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$E(2)$-LOCAL PICARD GRADED BETA ELEMENTS AT THE PRIME THREE

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Abstract. Let $E(2)$ be the second Johnson-Wilson spectrum at the prime 3. In this paper, we show that some beta elements exist in the homotopy groups of the $E(2)$-localized sphere spectrum with a grading over the Picard group of the stable homotopy category of $E(2)$-local spectra.

1. Introduction

Let $\mathcal{S}$ denote the stable homotopy category of spectra. For spectra A and B, we denote by $[A,B]$ the group of morphisms from A to B in $\mathcal{S}$, and $[A,B]_* = \bigoplus_{k\in\mathbb{Z}}[\Sigma^k A, B]$ where Σ is the suspension functor. For the n-th Johnson-Wilson spectrum $E(n)$ at a prime number p, we consider the $E(n)$-local stable homotopy category $\mathcal{L}_n = L_n(\mathcal{S})$, where $L_n\colon \mathcal{S}\to\mathcal{S}$ is the Bousfield localization functor with respect to $E(n)$.

A spectrum $X \in \mathcal{L}_n$ is *invertible* if there exists $Y \in \mathcal{L}_n$ such that $X \wedge Y = L_nS^0$. Hereafter, for $k\in\mathbb{Z}$, S^k denotes the k-dimensional sphere spectrum. The *Picard group* $\mathrm{Pic}(\mathcal{L}_n)$ *of* $\mathcal{L}_n$ is defined to be the collection of isomorphism classes of invertible spectra in $\mathcal{L}_n$. Throughout this paper, for a spectrum A, we denote

$$\pi^n_X(A) = [X, L_nA] \text{ for } X\in \mathrm{Pic}(\mathcal{L}_n) \quad\text{and}\quad \pi^n_\star(A) = \bigoplus_{X\in\mathrm{Pic}(\mathcal{L}_n)} \pi^n_X(A).$$

Remark that, for the ordinary homotopy group $\pi_k(L_nA)$ for $k\in\mathbb{Z}$, there exists an isomorphism $\pi_k(L_nA) = \pi^n_{L_nS^k}(A)$. Since any L_nS^k is in $\mathrm{Pic}(\mathcal{L}_n)$, we have a monomorphism

$$(1.1)\qquad \begin{array}{rcl} i^A_n\colon \pi_*(L_nA) & = & \bigoplus_{k\in\mathbb{Z}}[S^k, L_nA] \\ & = & \bigoplus_{k\in\mathbb{Z}}[L_nS^k, L_nA] \\ & \stackrel{\subset}{\to} & \bigoplus_{X\in\mathrm{Pic}(\mathcal{L}_n)}[X, L_nA] = \pi^n_\star(A). \end{array}$$

Note that we have natural transformations $\eta^n_k\colon L_n \to L_k$ for $k\le n$. They give rise to inverse systems $s(A) = \{\pi_*(L_nA) \xleftarrow{(\eta^{n+1}_n)_*} \pi_*(L_{n+1}A)\}_n$ and $s'(A) = \{\pi^n_\star(A) \xleftarrow{(\eta^{n+1}_n)_*} \pi^{n+1}_\star(A)\}_n$. From the homomorphism $(i^A_n)_n\colon s(A) \to$

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$s'(A)$ of these systems, we obtain a monomorphism

$$\lim_n (i_n^A)\colon \lim_n \pi_*(L_nA) \to \lim_n \pi_\star^n(A).$$

By the chromatic convergence theorem (*cf.* [9, Th. 7.5.7]), for a finite spectrum V, the universal homomorphism $u_V\colon \pi_*(V) \to \lim_n \pi_*(L_nV)$ is an isomorphism. The homotopy groups $\pi_*(V)$ are contained in $\lim_n \pi_\star^n(V)$ under the composite

$$(1.2)\qquad \pi_*(V) \xrightarrow[\sim]{u_V} \lim_n \pi_*(L_nV) \xrightarrow[\text{mono.}]{\lim_n(i_n^V)} \lim_n \pi_\star^n(V).$$

From this point of view, we expect that the groups $\pi_\star^n(V)$ have new information of $\pi_*(V)$. For example, at $(p,n) = (2.1)$, the element $\alpha_{4t+2/2}$ in $\pi_*(L_1S^0)$ is expressed as the product $2_QA_{4t+2/3}$ in $\pi_\star^1(S^0)$ [7, (1.3)].

