X_t^{N,(2),\geq} \right\|_{B_p^{1-\varepsilon/8}}^{2/3} \left\| P_N^{(1)} X_t^{N,(2),\geq} \right\|_{B_p^{1+\varepsilon}}^{1/3} \\ &\leq \delta \left\| P_N^{(1)} X_t^{N,(2),\geq} \right\|_{B_p^{1-\varepsilon/8}}^2 + C \delta^{-1/2} \left\| P_N^{(1)} X_t^{N,(2),\geq} \right\|_{B_p^{1+\varepsilon}}^{1/2}. \end{aligned}$$

In view of

$$\begin{aligned} &\left\| P_N^{(1)} X_t^{N,(2),\geq} \right\|_{B_p^{1-\varepsilon/8}}^2 \leq C \left\| P_N^{(1)} X_t^{N,(2),\geq} \right\|_{W^{1-\varepsilon/8,p}}^2 \\ &\leq C \left(\left\| P_N^{(1)} X_t^{N,(2)} \right\|_{L^p} + \left\| \nabla P_N^{(1)} X_t^{N,(2),\geq} \right\|_{L^p} \right)^2 + C \left\| P_N^{(1)} X_t^{N,(2),<} \right\|_{L^p}^2, \end{aligned}$$

from (3.1) and (3.2) we have

$$\begin{aligned} &\left\| (P_N^{(1)} X_t^{N,(2),\geq}) \ominus \mathcal{Z}_t^{(2,N)} \right\|_{B_p^{\varepsilon/8}} \\ &\leq \delta Q \left[\left(\left\| P_N^{(1)} X_t^{N,(2)} \right\|_{L^p} + \left\| \nabla P_N^{(1)} X_t^{N,(2),\geq} \right\|_{L^p} \right)^2 + \left\| P_N^{(1)} X_t^{N,(2),<} \right\|_{L^p}^2 \right] \\ &\quad + \delta^{-1/2} Q \left\| P_N^{(1)} X_t^{N,(2),\geq} \right\|_{B_p^{1+\varepsilon}}^{1/2} + Q, \\ &\leq \delta Q \left(\left\| P_N^{(1)} X_t^{N,(2)} \right\|_{L^p}^4 + \left\| \nabla P_N^{(1)} X_t^{N,(2),\geq} \right\|_{L^p}^2 \right) + \delta Q \left\| P_N^{(1)} X_t^{N,(2),<} \right\|_{L^p}^2 \\ &\quad + \delta \left\| P_N^{(1)} X_t^{N,(2),\geq} \right\|_{B_p^{1+\varepsilon}} + \delta^{-2} Q. \end{aligned}$$

Hence, by replacing δ and using the uniform boundedness of $P_N^{(1)}$ in N (see Proposition 2.5 in [1]) we obtain the assertion. \square

The following proposition is an improved version of Proposition 4.13 in [1], and actually the regularity of the Besov space is improved by α .

Proposition 3.4. For $t \in [0, T]$,

$$\begin{aligned}
 & E \left[\sup_{s', t' \in [0, t]; s' < t'} \frac{(s')^\eta \left\| X_{t'}^{N, (2)} - X_{s'}^{N, (2)} \right\|_{B_{4/3}^\alpha}}{(t' - s')^\gamma} \right] \\
 & \leq CE \left[\sup_{r \in [0, t]} r^\eta \left\| X_r^{N, (2), \geq} \right\|_{B_{4/3}^{\alpha+2\gamma}} \right] + CE \left[\sup_{r \in [0, t]} r^\eta \left\| X_r^{N, (2), <} \right\|_{B_{4/3}^{\alpha+2\gamma}} \right] \\
 & \quad + CE \left[\left\| X_t^{N, (2), <} \right\|_{L^2}^2 \right] + CE [\mathfrak{Y}_\varepsilon^N(t)] + C \sup_{s \in [0, t]} E \left[\left\| X_s^{N, (2)} \right\|_{B_1^{-1/2+\varepsilon}}^q \right] \\
 & \quad + C.
 \end{aligned}$$

