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ON HOOK FORMULAS FOR CYLINDRIC SKEW DIAGRAMS

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Abstract. We present a conjectural hook formula concerning the number of the standard tableaux on "cylindric" skew diagrams. Our formula can be seen as an extension of Naruse's hook formula for skew diagrams. Moreover, we prove our conjecture in some special cases.

1. Introduction

The hook formula gives the number of standard tableaux on Young diagrams and it was discovered in 1950's [FRT]. A generalization of the hook formula to skew diagrams was obtained relatively recently in [Nar], where Naruse gave the following formula by introducing excited diagrams.

Theorem 1.1 (Naruse [Nar]). *Let λ and μ be partitions with $\lambda \supset \mu$ and $|\lambda/\mu| = n$. Then the number $f^{\lambda/\mu}$ of standard tableaux on the skew diagram λ/μ is given by*

$$f^{\lambda/\mu} = n! \sum_{D \in \mathcal{E}_\lambda(\mu)} \prod_{x \in \lambda \setminus D} \frac{1}{h_\lambda(x)}, \tag{1.1}$$

where $\mathcal{E}_\lambda(\mu)$ denotes the set of all excited diagrams of μ in λ, and $h_\lambda(x)$ denotes the hook length at x.

For example, for the partitions $\lambda = (2,2)$ and $\mu = (1,0)$, the formula leads

$$2 = 3!\left(\frac{1}{2\cdot 2\cdot 1} + \frac{1}{3\cdot 2\cdot 2}\right)$$

(See Figure 1.) Several proofs and generalization have been known. Morales,

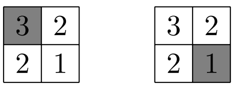


Figure 1. The excited diagrams of $\mu = (1,0)$ in $\lambda = (2,2)$. Here, the number in each cell expresses the hook length.

Pak and Panova gave a q-analogue of the skew hook formula ([MPP]). Naruse and Okada generalized the skew hook formula to the case where Young diagrams are replaced by general d-complete posets ([NO]).

In this paper, we will treat periodic or cylindric analogue of skew diagrams ([GK, Pos]) and standard tableaux on them.

Let $\omega \in \mathbb{Z}_{\geqq 1} \times \mathbb{Z}_{\leqq -1}$. A periodic skew diagram of period ω is a skew diagram consisting of infinitely many cells which is invariant under the parallel translation by ω. We will define a standard tableau on a periodic skew diagram as a periodic array of natural numbers whose entries increase in row and column directions. (See Section 3 for precise definition.)

Figure 2 indicates the periodic diagram

$$\hat{\lambda}/\hat{\mu} = \lambda/\mu + \mathbb{Z}\omega = \{u + k\omega \mid u \in \lambda/\mu,\ k \in \mathbb{Z}\}$$

of period $\omega = (2, -2)$ associated with the partitions $\lambda = (3, 1), \mu = (0, 0)$, and two standard tableaux on it. (In this case, these two tableaux exhaust all the periodic standard tableaux.)

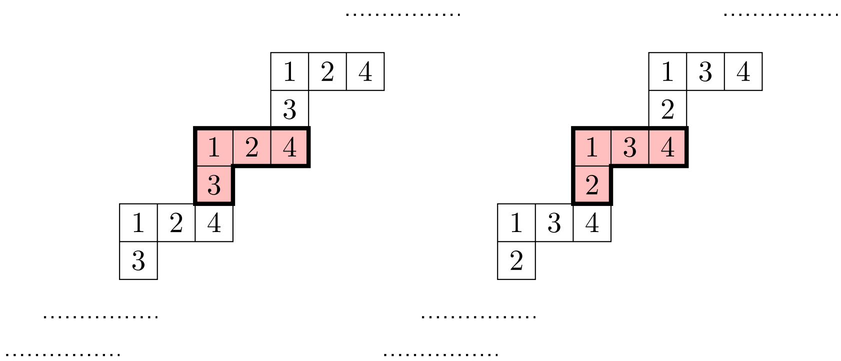


Figure 2.

The image of a periodic skew diagram of period ω under the projection $\pi : \mathbb{Z}^2 \to \mathbb{Z}^2/\mathbb{Z}\omega$ is called a cylindric skew diagram. The set of standard tableaux on a periodic skew diagram can be identified with the set of standard tableaux on the corresponding cylindric skew diagram.

We remark that the cylinder $\mathbb{Z}^2/\mathbb{Z}\omega$ has a poset structure induced from that of $\mathbb{Z}^2$, and cylindric skew diagrams can be seen as d-complete posets consisting of infinitely many cells (cf. [Str]).

We also note that periodic/cylindric skew diagrams parameterize a certain class of irreducible modules over the Cherednik algebras (double affine Hecke algebras) ([SV, Suz]) and the (degenerate) affine Hecke algebras [Kle, Ruff]) of type A, and cylindric standard tableaux also appear in those theories.

We will introduce excited diagrams for periodic skew diagrams, and present a conjectural hook formula (Conjecture 5.5) concerning the number of periodic standard tableaux on a periodic skew diagram. The formula in Conjecture 5.5 looks similar to Naruse's skew hook formula, but for periodic skew diagrams, there are infinitely many excited diagrams in general and the right hand side is an infinite sum. For example, in the case where $\lambda = (2), \mu = (0)$ and $m = \ell = 1$, our hook formula leads

$$1 = 2!\left(\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots\right)$$

We will prove that our conjecture is correct in the following cases (Theorem 5.8 and 5.9):

- (bar case) $\lambda = (n),\ \mu = (0)$ and $\omega = (1, -\ell)$.
- (hook case) $\lambda = (\overbrace{\ell+1, \dots, \ell+1}^{m}),\ \mu = (\overbrace{\ell, \dots, \ell}^{m-1}, 0)$ and $\omega = (m, -\ell)$.

Acknowledgments. We thank H. Tagawa for suggesting us a formula which leads a proof of Theorem 5.8. We also thank K. Nakada for discussion and valuable comments.