We note that $\mathrm{Pic}(\mathcal{L}_0) = \mathbb{Z}$ generated by L_0S^1. The natural transformation $\eta_0^n\colon L_n \to L_0$ induces the homomorphism

$$\ell_0\colon \mathrm{Pic}(\mathcal{L}_n) \to \mathrm{Pic}(\mathcal{L}_0) = \mathbb{Z}$$

of groups. Since this homomorphism admits a section $\mathbb{Z} \to \mathrm{Pic}(\mathcal{L}_n)$, which sends k to L_nS^k, the homomorphism ℓ_0 is a splitting epimorphism. Put $\mathrm{Pic}^0(\mathcal{L}_n) = \ker \ell_0$, and the group $\mathrm{Pic}^0(\mathcal{L}_n)$ is decomposed as

$$(1.3)\qquad \mathrm{Pic}(\mathcal{L}_n) = \mathbb{Z} \oplus \mathrm{Pic}^0(\mathcal{L}_n).$$

Here, the summand $\mathbb{Z}$ is generated by L_nS^1. The group $\mathrm{Pic}^0(\mathcal{L}_n)$ is known as follow.

Theorem 1.1 ([5, Th. A and Th. 6.1], [6, Cor. 1,4], [2, Th. 1.2])**.**

(1) *If* $p > 2$ *and* $2p-2 \geq n^2+n$, *then* $\mathrm{Pic}^0(\mathcal{L}_n) = 0$.
(2) *At* $p = 2$, $\mathrm{Pic}^0(\mathcal{L}_1) = \mathbb{Z}/2$.
(3) *At* $p = 3$, $\mathrm{Pic}^0(\mathcal{L}_2) = \mathbb{Z}/3 \oplus \mathbb{Z}/3$.

For the homology theory $BP_*(-)$ represented by the Brown-Peterson spectrum BP at p, we have

$$BP_* = BP_*(S^0) = \mathbb{Z}_{(p)}[v_1, v_2, \dots],$$
$$BP_*(BP) = BP_*[t_1, t_2, \dots]$$

with $|v_i| = |t_i| = 2(p^i-1)$. The homology theory $E(n)_*(-)$ represented by $E(n)$ satisfies that

$$E(n)_* = E(n)_*(S^0) = v_n^{-1}BP_*/(v_{n+1}, v_{n+2}, \dots) = \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_{n-1}, v_n^{\pm 1}],$$
$$E(n)_*(E(n)) = E(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E(n)_*$$

with $|v_i| = |t_i| = 2(p^i-1)$. The $E(n)$-based Adams spectral sequence for a spectrum A is of the form

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{E(n)_*(E(n))}(E(n)_*, E(n)_*(A)) \Longrightarrow \pi_{t-s}(L_nA).$$

Hereafter, we denote by $E(n)_r^{s,t}(A)$ the E_r-term of this spectral sequence. For an $E(n)_*(E(n))$-comodule M, we abbreviate

$$H^{*,*}M = \mathrm{Ext}^{*,*}_{E(n)_*(E(n))}(E(n)_*, M).$$

Let I_k denote the ideal $(v_0, v_1, \dots, v_{k-1})$ of $E(n)_*$, where $v_0 = p$. Consider the following $E(n)_*(E(n))$-comodules:

$$\begin{aligned} N_k^0 &= E(n)_*/I_k, \\ N_k^{i+1} &= \mathrm{Coker}\left(N_k^i \xrightarrow{\subset} M_k^i\right) \quad \text{and} \quad M_k^i = v_{k+i}^{-1} N_k^i \quad \text{for } i \geq 0. \end{aligned} \tag{1.4}$$

In particular, $N_k^i = M_k^i$ if $k + i = n$. The short exact sequence $N_0^i \to M_0^i \to N_0^{i+1}$ gives rise to the connecting homomorphism

$$\delta_i \colon H^* N_0^{i+1} \to H^{*+1} N_0^i. \tag{1.5}$$

For $k \leq n$, the *k-th algebraic Greek letter elements* are defined by

$$\overline{\alpha}^{(k)}_{e_k/e_{k-1},\dots,e_1,e_0} = \delta_0\delta_1 \cdots \delta_{k-1}\left(v_k^{e_k}/p^{e_0}v_1^{e_1}\cdots v_{k-1}^{e_{k-1}}\right) \in H^k N_0^0 = E(n)_2^k(S^0)$$

if $v_k^{e_k}/p^{e_0}v_1^{e_1}\cdots v_{k-1}^{e_{k-1}}$ is in $H^0 N_0^k$. In particular, we denote

$$\overline{\alpha}_{t/a} = \overline{\alpha}^{(1)}_{t/a}, \quad \overline{\beta}_{t/a,b} = \overline{\alpha}^{(2)}_{t/a,b}, \quad \overline{\beta}_{t/a} = \overline{\beta}_{t/a,1} \quad \text{and} \quad \overline{\beta}_t = \overline{\beta}_{t/1}.$$

By [6, Th. 1.1], for any invertible spectrum $X \in \mathrm{Pic}^0(\mathcal{L}_n)$, we have

$$E(n)_2^{*,*}(X) = E(n)_2^{*,*}(S^0)\{g_X\} \quad \text{with} \quad |g_X| = (0,0).$$