Proof. In view of (2.3) it follows that

$$\begin{aligned}
 & X_t^{N, (2), <} - e^{(t-s)(\Delta - m_0^2)} X_s^{N, (2), <} \\
 & = -3\lambda \int_s^t e^{(t-u)(\Delta - m_0^2)} P_N^{(1)} \left[\left(P_N^{(1)} X_u^{N, (2)} - \lambda P_N^{(1)} \mathcal{Z}_u^{(0,3,N)} \right) \odot \mathcal{Z}_u^{(2,N)} \right] du
 \end{aligned}$$

for $s, t \in [0, T]$ such that $s < t$. Hence, for $s', t' \in [0, T]$ such that $s' < t'$, the smoothing property of the heat semigroup and an estimate of the paraproduct (see Proposition 2.1 in [1]) imply

$$\begin{aligned}
 & \left\| X_{t'}^{N, (2), <} - X_{s'}^{N, (2), <} \right\|_{B_{4/3}^\alpha} \\
 & \leq \left\| e^{(t'-s')(\Delta - m_0^2)} - I \right\|_{B_{4/3}^{\alpha+2\gamma} \rightarrow B_{4/3}^\alpha} \left\| X_{s'}^{N, (2), <} \right\|_{B_{4/3}^{\alpha+2\gamma}} \\
 & \quad + 3\lambda \int_{s'}^{t'} \left\| e^{(t'-u)(\Delta - m_0^2)} P_N^{(1)} \right. \\
 & \quad \quad \left. \left[\left(P_N^{(1)} X_u^{N, (2)} - \lambda P_N^{(1)} \mathcal{Z}_u^{(0,3,N)} \right) \odot \mathcal{Z}_u^{(2,N)} \right] \right\|_{B_{4/3}^\alpha} du \\
 & \leq C(t' - s')^\gamma \left\| X_{s'}^{N, (2), <} \right\|_{B_{4/3}^{\alpha+2\gamma}} \\
 & \quad + C\lambda \int_{s'}^{t'} (t' - u)^{-(\alpha+1)/2 - \varepsilon/2} \\
 & \quad \quad \times \left\| \left(P_N^{(1)} X_u^{N, (2)} - \lambda P_N^{(1)} \mathcal{Z}_u^{(0,3,N)} \right) \odot \mathcal{Z}_u^{(2,N)} \right\|_{B_{4/3}^{-1-\varepsilon}} du \\
 & \leq C(t' - s')^\gamma \left\| X_{s'}^{N, (2), <} \right\|_{B_{4/3}^{\alpha+2\gamma}} \\
 & \quad + \lambda Q(t' - s')^\gamma \int_{s'}^{t'} (t' - u)^{-(\alpha+2\gamma+1+\varepsilon)/2}
 \end{aligned}$$

$$\times \left\| P_N^{(1)} X_u^{N,(2)} - \lambda P_N^{(1)} \mathcal{Z}_u^{(0,3,N)} \right\|_{L^{4/3}} du.$$

Thus, by applying Hölder's inequality we have for $t \in [0, T]$ and $\delta \in (0, 1]$

$$(3.3) \quad \begin{aligned} & (s')^\eta \left\| X_{t'}^{N,(2),<} - X_{s'}^{N,(2),<} \right\|_{B_{4/3}^\alpha} \\ & \sup_{s', t' \in [0, t]; s' < t'} \frac{\quad}{(t' - s')^\gamma} \\ & \leq C \sup_{r \in [0, t]} \left(r^\eta \left\| X_r^{N,(2),<} \right\|_{B_{4/3}^{\alpha+2\gamma}} \right) + \delta \lambda \int_0^t \left\| P_N^{(1)} X_u^{N,(2)} \right\|_{L^{4/3}}^4 du \\ & \quad + C_\delta Q. \end{aligned}$$

Similarly, from (2.3), for $s', t' \in [0, T]$ such that $s' < t'$, we have the estimate

