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A CHARACTERIZATION OF THE CLASS OF HARADA RINGS

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Abstract. There are many characterizations of Harada rings. In this paper, we characterize right co-Harada rings by giving a characterization of the class of basic right co-Harada rings.

1. Introduction

As is well-known, there are many characterizations of right co-Harada rings (equivalently, left Harada rings). The main purpose of this paper is to give a characterization of the class of basic right co-Harada rings.

Section 2 is the main part of this paper. We shall characterize the class of basic right co-Harada rings as a class of rings that is closed under certain operations (Theorem 2.1). Oshiro already determined the structure of right co-Harada rings as upper staircase factor rings of block extensions of QF rings (see, e.g. [3, Theorems 4.2.3 and 4.3.5]). Though the operations of Theorem 2.1 are special cases of the result, the theorem allows us to study and construct right co-Harada rings step by step and states that the operations are essential for right co-Harada rings. We also show that certain factor rings of right co-Harada rings are right co-Harada rings (Theorems 2.13, 2.14 and 2.16).

In Section 3 we show the uniqueness of QF rings associated with right co-Harada rings (Theorem 3.5). The result provides another description of the frame QF subrings of right co-Harada rings in the sense of Baba-Oshiro [3].

In Section 4 we illustrate the main result with some examples of right co-Harada rings represented as factor algebras of path algebras over a field. For this, we describe the quiver and the relations of the algebra R_e, a certain extension of R, for an algebra R over a field and a primitive idempotent e of R.

Throughout this paper, all rings have identity and all modules are unitary. Let R be a semiperfect ring. We denote by $\mathrm{pi}(R)$ a complete set of orthogonal primitive idempotents of R. For a right R-module M, the radical, the socle and the top of M (i.e., the factor module by its radical) are denoted by $J(M)$, $S(M)$ and $T(M)$, respectively. The symbol $S_i(M)$ denotes the i-th socle of M for $i = 0, 1, 2, \dots$.

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We first recall that a right artinian ring R is called a *right co-Harada ring* in case there exists a complete set of orthogonal primitive idempotents $\{e_{ij} \mid i = 1, 2, \ldots, m, j = 1, 2, \ldots, n(i)\}$ such that

(i) $e_{i1}R_R$ is injective for each $i = 1, 2, \ldots, m$;
(ii) $e_{i,j+1}R \cong e_{ij}R$ or $e_{i,j+1}R \cong J(e_{ij}R)$ for each $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n(i) - 1$.

Such a complete set of orthogonal primitive idempotents is called a *well-indexed set* of the right co-Harada ring R. As is well-known, serial rings and QF rings are right co-Harada rings. It should be noted that a ring is a *right co-Harada* ring if and only if it is a *left Harada* ring. Thus the terminology "left Harada rings" are mainly used recently. However, in this paper we shall use the terminology "right co-Harada rings" to emphasize properties of right co-Harada rings. For results about right co-Harada rings, refer to the book of Baba-Oshiro [3].

2. The class of right co-Harada rings

We denote by $\mathcal{H}$ the class of basic right co-Harada rings. In order to describe the characterization of $\mathcal{H}$, we need a notation.

For a ring R and an idempotent e of R, we define the ring R_e by

$$R_e = \begin{pmatrix} R & Re \\ eJ(R) & eRe \end{pmatrix}.$$

Note that if R is a basic semiperfect ring, then so is R_e. The ring R_e is a special case of block extensions of the ring R (see [3, Chapter 4]) and plays a very important role in the study of Harada rings. Oshiro proved that every basic right co-Harada ring can be represented as an upper staircase factor ring of a block extensions of a basic QF ring (cf. [3, Chapter 4]). We also note that, by taking factor rings of Theorem 2.1(3) and (4) of a block extension of a basic QF ring repeatedly, it follows that certain case of the upper staircase factor rings are right co-Harada rings (cf. Theorem 2.16).

The following is the main result of the paper.

Theorem 2.1. *The class of basic right co-Harada rings $\mathcal{H}$ satisfies the following conditions.*

(1) *$\mathcal{H}$ contains all basic QF rings.*
(2) *If $R \in \mathcal{H}$ and $e \in \mathrm{pi}(R)$, then $R_e \in \mathcal{H}$.*
(3) *If $R \in \mathcal{H}$ and $e, f \in \mathrm{pi}(R)$ such that*
  (a) *eR_R is injective with $S(eR_R) \cong T(fR_R)$,*
  (b) *fR_R is not injective,*
  then $R/S(eR_R) \in \mathcal{H}$.
(4) *If $R \in \mathcal{H}$, R is not a division ring, and $e, g \in \mathrm{pi}(R)$ such that*

(a) *eR_R is injective,*
(b) *$eR/S(eR_R) \cong J(gR_R)$,*
then $R/S(eR_R) \in \mathcal{H}$.

Moreover, $\mathcal{H}$ is the smallest class of basic one-sided artinian rings satisfying these four conditions.

To prove the main theorem, first we must verify that the class $\mathcal{H}$ satisfies the four conditions of the theorem. The condition (1) is clear and the condition (2) is verified in [4, Proposition 2.10]. Thus we must investigate the conditions (3) and (4).

Let R be a basic one-sided artinian ring. We recall that, for $e, f \in \mathrm{pi}(R)$, the pair (eR, Rf) is said to be an *i-pair* in case $S(eR_R) \cong T(fR_R)$ and $S({}_RRf) \cong T({}_RRe)$. For i-pairs, see [1, 31.3. Theorem] and [3, Chapter 2]. The following results, which we shall use freely, are well-known.

Lemma 2.2 (cf. [1, 31.3. Theorem] and [3, Lemma 2.1.1])**.** *Let R be a basic one-sided artinian ring and let $e, f \in \mathrm{pi}(R)$.*

(1) *The following are equivalent:*
 (i) *(eR, Rf) is an i-pair;*
 (ii) *eR_R is injective and $S(eR_R) \cong T(fR_R)$;*
 (iii) *${}_RRf$ is injective and $S({}_RRf) \cong T({}_RRe)$.*

(2) *If (eR, Rf) is an i-pair, then*

$$S(eR_R) = S({}_RRf).$$

Thus this is a two-sided ideal of R and is simple on both left and right side hands.

By Lemma 2.2, if R is a basic one-sided artinian ring and eR_R is injective for $e \in \mathrm{pi}(R)$, then $S(eR_R)$ is a two-sided ideal of R. Thus we can consider the factor ring $R/S(eR_R)$ of R in (3) and (4) of Theorem 2.1. We shall frequently denote by $\bar{R}$ the factor ring $R/S(eR_R)$. For $g \in \mathrm{pi}(R)$, $\bar{g}$ denotes the primitive idempotent of $\bar{R}$ corresponding to g. We also note that, for an i-pair (eR, Rf) $(e, f \in \mathrm{pi}(R))$, if eR_R is simple, then e is a central idempotent of R and $e = f$ because eR_R is projective and injective.

For basic right co-Harada rings, we note the following.

Lemma 2.3. *Let R be a basic right co-Harada ring and let $f \in \mathrm{pi}(R)$. Then ${}_RRf$ is injective if and only if $S(R_R)f \neq 0$.*

Proof. ($\Rightarrow$) This is clear from Lemma 2.2.

($\Leftarrow$) Since R is a right co-Harada ring, if $S(R_R)f \neq 0$, then there exists $e \in \mathrm{pi}(R)$ such that eR_R is injective and $eS(R_R)f \neq 0$. Thus by Lemma 2.2 ${}_RRf$ is injective. □

To show (3) and (4) of Theorem 2.1, we need to observe the relationship of i-pairs between basic right co-Harada rings R and certain factor rings $\bar{R}$ of R. The next lemma shows that i-pairs except (eR, Rf) of a basic one-sided artinian ring R are preserved to the factor ring $\bar{R} = R/S(eR) = R/S(Rf)$.

Lemma 2.4. *Let R be a basic one-sided artinian ring and let $e \in \mathrm{pi}(R)$ with eR_R injective. Set $\bar{R} = R/S(eR)$. If (gR, Rh) is an i-pair, then $(\bar{g}\bar{R}, \bar{R}\bar{h})$ is an i-pair, for $g, h \in \mathrm{pi}(R)$ with $g \neq e$,*

Proof. Let $f \in \mathrm{pi}(R)$ such that (eR, Rf) is an i-pair. We have $h \neq f$ by $g \neq e$. Thus we note $g \notin S(eR) = S(Rf)$ and $h \notin S(Rf) = S(eR)$. Then by assumption we have

$$S(\bar{g}\bar{R}) = S(gR) \cong T(hR) \cong T(\bar{h}\bar{R}),$$
$$S(\bar{R}\bar{h}) = S(Rh) \cong T(Rg) \cong T(\bar{R}\bar{g})$$

as R-modules. Thus $S(\bar{g}\bar{R}) \cong T(\bar{h}\bar{R})$ and $S(\bar{R}\bar{h}) \cong T(\bar{R}\bar{g})$ as $\bar{R}$-modules, i.e, $(\bar{g}\bar{R}, \bar{R}\bar{h})$ is an i-pair. □

Lemma 2.5 (cf. [6, Proposition 3.5] and [3, Lemma 3.3.1(1)]). *Let R be a basic right co-Harada ring and let $e, f \in \mathrm{pi}(R)$ such that (eR, Rf) is an i-pair. Let $e_1 = e, e_2, \ldots, e_n \in \mathrm{pi}(R)$ such that $J(e_iR) \cong e_{i+1}R$ for $i = 1, 2, \ldots, n-1$. Then, for each $i = 1, 2, \ldots, n$, the following hold.*

1. *$S_i({}_RRf) = S(e_1R_R) + S(e_2R_R) + \cdots + S(e_iR_R)$.*
2. *$S_i({}_RRf)/S_{i-1}({}_RRf) \cong T({}_RRe_i)$.*

Proof. (1) This is [6, Proposition 3.5].