If the element

$$\overline{\alpha}^{(k)}_{e_k/e_{k-1},\dots,e_1,e_0} g_X \in E(n)_2^{*,*}(X)$$

detects an element of $\pi_*(X)$, then we may consider that the element is in $\pi_\star^n(S^0)$ as follow:

$$\pi_*(X) = \bigoplus_k [S^k, X] = \bigoplus_k [\Sigma^k L_n S^0, X] = \bigoplus_k [\Sigma^k X^{-1}, L_n S^0] \subset \pi_\star^n(S^0).$$

In the case for $p > 2$ and $n = 1$, we have $\pi_*(L_1 S^0) = \pi_\star^1(S^0)$ since $\mathrm{Pic}(\mathcal{L}_1) = \{L_1 S^k \colon k \in \mathbb{Z}\} \cong \mathbb{Z}$. In this case, any nonzero $\overline{\alpha}_{t/a}$ in $E(1)_2^1(S^0)$ detects a nonzero element in $\pi_*(L_1 S^0) = \pi_\star^1(S^0)$. At $(p, n) = (2, 1)$, for a nonzero integer t, we define

$$\nu_2(t) = \max\{i \in \mathbb{Z} \colon 2^i \mid t\} \quad \text{and} \quad a(t) = \begin{cases} 1 & \nu_2(t) = 0, \\ \nu_2(t) + 2 & \nu_2(t) > 0. \end{cases}$$

The elements $\overline{\alpha}_{t/a} (\neq 0)$ for $a \leq a(t)$ are defined. (For any $a > 0$, the element $\overline{\alpha}_{0/a}$ is defined, and however this is 0.) For

$$b(t) = \begin{cases} a(t) - 1 & t \equiv 2 \bmod (4), \\ a(t) & \text{otherwise}, \end{cases}$$

the element $\overline{\alpha}_{t/a}$ survives to $\pi_*(L_1S^0)$ if and only if

$$(0 \neq)\, t \equiv 0,1,2 \bmod (4) \quad \text{and} \quad a \leq b(t).$$

This fact implies that some nonzero algebraic alpha elements don't survive to $\pi_*(L_1S^0)$ at $p=2$. The author calculated $\pi^1_\star(S^0)$ at $p=2$ [7, Th. 2]. In particular, for the generator Q of $\mathrm{Pic}^0(\mathcal{L}_1)=\mathbb{Z}/2$, the element $\overline{\alpha}_{t/a}g_Q \in E(1)^1_2(Q)$ survives to $\pi_*(Q)\cong [Q, L_1S^0]_* \subset \pi^1_\star(S^0)$ if and only if

$$t \neq 0 \quad \text{and} \quad a \leq b'(t) \quad \text{where} \quad b'(t)=\begin{cases} a(t)-1 & t\equiv 0,1 \bmod (4),\\ a(t) & t \equiv 2,3 \bmod (4).\end{cases}$$

This implies that, for any $t\neq 0$ and $a \leq a(t)$, at least one of $\overline{\alpha}_{t/a}$ and $\overline{\alpha}_{t/a}g_Q$ survives to $\pi^1_\star(S^0)$.

Conjecture 1.2 ([7, Conj. 4])**.** *For any algebraic Greek letter element $\overline{\alpha}^{(n)}_{t/e_{n-1},e_{n-2},\dots,e_0}$ with $t\neq 0$, there exists $X \in \mathrm{Pic}^0(\mathcal{L}_n)$ such that $\overline{\alpha}^{(n)}_{t/e_{n-1},e_{n-2},\dots,e_0}g_X$ survives to $\pi^n_\star(S^0)$.*

Conjecture 1.3. *If the element $\overline{\alpha}^{(n)}_{t/e_{n-1},e_{n-2},\dots,e_0}g_X$ survives to $A^{(n)}_{t/e_{n-1},e_{n-2},\dots,e_0}$ of $\pi^n_\star(S^0)$, then $A^{(n)}_{t/e_{n-1},e_{n-2},\dots,e_0}$ is in the image of $\lim_n \pi^n_\star(S^0)\to \pi^n_\star(S^0)$.*

If these conjectures hold, then every algebraic Greek letter element detects an element of $\lim_n \pi^n_\star(S^0)$, and we may express $\pi_*(S^0)$ as a subring of $\lim_n \pi^n_\star(S^0)$ under the monomorphism (1.2) at $V=S^0$.