$$\begin{aligned} & \left\| X_{t'}^{N,(2),\geq} - X_{s'}^{N,(2),\geq} \right\|_{B_{4/3}^\alpha} \\ & \leq C(t' - s')^\gamma \left\| X_{s'}^{N,(2),\geq} \right\|_{B_{4/3}^{\alpha+2\gamma}} \\ & \quad + C\lambda(t' - s')^\gamma \int_{s'}^{t'} (t' - u)^{-\alpha/2-\gamma} \left\| P_N^{(1)} X_u^{N,(2)} \right\|_{L^4}^3 du \\ & \quad + C\lambda(t' - s')^\gamma \int_{s'}^{t'} (t' - u)^{-\alpha/2-\gamma} \left\| \Phi_u^{(1)}(P_N^{(1)} X^{N,(2)}) \right\|_{L^{4/3}} du \\ & \quad + C\lambda(t' - s')^\gamma \int_{s'}^{t'} (t' - u)^{-\alpha/2-\gamma-1/4-\varepsilon/2} \left\| \Phi_u^{(2)}(P_N^{(1)} X^{N,(2)}) \right\|_{B_{4/3}^{-1/2-\varepsilon}} du \\ & \quad + C\lambda(t' - s')^\gamma \int_{s'}^{t'} (t' - u)^{-\alpha/2-\gamma-1/4-\varepsilon/2} \left\| \Phi_u^{(3)}(P_N^{(1)} X^{N,(2)}) \right\|_{B_{4/3}^{-1/2-\varepsilon}} du \\ & \quad + C\lambda(t' - s')^\gamma \int_{s'}^{t'} (t' - u)^{-\alpha/2-\gamma} \left\| (P_N^{(1)} X^{N,(2),\geq}) \ominus \mathcal{Z}_u^{(2,N)} \right\|_{L^{4/3}} du \\ & \quad + C\lambda(t' - s')^\gamma \int_{s'}^{t'} (t' - u)^{-\alpha/2-\gamma} \left\| \Psi_u^{(1)}(P_N^{(1)} X^{N,(2)}) \ominus \mathcal{Z}_u^{(2,N)} \right\|_{B_{4/3}^\varepsilon} du \\ & \quad + C\lambda(t' - s')^\gamma \int_{s'}^{t'} (t' - u)^{-\alpha/2-\gamma} \left\| \Psi_u^{(2)}(P_N^{(1)} X^{N,(2)}) \right\|_{B_{4/3}^\varepsilon} du. \end{aligned}$$

For $\delta \in (0, 1]$, applying Lemmas 4.4, 4.5 and 4.7 in [1] and Lemmas 3.2 and 3.3 with replacing δ by $(t' - u)^\beta$ with suitable β for each lemmas, and applying Lemma 3.1 and Hölder's inequality, we have for $\delta \in (0, 1]$ $s', t' \in [0, T]$ such that $s' < t'$

$$\left\| X_{t'}^{N,(2),\geq} - X_{s'}^{N,(2),\geq} \right\|_{B_{4/3}^\alpha}$$

$$\begin{aligned}
 &\leq C(t' - s')^\gamma \left\| X_{s'}^{N,(2),\geq} \right\|_{B_{4/3}^{\alpha+2\gamma}} \\
 &\quad + C\lambda(t' - s')^\gamma \int_{s'}^{t'} \left(\left\| \nabla X_u^{N,(2),\geq} \right\|_{L^2}^2 + \left\| P_N^{(1)} X_u^{N,(2)} \right\|_{L^4}^4 \right) du \\
 &\quad + C\lambda(t' - s')^\gamma \int_0^t \left(\left\| X_u^{N,(2)} \right\|_{B_2^{15/16}}^2 + \left\| P_N^{(1)} X_u^{N,(2)} \right\|_{L^4}^4 \right)^{7/8} du \\
 &\quad + C(t' - s')^\gamma \int_{s'}^{t'} \left\| X_u^{N,(2),<} \right\|_{L^{4/3}}^2 du + Q(t' - s')^\gamma \int_{s'}^{t'} \left\| X_u^{N,(2),\geq} \right\|_{B_{4/3}^{1+\varepsilon}} du \\
 &\quad + \delta(t' - s')^\gamma \sup_{u,v \in [0,t]; u < v} \frac{u^\eta \left\| P_N^{(1)} X_v^{N,(2)} - P_N^{(1)} X_u^{N,(2)} \right\|_{L^{4/3}}}{(v - u)^\gamma} du \\
 &\quad + C_\delta Q(t' - s')^\gamma.
 \end{aligned}$$

Here, we remark that applying Lemmas 3.2 and 3.3 instead of Lemmas 4.8 and 4.9 in [1] respectively, enables us to improve the regularity of the estimate by $\alpha \in [0, 1/2)$. It is also remarked that Lemma 3.1 is provided for the clarity of the proof.

From this inequality and (3.3) we obtain the conclusion by following the proof of Proposition 4.12 in [1]. \square

The following proposition is an improved version of Proposition 4.17 in [1], and again the regularity of the Besov space is improved by α . We need the version, because the supremum in time of the norms on $B_4^{\alpha+2\gamma}$ and $B_{4/3}^{\alpha+2\gamma}$ appeared in Proposition 3.4.