2. Cylindric diagrams

For $\omega \in \mathbb{Z}_{\geqq 1} \times \mathbb{Z}_{\leqq -1}$, we let $\mathbb{Z}\omega$ denote the subgroup of (the additive group) $\mathbb{Z}^2$ generated by ω, and define

$$\mathcal{C}_\omega = \mathbb{Z}^2/\mathbb{Z}\omega.$$

We regard $\mathbb{Z}^2$ as a poset with the following partial order

$$(a, b) \leq (a', b') \iff a \geqq a' \text{ and } b \geqq b' \text{ as integers.}$$

Then, the cylinder $\mathcal{C}_\omega$ admits an induced poset structure, namely,

$$x \leq y \iff \exists \widetilde{x}, \widetilde{y} \in \mathbb{Z}^2 \text{ such that } \pi(\widetilde{x}) = x,\ \pi(\widetilde{y}) = y \text{ and } \widetilde{x} \leq \widetilde{y},$$

where $\pi : \mathbb{Z}^2 \to \mathcal{C}_\omega$ is the natural projection. Note that the projection π is order preserving.

Definition 2.1. Let $(P, \leq)$ be a poset. A subset F of P is called an *order filter* if the following condition holds:

$$x \in F,\ x \leq y \ \Rightarrow\ y \in F.$$

An order filter F is said to be *non-trivial* if $F \neq \emptyset$ nor $F \neq P$.

Definition 2.2. Let $\omega \in \mathbb{Z}_{\geqq 1} \times \mathbb{Z}_{\leqq -1}$. A non-trivial order filter θ of $\mathcal{C}_\omega$ is called a *cylindric diagram*. The inverse image $\pi^{-1}(\theta) \subset \mathbb{Z}^2$ is called a *periodic diagram* of period ω. An element of a cylindric/periodic diagram is called a *cell*.

For a poset P and its order filter F, we denote by P/F the set difference $P \setminus F$.

Note that a cylindric (resp. periodic) diagram θ is a poset and its order filter η is a cylindric (resp. periodic) diagram such that $\eta \subset \theta$.

Definition 2.3. For a cylindric (resp. periodic) diagram θ and its order filter η, the set difference θ/η is called a *cylindric* (resp. *periodic*) *skew diagram.*

We sometimes parameterize periodic/cylindric diagrams by ℓ-restricted partitions:

Definition 2.4. Let $m, \ell \in \mathbb{Z}_{\geq 1}$. An non-increasing integer sequence $\lambda = (\lambda_1, \dots, \lambda_m)$ is called a *generalized partition of length* m, and it is said to be *ℓ-restricted* if it satisfies

$$\lambda_1 - \lambda_m \leqq \ell.$$

We denote by $\mathcal{P}_{m,\ell}$ the set of ℓ-restricted generalized partitions of length m. (Note that we allow λ_i to be negative).

For a generalized partition $\lambda = (\lambda_1, \dots, \lambda_m)$, we define

$$\begin{aligned} \boldsymbol{\lambda} &= \{(a, b) \in \mathbb{Z}^2 \mid 1 \leqq a \leqq m,\ b \leqq \lambda_a\}, \\ \hat{\lambda} &= \hat{\lambda}_{(m,-\ell)} = \boldsymbol{\lambda} + \mathbb{Z}(m, -\ell), \\ \mathring{\lambda} &= \mathring{\lambda}_{(m,-\ell)} = \pi(\hat{\lambda}). \end{aligned}$$

Note that $\boldsymbol{\lambda} = \hat{\lambda} \cap ([1, m] \times \mathbb{Z})$ and λ is a fundamental domain of $\hat{\lambda}$ with respect to the action of $\mathbb{Z}(m, -\ell)$ (see Figure 3).

It is easy to see that if $\lambda \in \mathcal{P}_{m,\ell}$ then $\hat{\lambda}$ is a periodic diagram of period $(m, -\ell)$ and $\mathring{\lambda}$ is a cylindric diagram. Moreover, any periodic (resp. cylindric) diagram of period $(m, -\ell)$ is of the form $\hat{\lambda}$ (resp. $\pi(\hat{\lambda})$) for some $\lambda \in \mathcal{P}_{m,\ell}$.

For generalized partitions $\lambda = (\lambda_1, \dots, \lambda_m)$ and $\mu = (\mu_1, \dots, \mu_m)$, we write $\lambda \supset \mu$ if $\lambda_i \geqq \mu_i$ for all i with $1 \leqq i \leqq m$. For $\lambda \supset \mu$, the set difference $\boldsymbol{\lambda}/\boldsymbol{\mu}$ is an ordinary skew diagram associated with λ and μ, which is denoted also by λ/μ.

3. Linear extensions

For two integers a, b, we use the following notation:

$$[a, b] = \{x \in \mathbb{Z} \mid a \leqq x \leqq b\}.$$

Definition 3.1. For a poset P such that $|P| = n$, a *linear extension* (or a *reverse standard tableau*) of P is a bijection $\varepsilon : P \to [1, n]$ satisfying

$$x < y \Longrightarrow \varepsilon(x) < \varepsilon(y).$$

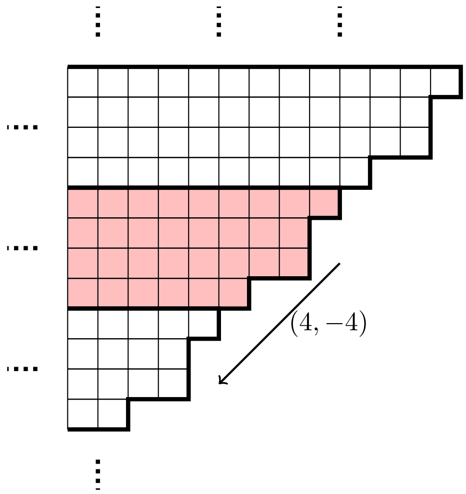


FIGURE 3. The periodic diagram of period $(4, -4)$ associate with $\lambda = (5, 4, 4, 2)$.

Let $\mathrm{RST}(P)$ denote the set of all linear extensions of P.

