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NON-MODULAR SOLUTION OF THE KANEKO-ZAGIER EQUATIONS WITH RESPECT TO FRICKE GROUPS OF LOW LEVELS

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Abstract. Pavel Guerzhoy show that the Kaneko-Zagier equation for $\mathrm{SL}_2(\mathbb{Z})$ has mixed mock mock modular solutions in certain weights. In this paper, we show that the Kaneko-Zagier equations for the Fricke groups of level 2 and 3 also have mixed mock modular solutions in certain weights.

1. Introduction

A modular linear differential equation is a differential equation on the complex upper half plane $\mathfrak{H}$ with an invariance property under the action of some arithmetic Fuchsian groups. In recent years, they are studied in several contexts in number theory, vertex operator algebra, and mathematical physics. The Kaneko-Zagier equation

$$(KZ_k) \qquad f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0,$$

which was introduced by Kaneko-Zagier [KZ98], is one of the most basic and interesting modular linear differetial equations. Here, $\tau \in \mathfrak{H}$, $' = \frac{1}{2\pi i}\frac{d}{d\tau}$, and $E_2(\tau)$ is the normalized Eisenstein series of weight 2, which is not a modular form but is a quasimodular form. The equation (KZ_k) has connections to supersingular polynomials, Atkin's orthogonal polynomials and trace functions of vertex operator algebra (see [KZ98], [KNS13]).

Recently, Guerzhoy [G15] described the non-modular solution of (KZ_k) and proved (KZ_k) has a mixed mock modular solution for certain weights k as a corollary of his theorem. In this paper, we will investigate non-modular solutions of the Kaneko-Zagier equations for the Fricke groups of level 2 and 3, and show they have mixed mock modular solutions for certain weights k. More precisely, the Kaneko-Zagier equations for the Fricke group of level 2 and 3 are given respectively by the following:

$$(KZ_{k,2}) \qquad f''(\tau) - \frac{k+1}{4}E_{2,2}(\tau)f'(\tau) + \frac{k(k+1)}{8}E_{2,2}'(\tau)f(\tau) = 0,$$

$$(KZ_{k,3}) \qquad f''(\tau) - \frac{k+1}{3}E_{2,3}(\tau)f'(\tau) + \frac{k(k+1)}{6}E_{2,3}'(\tau)f(\tau) = 0,$$

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where $E_{2,2}(\tau)$ and $E_{2,3}(\tau)$ are Eisenstein series of weight 2 with respect to the Fricke groups of level 2 and 3, respectively, precise definitions being given in the next section. The equations $(KZ_{k,2})$ and $(KZ_{k,3})$ were previously investigated in [S11], [SS15], and [ST12].

Our main results is the following theorem:

Theorem 1.1. (1) *The differential equation $(KZ_{k,2})$ has non-modular solutions of the form;*

$$\begin{cases} A_{k,2}(\tau)\mathcal{E}_2(\tau) + B_{k,2}(\tau) & \text{if } k \equiv 0 \mod 4, \\ C_{k,2}(\tau)F_{2,2}(\tau) + D_{k,2}(\tau) & \text{if } k \equiv 2 \mod 4, \end{cases}$$

where $\mathcal{E}_2(\tau) = 2\pi i \int_{i\infty}^{\tau} \eta(\tau)^2\eta(2\tau)^2 dz$, and $A_{k,2}(\tau)$, $B_{k,2}(\tau)$, $C_{k,2}(\tau)$, $D_{k,2}(\tau)$ are certain modular forms of weight k with respect to the Fricke group of level 2, $F_{2,2}(\tau)$ is an explicitly given solution of $(KZ_{2,2})$ in Proposition 4.2. In particular, when $k \equiv 2 \mod 4$, the solution is a mixed mock modular form.

(2) *The differential equation $(KZ_{k,3})$ has non-modular solution of the form;*

$$\begin{cases} A_{k,3}(\tau)\mathcal{E}_3(\tau) + B_{k,3}(\tau) & \text{if } k \equiv 0 \mod 3, \\ C_{k,3}(\tau)F_{1,3}(\tau) + D_{k,3}(\tau) & \text{if } k \equiv 1 \mod 3, \end{cases}$$

where $\mathcal{E}_3(\tau) = 2\pi i \int_{i\infty}^{\tau} \eta(\tau)^2\eta(3\tau)^2 d$, and $A_{k,3}(\tau)$, $B_{k,3}(\tau)$, $C_{k,3}(\tau)$, $D_{k,3}(\tau)$ are certain modular forms of weight k with respect to the Fricke group of level 3, $F_{1,3}(\tau)$ is an explicitly given solution of $(KZ_{1,3})$ in Proposition 4.2. In particular, when $k \equiv 1 \mod 3$, the solution is a mixed mock modular form.

The paper is organized as follows. In §2, we prepare some notations and terminology. In §3, we summarize the properties of mixed mock modular form arising from some modular elliptic curve. In §4, we state some results on $(KZ_{k,2})$ and $(KZ_{k,3})$, and prepare some lemmas. After that, we describe non-modular solutions of $(KZ_{k,2})$ and $(KZ_{k,3})$, and prove that their solutions are mixed mock modular forms under some conditions.

2. Preliminaries

For a function f on $\mathfrak{H}$, an integer k and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, we let $(c\tau + d)^{-k} f(\frac{a\tau+b}{c\tau+d})$ denote by $f|_k\gamma(\tau)$.