Proposition 3.5. *For $q \in (1, 8/7)$, $t \in [0, T]$ and $\delta \in (0, 1]$, we have*

$$\begin{aligned}
 &E \left[\sup_{r \in [0,t]} r^\eta \left\| X_r^{N,(2),<} \right\|_{B_4^{\alpha+2\gamma}}^3 \right] + E \left[\sup_{r \in [0,t]} r^\eta \left\| X_r^{N,(2),\geq} \right\|_{B_{4/3}^{\alpha+2\gamma}} \right] \\
 &\leq CE \left[\left\| X_0^{N,(2)} \right\|_{B_{4/3}^{\alpha+2\gamma-2\eta}} \right] + C\delta E [\mathfrak{X}_{\lambda,\eta,\gamma}^N(t)] + C\delta E [\mathfrak{Y}_\varepsilon^N(t)^q] + C_\delta.
 \end{aligned}$$

Proof. By Lemma 4.14(i) in [1] we have

$$\begin{aligned}
 &E \left[\sup_{r \in [0,t]} r^\eta \left\| X_r^{N,(2),<} \right\|_{B_4^{\alpha+2\gamma}}^3 \right] \\
 &\leq \lambda E \left[Q \sup_{r \in [0,t]} \left(\int_0^r (r - u)^{-(1+\alpha)/2-\gamma-\varepsilon/4} \right. \right. \\
 &\quad \left. \left. \times \left\| P_N^{(1)} X_u^{N,(2)} - \lambda P_N^{(1)} \mathcal{Z}_u^{(0,3,N)} \right\|_{L^4} du \right)^3 \right].
 \end{aligned}$$

Hence, by applying Hölder's inequality we have for $\delta \in (0, 1]$

$$(3.4) \quad E \left[\sup_{r \in [0, t]} r^\eta \left\| X_r^{N, (2), <} \right\|_{B_4^{\alpha+2\gamma}}^3 \right] \leq \delta \lambda E \left[\int_0^t \left\| P_N^{(1)} X_u^{N, (2)} \right\|_{L^4}^4 du \right] + C_\delta.$$

Similarly to the proof of Lemma 4.14(ii) in [1] we have for $s, t \in [0, T]$ such that $s < t$

$$\begin{aligned} & \left\| X_t^{N, (2), \geq} \right\|_{B_{4/3}^{\alpha+2\gamma}} \\ & \leq C(t-s)^{-\eta} \left\| X_s^{N, (2), \geq} \right\|_{B_{4/3}^{\alpha+2\gamma-2\eta}} \\ & \quad + C\lambda \int_s^t (t-u)^{-(\alpha+2\gamma)/2} \left\| P_N^{(1)} X_u^{N, (2)} \right\|_{L^4}^3 du \\ & \quad + C\lambda \int_s^t (t-u)^{-(\alpha+2\gamma)/2} \left\| \Phi_u^{(1)}(P_N^{(1)} X^{N, (2)}) \right\|_{L^{4/3}} du \\ & \quad + C\lambda \int_s^t (t-u)^{-(2\alpha+4\gamma+1+2\varepsilon)/4} \left\| \Phi_u^{(2)}(P_N^{(1)} X^{N, (2)}) \right\|_{B_{4/3}^{-1/2-\varepsilon}} du \\ & \quad + C\lambda \int_s^t (t-u)^{-(2\alpha+4\gamma+1+2\varepsilon)/4} \left\| \Phi_u^{(3)}(P_N^{(1)} X^{N, (2)}) \right\|_{B_{4/3}^{-1/2-\varepsilon}} du \\ & \quad + C\lambda \int_s^t (t-u)^{-(\alpha+2\gamma)/2} \left\| (P_N^{(1)} X_u^{N, (2), \geq}) \ominus \mathcal{Z}_u^{2, N} \right\|_{L^{4/3}} du \\ & \quad + C\lambda \int_s^t (t-u)^{-(\alpha+2\gamma-\varepsilon)/2} \left\| \Psi_u^{(1)}(P_N^{(1)} X^{N, (2)}) \ominus \mathcal{Z}_u^{2, N} \right\|_{B_{4/3}^\varepsilon} du \\ & \quad + C\lambda \int_s^t (t-u)^{-(\alpha+2\gamma-\varepsilon)/2} \left\| \Psi_u^{(2)}(P_N^{(1)} X^{N, (2)}) \right\|_{B_{4/3}^\varepsilon} du. \end{aligned}$$