In this paper, we consider Conjecture 1.2 for $\overline{\beta}_{t/a}=\overline{\alpha}^{(2)}_{t/a,1}$ at $(p,n)=(3,2)$. For a nonzero integer t, we define

$$\nu_3(t)=\max\{i\in\mathbb{Z}\colon 3^i \mid t\}, \qquad a_0(t)=\begin{cases}1 & 3\nmid t,\\ 4\cdot 3^{\nu_3(t)-1}-1 & 3\mid t,\end{cases} \tag{1.6}$$

and

$$b_0(t)=\begin{cases} a_0(t)-1 & t\equiv 3 \bmod (9),\\ a_0(t) & \text{otherwise.}\end{cases} \tag{1.7}$$

By [8, Th. 6.1], the element $v_2^t/3v_1^a$ is in $H^0N_0^2=H^0M_0^2$ if and only if $t=0$ or $a\leq a_0(t)$. Therefore,

$$\overline{\beta}_{t/a}(\neq 0) \text{ is in } E(2)^2_2(S^0) \text{ if and only if } a \leq a_0(t).$$

Remark that the element $\overline{\beta}_{0/a}\in E(2)^2_2(S^0)$ is defined for any $a>0$, and $\overline{\beta}_{0/a}=0$. By [11, Th. 2.13], the element $\overline{\beta}_{t/a}$ survives to an element $\beta_{t/a}$ in $\pi_*(L_2S^0)$ if and only if $0\neq t\equiv 0,1,2,3,5,6 \bmod (9)$ and $a\leq b_0(t)$. For an

$E(2)$-local spectrum A and an integer $u \geq 0$, we denote

$$A^0 = L_2S^0 \text{ and } A^u = \underbrace{A \wedge \cdots \wedge A}_{u} \text{ if } u > 0.$$

Recall (3) of Theorem 1.1, and we have

$$\mathrm{Pic}^0(\mathcal{L}_2) = \mathbb{Z}/3\{X_1\} \oplus \mathbb{Z}/3\{X_2\}$$

at $p = 3$. Here, X_1 is the invertible spectrum X given by Kamiya and Shimomura [6, Prop. 1.5].

Theorem 1.4. *At $(p,n) = (3,2)$, Conjecture 1.2 holds for the algebraic beta elements $\overline{\beta}_{t/a}$. More details, the element $\overline{\beta}_{t/a} g_{X_1^u}$ survive to $\pi_\star^2(S^0)$, where*

$$u = \begin{cases} 0 & 0 \neq t \equiv 0,1,2,5,6 \bmod (9), \\ 1 & t \equiv 4,8 \bmod (9), \\ 2 & t \equiv 3,7 \bmod (9). \end{cases}$$

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2. Algebraic beta elements $\overline{\beta}_{t/a}$

We fix $(p,n) = (3,2)$. For the mod 3 Moore spectrum $V(0)$, the Adams v_1-periodic map $\alpha\colon \Sigma^4V(0) \to V(0)$ exists. For $k \geq 1$, we consider the cofiber sequences

$$\Sigma^{4k}V(0) \xrightarrow{\alpha^k} V(0) \xrightarrow{i_1^{(k)}} V(1)_k \xrightarrow{j_1^{(k)}} \Sigma^{4k+1}V(0). \tag{2.1}$$

In particular, $V(1)_1$ is the first Smith-Toda spectrum $V(1)$. We then have

$$\begin{array}{ccccccc} \Sigma^{4\ell+4}V(1)_k & \xrightarrow{v_1^\ell} & \Sigma^4V(1)_{k+\ell} & \xrightarrow{\widetilde{i}_k} & \Sigma^4V(1)_\ell & \xrightarrow{\partial_{\ell,k}} & \Sigma^{4\ell+5}V(1)_k \\ \| & & {\scriptstyle v_1}\downarrow & & {\scriptstyle v_1}\downarrow & & \| \\ \Sigma^{4\ell+4}V(1)_k & \xrightarrow{v_1^{\ell+1}} & V(1)_{k+\ell+1} & \xrightarrow{\widetilde{i}_k} & V(1)_{\ell+1} & \xrightarrow{\partial_{\ell+1,k}} & \Sigma^{4\ell+5}V(1)_k \end{array}$$

Put

$$W = \mathrm{hocolim}_{v_1} V(1)_\ell, \tag{2.2}$$

and the diagram gives rise to the cofiber sequence

$$V(1)_k \xrightarrow{f^{(k)}} \Sigma^{4k}W \xrightarrow{v_1^k} W \xrightarrow{\partial_k} \Sigma V(1)_k. \tag{2.3}$$

By applying $E(2)_2^{*,*}(-)$, the cofiber sequence (2.3) at $k = 1$ induces the exact sequence

$$\cdots \xrightarrow{(\partial_1)_*} H^*M_2^0 \xrightarrow{f_*^{(1)}} H^*M_1^1 \xrightarrow{v_1} H^*M_1^1 \xrightarrow{(\partial_1)_*} H^{*+1}M_2^0 \xrightarrow{f_*^{(1)}} \cdots \tag{2.4}$$

of the Ext goups of the comodules in (1.4). We also have the short exact sequences

$$0 \to N_1^0 \to M_1^0 \to M_1^1 \to 0 \tag{2.5}$$

and

$$0 \to N_0^0 \xrightarrow{3} N_0^0 \to N_1^0 \to 0. \tag{2.6}$$

These short exact sequences give rise to the connecting homomorphims

$$\delta'\colon H^*M_1^1 \to H^{*+1}N_1^0 \text{ and } \delta\colon H^*N_1^0 \to H^{*+1}N_0^0 \left(= E(2)_2^{*+1}(S^0)\right), \tag{2.7}$$

respectively. For elements in $H^*M_1^1$, we use the notation of Behrens' type [1]: For $x \in H^*M_2^0$, the element $x_{t/a} \in H^*M_1^1$ for $a > 0$ is defined by

$$v_1^{a-1}x_{t/a} = v_2^t x/v_1.$$

By [8, Th. 5.3], for an integer t,

$$1_{t/a} \in H^0M_1^1 \text{ is defined if and only if } t = 0 \text{ or } a \le a_0(t)$$

where $a_0(t)$ is the integer in (1.6).