Let N be a positive integer and $\Gamma_0^*(N)$ be the discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ generated by the Hecke congruence subgroup $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \mod N\}$ and the Fricke involution $w_N = \begin{pmatrix} 0 & -\frac{1}{\sqrt{N}} \\ \sqrt{N} & 0 \end{pmatrix}$. We call $\Gamma_0^*(N)$

the Fricke group of level N. We note that all cusps of $\Gamma_0^*(2)$ and $\Gamma_0^*(3)$ are equivalent to $i\infty$. We put

$$E_{k,N}(\tau) = \frac{1}{1+N^{\frac{k}{2}}}(E_k(\tau) + N^{\frac{k}{2}}E_k(N\tau)) \qquad (k \geq 2),$$
$$E_{k,N,-}(\tau) = \frac{1}{1-N^{\frac{k}{2}}}(E_k(\tau) - N^{\frac{k}{2}}E_k(N\tau)) \qquad (k \geq 2),$$

where $E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sum_{d|n} d^{k-1}q^n$ is the Eisenstein series of weight k with respect to $\mathrm{SL}_2(\mathbb{Z})$, $q = e^{2\pi i\tau}$, and B_k is the k-th Bernoulli number. The function $E_{k,N}(\tau)$ is a modular form of weight k with respect to $\Gamma_0^*(N)$ if $k > 2$. The function $E_{k,N,-}(\tau)$ is a modular form of weight k with respect to $\Gamma_0(N)$ and an eigenfunction of the action of w_N with the eigenvalue -1. Also, $E_{2,N}(\tau)$ is not a modular form, but a quasimodular form of weight 2.

To describe our proof, we need the following modular forms and modular functions. In the case of $(KZ_{k,2})$, we need

$$\begin{aligned}
g_2(\tau) &:= \eta(\tau)^2\eta(2\tau)^2 \\
&= q^{\frac{1}{4}} - 2q^{\frac{5}{4}} - 3q^{\frac{9}{4}} + 6q^{\frac{13}{4}} + 2q^{\frac{17}{4}} + \cdots, \\
\Delta_2(\tau) &:= \eta(\tau)^8\eta(2\tau)^8 \\
&= q - 8q^2 + 12q^3 + 64q^4 - 210q^5 - 96q^6 + 1016q^7 + \cdots, \\
j_2^*(\tau) &:= \frac{E_{4,2}(\tau)^2}{\Delta_2(\tau)} \\
&= \frac{1}{q} + 104 + 4372q + 96256q^2 + 1240002q^3 + 10698752q^4 + \cdots.
\end{aligned}$$

Here, $\eta(\tau) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ is the Dedekind η-function. The function $g_2(\tau)$ is a modular form of weight 2 with respect to some subgroup of $\Gamma_0^*(2)$, and the function $\Delta_2(\tau)$ is a modular form of weight 8 with respect to $\Gamma_0(2)$. The function $j_2^*(\tau)$ is a Hauptmodul of $\Gamma_0^*(2)$ (see [S11]). In the case of $(KZ_{k,3})$ we need

$$\begin{aligned}
I_{3,3}(\tau) &:= \frac{\eta(3\tau)^9}{\eta(\tau)^3} \\
&= q + 3q^2 + 9q^3 + 13q^4 + 24q^5 + 27q^6 + 50q^7 + \cdots, \\
g_3(\tau) &:= \eta(\tau)^2\eta(3\tau)^2 \\
&= q^{\frac{1}{3}} - 2q^{\frac{4}{3}} - q^{\frac{7}{3}} + 5q^{\frac{13}{3}} + 4q^{\frac{16}{3}} - 7q^{\frac{19}{3}} + \cdots, \\
\Delta_3(\tau) &:= \eta(\tau)^6\eta(3\tau)^6 \\
&= q - 6q^2 + 9q^3 + 4q^4 + 6q^5 - 54q^6 - 40q^7 + 168q^8 + 81q^9 + \cdots,
\end{aligned}$$

$$\begin{aligned} I_{1,3}(\tau) &:= 1 + 6\sum_{n=1}^{\infty}\left(\sum_{m|n}\left(\frac{-3}{m}\right)\right)q^n \\ &= 1 + 6q + 6q^3 + 6q^4 + 12q^7 + 6q^9 + \cdots, \\ j_3^*(\tau) &= \frac{I_{1,3}(\tau)^6}{\Delta_3(\tau)} \\ &= \frac{1}{q} + 42 + 783q + 8672q^2 + 65367q^3 + 371520q^4 + \cdots. \end{aligned}$$

The function $g_3(\tau)$ is a modular form of weight 2 with respect to some subgroup of $\Gamma_0(3)$, and the function $\Delta_3(\tau)$ is a modular form of weight 6 with respect to $\Gamma_0^*(3)$. The functions $I_{1,3}(\tau)$ and $I_{3,3}(\tau)$ are modular forms of weights 1 and 3, respectively, with respect to $\Gamma_0(3)$ with the Kronecker character $\left(\frac{-3}{m}\right)$. The function $j_3^*(\tau)$ is a Hauptmodul of $\Gamma_0^*(3)$ (see [S11]).

Since $(E_{2,2,-}(\tau))^2 = E_{4,2}(\tau)$ and $(I_{1,3}(\tau))^4 = E_{4,3}(\tau)$, we would often write $E_{2,2,-}(\tau)$ and $I_{1,3}(\tau)$ as $E_{4,2}^{\frac{1}{2}}(\tau)$ and $E_{4,3}^{\frac{1}{4}}(\tau)$.

3. Mixed mock modular form

First, we recall the definition of the mixed mock modular form. For the detail, see [BFOR17].