(2) Though this is proved in [3, Lemma 3.3.1(1)], we give a proof here. We observe the left e_iRe_i-module $S(e_iR_R)$. Since R is basic, $S(e_iR_R)$ is simple as a left $\mathrm{End}(S(e_iR_R))$-module. Thus, since e_iR_R is quasi-injective, the restriction map $e_iRe_i \cong \mathrm{End}(e_iR_R) \to \mathrm{End}(S(e_iR_R))$ is a surjective ring homomorphism and hence $S(e_iR_R)$ is simple as a left e_iRe_i-module. Therefore by (1) we have

$$S_i({}_RRf)/S_{i-1}({}_RRf) = (S(e_iR_R) + S_{i-1}({}_RRf))/S_{i-1}({}_RRf) \cong T({}_RRe_i).$$

□

Lemma 2.6. *Let R be a basic right co-Harada ring and let $e, f, g \in \mathrm{pi}(R)$ such that (eR, Rf) is an i-pair and $gR \cong J(eR)$. Set $\bar{R} = R/S(eR) = R/S(Rf)$. Then $(\bar{g}\bar{R}, \bar{R}\bar{f})$ is an i-pair.*

Proof. By the assumption $gR \cong J(eR)$, eR_R is not simple and hence ${}_RRf$ is also not simple. Thus $f \notin S(Rf) = S(eR)$. We also note from $g \neq e$ that $g \notin S(eR) = S(Rf)$. Therefore, by Lemma 2.5(2),

$$T(\bar{R}\bar{g}) \cong T(Rg) \cong S_2(Rf)/S(Rf) \cong S(\bar{R}\bar{f})$$

as left R-modules. Thus $T(\bar{R}\bar{g}) \cong S(\bar{R}\bar{f})$ as left $\bar{R}$-modules. On the other hand,

$$S(\bar{g}\bar{R}) = S(gR) \cong S(eR) \cong T(fR) \cong T(\bar{f}\bar{R})$$

as right R-modules. Thus $S(\bar{g}\bar{R}) \cong T(\bar{f}\bar{R})$ as right $\bar{R}$-modules. Therefore $(\bar{g}\bar{R}, \bar{R}\bar{f})$ is an i-pair. □

Lemma 2.7. *Let R be a basic right co-Harada ring. For each $e \in \mathrm{pi}(R)$,*

$$Re/S(R_R)e \cong \mathrm{Hom}_R(J(eR), J(R))$$

canonically.

Proof. We first claim that the restriction map

$$\mathrm{Hom}_R(eR, gR) \to \mathrm{Hom}_R(J(eR), J(gR))$$

is surjective for any $g \in \mathrm{pi}(R)$. To verify this, let $\alpha : J(eR) \to J(gR)$ be a homomorphism. There exists an extension $\beta : eR \to E(gR)$ of α. It suffices to show that $\mathrm{Im}(\beta) \leq gR$. Assume to the contrary $\mathrm{Im}(\beta) \not\leq gR$. Then, since R is a right co-Harada ring, we have $gR \leq \mathrm{Im}(\beta)$, $gR \neq \mathrm{Im}(\beta)$ and hence $\mathrm{Im}(\beta)$ is projective. Thus $\beta : eR \to \mathrm{Im}(\beta)$ must be an isomorphism. Therefore we have $\mathrm{Im}(\beta)/J(gR) \cong eR/\beta^{-1}(J(gR))$. On the other hand, since β is an extension of α, we see $J(eR) \leq \beta^{-1}(J(gR))$. Thus we have a surjective homomorphism

$$T(eR) = eR/J(eR) \to eR/\beta^{-1}(J(gR)) \cong \mathrm{Im}(\beta)/J(gR).$$

So $\mathrm{Im}(\beta)/J(gR)$ is simple or 0. This contradicts the fact that $J(gR) < gR < \mathrm{Im}(\beta)$ are proper inclusions. Thus $\mathrm{Im}(\beta) \leq gR$ and $\beta : eR \to gR$ is an extension of α. Therefore we have a surjective homomorphism

$$gRe \cong \mathrm{Hom}_R(eR, gR) \to \mathrm{Hom}_R(J(eR), J(gR)).$$

The kernel of the homomorphism $gRe \to \mathrm{Hom}_R(J(eR), J(gR))$ is

$$gRe \cap l_R(J(eR)) = gRe \cap S(R_R) = S(gR_R)e,$$

where $l_R(J(eR))$ denotes the left annihilator of $J(eR)$ in R. Thus we have shown that

$$gRe/S(gR_R)e \cong \mathrm{Hom}_R(J(eR), J(gR))$$

canonically. That is,

$$g(Re/S(R_R)e) \cong g\,\mathrm{Hom}_R(J(eR), J(R))$$

canonically for any $g \in \mathrm{pi}(R)$. Therefore we obtain the isomorphism of the lemma. □

As the author proved and used in [4, Proposition 2.15], the following is a key result of the study of right co-Harada rings. Indeed, the result is closely related to (3) and (4) of Theorem 2.1 about factor rings of right co-Harada rings (see the proof of [4, Proposition 2.15]).

Proposition 2.8 ([4, Proposition 2.15]). *Let R be a basic right co-Harada ring and let $f \in \mathrm{pi}(R)$ with fR_R non-injective. Then $(1-f)R(1-f)$ is a right co-Harada ring.*

We cite the following proposition, which we shall frequently use in this paper.

Proposition 2.9 ([3, Proposition 7.1.11]). *Let R be a basic right co-Harada ring and let $e \in \mathrm{pi}(R)$. If $J(eR)$ is not projective, then the projective cover of $J(eR)$ is injective.*

Proof. This is proved in [3, Proposition 7.1.11]. However we can show the result easily by the first claim of the proof of Lemma 2.7. So we give a proof here. Let $\alpha : P \to J(eR)$ be a projective cover of $J(eR)$ and $P = \bigoplus_{i=1}^{n} P_i$ an indecomposable decomposition of P. To show that P is injective, assume to the contrary that P_i is not injective for some i. Then, since R is a right co-Harada ring, there exists an indecomposable projective right R-module Q_i such that $J(Q_i) = P_i$. By the first claim of the proof of Lemma 2.7, the canonical homomorphism $\mathrm{Hom}_R(Q_i, eR) \to \mathrm{Hom}_R(J(Q_i), J(eR))$ is surjective. Thus the restriction $P_i \to J(eR)$ of α can be extend to $\beta_i : Q_i \to eR$. If β_i is surjective, then β_i is an isomorphism and hence $J(eR) \cong J(Q_i) = P_i$ is projective, a contradiction. Thus β_i is not surjective and $\beta_i(Q_i) \leq J(eR)$. Therefore we have $\alpha(P_i) \leq J(J(eR))$. This contradicts the assumption that $\alpha : P \to J(eR)$ is a projective cover of $J(eR)$. □

Lemma 2.10. *Let R be a basic right co-Harada ring and let $e, f, g \in \mathrm{pi}(R)$ such that (eR, Rf) is an i-pair and $fR \cong J(gR)$. Then*

$$S_2(eR)/S(eR) \cong T(gR).$$

Proof. We first claim that $S_2(eR)/S(eR)$ is isomorphic to a direct sum of copies of $T(gR)$. To verify this, let M be a right R-submodule of $S_2(eR)$ such that $S(eR) \leq M$ and $M/S(eR)$ is simple. Let $\alpha : hR \to M/S(eR)$ be a projective cover for $h \in \mathrm{pi}(R)$ and let $\beta : hR \to M$ be a lift of α. Clearly β is surjective. Then the restriction $J(hR) \to S(eR)$ of β is also surjective. Let $\gamma : P \to J(hR)$ be a projective cover. Then there exists a split epimorphism $\delta : P \to fR$ that makes the following diagram commutative:

$$\begin{array}{ccccc} & & fR & \longrightarrow & T(fR) \\ & \overset{\delta}{\nearrow} & & & \downarrow \cong \\ P & \xrightarrow{\gamma} & J(hR) & \xrightarrow{\beta|_{J(hR)}} & S(eR). \end{array}$$

Since fR is not injective, P is also not injective. Thus by Proposition 2.9 $J(hR)$ must be indecomposable projective. Therefore we see from the surjective homomorphism $\beta|_{J(hR)} : J(hR) \to S(eR) \cong T(fR)$ that $J(hR) \cong fR$.