Lemma 2.1. $\delta\delta'(1_{t/a}) = \overline{\beta}_{t/a}$.

Proof. Consider the commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_1^0 & \longrightarrow & M_1^0 & \longrightarrow & M_1^1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle -/3} & & \downarrow{\scriptstyle -/3} & & \downarrow{\scriptstyle -/3} & & \\ 0 & \longrightarrow & N_0^1 & \longrightarrow & M_0^1 & \longrightarrow & M_0^2 & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_0^0 & \xrightarrow{3} & N_0^0 & \longrightarrow & N_1^0 & \longrightarrow & 0 \\ & & \Vert & & \downarrow{\scriptstyle -/3} & & \downarrow{\scriptstyle -/3} & & \\ 0 & \longrightarrow & N_0^0 & \longrightarrow & M_0^0 & \longrightarrow & N_0^1 & \longrightarrow & 0 \end{array}$$

From them, for δ_i in (1.5), we obtain $\delta\delta'(1_{t/a}) = \delta_0\left(\delta'(1_{t/a})/3\right) = \delta_0\delta_1\left((1_{t/a})/3\right) = \delta_0\delta_1\left(v_2^t/3v_1^a\right) = \overline{\beta}_{t/a}$. □

3. Recollection of $\mathrm{Pic}^0(\mathcal{L}_2)$

We recall the following result:

Theorem 3.1 ([10, Th. 5.8])**.** *Let $K(2)_* = E(2)_*/(3, v_1) = \mathbb{Z}/3[v_2^{\pm 1}]$. As a $K(2)_*$-module, we have an isomorphism*

$$E(2)_2^{*,*}(V(1)) = P(b_0) \otimes E(\zeta_2) \otimes \{1, h_0, h_1, b_1, \xi, \psi_0, b_1\xi\}.$$

Here, $P(-)$ and $E(-)$ are polynomial and exterior algebras, respectively. The generators satisfy that

$$|v_2| = (0, 16), \quad |h_0| = (1, 4), \quad |h_1| = (1, 12),$$
$$|b_0| = (2, 12), \quad |b_1| = (2, 36), \quad |\xi| = (2, 8),$$
$$|\psi_0| = (3, 16) \quad \textit{and} \quad |\psi_1| = (3, 24).$$

For the summand $\mathrm{Pic}^0(\mathcal{L}_2)$ in (1.3), we have the monomorphism

$$\varphi\colon \mathrm{Pic}^0(\mathcal{L}_2) \to E(2)_2^{5,4}(S^0) = \mathbb{Z}/3\{\chi_1\} \oplus \mathbb{Z}/3\{\chi_2\} \tag{3.1}$$

by [6, Th. 1.2]. Here, the generators χ_1 and χ_2 satisfy that

$$\iota(\chi_1) = v_2^{-2} b_0^2 h_1 \quad \text{and} \quad \iota(\chi_2) = v_2^{-1} b_0 \zeta_2 \xi, \tag{3.2}$$

where ι is a homomorphism $E(2)_2^{*,*}(S^0) \to E(2)_2^{*,*}(V(1))$ induced by the composite $S^0 \xrightarrow{i} V(0) \xrightarrow{i_1^{(1)}} V(1)$. Here, the first map i is given by the cofiber sequence

$$S^0 \xrightarrow{3} S^0 \xrightarrow{i} V(0) \xrightarrow{j} S^1, \tag{3.3}$$

and the second map $i_1^{(1)}$ is in (2.1). Note that (3) of Theorem 1.1 implies that the monomorphism (3.1) is an isomorphism. By this fact, we may consider that the generators X_1 and X_2 of $\mathrm{Pic}^0(\mathcal{L}_2)$ satisfy

$$\varphi(X_i) = \chi_i$$

and

$$X_i^3 = L_2 S^0, \quad E(2)_2^{*,*}(X_i) = E(2)_2^{*,*}(S^0)\{g_{X_i}\} \text{ with } |g_{X_i}| = (0,0), \\ \text{and} \quad d_5(g_{X_i}) = \chi_i g_{X_i} \tag{3.4}$$

where $i \in \{1, 2\}$, and d_5 is the 5-th Adams differential $E(2)_5^{0,0}(X_i) \to E(2)_5^{5,4}(X_i)$.