Let λ and μ be generalized partitions such that $\lambda \supset \mu$ and $|\lambda/\mu| = n$. It is easy to see that a bijection $\varepsilon : \lambda/\mu \to [1, n]$ is a linear extension on the finite skew diagram λ/μ if and only if the following conditions hold:

(1) $\varepsilon(a, b) > \varepsilon(a, b + 1)$ whenever $(a, b), (a, b + 1) \in \lambda/\mu$.
(2) $\varepsilon(a, b) > \varepsilon(a + 1, b)$ whenever $(a, b), (a + 1, b) \in \lambda/\mu$.

Fix $m, \ell \in \mathbb{Z}_{\geqq 1}$.

Definition 3.2. Let $\lambda, \mu \in \mathcal{P}_{m,\ell}$. A linear extension ε on λ/μ is said to be *ℓ-restricted* if it satisfies

$$\varepsilon(1, b) < \varepsilon(m, b - \ell) \quad \text{whenever } (1, b), (m, b - \ell) \in \lambda/\mu.$$

We denote by $\mathrm{RST}_\ell(\lambda/\mu)$ the set of all ℓ-restricted linear extensions of λ/μ.

Note that the projection $\pi : \mathbb{Z}^2 \to \mathcal{C}_\omega$ gives a bijection $\lambda/\mu \to \pi(\lambda/\mu) = \mathring{\lambda}/\mathring{\mu}$ and that a map $\varepsilon : \lambda/\mu \to [1, n]$ induces a map $\varepsilon : \mathring{\lambda}/\mathring{\mu} \to [1, n]$.

Lemma 3.3. *Let $\lambda, \mu \in \mathcal{P}_{m,\ell}$. A bijection $\varepsilon : \lambda/\mu \to [1, n]$ induces a linear extension on the cylindric skew diagram $\mathring{\lambda}/\mathring{\mu}$ if and only if $\varepsilon \in \mathrm{RST}_\ell(\lambda/\mu)$. Namely, the set $\mathrm{RST}(\mathring{\lambda}/\mathring{\mu})$ and $\mathrm{RST}_\ell(\lambda/\mu)$ are in one to one correspondence.*

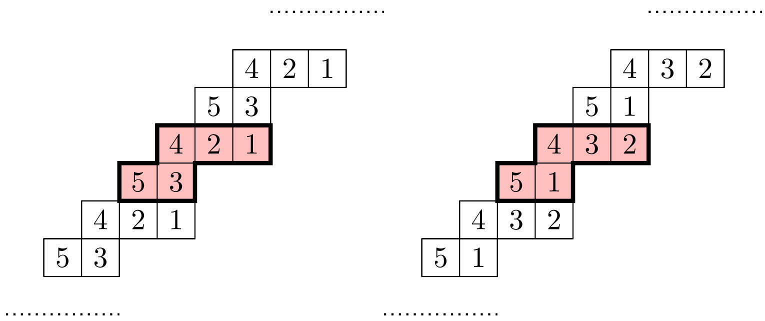


FIGURE 4. Two maps from $\mathring{\lambda}/\mathring{\mu}$ to $\{1,2,3,4,5\}$ with $\lambda = (4,2)$, $\mu = (1,0)$ and $\ell = 2$. The left is a linear extension, but the right is not a linear extension.

4. Excited diagrams

In this section, we fix $\ell, m \in \mathbb{Z}_{\geqq 1}$.

Definition 4.1. Let λ and μ be two partitions such that $\lambda \supset \mu$.

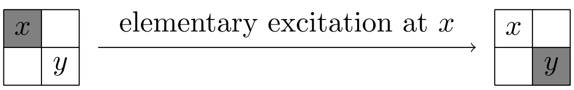
(1) Let D be a subset of λ. A cell $x = (a,b) \in D$ is said to be D-*active* if

$$(a+1,b),\ (a,b+1),\ (a+1,b+1) \in \lambda \setminus D.$$

(2) For a D-active cell $x = (a,b)$, we put

$$D_x = (D \setminus \{x\}) \cup \{y\},$$

where $y = (a+1,b+1)$. The replacement from D to D_x is called an *elementary excitation* at x.



(3) An *excited diagram* of μ in λ is a subset of λ obtained from μ after a sequence of elementary excitations on active cells. Let $\mathcal{E}_\lambda(\mu)$ denote the set of all excited diagrams of μ in λ.

We extend the concept of excited diagrams to the case of cylindric/periodic diagrams.

Let λ and μ be two ℓ-restricted partitions of length m such that $\lambda \supset \mu$. Put $\omega = (m, -\ell)$.

Definition 4.2. (1) Let D be a "periodic" subset of $\hat{\lambda}$ (i.e., $D + \omega = D$). A cell $(a,b) \in D$ is said to be D-*active* if

$$(a+1,b),(a,b+1),(a+1,b+1) \in \hat{\lambda} \setminus D.$$

(2) For a D-active cell (a,b), put

$$D_{(a,b)} := D \setminus ((a,b)+\mathbb{Z}\omega) \cup ((a+1,b+1)+\mathbb{Z}\omega).$$

The replacement from D to $D_{(a,b)}$ is called a *periodic elementary excitation* (see Figure 5).

(3) A *periodic excited diagram* of $\hat{\mu}$ in $\hat{\lambda}$ is a subset of $\hat{\lambda}$ obtained from $\hat{\mu}$ after a sequence of periodic elementary excitations on active cells, and the whole set is denoted by $\mathcal{E}_{\hat{\lambda}}(\hat{\mu})$.

Remark 4.3. It is easy to see that if $x \in D$ is D-active then $x + k\omega$ is also D-active for any $k \in \mathbb{Z}$ and D_x is periodic.

Definition 4.4. (1) Let D be a subset of the cylindric diagram $\mathring{\lambda}$. A cell $x = \pi(a,b) \in D$ is said to be D-active if (a,b) is $\pi^{-1}(D)$-active as a cell in $\hat{\lambda}$.

(2) For a D-active cell $x = \pi(a,b)$, put $D_x = (D \setminus \{x\}) \cup \{y\}$, where $y = \pi(a+1,b+1)$. The replacement from D to D_x is called a *cylindric elementary excitation*.

(3) A *cylindric excited diagram* of $\mathring{\mu}$ in $\mathring{\lambda}$ is a subset of $\mathring{\lambda}$ obtained from $\mathring{\mu}$ after a sequence of cylindric elementary excitations on active cells, and the whole set is denoted by $\mathcal{E}_{\mathring{\lambda}}(\mathring{\mu})$.