Hence by the assumption $J(gR) \cong fR$, we have $h = g$. Thus $M/S(eR) \cong T(gR)$. Therefore we have shown that $S_2(eR)/S(eR)$ is isomorphic to a direct sum of copies of $T(gR)$, that is, $S_2(eR)/S(eR) = (S_2(eR)/S(eR))g$.

Set $f' = 1 - f$. We note from Proposition 2.8 that $f'Rf'$ becomes a right co-Harada ring because fR is not injective. Since eR is injective, we see $e \neq f$ and $e \in f'Rf'$. Thus $S(eRf'_{f'Rf'})$ is simple. On the other hand, by $S_2(eR)/S(eR) = (S_2(eR)/S(eR))g$ we have

$$S_2(eR) = S_2(eR)g + S(eR) = S_2(eR)g + S(eR)f.$$

Here we note $S_2(eR)g \leq S(eRf'_{f'Rf'})$. Indeed, this follows from

$$S_2(eR)g \cdot f'J(R)f' = (S_2(eR)gJ(R))f' \leq S(eR)f' = S(Rf)f' = 0.$$

So, since $S(eRf'_{f'Rf'})$ is simple and $S_2(eR)g \neq 0$, we have $S_2(eR)g = S(eRf'_{f'Rf'})$. This implies that $S_2(eR)/S(eR)$ is simple and hence $S_2(eR)/S(eR) \cong T(gR)$. □

Lemma 2.11. *Let R be a basic right co-Harada ring and let $e, f, g \in \mathrm{pi}(R)$ such that (eR, Rf) is an i-pair and $fR \cong J(gR)$. Set $\bar{R} = R/S(eR) = R/S(Rf)$.*

(1) *In case ${}_RRg$ is not injective, $(\bar{e}\bar{R}, \bar{R}\bar{g})$ is an i-pair.*

(2) *In case ${}_RRg$ is injective, let $h_1, h_2, \ldots, h_n \in \mathrm{pi}(R)$ such that*

(a) *(h_1R, Rg) is an i-pair,*

(b) *$J(h_iR) \cong h_{i+1}R$ for $i = 1, 2, \ldots, n-1$ and $J(h_nR)$ is not projective.*

Then $eR/S(eR) \cong J(h_nR)$ as right R-modules and $\bar{e}\bar{R} \cong J(\overline{h_n}\bar{R})$ as right $\bar{R}$-modules.

Proof. (1) We first note that $e, g \notin S(eR) = S(Rf)$. Indeed, since fR is not injective, we see $e \neq f$ and hence $e \notin S(Rf) = S(eR)$. Since gR_R is not simple, $g \notin S(eR) = S(Rf)$. By Lemma 2.10,

$$S(\bar{e}\bar{R}) = S_2(eR)/S(eR) \cong T(gR) \cong T(\bar{g}\bar{R})$$

as right R-modules. Thus $S(\bar{e}\bar{R}) \cong T(\bar{g}\bar{R})$ as right $\bar{R}$-modules. Since ${}_RRg$ is not injective, by Lemmas 2.3 and 2.7 and by assumption,

$$Rg \cong \mathrm{Hom}_R(J(gR), J(R)) \cong \mathrm{Hom}_R(fR, J(R)) \cong J(Rf)$$

as left R-modules. So, since $J(Rf)$ is essential in Rf and $g \neq f$,

$$S(\bar{R}\bar{g}) = S(Rg) \cong S(J(Rf)) = S(Rf) \cong T(Re) \cong T(\bar{R}\bar{e})$$

as left R-modules. Hence $S(\bar{R}\bar{g}) \cong T(\bar{R}\bar{e})$ as left $\bar{R}$-modules. Therefore $(\bar{e}\bar{R}, \bar{R}\bar{g})$ is an i-pair.

(2) By Lemma 2.10 and assumption, we have

$$S(eR/S(eR)) = S_2(eR)/S(eR) \cong T(gR) \cong S(h_1R)$$

as right R-modules. Since h_1R_R is injective, there exists an exact sequence of right R-modules

$$0 \to S(eR) \to eR \to h_1R.$$

Thus by assumption there exists an exact sequence of right R-modules

$$0 \longrightarrow S(eR) \longrightarrow eR \xrightarrow{\alpha} J(h_nR).$$

We claim that α is an epimorphism. To prove this, apply the functor $\mathrm{Hom}_R((1-f)R, -)$ to the exact sequence above. Then, since

$$\mathrm{Hom}_R((1-f)R, S(eR)) \cong \mathrm{Hom}_R((1-f)R, T(fR)) = 0,$$

we have the following commutative diagram with exact rows

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{Hom}_R((1-f)R, eR) & \xrightarrow{\alpha_*} & \mathrm{Hom}_R((1-f)R, J(h_nR)) \\ & & \cong\downarrow & & \cong\downarrow \\ 0 & \longrightarrow & eR(1-f) & \longrightarrow & J(h_nR)(1-f), \end{array}$$

where $\alpha_* = \mathrm{Hom}_R((1-f)R, \alpha)$. Since the homomorphism of the bottom row is an isomorphism by [4, Lemma 2.13(4)], α_* is also an isomorphism. On the other hand, there exists an epimorphism $((1-f)R)^{(m)} \to J(h_nR)$ for some $m \geq 1$ because the projective cover of $J(h_nR)$ is injective by Proposition 2.9 and fR is not injective. Thus, there exist homomorphisms $\beta_1, \beta_2, \ldots, \beta_m : (1-f)R \to J(h_nR)$ such that $\bigoplus_{i=1}^m \beta_i : ((1-f)R)^{(m)} \to J(h_nR)$ is an epimorphism. Since $\alpha_* = \mathrm{Hom}_R((1-f)R, \alpha)$ is an isomorphism, there exist homomorphisms $\beta_i' : (1-f)R \to eR$ such that $\beta_i = \alpha\beta_i'$ for $i = 1, 2, \ldots, m$. Then

$$J(h_nR) = \sum_{i=1}^m \beta_i((1-f)R) = \sum_{i=1}^m \alpha\beta_i'((1-f)R) = \alpha(\sum_{i=1}^m \beta_i'((1-f)R)).$$

Thus $\alpha : eR \to J(h_nR)$ is an epimorphism. Therefore we have $J(h_nR) \cong eR/S(eR)$ as right R-modules and hence $J(\overline{h_n}\bar{R}) \cong \bar{e}\bar{R}$ as right $\bar{R}$-modules. □

We have prepared results to prove (3) and (4) of Theorem 2.1. We also cite one more result.

Lemma 2.12 ([4, Lemma 2.4]). *A right artinian ring R is a right co-Harada ring if and only if, for each $e \in \mathrm{pi}(R)$, eR_R is injective or $eR \cong J(fR)$ for some $f \in \mathrm{pi}(R)$.*

We can show Theorem 2.1(4) as the following form.

Theorem 2.13. *Let R be a basic right co-Harada ring and let $e \in \mathrm{pi}(R)$ with eR_R injective. Assume that R is not a division ring. If $eR/S(eR) \cong J(gR)$ for some $g \in \mathrm{pi}(R)$, then the factor ring $\bar{R} = R/S(eR_R)$ is a right co-Harada ring.*

Proof. We first consider the case $g = e$, that is, $eR/S(eR) \cong J(eR)$. It follows that eR is a uniserial module whose each composition factor is isomorphic to $T(eR)$. Thus e is a central idempotent of R and eR is a local uniserial ring, which is a ring direct summand of R. Since R is not a division ring by assumption, if $S(eR) = eR$ then $(1 - e)R \neq 0$. Thus, for the case $g = e$, the statement of the theorem is clear. Therefore we may assume that $g \neq e$ and eR_R is not simple by the observation above.