A real analytic function f is called a harmonic Maass form of weight k with respect to Γ with a character χ if (i) $f|_k\gamma = \chi(\gamma)f$ for any $\gamma \in \Gamma$, (ii) f is an eignenfunction of $\mathrm{Im}(\tau)^2\frac{\partial}{\partial\tau}\frac{\partial}{\partial\bar{\tau}} + 2ik\frac{\partial}{\partial\bar{\tau}}$ with an eigenvalue 0, and (iii) f is an exponential growth at every cusp.

A harmonic Maass form f has the following Fourier expansion

$$f = f^+ + f^-,$$

where

$$\begin{aligned} f^+ &= \sum_{\substack{n\in\mathbb{Q}\\ n\gg-\infty}} a_n q^n, \\ f^- &= \sum_{\substack{n\in\mathbb{Q}\\ n<0}} b_n\Gamma(1-k, -2\pi n\mathrm{Im}(\tau))q^n. \end{aligned}$$

Here, $\Gamma(\alpha, x) = \int_x^\infty e^{-t}t^{\alpha-1}dt$ is the incomplete gamma function. The holomorphic part f^+ is called a mock modular form of weght k with respect to Γ with a character χ.

A mixed harmonic Maass form of weight k is a function of the form

$$h = \sum_i f_i g_i,$$

where f_i is a modular form of weight k_i with respect to Γ with a character ϕ_i and g_i is a harmonic Maass form of weght l_i with respect to Γ with a character ψ_i, satisfying $k_i + l_i = k$ and $\phi_i\psi_i = \chi$.

The holomorphic part $h^+ = \sum_i f_i g_i^+$ is called a mixed mock modular form of weght k with respect to Γ with a character χ if h has at most polynomial growth at every cusp.

Next we introduce a particular mock modular form for later use. Let L be a lattice in $\mathbb{C}$ and put $E = \mathbb{C}/L$. According to [W], the function

$$\hat{\zeta}(z, E) = \zeta(z, E) + e_2(L)z + \frac{\pi}{\mathrm{Vol}(E)}\bar{z} \tag{3.1}$$

is a doubly periodic real analytic function with period L. Here, $\mathrm{Vol}(E)$ is the volume of E, $e_2(L) = \lim_{s\to 0} \sum_{\omega\in L\setminus\{0\}} \frac{1}{\omega^2|\omega|^s}$. For modular elliptic curve E, let $z(\tau)$ be the integral from $i\infty$ to $\tau \in \mathfrak{H}$ of the normalized cusp form of weight two associated with E. For the modular elliptic curve $X_0(N)$, there exists a lattice L_N such that $\mathbb{C}/L_N$ is analytically isomorphic to $X_0(N)$, and the isomorphism is given by $2\pi i z(\tau)$. We put $E_N = \mathbb{C}/L_N$. Then $\hat{\zeta}(z(\tau), E_N)$ is a harmonic Maass form of weight 0.

We need the case of $N = 32$, 27. Then E_{32} is $y^2 = x^3 + 4x$ and E_{27} is $y^2 + y = x^3 - 7$. It is well-known that E_{32} and E_{27} have complex multiplication, moreover $\mathrm{End}(E_{32}) = \mathbb{Z}[i]$ and $\mathrm{End}(E_{27}) = \mathbb{Z}[\frac{1+\sqrt{3}i}{2}]$(see [LMFDB]). Then, for $N = 32$, 27, there is a root of unity $\lambda_N \neq \pm 1$ such that $\lambda_N L_N = L_N$. Therefore,

$$\begin{aligned} e_2(L_N) &= \lim_{s\to 0} \sum_{\omega\in L\setminus\{0\}} \frac{1}{\omega^2|\omega|^s} \\ &= \lim_{s\to 0} \sum_{\omega\in L\setminus\{0\}} \frac{1}{(\lambda_N\omega)^2|\lambda_N\omega|^s} \\ &= \lambda_N^{-2} \lim_{s\to 0} \sum_{\omega\in L\setminus\{0\}} \frac{1}{\omega^2|\omega|^s} \\ &= \lambda_N^{-2} e_2(L_N), \end{aligned}$$

we get $e_2(L_N) = 0$. Specifically, $z(\tau)$ equals $2\pi i \int_{i\infty}^{\tau} \eta(4\tau)^2\eta(8\tau)^2 d\tau$ if $N = 32$, and $2\pi i \int_{i\infty}^{\tau} \eta(3\tau)^2\eta(9\tau)^2 d\tau$ if $N = 27$. Therefore, from (3.1), we have the following proposition.

Proposition 3.1. (1) *The function*

$$\zeta\left(2\pi i \int_{i\infty}^{\tau} \eta(4\tau)^2\eta(8\tau)^2 d\tau, E_{32}\right) = \frac{1}{q} + \frac{2}{3}q^3 + \frac{1}{7}q^7 - \frac{2}{11}q^{11} + \cdots$$

is a mock modular form of weight zero with respect to $\Gamma_0(32)$.

(2) *The function*

$$\zeta\left(2\pi i\int_{i\infty}^{\tau}\eta(3\tau)^2\eta(9\tau)^2d\tau, E_{27}\right)=\frac{1}{q}+\frac{1}{2}q^2+\frac{1}{5}q^5+\frac{3}{4}q^8-\frac{6}{11}q^{11}+\cdots$$

is a mock modular form of weight zero with respect to $\Gamma_0(27)$.

We further need the following.