Similarly to the proof of Proposition 3.4, for $\delta \in (0, 1]$, applying Lemmas 4.4, 4.5 and 4.7 in [1] and Lemmas 3.2 and 3.3 with replacing δ by $\delta(t-u)^\beta$ with suitable β for each lemmas, and applying Lemma 3.1 and Hölder's inequality, we have

$$\begin{aligned} & E \left[\sup_{r \in [0, t]} r^\eta \left\| X_r^{N, (2), \geq} \right\|_{B_{4/3}^{\alpha+2\gamma}} \right] \\ & \leq CE \left[\left\| X_0^{N, (2)} \right\|_{B_{4/3}^{\alpha+2\gamma-2\eta}} \right] + \delta E [\mathfrak{X}_{\lambda, \eta, \gamma}^N(t)] + \delta E [\mathfrak{Y}_\varepsilon^N(t)^q] + C_\delta. \end{aligned}$$

Here, we use the assumptions of the parameters α , γ and ε . Therefore, by this inequality and (3.4) we have the assertion. \square

Now we obtain the following uniform estimate in N .

Theorem 3.6. *Let $\alpha \in [0, 1/2)$ and choose $\varepsilon \in (0, 1/16]$, $\gamma \in (0, 1/8)$ and $\eta \in (1/2, 1)$ such that $4\varepsilon < \gamma$, $\eta > \alpha + 2\gamma$ and $2\alpha + 4\gamma + \varepsilon < 1$, and let $q \in (1, 8/7)$. Then, we have*

$$\begin{aligned} & E \left[\sup_{s,t \in [0,T]; s < t} \frac{s^\eta \left\| X_t^{N,(2)} - X_s^{N,(2)} \right\|_{B_{4/3}^\alpha}}{(t-s)^\gamma} \right] + E [\mathfrak{X}_{\lambda,\eta,\gamma}^N(T)] + E [\mathfrak{Y}_\varepsilon^N(T)^q] \\ & + E \left[\sup_{r \in [0,T]} r^\eta \left\| X_r^{N,(2), <} \right\|_{B_4^{\alpha+2\gamma}}^3 \right] + E \left[\sup_{r \in [0,T]} r^\eta \left\| X_r^{N,(2), \geq} \right\|_{B_{4/3}^{\alpha+2\gamma}} \right] \\ & \leq C. \end{aligned}$$

Proof. By following the proof of Theorem 4.18 in [1] with applying Propositions 3.4 and 3.5 instead of Propositions 4.13 and 4.17 in [1], we obtain the assertion. \square

Theorem 3.6 improves the regularity of Besov norms in Theorem 4.18 in [1] by α . By using the improvement we are able to show the tightness of the laws of $\{X^N\}$ in the spaces smaller than that in Theorem 4.19 in [1] as follows.

Theorem 3.7. *For $\tilde{\varepsilon} \in (0, 1/16]$, the laws of $\{X^N\}$ are tight on $C([0, \infty); B_{12/5}^{-1/2-\tilde{\varepsilon}})$. Moreover, if X is a limit in law of a subsequence $\{X^{N(k)}\}$ of $\{X^N\}$ on $C([0, \infty); B_{12/5}^{-1/2-\tilde{\varepsilon}})$, then X is a continuous process on $B_{12/5}^{-1/2-\tilde{\varepsilon}}$, the limit measure μ of the associated subsequence $\{\mu_{N(k)}\}$ is a stationary measure with respect to X and it holds that*

$$(3.5) \quad \int \|\phi\|_{B_\infty^{-1/2-\tilde{\varepsilon}}}^2 \mu(d\phi) < \infty.$$

Proof. We follow the proof of Theorem 4.19 in [1]. Choose $\alpha \in [0, 1/2)$ sufficiently close to $1/2$ so that $\alpha + \tilde{\varepsilon} > 1/2$, choose $\gamma \in (0, 1/8)$ and $\varepsilon \in (0, 1/16)$ sufficiently small, and choose $\eta \in (1/2, 1)$ sufficiently large so that the assumptions in Theorem 3.6 and $\varepsilon < \tilde{\varepsilon}$ hold. Let $T \in (0, \infty)$ and $t_0 \in (0, T)$. For $h \in (0, 1]$ and $\varepsilon' \in (0, 1]$, Chebyshev's inequality implies that