Let $f \in \mathrm{pi}(R)$ such that (eR, Rf) is an i-pair. Since eR_R is not simple, we see $e, f \notin S(eR) = S(Rf)$. By using Lemma 2.12, we shall check that $\bar{R} = R/S(eR) = R/S(Rf)$ is a right co-Harada ring. Let $h \in \mathrm{pi}(R)$. First we consider the case $S(hR) \not\cong S(eR)$. If hR_R is injective, then so is $\bar{h}\bar{R}_{\bar{R}}$ by Lemma 2.4. If $hR_R \cong J(kR_R)$ for some $k \in \mathrm{pi}(R)$, then by $h \neq e$ and $k \neq e$ we have
$$\bar{h}\bar{R} = hR \cong J(kR) = J(\bar{k}\bar{R})$$
as right R-modules. Thus $\bar{h}\bar{R} \cong J(\bar{k}\bar{R})$ as right $\bar{R}$-modules. Second we consider the case $S(hR) \cong S(eR)$. If hR_R is injective, i.e., $h = e$, then by assumption and $g \neq e$ we have
$$\bar{h}\bar{R} = \bar{e}\bar{R} \cong J(gR) = J(\bar{g}\bar{R})$$
as right R-modules. Thus $\bar{h}\bar{R} \cong J(\bar{g}\bar{R})$ as right $\bar{R}$-modules. If $hR \cong J(eR)$, then $\bar{h}\bar{R}_{\bar{R}}$ is injective by Lemma 2.6. If $hR \cong J(kR)$ for some $k \in \mathrm{pi}(R)$ with $k \neq e$, then by $h \neq e$ and $k \neq e$ we have
$$\bar{h}\bar{R} = hR \cong J(kR) = J(\bar{k}\bar{R})$$
as right R-modules. Thus $\bar{h}\bar{R} \cong J(\bar{k}\bar{R})$ as right $\bar{R}$-modules. □

Now we can show Theorem 2.1(3) as the following form.

Theorem 2.14. *Let R be a basic right co-Harada ring and let $f \in \mathrm{pi}(R)$ with ${}_RRf$ injective. If fR_R is not injective, then the factor ring $\bar{R} = R/S({}_RRf)$ is a right co-Harada ring.*

Proof. Let $e \in \mathrm{pi}(R)$ such that (eR, Rf) is an i-pair. Since fR_R is not injective, $fR \cong J(gR)$ for some $g \in \mathrm{pi}(R)$. If ${}_RRg$ is injective, then $eR/S(eR) \cong J(kR)$ for some $k \in \mathrm{pi}(R)$ by Lemma 2.11(2). So by Theorem 2.13 $\bar{R} = R/S(eR) = R/S(Rf)$ is a right co-Harada ring. Thus we may assume that ${}_RRg$ is not injective. We shall also check that, for any $h \in \mathrm{pi}(R)$, $\bar{h}\bar{R}_{\bar{R}}$ is injective or $\bar{h}\bar{R}_{\bar{R}} \cong J(\bar{k}\bar{R}_{\bar{R}})$ for some $k \in \mathrm{pi}(R)$. For the case $S(hR) \not\cong S(eR)$, it is similar to the case of the proof of Theorem 2.13.

So we may assume $S(hR) \cong S(eR)$. Since ${}_RRg$ is not injective, if hR_R is injective, i.e, $h = e$, then $\bar{h}\bar{R}_{\bar{R}} = \bar{e}\bar{R}_{\bar{R}}$ is injective by Lemma 2.11(1). As is similar to the cases of the proof of Theorem 2.13, if $hR \cong J(eR)$ then $\bar{h}\bar{R}_{\bar{R}}$ is injective, and if $hR \cong J(kR)$ for some $k \in \mathrm{pi}(R)$ with $k \neq e$, then $\bar{h}\bar{R} \cong J(\bar{k}\bar{R})$ as right $\bar{R}$-modules.

□

Remark 2.15. We record here inheritances of well-indexed sets of the factor rings from the right co-Harada rings in Theorems 2.13 and 2.14. Let R be a basic right co-Harada ring and $E = \{e_{ij} \mid i = 1, 2, \ldots, m, j = 1, 2, \ldots, n(i)\}$ a well-indexed set of R. We may assume that the idempotents of R in Theorems 2.13 and 2.14 are in E. Set

$$E_1 = \{e_{i1} \mid i = 1, 2, \ldots, m\} = \{e \in E \mid eR \text{ is injective}\}.$$

We define a map $\mu : E_1 \to E$ as $(eR, R\mu(e))$ is an i-pair for $e \in E_1$. Let $\bar{R}$ be the factor ring of R in Theorems 2.13 or 2.14. The symbol F will denote the well-indexed set of $\bar{R}$ induced by E. The symbols F_1 and ν will denote the subset of F and the map $F_1 \to F$ which are similar to E and μ, respectively. We describe the relationship between E and F and the relationship between μ and ν.

(1) Let R be a basic right co-Harada ring in Theorem 2.13 and let $e, g \in \mathrm{pi}(R)$ such that eR is injective and $eR/S(eR) \cong J(gR)$. R is not a division ring. Set $\bar{R} = R/S(eR)$. We divide observations into the two cases $g = e$ and $g \neq e$.

Case $g = e$: Assume $e = e_{11}$. As we noted in the proof of Theorem 2.13, e is a central idempotent. Thus

$$F = \begin{cases} \{\overline{e_{ij}} \mid i = 1, 2, \ldots, m, j = 1, 2, \ldots, n(i)\} & \text{if } S(eR) \neq eR, \\ \{\overline{e_{ij}} \mid i = 2, 3, \ldots, m, j = 1, 2, \ldots, n(i)\} & \text{if } S(eR) = eR \end{cases}$$

become well-indexed sets naturally.

Case $g \neq e$: Then $m \geq 2$. Assume $e = e_{m1}$. Since $eR/S(eR) \cong J(gR)$, $J(gR)$ is not projective. Thus, by renumbering the indices, we may assume $g = e_{m-1,n(m-1)}$. Then

$$eR/S(eR) = e_{m1}R/S(e_{m1}R) \cong J(e_{m-1,n(m-1)}R) = J(gR).$$

Define idempotents f_{ij} of $\bar{R}$ by

$(f_{11}, \ldots, f_{1,n(1)}) = (\overline{e_{11}}, \ldots, \overline{e_{1,n(1)}})$,

$\cdots$,

$(f_{m-2,1}, \ldots, f_{m-2,n(m-2)}) = (\overline{e_{m-2,1}}, \ldots, \overline{e_{m-2,n(m-2)}})$,

$(f_{m-1,1}, \ldots, f_{m-1,n(m-1)}, f_{m-1,n(m-1)+1}) = (\overline{e_{m-1,1}}, \ldots, \overline{e_{m-1,n(m-1)}}, \overline{e_{m,1}})$,

$(f_{m1}, \ldots, f_{m,n(m)-1}) = (\overline{e_{m2}}, \ldots, \overline{e_{m,n(m)}})$.

Then by the proof of Theorem 2.13 $F = \{f_{ij}\}$ is a well-indexed set of $\bar{R}$ with the subset

$$F_1 = \{f_{i1} \mid i = 1, 2, \ldots, m\} = \{\overline{e_{11}}, \ldots, \overline{e_{m-1,1}}, \overline{e_{m2}}\}$$

and by Lemmas 2.4 and 2.6 the map $\nu : F_1 \to F$ is given by

$$\begin{aligned}
&\nu(f_{11}) = \nu(\overline{e_{11}}) = \overline{\mu(e_{11})},\\
&\cdots,\\
&\nu(f_{m-2,1}) = \nu(\overline{e_{m-2,1}}) = \overline{\mu(e_{m-2,1})},\\
&\nu(f_{m-1,1}) = \nu(\overline{e_{m-1,1}}) = \overline{\mu(e_{m-1,1})},\\
&\nu(f_{m1}) = \nu(\overline{e_{m2}}) = \overline{\mu(e_{m1})}.
\end{aligned}$$

(2) Let R be a basic right co-Harada ring in Theorem 2.14 and let $e, f \in \mathrm{pi}(R)$ such that (eR, Rf) is an i-pair. Assume that fR is not injective. Set $\bar{R} = R/S(eR) = R/S(Rf)$. Let $g \in \mathrm{pi}(R)$ such that $fR \cong J(gR)$.

Case Rg is injective: Let $h \in \mathrm{pi}(R)$ such that (hR, Rg) is an i-pair. Then $e \neq h$ by $f \neq g$. Thus we may assume $e = e_{m1}$ and $h = e_{m-1,1}$. By Lemma 2.11(2)

$$eR/S(eR) = e_{m1}R/S(e_{m1}R) \cong J(e_{m-1,n(m-1)}R).$$

Therefore the well-indexed set F of $\bar{R}$ with the subset F_1 and the map $\nu : F_1 \to F$ are the same as in the case $g \neq e$ of (1).

Case Rg is not injective: We may assume $e = e_{m1}$. Define idempotents f_{ij} of $\bar{R}$ by

$$\begin{aligned}
&(f_{11}, \ldots, f_{1,n(1)}) = (\overline{e_{11}}, \ldots, \overline{e_{1,n(1)}}),\\
&\cdots,\\
&(f_{m-1,1}, \ldots, f_{m-1,n(m-1)}) = (\overline{e_{m-1,1}}, \ldots, \overline{e_{m-1,n(m-1)}}),\\
&(f_{m1}) = (\overline{e_{m1}}),\\
&(f_{m+1,1}, \ldots, f_{m+1,n(m)-1}) = (\overline{e_{m2}}, \ldots, \overline{e_{m,n(m)}}).
\end{aligned}$$

Then by the proof of Theorem 2.14 $F = \{f_{ij}\}$ is a well-indexed set of $\bar{R}$ with the subset

$$F_1 = \{f_{i1} \mid i = 1, 2, \ldots, m+1\} = \{\overline{e_{11}}, \ldots, \overline{e_{m-1,1}}, \overline{e_{m1}}, \overline{e_{m2}}\}$$

and by Lemmas 2.4, 2.6 and 2.11(1) the map $\nu : F_1 \to F$ is given by

$$\begin{aligned}
&\nu(f_{11}) = \nu(\overline{e_{11}}) = \overline{\mu(e_{11})},\\
&\cdots,\\
&\nu(f_{m-1,1}) = \nu(\overline{e_{m-1,1}}) = \overline{\mu(e_{m-1,1})},\\
&\nu(f_{m1}) = \nu(\overline{e_{m1}}) = \overline{g},
\end{aligned}$$

$$\nu(f_{m+1,1}) = \nu(\overline{e_{m2}}) = \overline{f} = \overline{\nu(e_{m1})}.$$

Combining Theorems 2.13 and 2.14, we also obtain the following theorem.