Proposition 3.2. *There exist weak harmonic Maass forms* $M_{32}(\tau)$ *and* $M_{27}(\tau)$ *such that*

$$\frac{1}{2\pi i}\frac{dM_{32}}{d\tau}(\tau)=\frac{E_{4,2}(4\tau)}{g_2(4\tau)},\qquad \frac{1}{2\pi i}\frac{dM_{32}}{d\bar\tau}(\tau)=t_{32}\,\overline{g_2}(4\tau),$$
$$\frac{1}{2\pi i}\frac{dM_{27}}{d\tau}(\tau)=\frac{E_{4,3}(3\tau)}{g_3(3\tau)},\qquad \frac{1}{2\pi i}\frac{dM_{27}}{d\bar\tau}(\tau)=t_{27}\,\overline{g_3}(3\tau),$$

where t_{32}, t_{27} *are non-zero constants.*

Proof. We show the existence of M_{32}. Set $\mathcal{E}_2=2\pi i\int_{i\infty}^{\tau}\eta(4z)^2\eta(8z)^2dz$. We see the relation (up to additive constant)

$$2\pi i\int\frac{E_{4,2}(4\tau)}{g_2(4\tau)}d\tau+8\zeta(\mathcal{E}_2,E_{32})$$
$$=\frac{1}{g_2(4\tau)}\left(\frac{-1}{3}E_{2,2,-}(4\tau)-2E_{2,4,-}(4\tau)+\frac{28}{3}E_{2,8,-}(4\tau)\right)$$

holds by comparing first several Fourier coefficients of the derivatives of both sides, which are weakly modular forms of weight 2 lying in a finite dimensional vector space. Therefore, if we put

$$M_{32}(\tau)=\frac{1}{g_2(4\tau)}\left(\frac{-1}{3}E_{2,2,-}(4\tau)-2E_{2,4,-}(4\tau)+\frac{28}{3}E_{2,8,-}(4\tau)\right)$$
$$-8\hat{\zeta}(\mathcal{E}_2,E_{32}),$$

then $M_{32}(\tau)$ satisfies our required properties.

Similarly, we can have the relation

$$2\pi i\int\frac{E_{4,3}(3\tau)}{g_3(3\tau)}d\tau+6\zeta(\mathcal{E}_3,E_{27})=\frac{1}{g_3(3\tau)}\big(-E_{2,3,-}(3\tau)+6E_{2,9,-}(3\tau)\big),$$

where $\mathcal{E}_3=2\pi i\int_{i\infty}^{\tau}\eta(3z)^2\eta(9z)^2dz$. Therefore, we can construct $M_{27}(\tau)$ similarly. $\square$

4. The Kaneko-Zagier equation with respect to $\Gamma_0^*(N)$

In this section, we recall some facts on the Kaneko-Zagier equation and prove Theorem 1.1. The Kaneko-Zagier equation with respect to $\Gamma_0^*(2)$ and $\Gamma_0^*(3)$ of weight k is the following second-order ordinary differential equations respectively:

$$(KZ_{k,2}) \qquad f''(\tau) - \frac{k+1}{4}E_{2,2}(\tau)f'(\tau) + \frac{k(k+1)}{8}E'_{2,2}(\tau)f(\tau) = 0,$$

and

$$(KZ_{k,3}) \qquad f''(\tau) - \frac{k+1}{3}E_{2,3}(\tau)f'(\tau) + \frac{k(k+1)}{6}E'_{2,3}(\tau)f(\tau) = 0,$$

where $'$ denotes $\frac{1}{2\pi i}\frac{d}{d\tau}$. Define the Serre operators $\partial_{k,2}$ and $\partial_{k,3}$ by

$$\partial_{k,2}(f)(\tau) = f'(\tau) - \frac{k}{8}E_{2,2}(\tau)f(\tau)$$

and

$$\partial_{k,3}(f)(\tau) = f'(\tau) - \frac{k}{6}E_{2,3}(\tau)f(\tau).$$

Then $(KZ_{k,2})$ and $(KZ_{k,3})$ are equivalent to the following equations:

$$(KZ'_{k,2}) \qquad \partial_{k+2,2} \circ \partial_{k,2}(f(\tau)) = \frac{k(k+2)}{64}E_{4,2}(\tau)f(\tau)$$

and

$$(KZ'_{k,3}) \qquad \partial_{k+2,3} \circ \partial_{k,3}(f(\tau)) = \frac{k(k+2)}{36}E_{4,3}(\tau)f(\tau).$$

Since $\partial_{k,N}(f|_k\gamma) = \partial_{k,N}(f)|_k\gamma$ for any $\gamma \in \Gamma_0^*(N)$ $(N = 2, 3)$, the solution space of $(KZ_{k,N})$ is $\Gamma_0^*(N)$ invariant.

In the case of $\mathrm{SL}_2(\mathbb{Z})$, the modular solutions of the Kaneko-Zagier equation (KZ_k) are explicitly described in terms of the hypergeometric series. In the case of $\Gamma_0^*(2)$ and $\Gamma_0^*(3)$, their modular solutions have similar expression as follows.

Theorem 4.1 (Sakai [S11], Proposition 4). (1) *If k is a positive even integer, then the modular solution $f_{k,2}(\tau)$ of $(KZ_{k,2})$ whose 0-th Fourier coefficient is equal to 1 has the following form;*

(a) *for $k \equiv 0, 2 \mod 8$,*

$$f_{k,2}(\tau) = E_{4,2}(\tau)^{\frac{k}{4}}\,{}_2F_1\left(-\frac{k}{8}, -\frac{k-2}{8}; -\frac{k-3}{4}; \frac{256}{j_2^*(\tau)}\right),$$

(b) *for $k \equiv 4, 6 \mod 8$*

$$f_{k,2}(\tau) = E_{6,2}(\tau)E_{4,2}(\tau)^{\frac{k-6}{4}}\,{}_2F_1\left(-\frac{k-6}{8}; -\frac{k-4}{8} - \frac{k-3}{4}; \frac{256}{j_2^*(\tau)}\right).$$

In particular, $f_{2,2}(\tau)=E_{4,2}(\tau)^{\frac{1}{2}}$.