4. On the elements $\overline{\beta}_{t/a} g_{X_1}$ and $\overline{\beta}_{t/a} g_{X_1^2}$

For the generator $X_i \in \mathrm{Pic}^0(\mathcal{L}_2)$, we have

$$E(2)_2^{0,0}(X_i^2) = E(2)_2^{0,0}(S^0)\{g_{X_1^2}\}.$$

Note that

$$g_{X_i^2} = (g_{X_i})^2$$

under the paring $E(2)_2^{*,*}(X_i) \otimes E(2)_2^{*,*}(X_i) \to E(2)_2^{*,*}(X_i^2)$, and $g_{S^0} = 1 \in E(2)_2^{0,0}(S^0)$.

Lemma 4.1. *Let $u \in \{0, 1, 2\}$. For the spectrum W in (2.2), if $(g_{X_i^u})_{t/a} \in E(2)_2^0(W \wedge X_i^u)$ is a permanent cycle, then $\overline{\beta}_{t/a} g_{X_i^u} \in E(2)_2^2(X_i^u)$ is a permanent cycle.*

Proof. We note that the short exact sequences (2.5) and (2.6) are obtained from the cofiber sequences

$$V(0) \to L_1V(0) \to W \xrightarrow{\partial'} \Sigma V(0) \tag{4.1}$$

and (3.3), respectively. Therefore, by Lemma 2.1 and the geometric boundary theorem, our claim at $u = 0$ is shown. Similarly, our claim holds at $u = 1, 2$. □

Theorem 4.2 ([11, Th. 2.8])**.** *The element $1_{t/a} \in E(2)^0_2(W) = H^0M^1_1$ is a permanent cycle if $t \equiv 0, 1, 2, 3, 5, 6 \bmod (9)$ and $a \le b_0(t)$ in (1.7).*

Proposition 4.3. *If $v_2^t \in E(2)^0_2(V(1))$ is a permanent cycle, then $(g_{X_1})_{t+3/1} \in E(2)^0_2(W \wedge X_1)$ and $(g_{X_1^2})_{t+6/1} \in E(2)^0_2(W \wedge X_1^2)$ are permanent cycles.*

Proof. Consider the cofiber sequence

$$\Sigma^4V(1) \xrightarrow{v_1} V(1)_2 \to V(1) \to \Sigma^5V(1).$$

If $v_2^t \in E(2)^0_2(V(1))$ is a permanent cycle, then the element $v_1v_2^t \in E^0_2(V(1)_2)$ is a permanent cycle. Since $V(1)_2$ is a ring spectrum, we have the paring

$$E(2)^{*,*}_r(V(1)_2) \otimes E(2)^{*,*}_r(V(1)_2 \wedge X_1) \to E(2)^{*,*}_r(V(1)_2 \wedge X_1).$$

By [3, Lemma 3.4],

$$v_2^3g_{X_1} \in E(2)^0_2(V(1)_2 \wedge X_1) \text{ is a permanent cycle.} \tag{4.2}$$

Therefore,

$$v_1v_2^{t+3}g_{X_1} = (v_1v_2^t)(v_2^3g_{X_1}) \in E(2)^0_2(V(1)_2 \wedge X_1) \text{ is permanent.} \tag{4.3}$$

For the map $f^{(2)}$ in (2.3), we have

$$d_r((g_{X_1})_{t+3/1}) = d_rf^{(2)}_*(v_1v_2^{t+3}g_{X_1}) = f^{(2)}_*d_r(v_1v_2^{t+3}g_{X_1}) = 0$$

for any r. We also have the pairing

$$E(2)^{*,*}_r(V(1)_2 \wedge X_1) \otimes E(2)^{*,*}_r(V(1)_2 \wedge X_1) \to E(2)^{*,*}_r(V(1)_2 \wedge X_1^2).$$

Therefore, by [3, Lemma 3.4] and (4.3),

$$d_r((g_{X_1^2})_{t+6/1}) = d_rf^{(2)}_*(v_1v_2^{t+6}g^2_{X_1}) = f^{(2)}_*d_r((v_1v_2^{t+3}g_{X_1})(v_2^3g_{X_1})) = 0$$

for any r. □

By [10, Th. A],

$$t \equiv 0, 1, 5 \bmod (9) \Rightarrow v_2^t \in E(2)^0_2(V(1)) \text{ survives to } \pi_*(L_2V(1)). \tag{4.4}$$

Therefore, by Lemma 4.1 and Proposition 4.3, we have the following:

Corollary 4.4. (1) *If $t \equiv 3, 4, 8 \bmod (9)$, then $\overline{\beta}_tg_{X_1}$ survives to $\pi^2_\star(S^0)$.*

(2) *If $t \equiv 2, 6, 7 \bmod (9)$, then $\overline{\beta}_tg_{X_1^2}$ survives to $\pi^2_\star(S^0)$.*

Lemma 4.5. $\pi_{31}(W \wedge X_1^2) = 0$.