Theorem 2.16. *Let R be a basic right co-Harada ring, let $e, f \in \mathrm{pi}(R)$ such that (eR, Rf) is an i-pair and let $e_1 = e, e_2, \dots, e_n \in \mathrm{pi}(R)$ such that $J(e_iR) \cong e_{i+1}R$ for $i = 1, 2, \dots, n-1$. If fR_R is not injective, then $R/S_i({}_RRf)$ are right co-Harada rings for all $i = 1, 2, \dots, n$.*

Proof. Note by Lemma 2.5(1)

$$S_i({}_RRf) = S(e_1R_R) + S(e_2R_R) + \cdots + S(e_iR_R)$$

for $i = 1, 2, \dots, n$. We show the statement by induction. First, the factor ring $R/S({}_RRf)$ is a right co-Harada ring by Theorem 2.14. Assume that $\bar{R} = R/S_{i-1}({}_RRf)$ is a right co-Harada ring for $2 \leq i < n$. Then by Lemmas 2.5(1) and 2.6 $\overline{e_i}\bar{R}_{\bar{R}}$ is injective and $\overline{e_i}\bar{R}/S(\overline{e_i}\bar{R}) \cong J(\overline{e_{i-1}}\bar{R})$. Thus by Theorem 2.13 $\bar{R}/S(\overline{e_i}\bar{R}) \cong R/S_i({}_RRf)$ is also a right co-Harada ring. Therefore we have shown the statement of the theorem by induction. □

Combining Proposition 2.8 with Theorem 2.13, we have the following.

Proposition 2.17. *Let R be a basic non-local right co-Harada ring and let $e \in \mathrm{pi}(R)$ with eR_R injective. If $eR/S(eR) \cong J(gR)$ for some $g \in \mathrm{pi}(R)$, then $(1-e)R(1-e)$ is a right co-Harada ring.*

Proof. As is similar to the proof of Theorem 2.13, we may assume that $g \neq e$. By Theorem 2.13 the factor ring $\bar{R} = R/S(eR)$ is a right co-Harada ring and $\bar{e}\bar{R}_{\bar{R}}$, which is isomorphic to $J(\bar{g}\bar{R}_{\bar{R}})$, is not injective. Then by Proposition 2.8 $(1-\bar{e})\bar{R}(1-\bar{e})$ is a right co-Harada ring. Thus, so is $(1-e)R(1-e) \cong (1-\bar{e})\bar{R}(1-\bar{e})$. □

To complete the proof of Theorem 2.1, we need two more lemmas. Let R be a basic right co-Harada ring and let $e, f \in \mathrm{pi}(R)$ such that $fR \cong J(eR)$. Set $f' = 1-f$, $R' = f'Rf' = (1-f)R(1-f)$ and $\tilde{R} = R'_e$. Then

$$\tilde{R} = R'_e = \begin{pmatrix} R' & R'e \\ eJ(R') & eR'e \end{pmatrix} = \begin{pmatrix} f'Rf' & f'Re \\ eJ(R)f' & eRe \end{pmatrix}.$$

We should note that $\tilde{R}$ is a right co-Harada ring. Indeed $R' = (1-f)R(1-f)$ is a right co-Harada ring by Proposition 2.8. Thus, as we stated in Theorem 2.1 (2), $\tilde{R} = R'_e$ is also a right co-Harada ring by [4, Proposition 2.10].

Lemma 2.18 (cf. [4, Lemmas 2.6 and 2.7])**.** *With the setting above, there exists a surjective ring homomorphism $\phi_e : \tilde{R} \to R$ such that*

$$\mathrm{Ker}(\phi_e) = \begin{pmatrix} 0 & S(f'R_R)e \\ 0 & S(eR_R)e \end{pmatrix}.$$

Then ϕ_e is an isomorphism if and only if ${}_RRe$ is not injective.

Proof. [4, Lemma 2.6] states the existence of the ring homomorphism ϕ_e. Since R is a basic right co-Harada ring, we can apply [4, Lemma 2.7] to R. Thus ϕ_e is surjective. Hence by [4, Lemma 2.6(3)] ϕ_e is an isomorphism if and only if $\mathrm{Hom}_R(T(eR),(1-f)R)=0$. Since $fR \cong J(eR) < eR \leq (1-f)R$, we note that $\mathrm{Hom}_R(T(eR),(1-f)R)=0$ if and only if $\mathrm{Hom}_R(T(eR),R)=0$. On the other hand, we have canonical isomorphisms

$$\mathrm{Hom}_R(T(eR),R) \cong \mathrm{Hom}_R(T(eR),S(R_R)) \cong \mathrm{Hom}_R(eR,S(R_R)) \cong S(R_R)e.$$

Therefore ϕ_e is an isomorphism if and only if $S(R_R)e=0$, which is equivalent to ${}_RRe$ being non-injective by Lemma 2.3. □

To give the proof of Theorem 2.1, we provide the following lemma, which describes $\mathrm{Ker}(\phi_e)$ in Lemma 2.18 in terms of a well-indexed set of the right co-Harada ring $\tilde{R}$. For $g \in \mathrm{pi}(R') = \mathrm{pi}(f'Rf')$ and $e \in \mathrm{pi}(eR'e) = \mathrm{pi}(eRe)$, we put

$$\tilde{g} = \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \hat{e} = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}.$$

Then

$$\mathrm{pi}(\tilde{R}) = \{\tilde{g} \mid g \in \mathrm{pi}(R), g \neq f\} \cup \{\hat{e}\}.$$

We also note the fact that, for a basic right co-Harada ring R, if $g_1, g_2, \ldots, g_n \in \mathrm{pi}(R)$ satisfy the conditions (a) and (b) of Lemma 2.19, then $S(kR) \cong T(eR)$ iff $S(kR)e = S(kR)$ iff $k \in \{g_1, g_2, \ldots, g_n\}$ for any $k \in \mathrm{pi}(R)$. Thus we may assume $e = g_{j-1}$ and $f = g_j$ in (ii) of the lemma below.

Lemma 2.19. *With the same setting as in Lemma 2.18, assume that ${}_RRe$ is injective. Let $g_1, g_2, \ldots, g_n \in \mathrm{pi}(R)$ such that*

(a) *(g_1R, Re) is an i-pair;*
(b) *$J(g_iR) \cong g_{i+1}R$ for $i = 1, 2, \ldots, n-1$ and $J(g_nR)$ is not projective.*

Define $h_1, h_2, \ldots, h_n \in \mathrm{pi}(\tilde{R})$ as the following manner:

(i) *In case $S(eR) \not\cong T(eR)$, set $h_i = \tilde{g}_i$ for $i = 1, 2, \ldots, n$;*
(ii) *In case $S(eR) \cong T(eR)$, let $e = g_{j-1}$ and $f = g_j$ and set $h_i = \tilde{g}_i$ for $i = 1, 2, \ldots, j-1, j+1, \ldots, n$ and $h_j = \hat{e}$.*

Then

(1) *$(h_1\tilde{R}, \tilde{R}\hat{e}) = (\tilde{g}_1\tilde{R}, \tilde{R}\hat{e})$ is an i-pair.*
(2) *$J(h_i\tilde{R}) \cong h_{i+1}\tilde{R}$ for $i = 1, 2, \ldots, n-1$.*
(3) *$\mathrm{Ker}(\phi_e) = \sum_{i=1}^n S(h_i\tilde{R}) = S_n(\tilde{R}\hat{e})$.*

Proof. (1) By the assumption (a), $S(g_1R) \cong T(eR)$ and g_1R_R is injective. Thus the left annihilator $l_{g_1R}(R(1-f))$ of $R(1-f)$ in g_1R must be 0 and hence by [1, 31.2. Lemma] $g_1R(1-f)_{(1-f)R(1-f)} = g_1R'_{R'}$ is injective. It also follows from $S(g_1R) \cong T(eR)$ that $S(g_1R'_{R'})e \neq 0$. Thus we see from

Lemma 2.2 that $(g_1R', R'e)$ is an i-pair. Therefore, by using Lemma 2.2, [4, Lemma 2.9] and its proof, we can verify that $(\tilde{g}_1\tilde{R}, \tilde{R}\hat{e})$ is an i-pair.