(2) *If k is a positive integer with $k\not\equiv 2 \mod 3$, then the modular solution $f_{k,3}(\tau)$ of $(KZ_{k,3})$ whose 0-th Fourier coefficient is equal to 1 has the following form;*

(a) *for $k\equiv 0,1 \mod 6$,*

$$f_{k,3}(\tau)=E_{4,3}(\tau)^{\frac{k}{4}}{}_2F_1\left(-\frac{k}{6},-\frac{k-1}{6};-\frac{k-2}{3};\frac{108}{j_3^*(\tau)}\right),$$

(b) *for $k\equiv 3,4 \mod 6$,*

$$f_{k,3}(\tau)=E_{6,3}(\tau)E_{4,3}(\tau)^{\frac{k-6}{4}}{}_2F_1\left(-\frac{k-4}{6};-\frac{k-3}{6}-\frac{k-2}{3};\frac{108}{j_3^*(\tau)}\right).$$

In particular, $f_{1,3}(\tau)=E_{4,3}(\tau)^{\frac{1}{4}}$.

Next two propositions give the non-modular solutions of $(KZ_{k,2})$ and $(KZ_{k,3})$.

Proposition 4.1. *There exist meromprphic functions $h_{k,2}$ (for non-negative even integer k) and $h_{k,3}$ (for non-negative integer $k\not\equiv 2 \mod 3$) satisfying*

$$h'_{k,2}(\tau)=\frac{g_2(\tau)^{k+1}}{f_{k,2}(\tau)^2}\quad \text{and}\quad h'_{k,3}(\tau)=\frac{g_3(\tau)^{k+1}}{f_{k,3}(\tau)^2}.$$

Proof. Because the proof of the second equality is similar to the first one, we only prove the first equality. Let z be a zero of $f_{k,2}(\tau)$. Then, the order of $f_{k,2}(\tau)$ at z is 1 because $f_{k,2}(\tau)$ is a solution of $(KZ_{k,2})$.

The function $h(\tau):=g_2(\tau)^{k+1}/f_{k,2}(\tau)^2$ has the following Laurant expansion at z:

$$h(\tau)=\frac{c_{-2}}{(\tau-z)^2}+\frac{c_{-1}}{\tau-z}+c_0+O((\tau-z)).$$

We will prove that c_{-1} is equal to zero. We see the following estimate by an elementary calculation:

$$2(\tau-z)h(\tau)+2\pi i\,(\tau-z)^2h'(\tau)=c_{-1}+O((\tau-z)).$$

Thus, we obtain $c_{-1}=\big(2(\tau-z)h(\tau)+2\pi i(\tau-z)^2h'(\tau)\big)|_{\tau=z}$. Since $4g_2'/g_2=E_{2,2}$,

$$\frac{h'(\tau)}{h(\tau)}=-2\frac{f'_{k,2}(\tau)}{f_{k,2}(\tau)}+\frac{k+1}{4}E_{2,2}(\tau).$$

Therefore,

$$c_{-1}=2\pi i\,(\tau-z)^2h(\tau)\left(\frac{1}{\pi i(\tau-z)}-2\frac{f'_{k,2}(\tau)}{f_{k,2}(\tau)}+\frac{k+1}{4}E_{2,2}(\tau)\right)\Bigg|_{\tau=z}.$$

Write $f_{k,2}(\tau) = (\tau - z)\phi(\tau)$ with a function $\phi(\tau)$. Then we have

$$\frac{f'_{k,2}(\tau)}{f_{k,2}(\tau)} = \frac{1}{2\pi i}\frac{1}{(\tau - z)} + \frac{\phi'(\tau)}{\phi(\tau)}$$

and thus

$$c_{-1} = 2\pi i\,(\tau - z)^2 h(\tau)\left(-2\frac{\phi'(\tau)}{\phi(\tau)} + \frac{k+1}{4}E_{2,2}(\tau)\right).$$

From $f'_{k,2}(\tau) = \frac{1}{2\pi i}\phi(\tau) + (\tau - z)\phi'(\tau)$, $f''_{k,2}(\tau) = \frac{1}{\pi i}\phi'(\tau) + (\tau - z)\phi''(\tau)$, and

$$f''_{k,2}(\tau) - \frac{k+1}{4}E_{2,2}(\tau)f'_{k,2}(\tau) + \frac{k(k+1)}{8}E'_{2,2}(\tau)f_{k,2}(\tau) = 0,$$

we obtain by letting $\tau \to z$

$$2\phi'(z) = \frac{k+1}{4}E_{2,2}(z)\phi(z).$$

Therefore, we conclude $c_{-1} = 0$. $\square$

Proposition 4.2. *Under the same notation as in Theorem 4.1 and Proposition 4.1, we have the following.*

(1) *For non-negative even integer k, the function $F_{k,2}(\tau) := f_{k,2}(\tau)\,h_{k,2}(\tau)$ is a solution of $(KZ_{k,2})$.*
(2) *For non-negative integer $k \not\equiv 2 \mod 3$, the function $F_{k,3}(\tau) := f_{k,3}(\tau)\,h_{k,3}(\tau)$ is a solution of $(KZ_{k,3})$.*