Proof. By [11, Th. 2.5], we have $\bigoplus_{t-s=31} E(2)_2^{s,t}(W) = \mathbb{Z}/3\left\{(b_0^2h_0)_{1/2}, (b_0^4h_1)_{-1/1}\right\}$. (In [11, Th. 2.5], $(b_0^2h_0)_{1/2}$ and $(b_0^4h_1)_{-1/1}$ are denoted by $v_2b_{10}^2h_{10}/v_1^2$ and $v_2^{-1}b_{10}^4h_{11}/v_1$ in $F \otimes \mathbb{Z}/3[b_{10}]$, respectively.) This implies that

$$\bigoplus_{t-s=31} E(2)_2^{s,t}(W \wedge X_1^2) = \mathbb{Z}/3\left\{(b_0^2h_0g_{X_1^2})_{1/2}, (b_0^4h_1g_{X_1^2})_{-1/1}\right\}.$$

From [11, (8.3) and Prop. 8.9] and [3, Lemma 3.4], we obtain

$$\begin{aligned}
v_1d_9((b_0^2h_0g_{X_1^2})_{1/2}) &= v_1f_*^{(2)}d_9(v_2^{-5}b_0^2h_0(v_2^3g_{X_1})^2) \\
&= f_*^{(1)}(\widetilde{i}_1)_*(d_9(v_2^{-5}b_0^2h_0)(v_2^3g_{X_1})^2) \\
&= f_*^{(1)}(v_2^{-8}b_0^7)(v_2^3g_{X_1})^2 \\
&= (b_0^7g_{X_1^2})_{-2/1} \\
&\neq 0, \\
(b_0^4h_1g_{X_1^2})_{-1/1} &= f_*^{(1)}(v_2^{-7}b_0^4h_1(v_2^3g_{X_1})^2) \\
&= f_*^{(1)}d_5(v_2^{-5}b_0^2(v_2^3g_{X_1})^2) \\
&= d_5f_*^{(1)}(v_2^{-5}b_0^2(v_2^3g_{X_1})^2) \\
&= d_5((b_0^2g_{X_1^2})_{1/1}).
\end{aligned}$$

Therefore, both $(b_0^2h_0g_{X_1^2})_{1/2}$ and $(b_0^4h_1g_{X_1^2})_{-1/1}$ don't survive to $\pi_{31}(W \wedge X_1^2)$. □

By [4, Th. 2.24], $\pi_*(L_2V(1)_2)$ contains the part $huP(5)$. In particular, we have the element $hu \in \pi_*(L_2V(1)_2)$. By [4, (2.13)] and [4, p.3], this element is detected by $uh = \overline{h}_0 = v_2^5h_0$ in $E(2)_2^1(V(1)_2)$. We also note that v_2^{-9} and $v_2^3g_{X_1}$ are permanent cycles by [3, Lemma 1.6] and (4.2), respectively. Thus, the element

$$\overline{y} = v_2^{-9}(v_2^5h_0)(v_2^3g_{X_1})^2 \in E(2)_2^1(V(1)_2 \wedge X_1^2)$$

is a permanent cycle. We denote by $y \in \pi_*(V(1)_2 \wedge X_1^2)$ an element detected by $\overline{y}$.

Proposition 4.6. $(g_{X_1^2})_{3/3} \in E(2)_2^0(W \wedge X_1^2)$ *is a permanent cycle.*

Proof. Consider the cofiber sequence

$$V(1) \xrightarrow{f^{(1)}} \Sigma^4W \xrightarrow{v_1} W \xrightarrow{\partial_1} \Sigma V(1).$$

By [8, Prop. 5.4], we have

$$(\partial_1)_*((g_{X_1^2})_{3/3}) = v_2^2h_0g_{X_1^2} = (\widetilde{i}_1)_*(\overline{y}),$$

which detects $(\widetilde{i}_1 \wedge 1_{X_1^2})y$. By Lemma 4.5, the element $f_*^{(1)}((\widetilde{i}_1 \wedge 1_{X_1^2})y) \in \pi_{31}(W \wedge X_1^2)$ is trivial. Therefore, there exists $\xi \in \pi_{36}(W \wedge X_1^2)$ such that $\partial_1 \xi = (\widetilde{i}_1 \wedge 1_{X_1^2})y$. Since $E(2)_2^{0,36}(W \wedge X_1^2) = \mathbb{Z}/3\{(g_{X_1^2})_{3/3}\}$ by [11, Th. 2.5], the element ξ is detected by $\pm (g_{X_1^2})_{3/3}$. □

Proof of Theorem 1.4. By [11, Th. 2.13], for $0 \neq t \equiv 0, 1, 2, 5, 6 \bmod (9)$, we know that $\overline{\beta}_{t/a}$ for $a \leq a_0(t)$ survives to $\pi_*(L_2 S^0) \subset \pi_\star^2(S^0)$.