Note that we have $\mathcal{E}_{\mathring{\lambda}}(\mathring{\mu}) = \{\pi(D) \mid D \in \mathcal{E}_{\hat{\lambda}}(\hat{\mu})\}$.

5. Conjectural hook formula for cylindric skew diagrams

Definition 5.1. Let λ be a partition. For a cell x of the corresponding finite Young diagram λ, the *hook* $H_\lambda(x)$ of x in λ is given by

$$H_\lambda(x) = \lambda \cap \Big(\{x+(k,0) \mid k \in \mathbb{Z}_{\geqq 0}\} \cup \{x+(0,k) \mid k \in \mathbb{Z}_{\geqq 1}\}\Big).$$

and the *hook length* $h_\lambda(x)$ is the number of cells of $H_\lambda(x)$.

Fix $m, \ell \in \mathbb{Z}_{\geqq 1}$ and put $\omega = (m, -\ell)$.

Definition 5.2. Let $\lambda \in \mathcal{P}_{m,\ell}$. For a cell $x \in \hat{\lambda}$, define the hook $H_{\hat{\lambda}}(x)$ of x in $\hat{\lambda}$ by

$$H_{\hat{\lambda}}(x) = \hat{\lambda} \cap \Big(\{x+(k,0) \mid k \in \mathbb{Z}_{\geqq 0}\} \cup \{x+(0,k) \mid k \in \mathbb{Z}_{\geqq 1}\}\Big).$$

The number $h_{\hat{\lambda}}(x)$ of cells of $H_{\hat{\lambda}}(x)$ is called the *hook length* in $\hat{\lambda}$ (see Figure 6).

Definition 5.3. For a cell $x \in \mathring{\lambda}$, define

$$h_{\mathring{\lambda}}(x) := h_{\hat{\lambda}}(y),$$

where $y \in \pi^{-1}(x)$.

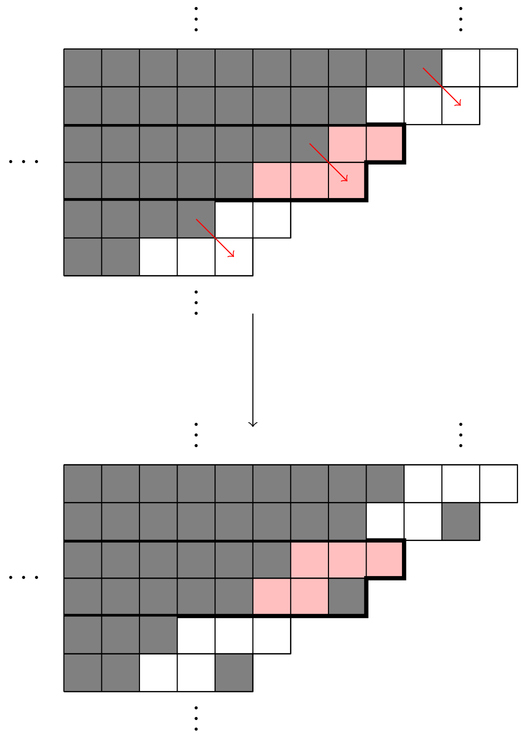


FIGURE 5. A periodic elementary excitation.

Note that $h_{\mathring{\lambda}}(x)$ is well-defined since

$$h_{\hat{\lambda}}(y+\omega) = h_{\hat{\lambda}}(y).$$

For a skew diagram λ/μ, we denote by $f^{\lambda/\mu}$ the number of linear extensions of λ/μ:

$$f^{\lambda/\mu} = |\mathrm{RST}(\lambda/\mu)|.$$

Theorem 5.4 ([Nar])**.** *Let λ and μ be partitions with $\lambda \supset \mu$ and $|\lambda/\mu| = n$. Then,*

$$f^{\lambda/\mu} = n! \sum_{D \in \mathcal{E}_\lambda(\mu)} \prod_{x \in \lambda \setminus D} \frac{1}{h_\lambda(x)}, \tag{5.1}$$

where $\mathcal{E}_\lambda(\mu)$ is the set of all excited diagrams of μ in λ, and $h_\lambda(x)$ is the hook length of x in λ.

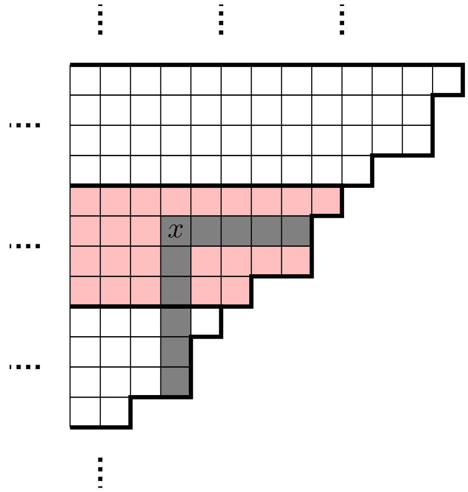


FIGURE 6. The hook $H_{\hat{\lambda}}(x)$ of x in the periodic diagram $\hat{\lambda}$. The hook length $h_{\mathring{\lambda}}(\pi(x)) = 10$.

For a cylindric skew diagram $\mathring{\lambda}/\mathring{\mu}$ of period $(m, -\ell)$, we denote by $f^{\mathring{\lambda}/\mathring{\mu}}$ the number of linear extensions of $\mathring{\lambda}/\mathring{\mu}$:

$$f^{\mathring{\lambda}/\mathring{\mu}} = |\mathrm{RST}(\mathring{\lambda}/\mathring{\mu})| = |\mathrm{RST}_\ell(\lambda/\mu)|.$$

Conjecture 5.5. *Let $m, \ell \in \mathbb{Z}_{\geqq 1}$ and $\lambda, \mu \in \mathcal{P}_{m,\ell}$ such that $\lambda \supset \mu$. Put $n = |\lambda/\mu| = |\mathring{\lambda}/\mathring{\mu}|$. Then,*

$$f^{\mathring{\lambda}/\mathring{\mu}} = n! \sum_{D \in \mathcal{E}_{\mathring{\lambda}}(\mathring{\mu})} \prod_{x \in \mathring{\lambda} \setminus D} \frac{1}{h_{\mathring{\lambda}}(x)}, \tag{5.2}$$

where $\mathcal{E}_{\mathring{\lambda}}(\mathring{\mu})$ is the set of all cylindric excited diagrams of $\mathring{\mu}$ in $\mathring{\lambda}$, and $h_{\mathring{\lambda}}(x)$ is the hook length of x in $\mathring{\lambda}$.