(2) For the case (i), it is clear from the assumption (b) that $J(g_iR') \cong g_{i+1}R'$ for $i = 1, 2, \dots, n-1$. Thus by $e \neq g_i$ and by the form of $J(\tilde{R}) = J(R'_e)$, which is described in Lemma 4.1, we have $J(\tilde{g}_i\tilde{R}) \cong \widetilde{g_{i+1}}\tilde{R}$, that is, $J(h_i\tilde{R}) \cong h_{i+1}\tilde{R}$ for $i = 1, 2, \dots, n-1$.

For the case (ii), since R is basic, $J(eR) \cong fR$ and $J(fR) = J(g_jR) \cong g_{j+1}R$, we have isomorphisms

$$eJ(R)f' \cong fRf' = fJ(R)f' \cong g_{j+1}Rf'$$

as right R'-modules. In particular, $eJ(R)e \cong g_{j+1}Re$ as right eRe-modules. Thus it follows from the form of $J(\tilde{R})$ that $J(\hat{e}\tilde{R}) \cong \widetilde{g_{j+1}}\tilde{R}$. It also holds that $J(\tilde{e}\tilde{R}) \cong \hat{e}\tilde{R}$. Therefore, similar to the case of (i), we have $J(h_i\tilde{R}) \cong h_{i+1}\tilde{R}$ for $i = 1, 2, \dots, n-1$.

(3) As we noted above the lemma, for $k \in \mathrm{pi}(R)$, $S(kR) \cong T(eR)$ iff $S(kR)e = S(kR)$ iff $k \in \{g_1, g_2, \dots, g_n\}$. Therefore, in case $S(eR) \not\cong T(eR)$, by Lemma 2.18 and the definition of h_i we have

$$\begin{aligned}\mathrm{Ker}(\phi_e) &= \begin{pmatrix} 0 & \sum_{i=1}^{n} S(g_iR_R) \\ 0 & 0 \end{pmatrix} = \sum_{i=1}^{n} \begin{pmatrix} 0 & S(g_iR_R) \\ 0 & 0 \end{pmatrix} \\ &= \sum_{i=1}^{n} S(\tilde{g}_i\tilde{R}_{\tilde{R}}) = \sum_{i=1}^{n} S(h_i\tilde{R}_{\tilde{R}}),\end{aligned}$$

because each $\begin{pmatrix} 0 & S(g_iR_R) \\ 0 & 0 \end{pmatrix}$ is a simple submodule of $\tilde{g}_i\tilde{R}_{\tilde{R}}$ and $\tilde{R}$ is a right co-Harada ring. In case $S(eR) \cong T(eR)$, we recall $e = g_{j-1}$ and $f = g_j$ as in (ii). Thus, as is similar to the case above, we have

$$\begin{aligned}\mathrm{Ker}(\phi_e) &= \begin{pmatrix} 0 & \sum_{1\le i\le n, i\neq j} S(g_iR_R) \\ 0 & S(eR_R) \end{pmatrix} \\ &= \sum_{1\le i\le n, i\neq j} \begin{pmatrix} 0 & S(g_iR_R) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & S(eR_R) \end{pmatrix} \\ &= \sum_{1\le i\le n, i\neq j} S(\tilde{g}_i\tilde{R}_{\tilde{R}}) + S(\hat{e}\tilde{R}_{\tilde{R}}) = \sum_{i=1}^{n} S(h_i\tilde{R}_{\tilde{R}}).\end{aligned}$$

Furthermore, by (1), (2) and Lemma 2.5 (1), we have $\sum_{i=1}^{n} S(h_i\tilde{R}) = S_n(\tilde{R}\hat{e})$. □

Now we can complete a proof of Theorem 2.1.

Proof of Theorem 2.1. We have already shown that the class $\mathcal{H}$ satisfies the conditions (1)–(4) of Theorem 2.1. Indeed, as we noted before, the condition (1) is clear and the condition (2) is verified in [4, Proposition 2.10]. The conditions (3) and (4) are proved as Theorems 2.14 and 2.13, respectively. In order to prove the smallestness of $\mathcal{H}$, let $\mathcal{H}'$ be a class of rings satisfying the conditions (1)–(4). Let $R \in \mathcal{H}$. We shall prove $R \in \mathcal{H}'$ by induction on the composition length of R. If the composition length of R is one, that is, R is a division ring, then R is a QF ring. Thus by the condition (1) $R \in \mathcal{H}'$. We assume that R is not a division ring. In case R is a QF ring, $R \in \mathcal{H}'$ by (1). In case R is not a QF ring, since R is a right co-Harada ring, there exist $e, f \in \mathrm{pi}(R)$ such that $fR \cong J(eR)$. Set $f' = 1 - f$ and $R' = f'Rf'$. By Proposition 2.8 $R' \in \mathcal{H}$. Then by induction hypothesis, we have $R' \in \mathcal{H}'$. Thus by the condition (2), we have $R'_e \in \mathcal{H}'$. Set $\tilde{R} = R'_e$. In case ${}_RRe$ is not injective, $R \cong \tilde{R} \in \mathcal{H}'$ by Lemma 2.18. In case ${}_RRe$ is injective, by Lemmas 2.18 and 2.19 there exist $h_1, h_2, \ldots, h_n \in \mathrm{pi}(\tilde{R})$ such that $(h_1\tilde{R}, \tilde{R}\hat{e})$ is an i-pair, $J(h_i\tilde{R}) \cong h_{i+1}\tilde{R}$ $(i = 1, 2, \ldots, n-1)$ and $R \cong \tilde{R}/S_n(\tilde{R}\hat{e})$. Since $\hat{e}\tilde{R} \cong J(\tilde{e}\tilde{R})$, $\hat{e}\tilde{R}$ is not injective. Therefore by Theorem 2.16, which is proved by the conditions (3) and (4), we have $R \cong \tilde{R}/S_n(\tilde{R}\hat{e}) \in \mathcal{H}$. □

Concluding this section, we provide the almost self-duality of right co-Harada rings as an example of Theorem 2.1.

Example 2.20. Recall that a right artinian ring R is (right) *Morita dual* to a left artinian ring S in case there exists a duality between the category of finitely generated right R-modules and the category of finitely generated left S-modules. An artinian ring R is said to have a *self-duality* if R is Morita dual to R itself and R is said to have an *almost self-duality* if there exist artinian rings $R_0 = R, R_1, \ldots, R_{n-1}, R_n = R$ such that each R_i is right Morita dual to R_{i+1}. Clearly, the concept of almost self-duality is a generalization of that of self-duality. (For almost self-duality in detail, see [4].)

Let $\mathcal{A}$ be the class of basic artinian rings with almost self-duality. Clearly $\mathcal{A}$ contains all basic QF rings. That is, $\mathcal{A}$ satisfies the condition (1) of Theorem 2.1. By [4, Proposition 1.14(2)] $\mathcal{A}$ satisfies the condition (2) of Theorem 2.1. It follows from [4, Lemma 1.9(2)] and the proof of [4, Theorem 3.2] that $\mathcal{A}$ satisfies the conditions (3) and (4) of Theorem 2.1. Therefore $\mathcal{H} \subset \mathcal{A}$ by Theorem 2.1. In other words, every basic right co-Harada ring has an almost self-duality ([4, Theorem 3.2]).

3. Uniqueness of the QF rings reduced from right co-Harada rings

Oshiro proved that every basic right co-Harada ring R can be constructed from a QF ring. The QF ring has the form eRe for some idempotent e of R. He called the QF ring eRe the *frame QF subring* of R. However the definition of eRe is somewhat complicated. (See [3, Chapter 4].) In this section, we provide another description of the frame QF subring of a right co-Harada ring.

Let R be a right co-Harada ring. If R is not a QF ring, there exists $e_1 \in \mathrm{pi}(R)$ such that e_1R_R is not injective. Then by Proposition 2.8 the ring $R_1 = (1-e_1)R(1-e_1)$ is a right co-Harada ring again. Similarly, if R_1 is not a QF ring, there exists $e_2 \in \mathrm{pi}(R_1) = \mathrm{pi}(1-e_1)R(1-e_1)$ such that the ring

$$R_2 = (1-e_2)R_1(1-e_2) = (1-e_1-e_2)R(1-e_1-e_2)$$

is a right co-Harada ring. Iterating such processes, we shall reach a QF ring for any right co-Harada ring. For these processes, we notice the following lemma, which follows from the proofs of [4, Proposition 2.10 and Lemmas 2.12 and 2.14].

Lemma 3.1. *Let R be a basic right co-Harada ring and let $f \in \mathrm{pi}(R)$ with fR_R non-injective. Set $R' = (1-f)R(1-f)$. If eR_R is non-injective, then $eR'_{R'}$ is non-injective, for $e \in \mathrm{pi}(R)$ with $e \neq f$.*

For the lemma above, we should note that $eR'_{R'}$ might not be injective even if eR_R is injective. Thus there are many processes of removing idempotents f with fR non-injective. So it is not trivial that all processes provide the same QF ring. The main purpose of this section is to show the uniqueness of the QF ring and that the QF ring is just the frame QF subring.