By Corollary 4.4, if $t \equiv 4, 8 \bmod (9)$, then $\overline{\beta}_{t/a_0(t)} g_{X_1} = \overline{\beta}_t g_{X_1}$ survives to $\pi_\star^2(S^0)$. Corollary 4.4 also implies that if $t \equiv 7 \bmod (9)$, then $\overline{\beta}_{t/a_0(t)} g_{X_1^2} = \overline{\beta}_t g_{X_1^2}$ survives to $\pi_\star^2(S^0)$.

We turn to the last case $\overline{\beta}_{t/a}$ for $t \equiv 3 \bmod (9)$ and $a \leq 3$. Proposition 4.6 implies that the element $(g_{X_1^2})_{3/a} = v_1^{3-a}(g_{X_1^2})_{3/3}$ detects an element in $\pi_*(W \wedge X_1^2)$. Put $t = 9s + 3$, and

$$\begin{aligned} d_r((g_{X_1^2})_{t/a}) &= d_r f_*^{(a)}(v_2^{9s+3} g_{X_1^2}) = f_*^{(a)} d_r(v_2^{9s}(v_2^3 g_{X_1^2})) \\ &= f_*^{(a)}(v_2^{9s} d_r(v_2^3 g_{X_1^2})) = (v_2^{9s} d_r(v_2^3 g_{X_1^2}))/v_1^a \\ &= v_2^{9s}(d_r(v_2^3 g_{X_1^2})/v_1^a) = v_2^{9s} d_r((g_{X_1^2})_{3/a}) \\ &= 0 \end{aligned}$$

for any $r > 1$. Therefore, by Lemma 4.1, the element $\overline{\beta}_{t/3} g_{X_1^2}$ survives to $\pi_\star^2(S^0)$. □

5. A note on $\pi_\star^2(V(0))$

Note that

$$E(2)_2^{*,*}(V(0) \wedge X_1) = E(2)_2^{*,*}(V(0))\{g'\}.$$

Here, $g' = i_*(g_1)$ where i_* is induced by i in (3.3). In this section, we consider the element $v_1 g'$ in the E_2-term.

The cofiber sequence (4.1) induces the long exact sequence

$$0 \to H^0 N_1^0 \to H^0 M_1^0 \to H^0 M_1^1 \xrightarrow{\delta'} H^1 N_1^0 \to \cdots.$$

Note that v_1 survives to $\pi_*(L_2 V(0))$, and $d_5(g_{X_1}) = \chi_1 g_{X_1} = \eta(v_2^{-1} h_1 b_0/3v_1) g_{X_1}$. Here, η is the composite $H^* M_0^2 \xrightarrow{\delta''} H^{*+1} N_0^1 \xrightarrow{\delta} H^{*+2} N_0^0$ where δ'' is the connecting homomorphism associated with the short exact sequence $N_0^1 \to M_0^1 \to M_0^2$. We then have

$$d_5(v_1 g') = v_1(i_* d_5(g_{X_1})) = v_1 i_*(\chi_1) g_{X_1}.$$

We denote by $B_{t/a}$ an element of $\pi_\star^2(S^0)$ detected by $\overline{\beta}_{t/a} g_{X_1^u}$ in Theorem 1.4.

Conjecture 5.1. (1) *The element* $v_1 g' \in E(2)^0_2(V(0) \wedge X_1)$ *detects a nonzero element* $w_1 \in \pi^2_\star(V(0))$.

(2) $i_*(\overline{\beta}_{t/a}) \neq 0$ *for* $a \leq a_0(t)$.

As an analogue of [7, (1.3)], we see the following.

Proposition 5.2. *If Conjecture 5.1 holds, then the homomorphism* $i_2^{V(0)} \colon \pi_*(L_2V(0)) \to \pi^2_\star(V(0))$ *in (1.1) satisfies that*

$$i_2^{V(0)} i_*(\beta_{t/a}) = \begin{cases} w_1 i_*(B_{t/a+1}) & 3 \neq t \equiv 3 \bmod (9), \\ i_*(B_{t/a}) & \textit{otherwise}, \end{cases}$$

up to higher filtration.

Proof. Let $t = 9s+3$, and suppose that $v_1 g'$ converges to $w_1 \in \pi_4(V(0) \wedge X_1) = [\Sigma^4 X_1^2, L_2V(0)] \subset \pi^2_\star(V(0))$. We note that

$$(v_1 g') i_*(\overline{\beta}_{t/a+1} g_{X_1^2}) = i_*((v_1 g_{X_1})\overline{\beta}_{t/a+1} g_{X_1^2}) = i_*(v_1^{3-a} v_2^{t-3} b_1) = i_*(\overline{\beta}_{t/a}).$$

Therefore, if $i_*(\overline{\beta}_{t/a}) \neq 0$, then $w_1 i_*(B_{t/a+1}) = i_*(\beta_{t/a})$ up to higher filtration. □

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