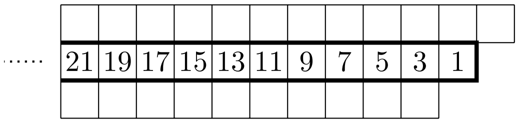
Remark 5.6. If $\ell \geqq \lambda_1$, then

$$\mathcal{E}_{\mathring{\lambda}}(\mathring{\mu}) = \mathcal{E}_\lambda(\mu),$$
$$h_{\mathring{\lambda}}(\pi(x)) = h_\lambda(x),$$

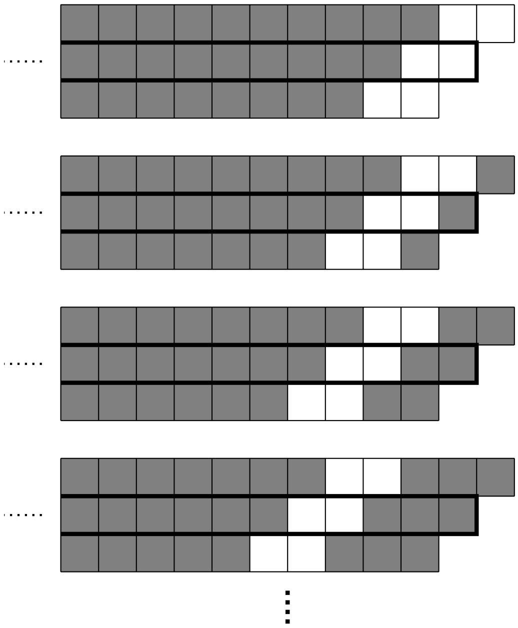
and hence Conjecture 5.5 follows from Theorem 5.4. ◊

Example 5.7. Let us see the simplest non-trivial example. Let $\lambda = (2)$, $\mu = (0) \in \mathcal{P}_{1,1}$. Then $\mathring{\lambda}/\mathring{\mu}$ has just one linear extension.

The hook length on fundamental domain of $\mathring{\lambda}$ are as follows:



The excited diagrams of $\mathring{\mu}$ in $\mathring{\lambda}$ are as follows:



Therefore, by computing the right hand side of (5.2),

$$\begin{aligned} f^{\mathring{\lambda}/\mathring{\mu}} &= 2! \left(\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \cdots \right) \\ &= 2! \cdot \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)} \\ &= 2! \cdot \frac{1}{2} \cdot \sum_{k=0}^{\infty} \left(\frac{1}{2k+1} - \frac{1}{2k+3} \right) = 1. \end{aligned}$$

$\Diamond$

By similar case by case compilation, we have confirmed Conjecture 5.5 for any shape with $n \leqq 4$.

In the rest, we denote the right hand side of (5.2) by $g^{\mathring{\lambda}/\mathring{\mu}}$:

$$g^{\mathring{\lambda}/\mathring{\mu}} = n! \sum_{D \in \mathcal{E}_{\mathring{\lambda}}(\mathring{\mu})} \prod_{x \in \mathring{\lambda} \setminus D} \frac{1}{h_{\mathring{\lambda}}(x)}.$$

The proofs for the following two theorems will be given in the later sections:

Theorem 5.8. (Bar cases) *Let $n, \ell \in \mathbb{Z}_{\geqq 1}$. Put $\lambda = (n)$ and $\mu = (0)$, which belong to $\mathcal{P}_{1,\ell}$, and let $\mathring{\lambda} = \mathring{\lambda}_{(1,-\ell)}$ and $\mathring{\mu} = \mathring{\mu}_{(1,-\ell)}$ be the corresponding cylindric diagrams. Then*

$$f^{\mathring{\lambda}/\mathring{\mu}} = 1 = g^{\mathring{\lambda}/\mathring{\mu}}. \tag{5.3}$$

Theorem 5.9. (Hook cases) *Let $\ell, m \in \mathbb{Z}_{\geqq 1}$. Put $\lambda = ((\ell+1)^m)$ and $\mu = (\ell^{m-1}, 0)$, which belong to $\mathcal{P}_{m,\ell}$, and let $\mathring{\lambda} = \mathring{\lambda}_{(m,-\ell)}$ and $\mathring{\mu} = \mathring{\mu}_{(m,-\ell)}$ be the corresponding cylindric diagrams. Then*

$$f^{\mathring{\lambda}/\mathring{\mu}} = \binom{\ell+m-2}{m-1} = g^{\mathring{\lambda}/\mathring{\mu}}. \tag{5.4}$$

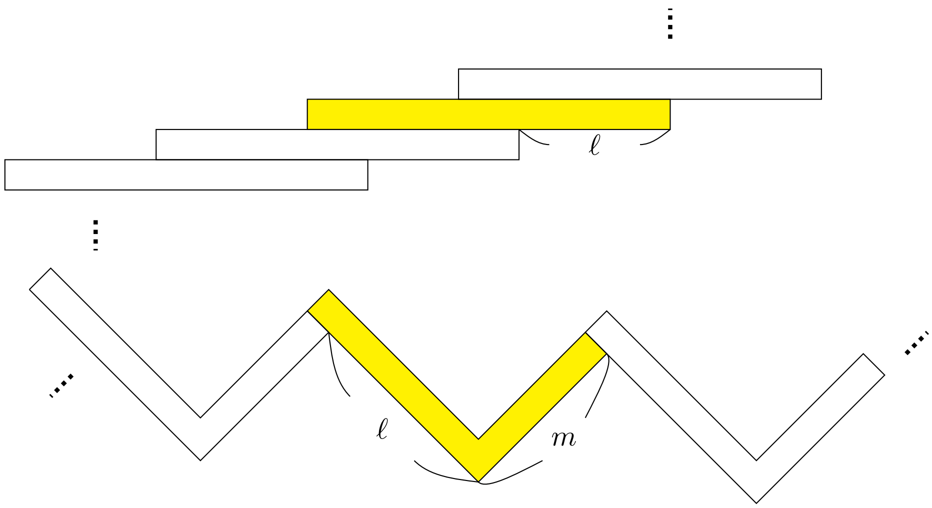


Figure 7. The shapes indicated by Theorem 5.8 and 5.9, respectively.