$$\begin{aligned} & \sup_{N \in \mathbb{N}} P \left(\sup_{s,t \in [t_0, T]; |s-t| < h} \left\| X_t^{N,(2)} - X_s^{N,(2)} \right\|_{B_{4/3}^\alpha} > \varepsilon' \right) \\ & \leq \frac{h^\gamma}{\varepsilon' t_0^\eta} E \left[\sup_{s,t \in [t_0, T]; s < t, t-s < h} \frac{s^\eta \left\| X_t^{N,(2)} - X_s^{N,(2)} \right\|_{B_{4/3}^\alpha}}{(t-s)^\gamma} \right] \end{aligned}$$

Hence, from Theorem 3.6 we obtain

$$(3.6) \quad \lim_{h \downarrow 0} \sup_{N \in \mathbb{N}} P \left(\sup_{s, t \in [t_0, T]; |s-t| < h} \left\| X_t^{N, (2)} - X_s^{N, (2)} \right\|_{B_{4/3}^\alpha} > \varepsilon' \right) = 0$$

for $\varepsilon' \in (0, 1]$. On the other hand, Chebyshev's inequality implies that, for any $R > 0$,

$$\sup_{N \in \mathbb{N}} P \left(\left\| X_{t_0}^{N, (2)} \right\|_{B_{4/3}^{\alpha+2\gamma}} > R \right) \leq \frac{1}{R t_0^\eta} \sup_{N \in \mathbb{N}} E \left[\sup_{r \in [0, T]} r^\eta \left\| X_r^{N, (2)} \right\|_{B_{4/3}^{\alpha+2\gamma}} \right].$$

Hence, by Theorem 3.6 we obtain

$$(3.7) \quad \lim_{R \rightarrow \infty} \sup_{N \in \mathbb{N}} P \left(\left\| X_{t_0}^{N, (2)} \right\|_{B_{4/3}^{\alpha+2\gamma}} > R \right) = 0.$$

In view of the fact that the unit ball in $B_{4/3}^{\alpha+2\gamma}$ is compactly embedded in $B_{4/3}^\alpha$ (see Theorem 2.94 in [2]), the tightness of the laws of $\{X^{N, (2)}\}$ on $C([t_0, T]; B_{4/3}^\alpha)$ follows from (3.6) and (3.7). By the Besov embedding theorem (see Proposition 2.1 in [1]) we have $B_{4/3}^\alpha \subset B_{12/5}^{-1/2-\tilde{\varepsilon}}$. Hence, we have the tightness of the laws of $\{X^{N, (2)}\}$ on $C([t_0, T]; B_{12/5}^{-1/2-\tilde{\varepsilon}})$. The rest of the proofs are completely same as that of Theorem 4.19 in [1] except (3.5).

Now we prove (3.5). The stationarity of X^N implies

$$\begin{aligned} \int \|\phi\|_{B_\infty^{-1/2-\tilde{\varepsilon}}}^2 \mu(d\phi) &\leq \liminf_{k \rightarrow \infty} \int \|\phi\|_{B_\infty^{-1/2-\tilde{\varepsilon}}}^2 \mu_{N(k)}(d\phi) \\ &= \liminf_{k \rightarrow \infty} E \left[\left\| X_0^{N(k)} \right\|_{B_\infty^{-1/2-\tilde{\varepsilon}}}^2 \right] \\ &= \frac{1}{T} \liminf_{k \rightarrow \infty} \int_0^T E \left[\left\| X_t^{N(k)} \right\|_{B_\infty^{-1/2-\tilde{\varepsilon}}}^2 \right] dt \\ &\leq C \liminf_{k \rightarrow \infty} \int_0^T E \left[\left\| X_t^{N(k), (2)} \right\|_{B_\infty^{-1/2-\tilde{\varepsilon}}}^2 \right] dt + C. \end{aligned}$$

Since the Besov embedding theorem implies

$$\left\| X_t^{N(k), (2)} \right\|_{B_\infty^{-1/2-\tilde{\varepsilon}}}^2 \leq C \left\| X_t^{N(k), (2)} \right\|_{B_2^{1-\tilde{\varepsilon}}}^2,$$

we have

$$\begin{aligned} \int \|\phi\|_{B_\infty^{-1/2-\varepsilon}}^2 \mu(d\phi) &\leq C \liminf_{N \rightarrow \infty} \int_0^T E \left[\left\| X_t^{N, (2)} \right\|_{B_2^{1-\tilde{\varepsilon}}}^2 \right] dt + C \\ &\leq C \liminf_{N \rightarrow \infty} E [\mathfrak{X}_{\lambda, \eta, \gamma}^N(T)] + C \liminf_{N \rightarrow \infty} E [\mathfrak{Y}_\varepsilon^N(T)] + C. \end{aligned}$$

Therefore, we obtain (3.5) from Theorem 3.6. \square

Acknowledgements. The author thanks the anonymous referee for helpful comments. The comments improved the quality of the present paper.