Proof. By Proposition 4.1, all poles of $h_{k,N}$ lie on the zeros of $f_{k,N}$ and are simple. Thus, $F_{k,N}$ is holomorphic on $\mathfrak{H}$. We show that $F_{k,N}$ satisfies $(KZ_{k,N})$. When $N = 2$,

$$\begin{aligned}
&F''_{k,2}(\tau) - \frac{k+1}{4}E_{2,2}(\tau)F'_{k,2}(\tau) + \frac{k(k+1)}{8}E'_{2,2}(\tau)F_k(\tau)\\
&\quad= (f''_{k,2}(\tau) - \frac{k+1}{4}E_{2,2}(\tau)f'_{k,2}(\tau) + \frac{k(k+1)}{8}E'_{2,2}(\tau)f_{k,2}(\tau))h_{k,2}(\tau)\\
&\qquad\quad + f_{k,2}(\tau)h''_{k,2}(\tau) + (2f'_{k,2}(\tau) - \frac{k+1}{4}E_{2,2}(\tau)f_{k,2}(\tau))h'_{k,2}(\tau) = 0.
\end{aligned}$$

The proof of the case $N = 3$ is similar and will be omitted. $\square$

Next we show the mixed mock modularity of the non-modular solutions of $(KZ_{2,2})$ and $(KZ_{1,3})$.

Proposition 4.3. *The functions $F_{2,2}(\tau)$ and $F_{1,3}(\tau)$ are mixed mock modular forms.*

Proof. We can obtain the relation

$$h_{2,2}(4\tau)=\frac{1}{256}\left(\frac{E_{4,2,-}(4\tau)}{E_{2,2,-}(4\tau)g_2(4\tau)}+2\pi i\int\frac{E_{4,2}(4\tau)}{g_2(4\tau)}d\tau\right),$$

by comparing several Fourier coefficients, where $\int\frac{E_{4,2}(4\tau)}{g_2(4\tau)}d\tau$ is a primitive function of $\frac{E_{4,2}(4\tau)}{g_2(4\tau)}$, because the derivatives of both sides multiplied by $(E_{2,2,-}(4\tau)g_2(4\tau))^2$ are modular forms of weight 10. By Theorem 3.2, $h_{2,2}(4\tau)$ and thus $h_{2,2}(\tau)$ is a sum of a meromorphic modular form and a mock modular form. Since $F_{2,2}$ is holomorphic on $\mathfrak{H}$ as shown in the proof of Proposition 4.2, $F_{2,2}(\tau)$ is sum of a modular form and a product of a modular form and a mock modular form.

By Proposition 4.1, $F_{2,2}(\tau)=O(1)$ at $i\infty$, that is, $F_{2,2}(\tau)$ is holomorphic at $i\infty$. By Proposition 4.2, for any $\gamma\in\Gamma^*(2)$, $F_{2,2}|_2\gamma$ is a linear combination of $f_{2,2}$ and $F_{2,2}$. Therefore, $F_{2,2}$ is holomorphic at every cusp and we conclude that $F_{2,2}$ is a mixed mock modular form.

Similarly, we obtain the relation

$$h_{1,3}(3\tau)=\frac{-1}{54}\left(\frac{I_{1,3}(3\tau)^3-54I_{3,3}(3\tau)}{I_{1,3}(3\tau)g_{27}(\tau)}+2\pi i\int\frac{E_{4,3}(3\tau)}{g_{27}(\tau)}d\tau\right),$$

where $\int\frac{E_{4,3}(3\tau)}{g_{27}(\tau)}d\tau$ is a primiteive function of $\frac{E_{4,3}(3\tau)}{g_{27}(\tau)}$, and from this we conclude that $h_{1,3}(\tau)I_{1,3}(\tau)$ is a mixed mock modular form. $\square$

Denote by $L_{k,2}$ (resp. $L_{k,3}$) the space of solutions of $(KZ_{k,2})$ (resp. $KZ_{k,3}$). There exist the following isomorphisms from $L_{k,2}$ (resp. $L_{k,3}$) to $L_{k-4,2}$ (resp. $L_{k-3,3}$). To prove Theorem 1.1, we will give the relation between the solution space $L_{k,2}$ and $L_{k-4,2}$ and between $L_{k,3}$ and $L_{k-3,3}$.

Proposition 4.4. (1) *For a positive even integer $k>2$, define*

$$\mu_{k,2}(f(\tau))=\frac{[f(\tau),E_{4,2}^{\frac{1}{2}}(\tau)]_1^{(k,2)}}{\Delta_2(\tau)}.$$

Then $\mu_{k,2}$ is an isomorphism from $L_{k,2}$ to $L_{k-4,2}$ sending a modular solution to a modular solution.

(2) *For a positive integer $k>1$ which is not congruent 2 modulo 3, define*

$$\mu_{k,3}(f)=\frac{[f(\tau),E_{4,3}^{\frac{1}{4}}(\tau)]_1^{(k,1)}}{\Delta_3(\tau)}.$$

Then $\mu_{k,3}$ is an isomorphism from $L_{k,3}$ to $L_{k-3,3}$ sending a modular solution to a modular solution.

Here $[f(\tau),g(\tau)]_1^{(k,l)}:=kf(\tau)g'(\tau)-lf'(\tau)g(\tau)$ (k, l are integers) is the Rankin-Cohen bracket.