It is easy to see the following:

Proposition 5.10. *Let $m, \ell \in \mathbb{Z}_{\geqq 1}$ and $\lambda, \mu \in \mathcal{P}_{m,\ell}$ with $\lambda \supset \mu$. For $u \in \mathbb{Z}^2$, put*

$$\mathring{\eta} = \pi(\hat{\lambda} + u), \quad \mathring{\nu} = \pi(\hat{\mu} + u).$$

Then $\mathring{\eta}$ and $\mathring{\nu}$ are cylindric diagrams in $\mathbb{Z}^2/\mathbb{Z}(m, -\ell)$, and

$$f^{\mathring{\lambda}/\mathring{\mu}} = f^{\mathring{\eta}/\mathring{\nu}}, \quad g^{\mathring{\lambda}/\mathring{\mu}} = g^{\mathring{\eta}/\mathring{\nu}}.$$

By Proposition 5.10, Theorem 5.9 implies that Conjecture 5.5 is also true for $\lambda = (\ell+1, 1^{m-1})$ and $\mu = (0^m)$.

6. Proof of Theorem 5.8

Fix $\ell \geqq 1$ and $n \geqq 1$. Let $\lambda = (n)$, $\mu = (0)$, $\mathring{\lambda} = \mathring{\lambda}_{(1,-\ell)}$ and $\mathring{\mu} = \mathring{\mu}_{(1,-\ell)}$.

The cylindric skew diagram $\mathring{\lambda}/\mathring{\mu}$ has a unique linear extension, in which $1, 2, \ldots, n$ are arranged in order from right to left. Hence the first equality in Theorem 5.8 holds:

$$f^{\mathring{\lambda}/\mathring{\mu}} = 1.$$

We will show $g^{\mathring{\lambda}/\mathring{\mu}} = 1$ in the rest of this section.

For $i \in \mathbb{Z}_{\geqq 1}$, we denote the cell $\pi(n-i+1, 1)$ by p_i. Note that for a subset D of $\mathring{\lambda}$, a cell p_i is D-active if and only if

$$p_i \in D \text{ and } p_{i-1}, p_{i-\ell}, p_{i-\ell-1} \in \mathring{\lambda} \setminus D,$$

and hence

$$D_{p_i} = (D \setminus \{p_i\}) \cup \{p_{i-\ell-1}\}.$$

....	$p_{\ell+5}$	$p_{\ell+4}$	$p_{\ell+3}$	$p_{\ell+2}$	$p_{\ell+1}$	p_ℓ	$p_{\ell-1}$	....	p_3	p_2	p_1
....	p_5	p_4	p_3	p_2	p_1						

$\vdots$

Take $q, r \in \mathbb{Z}$ such that

$$n = (\ell+1)q + r, \quad q \geqq 0, \quad 0 \leqq r \leqq \ell. \tag{6.1}$$

Put

$$\mathcal{E}_{\ell;n} := \{(i_1, \ldots, i_q) \in \mathbb{Z}^q \mid i_1 \geqq r+1,\ i_{k+1} - i_k \geqq \ell + 1\ (1 \leqq k \leqq q-1)\}.$$

For $(i_1, \ldots, i_q) \in \mathcal{E}_{\ell;n}$, define

$$\psi(i_1, \ldots, i_q) = \mathring{\lambda} \setminus \left([p_1, p_r] \sqcup \left(\bigsqcup_{k=1}^{q} [p_{i_k}, p_{i_k+\ell}] \right) \right),$$

where $[p_i, p_j] = \{p_i, p_{i+1}, \ldots, p_j\} \subset \mathring{\lambda}$ for $i \leqq j$ and $[p_i, p_j] = \emptyset$ for $i > j$.

Proposition 6.1. *The map ψ gives a bijection from $\mathcal{E}_{\ell;n}$ to $\mathcal{E}_{\mathring{\lambda}}(\mathring{\mu})$.*

Proof. First, we show that $\psi(i_1, \ldots, i_q) \in \mathcal{E}_{\mathring{\lambda}}(\mathring{\mu})$. We proceed by induction on

$$M = \sum_{k=1}^{q} i_k.$$

The number M takes the minimum value

$$M_{\min} := \frac{1}{2} q(q-1)(\ell+1) + q(r+1)$$

when $i_1 = r+1$, $i_2 = (\ell+1)+r+1, \dots, i_q = (q-1)(\ell+1)+r+1$. For such $(i_1,\dots,i_q)$, we have

$$\psi(i_1,\dots,i_q) = \mathring{\mu} \in \mathcal{E}_{\mathring{\lambda}}(\mathring{\mu}).$$

Let $M > M_{\min}$ and suppose that $\psi(i_1,\dots,i_q) \in \mathcal{E}_{\mathring{\lambda}}(\mathring{\mu})$ for all $(i_1,\dots,i_q) \in \mathcal{E}_{\ell;n}$ such that $\sum_{k=1}^{q} i_k \leqq M-1$. Take $\boldsymbol{i} = (i_1,\dots,i_q) \in \mathcal{E}_{\ell;n}$ with $\sum_{k=1}^{q} i_k = M$. As $M > M_{\min}$, there exists $g \in [2, q+1]$ such that $i_g - i_{g-1} > \ell+1$ (we consider "$i_{q+1} = +\infty$"). For such g, we have $(i_1,\dots,i_g-1,\dots,i_q) \in \mathcal{E}_{\ell;n}$. Put $D = \psi(i_1,\dots,i_g-1,\dots,i_q)$. By induction hypothesis, $D \in \mathcal{E}_{\mathring{\lambda}}(\mathring{\mu})$. Now $p_{i_g+\ell} \in D$. Note that $[p_{i_g-1}, p_{i_g-1+\ell}] \subset \mathring{\lambda} \setminus D$. In particular, $p_{i_g-1}, p_{i_g}, p_{i_g+\ell-1} \in \mathring{\lambda} \setminus D$, and hence the cell $p_{i_g+\ell}$ is D-active. Hence

$$D_{p_{i_g+\ell}} = \big(D \setminus \{p_{i_g+\ell}\}\big) \cup \{p_{i_g-1}\} = \psi(i_1,\dots,i_g,\dots,i_q) \in \mathcal{E}_{\mathring{\lambda}}(\mathring{\mu}).$$