REFERENCES

- [1] S. Albeverio and Sei. Kusuoka, The invariant measure and the flow associated to the Φ_3^4 -quantum field model, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **20** (2020), 1359–1427. https://doi.org/10.2422/2036-2145.201809_008
- [2] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011. <https://doi.org/10.1007/978-3-642-16830-7>
- [3] N. Barashkov and M. Gubinelli, A variational method for Φ_3^4 , *Duke Math. J.* **169** (2020), 3339–3415. <https://doi.org/10.1215/00127094-2020-0029>
- [4] N. Barashkov and M. Gubinelli, The Φ_3^4 measure via Girsanov’s theorem, *Electron. J. Probab.* **26** (2021), Paper No. 81, 29pp. <https://doi.org/10.1214/21-EJP635>
- [5] R. Catellier and K. Chouk, Paracontrolled distributions and the 3-dimensional stochastic quantization equation, *Ann. Probab.* **46** (2018), no. 5, 2621–2679. <https://doi.org/10.1214/17-AOP1235>
- [6] M. Gubinelli and M. Hofmanova, Global solutions to elliptic and parabolic Φ^4 models in Euclidean space, *Comm. Math. Phys.* **368** (2019), no. 3, 1201–1266. <https://doi.org/10.1007/s00220-019-03398-4>
- [7] M. Gubinelli and M. Hofmanova, A PDE construction of the Euclidean Φ_3^4 quantum field theory, *Comm. Math. Phys.* **384** (2021), 1–75. <https://doi.org/10.1007/s00220-021-04022-0>
- [8] M. Gubinelli, P. Imkeller and N. Perkowski, Paracontrolled distributions and singular PDEs, *Forum Math. Pi* **3** (2015), e6, 75 pp. <https://doi.org/10.1017/fmp.2015.2>
- [9] M. Hairer, A theory of regularity structures, *Invent. Math.* **198** (2014), no. 2, 269–504. <https://doi.org/10.1007/s00222-014-0505-4>
- [10] M. Hoshino, Global well-posedness of complex Ginzburg-Landau equation with a space-time white noise, *Ann. Inst. Henri Poincaré Probab. Stat.* **54** (2018), no. 4, 1969–2001. <https://doi.org/10.1214/17-AIHP862>
- [11] M. Hoshino, Y. Inahama and N. Naganuma, Stochastic complex Ginzburg-Landau equation with space-time white noise, *Electron. J. Probab.* **22** (2017), Paper No. 104, 68 pp. <https://doi.org/10.1214/17-EJP125>
- [12] A. Moinat and H. Weber. Space-time localisation for the dynamic model. *Communications on Pure and Applied Mathematics*, **73** (2020), 2519–2555. <https://doi.org/10.1002/cpa.21925>
- [13] J.-C. Mourrat and H. Weber, The dynamic Φ_3^4 model comes down from infinity, *Commun. Math. Phys.*, **356**(3) (2017), 673–753. <https://doi.org/10.1007/s00220-017-2997-4>
- [14] R. Zhu and X. Zhu, Lattice approximation to the dynamical Φ_3^4 model, *Ann. Probab.* **46** (2018), no. 1, 397–455. <https://doi.org/10.1214/17-AOP1188>
- [15] R. Zhu and X. Zhu, Dirichlet form associated with the Φ_3^4 model, *Electron. J. Probab.* **23** (2018), Paper No. 78, 31 pp. <https://doi.org/10.1214/18-EJP207>

SEIICHIRO KUSUOKA
DEPARTMENT OF MATHEMATICS
GRADUATE SCHOOL OF SCIENCE
KYOTO UNIVERSITY
KYOTO 606-8502, JAPAN

e-mail address: kusuoka@math.kyoto-u.ac.jp

(Received October 22, 2021)

(Accepted April 15, 2022)