Proof. Let $g_{k,2}$ (resp. $g_{k,3}$) be the modular solution of $(KZ_{k,2})$ (resp. $(KZ_{k,3})$). Let N be 2 or 3 and put $\alpha = 2$ if $N = 2$, $\alpha = 1$ if $N = 3$. We remark that $E_{4,N}^{\frac{\alpha}{2}}(\tau)$ is a solution of $(KZ_{N,\alpha})$ by Theorem 4.1. First we show $\partial_{k+\alpha+2,N}(\Delta_N \mu_{k,2}(g_{k,N}(\tau))) = \partial_{k+\alpha+2,N}([g_{k,N}(\tau), E_{4,N}^{\frac{\alpha}{4}}(\tau)]_1^{(k,\alpha)})$.

$$
\begin{aligned}
&\partial_{k+\alpha+2,N}([g_{k,N}(\tau), E_{4,N}^{\frac{\alpha}{4}}(\tau)]_1^{(k,\alpha)})\\
=&\partial_{k+\alpha+2,N}(k g_{k,N}(\tau)\partial_{\alpha,N}(E_{4,N}^{\frac{\alpha}{4}}(\tau)) - \alpha\partial_{\alpha,N}(g_{k,N}(\tau))E_{4,N}^{\frac{\alpha}{4}}(\tau))\\
=&k(\partial_{k,N}(g_{k,N}(\tau))\partial_{\alpha,N}(E_{4,N}^{\frac{\alpha}{4}}(\tau)) + g_{k,N}(\tau)\partial_{\alpha+2,N}(\partial_{\alpha,N}(E_{4,N}^{\frac{\alpha}{4}}(\tau))))\\
&-\alpha(\partial_{k+2,N}\partial_k(g_{k,N}(\tau))E_{4,N}^{\frac{\alpha}{4}}(\tau) + \partial_{k,N}(g_{k,N}(\tau))\partial_{\alpha,N}(E_{4,N}^{\frac{\alpha}{4}}(\tau)))\\
=&\left(k - \alpha\frac{k(k+2)}{\alpha(\alpha+2)}\right) g_{k,N}(\tau)\partial_{\alpha+2}(\partial_\alpha(E_{4,N}^{\frac{\alpha}{4}}(\tau)))\\
&+(k-\alpha)\partial_k(g_{k,N}(\tau))\partial_{\alpha+2,N}\partial_{\alpha,N}(E_{4,N}^{\frac{\alpha}{4}}(\tau))\\
=&\frac{k(\alpha-k)}{\alpha+2} g_{k,N}(\tau)\partial_{\alpha+2,N}(\partial_{\alpha,N}(E_{4,N}^{\frac{\alpha}{4}}(\tau)))\\
&+(k-\alpha)\partial_{k,N}(g_{k,N}(\tau))\partial_{\alpha,N}(E_{4,N}^{\frac{\alpha}{4}}(\tau))\\
=&\frac{\alpha-k}{\alpha+2}[g_{k,N}(\tau), \partial_{\alpha,N}(E_{4,N}^{\frac{\alpha}{4}}(\tau))]_1^{(k,\alpha+2)}.
\end{aligned}
$$

Next we show that

$$
\partial_{k+\alpha+4,N}([g_{k,N}, \partial_{\alpha,N}(E_{4,N}^{\frac{\alpha}{4}})]_1^{(k,\alpha+2)}) = -\frac{(k-\alpha-2)(\alpha+2)}{4(6-N)^2} E_{4,N}[g_{k,N}, E_{4,N}^{\frac{\alpha}{4}}]_1^{(k,\alpha)}.
$$

By noting $\partial_{\alpha+4,N}(E_{4,N}^{\frac{\alpha+4}{4}}) = \frac{\alpha+4}{\alpha} E_{4,N}\partial_{\alpha,N}(E_{4,N}^{\frac{\alpha}{4}})$, we have

$$
\begin{aligned}
&\partial_{k+\alpha+4,N}([g_{k,N}, \partial_{\alpha,N}(E_{4,N}^{\frac{\alpha}{4}})]_1^{(k,\alpha+2)})\\
&\quad= \partial_{k+\alpha+4,N}(k g_{k,N}\partial_{\alpha+2,N}(\partial_{\alpha,N}(E_{4,N}^{\frac{\alpha}{4}})) - (\alpha+2)\partial_{k,N}(g_{k,N})\partial_{\alpha,N}(E_{4,N}^{\frac{\alpha}{4}}))\\
&\quad= \frac{\alpha+2}{4(6-N)^2} E_{4,N}(\,k\,(\alpha\partial_{k,N}(g_{k,N})E_{4,N}^{\frac{\alpha}{4}} + (\alpha+4)g_{k,N}\partial_{\alpha,N}(E_{4,N}^{\frac{\alpha}{4}}))\\
&\qquad\quad -(k(k+2)g_{k,N}\partial_{\alpha,N}(E_{4,N}^{\frac{\alpha}{4}}) + \alpha(\alpha+2)\partial_{k,N}(g_{k,N})E_{4,N}^{\frac{\alpha}{4}}))\\
&\quad= -\frac{(k-\alpha-2)(\alpha+2)}{4(6-N)^2} E_{4,N}[g_{k,N}, E_{4,N}^{\frac{\alpha}{4}}]_1^{(k,\alpha)}.
\end{aligned}
$$

Also, since $\partial_{2(6-N),N}(\Delta_N) = 0$ and the Serre operator satisfies the Leibniz rule, $\mu_{k,N}(f_{k\,N}(\tau))$ is a solution of $(KZ_{k-N\alpha,N})$. The property that a modular solution goes to a modular solution follows from the property of the Rankin-Cohen bracket.