Next, we show that the map ψ is surjective (injectivity is obvious). It is obvious that $\mathring{\mu} = \psi(i_1,\dots,i_q)$ as $i_k = (\ell+1)(k-1)+r$. Take $D \in \mathcal{E}_{\mathring{\lambda}}(\mathring{\mu})$. Suppose that there exists $(i_1,\dots,i_q) \in \mathcal{E}_{\ell;n}$ such that $D = \psi(i_1,\dots,i_q)$. Any D-active cell is of the form $p_{i_g+\ell+1}$ for some $g \in [1,q]$ such that $i_{g+1} - i_g > \ell+1$. Since $i_{g+1} - i_g > \ell+1$, $(i_1,\dots,i_g+1,\dots,i_q) \in \mathcal{E}_{\ell;n}$. We have

$$D_{p_{i_g+\ell+1}} = \psi(i_1,\dots,i_g+1,\dots,i_q) \in \mathcal{E}_{\mathring{\lambda}}(\mathring{\mu}).$$

Hence ψ is surjective. □

We will use the following lemma, which was suggested by H. Tagawa.

Lemma 6.2. *Let $\ell, c, q \in \mathbb{Z}_{\geqq 1}$, $r \in \mathbb{Z}_{\geqq 0}$ and let $(a_i)_{i \geqq 1}$ be a numerical sequence such that*

- *$a_i \neq 0$ for all $i \geqq 1$,*
- *$\lim_{i\to\infty} a_i = +\infty$,*
- *$a_{i+\ell} - a_i = c$ for all $i \geqq 1$.*

Then

$$\sum_{\substack{(i_1,\dots,i_q);\\ i_1 \geqq r+1\\ i_{k+1}-i_k \geqq \ell+1\ (k=1,2,\dots,q-1)}} \frac{1}{\prod_{v=1}^{q}\prod_{u=0}^{\ell} a_{i_v+u}} = \frac{1}{q!c^q \prod_{u=0}^{\ell q-1} a_{r+1+u}}. \tag{6.2}$$

Proof. By assumption, for $j, t \in \mathbb{Z}_{\geqq 1}$, we have

$$\frac{1}{\prod_{u=0}^{\ell t-1} a_{j+u}} - \frac{1}{\prod_{u=1}^{\ell t} a_{j+u}} = \frac{a_{j+\ell t} - a_j}{\prod_{u=0}^{\ell t} a_{j+u}} = \frac{ct}{\prod_{u=0}^{\ell t} a_{j+u}}. \tag{6.3}$$

We proceed by induction on q to prove (6.2).

If $q = 1$, then we have

$$\begin{aligned}
\sum_{i_1=r+1}^{\infty} \frac{1}{\prod_{u=0}^{\ell} a_{i_1+u}} &= \frac{1}{c} \sum_{i_1=r+1}^{\infty} \frac{c}{\prod_{u=0}^{\ell} a_{i_1+u}} \\
&= \frac{1}{c} \sum_{i_1=r+1}^{\infty} \left(\frac{1}{\prod_{u=0}^{\ell-1} a_{i_1+u}} - \frac{1}{\prod_{u=1}^{\ell} a_{i_1+u}} \right) \quad \text{(by (6.3))} \\
&= \frac{1}{c} \sum_{i_1=r+1}^{\infty} \left(\frac{1}{\prod_{u=0}^{\ell-1} a_{i_1+u}} - \frac{1}{\prod_{u=0}^{\ell-1} a_{i_1+1+u}} \right) \\
&= \frac{1}{c} \lim_{N\to\infty} \sum_{i_1=r+1}^{N} \left(\frac{1}{\prod_{u=0}^{\ell-1} a_{i_1+u}} - \frac{1}{\prod_{u=0}^{\ell-1} a_{i_1+1+u}} \right) \\
&= \frac{1}{c\prod_{u=0}^{\ell-1} a_{r+1+u}} - \lim_{N\to\infty} \frac{1}{c\prod_{u=0}^{\ell-1} a_{N+1+u}} \\
&= \frac{1}{c\prod_{u=0}^{\ell-1} a_{r+1+u}}.
\end{aligned}$$

Suppose $q > 1$. We have

$$\begin{aligned}
&\sum_{\substack{(i_1,\dots,i_q);\\ i_1 \geqq r+1\\ i_{k+1}-i_k \geqq \ell+1\ (k=1,2,\dots,q-1)}} \frac{1}{\prod_{v=1}^{q} \prod_{u=0}^{\ell} a_{i_v+u}} \\
&= \sum_{i_1=r+1}^{\infty} \frac{1}{\prod_{u=0}^{\ell} a_{i_1+u}} \left(\sum_{\substack{(i_2,\dots,i_q);\\ i_2 \geqq i_1+\ell+1\\ i_{k+1}-i_k \geqq \ell+1\ (k=2,3,\dots q-1)}} \frac{1}{\prod_{v=2}^{q} \prod_{u=0}^{\ell} a_{i_v+u}} \right) \\
&= \sum_{i_1=r+1}^{\infty} \frac{1}{\prod_{u=0}^{\ell} a_{i_1+u}} \cdot \frac{1}{(q-1)!c^{q-1} \prod_{u=0}^{\ell(q-1)-1} a_{i_1+\ell+1+u}} \\
&\text{(by induction hypothesis)} \\
&= \frac{1}{(q-1)!c^{q-1}} \sum_{i_1=r+1}^{\infty} \frac{1}{\prod_{u=0}^{\ell q} a_{i_1+u}} \\
&= \frac{1}{q!c^{q}} \sum_{i_1=r+1}^{\infty} \left(\frac{1}{\prod_{u=0}^{\ell q-1} a_{i_1+u}} - \frac{1}{\prod_{u=1}^{\ell q} a_{i_1+u}} \right) \quad \text{(by (6.3))} \\
&= \frac{1}{q!c^{q} \prod_{u=0}^{\ell q-1} a_{r+1+u}}.
\end{aligned}$$