Let R be a basic one-sided artinian ring with $E = \mathrm{pi}(R)$. For a non-empty subset F of E, set $e_F = \sum_{e\in F} e$ and $R(F) = e_FRe_F$.

Definition 3.2. Let R be a basic right co-Harada ring with $E = \mathrm{pi}(R)$ and F a non-empty subset of E. For distinct elements $e_1, e_2, \ldots, e_n$ of E, we say that the sequence $(e_1, e_2, \ldots, e_n)$ is a *route* from E to F if the following conditions hold:

(i) $E - \{e_1, e_2, \ldots, e_n\} = F$;
(ii) For each $i = 1, 2, \ldots, n$, the right R_{i-1}-module e_iR_{i-1} is not injective, where $R_0 = R$ and

$$R_i = R(E - \{e_1, e_2, \ldots, e_i\}) = (1 - \sum_{j=1}^{i} e_j)R(1 - \sum_{j=1}^{i} e_j).$$

In this case, we call n the *length* of the route $(e_1, e_2, \dots, e_n)$. When $E = F$, we consider that there is the trivial route from E to E itself and that the length of the trivial route is 0.

Remark 3.3. (1) In the setting of Definition 3.2, if there is a route from E to F, then $R(F) = e_F R e_F$ is a right co-Harada ring by Proposition 2.8.

(2) For a non-empty subset G of F, if there exist a route from E to F and a route from F to G, then by definition there exists a route from E to G.

The following is a key lemma.

Lemma 3.4. *Let R be a basic right co-Harada ring with $E = \mathrm{pi}(R)$ and F a non-empty subset of E. Assume that there exists a route from E to F such that $R(F)$ is a QF ring. For any $e \in E$ with eR_R non-injective, the following hold.*

(1) *$F \subset E - \{e\}$.*

(2) *There exists a route from $E - \{e\}$ to F.*

Proof. Let $(e_1, \dots, e_n)$ be a route from E to F and set $R_0 = R$ and

$$R_i = R(E - \{e_1, e_2, \dots, e_i\}) = (1 - \sum_{j=1}^{i} e_j) R (1 - \sum_{j=1}^{i} e_j)$$

for $i = 1, 2, \dots, n$.

(1) To show $F \subset E - \{e\}$, assume to the contrary $e \in F$. Then $e \neq e_i$ for any $i = 1, 2, \dots, n$. Thus by Lemma 3.1 $eR_{1R_1}, eR_{2R_2}, \dots, eR_{nR_n}$ are non-injective. However this contradicts the fact that $R_n = R(F)$ is a QF ring.

(2) We prove the statement by induction on the length n of the route $(e_1, \dots, e_n)$ from E to F. In case $n = 0$, the statement is trivial. We assume that the statement holds in case that the length of the route is less than n. That is, we assume that if R' is a basic right co-Harada ring with a complete set of orthogonal primitive idempotents $E' = \mathrm{pi}(R')$ containing F and if there exists a route of length $< n$ from E' to F, then there exists a route from $E' - \{e'\}$ to F for any $e' \in E'$ with $e'R'_{R'}$ non-injective. Since $(e_1, e_2, \dots, e_n)$ is a route from E to F, $(e_2, \dots, e_n)$ is a route from $E - \{e_1\}$ to F. In case $e = e_1$, there is a route from $E - \{e\}$ to F. In case $e \neq e_1$, since eR_R is non-injective, eR_{1R_1} is non-injective by Lemma 3.1. Then by the induction hypothesis on the basic right co-Harada ring $R_1 = (1 - e_1)R(1 - e_1)$ with a complete set of orthogonal primitive idempotents $E - \{e_1\}$ and the route $(e_2, \dots, e_n)$ from $E - \{e_1\}$ to F, there exists a route from $(E - \{e_1\}) - \{e\}$ to F. Since $(E - \{e_1\}) - \{e\} = (E - \{e\}) - \{e_1\}$, we have a route from $(E - \{e\}) - \{e_1\}$ to F. On the other hand, since $e_1 R(E - \{e\})$ is non-injective

by Lemma 3.1, we have the route (e_1) from $E - \{e\}$ to $(E - \{e\}) - \{e_1\}$. Therefore, composing the route from $E - \{e\}$ to $(E - \{e\}) - \{e_1\}$ and the route from $(E - \{e\}) - \{e_1\}$ to F, we obtain a route from $E - \{e\}$ to F (see Remark 3.3(2)). □

We can now prove the main result of this section easily.

Theorem 3.5. *Let R be a basic right co-Harada ring with $E = \mathrm{pi}(R)$ and let F and G be non-empty subsets of E such that $R(F)$ and $R(G)$ are QF rings. If there exist a route from E to F and a route from E to G, then $F = G$.*

Proof. Let $(e_1, e_2, \dots, e_n)$ be a route from E to F. Since there is a route from E to G, we have $G \subset E - \{e_1\}$ and a route from $E - \{e_1\}$ to G by Lemma 3.4. Again by Lemma 3.4, we have $G \subset E - \{e_1, e_2\}$ and a route from $E - \{e_1, e_2\}$ to G. By iteration, we obtain $G \subset E - \{e_1, e_2, \dots, e_n\} = F$. Similarly, we obtain $F \subset G$. Thus $F = G$ as required. □

Remark 3.6. Let R be a basic right co-Harada ring with $E = \mathrm{pi}(R)$. As we stated before, in case R is not a QF ring, there exist $e_1, e_2, \dots, e_n \in E$ such that each $e_i R_{i-1 R_{i-1}}$ is non-injective for $i = 1, 2, \dots, n$ and R_n is a QF ring, where $R_0 = R$ and $R_i = (1 - \sum_{j=1}^{i} e_j) R (1 - \sum_{j=1}^{i} e_j)$. Theorem 3.5 shows the uniqueness of the set $E - \{e_1, \dots, e_n\}$ and the QF ring R_n. Thus, such a QF ring R_n does not depend on choices of removing idempotents of E. This shows that the ring R_n is just the frame QF subring of the right co-Harada ring R. (See [3, Theorem 4.3.11(2)].)

4. Examples – Quiver of R_e –

In the final section, we provide several examples of right co-Harada rings represented by factor algebras of path algebras over a field. For this, we begin with the following.

Lemma 4.1. *Let R be a basic artinian ring with $J = J(R)$. For $e \in \mathrm{pi}(R)$, set*

$$\tilde{R} = R_e = \begin{pmatrix} R & Re \\ eJ & eRe \end{pmatrix}$$

and $\tilde{J} = J(\tilde{R})$. Then

$$\tilde{J} = \begin{pmatrix} J & Re \\ eJ & eJe \end{pmatrix}.$$

Thus

$$\tilde{J}/\tilde{J}^2 = \begin{pmatrix} J/(J^2 + ReJ) & Re/Je \\ eJ/eJ^2 & 0 \end{pmatrix}.$$

In particular, for $f, g \in \mathrm{pi}(R)$, the following hold.

$$(1)\ \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} (\tilde{J}/\tilde{J}^2) \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} = \begin{cases} 0 & \text{if } f = e \\ \begin{pmatrix} f(J/J^2)g & 0 \\ 0 & 0 \end{pmatrix} & \text{if } f \neq e \end{cases}.$$

$$(2)\ \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} (\tilde{J}/\tilde{J}^2) \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & eRe/eJe \\ 0 & 0 \end{pmatrix} & \text{if } f = e \\ 0 & \text{if } f \neq e \end{cases}.$$

$$(3)\ \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} (\tilde{J}/\tilde{J}^2) \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ e(J/J^2)f & 0 \end{pmatrix}.$$

$$(4)\ \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} (\tilde{J}/\tilde{J}^2) \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} = 0.$$

Proof. Since $J = J(R)$ is nilpotent, it is easy to check that

$$\begin{pmatrix} J & Re \\ eJ & eJe \end{pmatrix}$$

is also a nilpotent ideal of $\tilde{R}$. On the other hand, it is clear that the factor ring of $\tilde{R}$ by the ideal above is semisimple. Therefore we have the form of $\tilde{J} = J(\tilde{R})$ as in the lemma. It is routine to check the rest of the statements. □

From this lemma, we have the following. (For the definition of quivers and relations of algebras, see e.g. [2, Chapter III].)