Finally, we will check $\ker \mu_{k,N} = 0$. If $\mu_{k,N}(f) = \{0\}$, a function f is a solution of $kf(\tau)(E_{4,N}^{\frac{\alpha}{4}})'(\tau) - lf'(\tau)E_{4,N}^{\frac{\alpha}{4}}(\tau) = 0$. Since $E_{4,N}^{\frac{k}{4}}$ is a solution of this first order differential equation, the element of $\ker \mu_{k,N}$ is equal to $E_{4,N}^{\frac{k}{4}}$ up to a constant multiple. But, if $E_{4,N}^{\frac{k}{4}}$ is a solution of $(KZ_{k,N})$ when $k > \alpha$, it contradicts Theorem 4.1. That is, $E_{4,N}^{\frac{k}{4}}$ is not contained in $L_{k,N}$. Therefore, we conclude that $\ker \mu_{k,N} = \{0\}$ and we have completed the proof. □

The proof of Theorem 1.1. We will prove the case of $(KZ_{k,2})$. The case of $(KZ_{k,3})$ can be proved in a similarly way.

The function $F_{k,2}(\tau)$ is holomorphic on $\mathfrak{H}$ by Proposition 4.4. There are α_k, γ_k, $\delta_k \in \mathbb{C}$ such that $\mu_{k,2}(f_{k,2}) = \alpha_k f_{k-4,2}(\tau)$, $\mu_{k,2}(F_{k,2}(\tau)) = \gamma_k f_{k-4,2}(\tau) + \delta_k F_{k-4,2}(\tau)$ and $\alpha_k\delta_k \neq 0$. Then,

$$\begin{aligned}\mu_{k,2}(F_{k,2}(\tau)) &= \mu_{k,2}(f_{k,2}(\tau)h_{k,2}(\tau)) \\ &= \frac{[f_{k,2}(\tau), E_{4,2}^{\frac{1}{2}}(\tau)]_1^{(k,2)} h_{k,2}(\tau) - 2f_{k,2}(\tau)h'_{k,2}(\tau)E_{4,2}^{\frac{1}{2}}(\tau)}{\Delta_2(\tau)} \\ &= \mu_{k,2}(f_{k,2}(\tau))\frac{F_{k,2}(\tau)}{f_{k,2}(\tau)} - 2\frac{E_{2,2}^{\frac{1}{2}}(\tau)g_2(\tau)^{k-3}}{f_{k,2}(\tau)}.\end{aligned}$$

Hence $\mu_{k,2}(f_{k,2})(\tau) = \alpha_k f_{k-4,2}(\tau)$, and we obtain that

$$\alpha_k F_{k,2}(\tau) = \gamma_k f_{k,2}(\tau) + \delta_k F_{k-4,2}(\tau)\frac{f_{k,2}(\tau)}{f_{k-4,2}(\tau)} + 2\frac{E_{4,2}^{\frac{1}{2}}(\tau)(g_2(\tau))^{k-3}}{f_{k-4,2}(\tau)}. \tag{4.1}$$

By Propositions 4.1 and 4.2, $f_{0,2}(\tau) = 1$, $F_{0,2}(\tau) = \mathcal{E}_2(\tau)$, $f_{2,2}(\tau) = E_{4,2}^{\frac{1}{2}}(\tau)$ and $F_{2,2}(\tau) = E_{4,2}^{\frac{1}{2}}(\tau)h_{2,2}(\tau)$. Therefore there exist meromorphic modular forms $A_k(\tau)$, $B_k(\tau)$, $C_k(\tau)$, $D_k(\tau)$ of weight k such that

$$F_{k,2}(\tau) = \begin{cases} A_k(\tau)\mathcal{E}_2(\tau) + B_k(\tau) & (k \equiv 0 \mod 4) \\ C_k(\tau)h_{2,2}(\tau) + D_k(\tau) & (k \equiv 2 \mod 4). \end{cases}$$

Here, $A_k(\tau)$ and $C_k(\tau)$ are equal to $f_{k,2}(\tau)$ up to non-zero constant multiples. The function $B_k(\tau)$ is a holomorphic function because $F_{k,2}(\tau)$ and $A_k(\tau)\mathcal{E}_2(\tau)$ are holomorphic.

For $k \equiv 2 \mod 4$, the function $\frac{f_{k,2}(\tau)}{E_{4,2}^{\frac{1}{2}}(\tau)}$ is holomorphic by Proposition 4.1. Thus $f_{k,2}(\tau)h_{2,2}(\tau) = \frac{f_{k,2}(\tau)}{E_{4,2}^{\frac{1}{2}}(\tau)}F_{2,2}(\tau)$ is holomorphic. Therefore, we conclude that $D_k(\tau)$ is a holomorphic function. The holomorphy of $F_{k,2}$ at $i\infty$ follows from (4.1) by induction. By Proposition 4.2, for any $\gamma \in \Gamma_0^*(2)$, $F_{2,2}|_2\gamma$ is a linear combination of $f_{2,2}$ and $F_{2,2}$. Therefore, $F_{k,2}$ is holomorphic at every cusp. Since $F_{k,2}(\tau)$, $A_k(\tau)\mathcal{E}_2(\tau)$ and $C_k(\tau)h_{2,2}(\tau)$ are holomorphic at every cusp, $B_k(\tau)$ and $D_k(\tau)$ are holomorphic at every cusp. Moreover, by Proposition 4.3, $f_{k,2}(\tau)h_{2,2}(\tau) = \frac{f_{k,2}(\tau)}{E_{2,2}^{\frac{1}{2}}(\tau)}F_{2,2}(\tau)$ is a product of a modular form and a mixed mock modular form. Thus $f_{k,2}(\tau)h_{2,2}(\tau)$ is a mixed mock modular form. Since $F_{k,2}(\tau)$ is a sum of a modular form and a mixed mock modular form, $F_{k,2}(\tau)$ is a mixed mock modular form. The proof is complete. $\square$

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