Proposition 4.2. *Let K be a field, $\Gamma = (\Gamma_0, \Gamma_1, s, t)$ a finite quiver, I an admissible ideal of the path algebra $K\Gamma$, and ρ a set of relations of $K\Gamma$ that generates I. Set $R = K\Gamma/I = K\Gamma/\langle\rho\rangle$. For a fixed vertex $i \in \Gamma_0$, let e_i be the primitive idempotent of R corresponding to i. Set $\tilde{R} = R_{e_i}$. Then the quiver $\tilde{\Gamma} = (\tilde{\Gamma}_0, \tilde{\Gamma}_1)$ of $\tilde{R}$ and the admissible ideal $\tilde{I}$ of $K\tilde{\Gamma}$ are given by the following manner:*

Vertices: *The vertices of $\tilde{\Gamma}$ is obtained by adding a "copy" $\hat{i}$ of i to the vertices of Γ. That is,*

$$\tilde{\Gamma}_0 = \Gamma_0 \cup \{\hat{i}\}.$$

Arrows: *The arrows in $\tilde{\Gamma}$ are defined as the following:*

(i) *Any arrow $\alpha : j \to k$ in Γ with $k \neq i$ is also an arrow $j \to k$ in $\tilde{\Gamma}$;*
(ii) *for any arrow $\beta : j \to i$ in Γ, there exists a corresponding arrow $\hat{\beta} : j \to \hat{i}$ in $\tilde{\Gamma}$;*
(iii) *there exists a unique arrow $\omega : \hat{i} \to i$ in $\tilde{\Gamma}$.*

That is,

$$\tilde{\Gamma}_1 = \{\alpha \mid \alpha \in \Gamma_1, t(\alpha) \neq i\} \cup \{\hat{\beta} \mid \beta \in \Gamma_1, t(\beta) = i\} \cup \{\omega\}.$$

Relations: *For an arrow* $\beta : j \to i$ *in* Γ, *the path* $\omega\hat{\beta} : j \to i$ *in* $\tilde{\Gamma}$ *is denoted by the same* β. *For a path* $q : k \to i$ *in* Γ *with* $q = \beta p$, *where* $\beta : j \to i$ *is an arrow and* $p : k \to j$ *is a path in* Γ, *the path* $\hat{\beta}p : k \to \hat{i}$ *in* $\tilde{\Gamma}$ *is denoted by* $\hat{q}$. *Then*

$$\tilde{\rho} = \{u \mid u \in \rho, t(u) \neq i\} \cup \{\hat{v} \mid v \in \rho, t(v) = i\}$$

is a set of relations of $K\tilde{\Gamma}$ *that generates* $\tilde{I}$, *where* $\hat{v} = \sum_l a_l \hat{q}_l$ *for* $v = \sum_l a_l q_l \in \rho$ *with* $a_l \in K$ *and paths* $q_l : k \to i$ *in* Γ.

Proof. It follows from Lemma 4.1 that the quiver $\tilde{\Gamma} = (\tilde{\Gamma}_0, \tilde{\Gamma}_1)$ of the algebra $\tilde{R} = R_{e_i}$ has the vertices and the arrows in the proposition. To observe the relations, let e_j be the idempotent of R or $K\tilde{\Gamma}$ corresponding to a vertex $j \in \Gamma_0 \subset \tilde{\Gamma}_0$ and let $\hat{e_i} = e_{\hat{i}}$ be the idempotent of $K\tilde{\Gamma}$ corresponding to the vertex $\hat{i} \in \tilde{\Gamma}_0$. We define a K-algebra homomorphism $\Phi : K\tilde{\Gamma} \to \tilde{R}$ by

$$e_j \mapsto \begin{pmatrix} e_j & 0 \\ 0 & 0 \end{pmatrix} \ (j \in \Gamma_0),\ \hat{e_i} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & e_i \end{pmatrix},\ \omega \mapsto \begin{pmatrix} 0 & e_i \\ 0 & 0 \end{pmatrix},$$
$$\alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \ (\alpha \in \Gamma_1, t(\alpha) \neq i),\ \hat{\beta} \mapsto \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} \ (\beta \in \Gamma_1, t(\beta) = i).$$

Then it is routine to check that Φ is surjective and $\mathrm{Ker}(\Phi)$, which is just the admissible ideal $\tilde{I}$, is generated by $\tilde{\rho}$ in the proposition. □

For a concrete quiver with relations of R_{e_i}, see Example 4.3 below. We should also note that the quiver with relations of a right co-Harada algebra is described in Yamaura [7].

Concluding the paper, we illustrate Theorem 2.1 with the following examples.

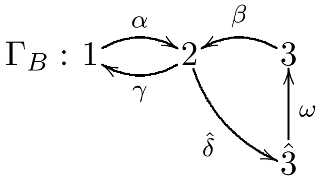
Example 4.3. (1) Let K be a field and let A be the factor algebra of the path algebra over K defined by the quiver and the relations

$$\Gamma_A : 1 \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\delta}{\overset{\beta}{\leftrightarrows}} 3 \quad \text{and} \quad \rho_A = \{\delta\alpha,\ \gamma\beta,\ \alpha\gamma - \beta\delta\}.$$

Then A is a QF algebra and hence A is a right co-Harada ring (by Theorem 2.1(1)). Let e_i be the primitive idempotent of A corresponding to the vertex i for $i = 1, 2, 3$. We denote by "i" the simple right A-module $T(e_iR)$. Then the Loewy series of the indecomposable projective right A-modules are the following:

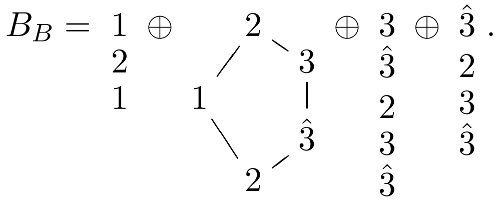
$$A_A = \begin{matrix} 1 \\ 2 \\ 1 \end{matrix} \oplus \begin{matrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{matrix} \oplus \begin{matrix} 3 \\ 2 \\ 3 \end{matrix}.$$

(2) We consider the algebra $B = A_{e_3}$. Theorem 2.1(2) claims that B is a right co-Harada ring. By Proposition 4.2 the quiver and the relations of B are the following:



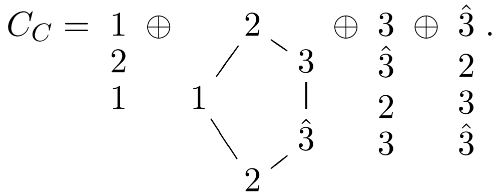
and $\rho_B = \tilde{\rho_A} = \{\hat{\delta}\alpha,\ \gamma\beta,\ \alpha\gamma - \beta\delta\,(= \alpha\gamma - \beta\omega\hat{\delta})\}$,

where δ denotes the path $\omega\hat{\delta}$. Then the Loewy series of the indecomposable projective right B-modules are the following:



Let $\hat{e_3} = e_{\hat{3}}$ be the primitive idempotent of B corresponding to the vertex $\hat{3}$. Then e_iB_B $(i = 1, 2, 3)$ are injective and $J(e_3B) \cong e_{\hat{3}}B$. Therefore B satisfies the definition of right co-Harada rings. The frame QF subring of B is just the QF algebra A.

(3) For the right co-Harada ring B, e_3B is injective, $S(e_3B) \cong T(e_{\hat{3}}B)$ and $e_{\hat{3}}B$ is not injective. Thus by Theorem 2.1(3) the factor ring $C = B/S(e_3B) = B/S(Be_{\hat{3}})$ is also a right co-Harada ring. Actually, this is a QF ring. So the frame QF subring of C is just C itself but not A. The quiver of C is the same as B, and the relations of C are that of B adding by $\omega\hat{\delta}\beta\omega$. That is, $\Gamma_C = \Gamma_B$ and $\rho_C = \rho_B \cup \{\omega\hat{\delta}\beta\omega\}$. Then the Loewy series of the indecomposable projective right C-modules are the following:



(4) For the right co-Harada ring C, $e_{\hat{3}}C$ is injective and $e_{\hat{3}}C/S(e_{\hat{3}}C) \cong J(e_3C)$. By Theorem 2.1(4) (or by Theorem 2.16) the factor ring

$$D = C/S(e_{\hat{3}}C) = B/S_2(Be_{\hat{3}}) = B/(S(e_3B) \oplus S(e_{\hat{3}}B))$$

is a right co-Harada ring. The frame QF subring of D is the QF algebra A. The quiver of D is the same as B, and the relations of D are that of B adding by $\omega\hat{\delta}\beta\omega$ and $\hat{\delta}\beta\omega$. That is, $\Gamma_D = \Gamma_B$ and $\rho_D = \rho_B \cup \{\omega\hat{\delta}\beta\omega, \hat{\delta}\beta\omega\}$.

Then the Loewy series of the indecomposable projective right D-modules are the following:

$$D_D = \begin{array}{c} 1 \\ 2 \\ 1 \end{array} \oplus \begin{array}{ccc} & 2 & \\ \diagup & & \diagdown \; 3 \\ 1 & & \mid \\ \diagdown & & \diagup \; \hat{3} \\ & 2 & \end{array} \oplus \begin{array}{c} 3 \\ \hat{3} \\ 2 \\ 3 \end{array} \oplus \begin{array}{c} \hat{3} \\ 2 \\ 3 \end{array}.$$

We note that the ring D is just the case of $n = 3$ of [5, Example 2.2].

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