This completes the proof. □

The hook length for each cell p_i is

$$h_{\mathring{\lambda}}(p_{\ell t+j}) = (\ell+1)t + j \quad \text{for} \quad t \in \mathbb{Z}_{\geqq 0},\ j = 1, 2, \ldots, \ell.$$

Put $h_i := h_{\mathring{\lambda}}(p_i)$. By Proposition 6.1, we have

$$g^{\mathring{\lambda}/\mathring{\mu}} = \frac{n!}{\prod_{i=1}^{r} h_i} \times \sum_{(i_1,\ldots,i_q)\in\mathcal{E}_{\ell;n}} \frac{1}{\prod_{v=1}^{q}\prod_{u=0}^{\ell} h_{i_v+u}}.$$

By applying Lemma 6.2 with $a_i = h_i$, $c = \ell + 1$ and q, r as in (6.1), we have

$$\begin{aligned}
n!/g^{\mathring{\lambda}/\mathring{\mu}} &= q!(\ell+1)^q \prod_{u=1}^{\ell q+r} h_u \\
&= \left(\prod_{k=0}^{q-1}\left((\ell+1)(k+1)\prod_{j=1}^{\ell} h_{\ell k+j}\right)\right) h_{\ell q+1}h_{\ell q+2}\cdots h_{\ell q+r} \\
&= \prod_{k=0}^{q-1}\Big(((\ell+1)k+1)((\ell+1)k+2)\cdots((\ell+1)k+\ell)((\ell+1)(k+1))\Big) \\
&\qquad \times ((\ell+1)q+1)((\ell+1)q+2)\cdots((\ell+1)q+r) \\
&= ((\ell+1)q)!((\ell+1)q+1)((\ell+1)q+2)\cdots((\ell+1)q+r) \\
&= ((\ell+1)q+r)! = n!.
\end{aligned}$$

Therefore $g^{\mathring{\lambda}/\mathring{\mu}} = 1$ and Theorem 5.8 has been proved.

7. Proof of Theorem 5.9

Let $\ell, m \in \mathbb{Z}_{\geqq 1}$ and let $\lambda = ((\ell+1)^m)$, $\mu = (\ell^{m-1}, 0)$, $\mathring{\lambda} = \mathring{\lambda}_{(m,-\ell)}$ and $\mathring{\mu} = \mathring{\mu}_{(m,-\ell)}$. Put $n = |\lambda/\mu| = \ell + m$. Any ℓ-restricted linear extension ε of λ/μ satisfies

$$\varepsilon^{-1}(1) = (m, \ell+1), \quad \varepsilon^{-1}(n) = (m, 1).$$

Hence ε is uniquely determined by choosing $(m-1)$ vertical components from $[2, n-1]$. By Lemma 3.3, the first equality in Theorem 5.9 holds:

$$f^{\mathring{\lambda}/\mathring{\mu}} = \binom{n-2}{m-1} = \binom{\ell+m-2}{m-1}.$$

We will prove the second equality

$$g^{\mathring{\lambda}/\mathring{\mu}} = \binom{\ell+m-2}{m-1}$$

in the rest of this section.

For $u, v \in \mathbb{Z}^2$, we write $u \to v$ if $v - u = (1, 0)$ or $(0, -1)$.

Definition 7.1. For $u = (a, b), v = (c, d) \in \mathbb{Z}^2$ with $a < c$ and $b > d$, a subset

$$\mathfrak{p} = \{u = u_1, u_2, \ldots, u_r = v\}$$

of $\mathbb{Z}^2$ is called a *lattice path* from u to v if

$$u_1 \to u_2 \to \cdots \to u_r,$$

and the whole set is denoted by $\mathcal{L}(u, v)$.

Let $m, \ell \in \mathbb{Z}_{\geqq 1}$ and let π denote the natural projection $\mathbb{Z}^2 \to \mathbb{Z}^2/\mathbb{Z}(m, -\ell) = \mathcal{C}_{(m,-\ell)}$ as before. For $u, v \in \mathcal{C}_{(m,-\ell)}$, we write $u \to v$ if there exist $\widetilde{u} \in \pi^{-1}(u)$ and $\widetilde{v} \in \pi^{-1}(v)$ such that $\widetilde{u} \to \widetilde{v}$.

Definition 7.2. A subset

$$\mathfrak{p} = \{u_1, u_2, \ldots, u_n\}$$

of $\mathcal{C}_{(m,-\ell)}$ is called a *non-intersecting loop* in $\mathcal{C}_{(m,-\ell)}$ if $n = \ell + m$ and

$$u_1 \to u_2 \to \cdots \to u_{n-1} \to u_n \to u_1.$$

The whole set of non-intersecting loops is denoted by $\mathring{\mathcal{L}}$.

Let $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{P}_{m,\ell}$ and let $\boldsymbol{\lambda}$ denote the semi-infinite diagram

$$\boldsymbol{\lambda} = \{(a, b) \in \mathbb{Z}^2 \mid 1 \leqq a \leqq m,\ b \leqq \lambda_a\}$$

as before. Note that $\boldsymbol{\lambda}$ is in one-to-one correspondence with the cylindric diagram $\mathring{\lambda}$ via the projection π.

Define

$$\mathcal{L}_\lambda(u, v) = \{\mathfrak{p} \in \mathcal{L}(u, v) \mid \mathfrak{p} \subset \boldsymbol{\lambda}\} \quad (u, v \in \boldsymbol{\lambda}),$$
$$\mathring{\mathcal{L}}_\lambda = \{\mathfrak{p} \in \mathring{\mathcal{L}} \mid \mathfrak{p} \subset \mathring{\lambda}\}.$$

Lemma 7.3. *Let $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{P}_{m,\ell}$. Then the projection π induces a bijection*

$$\bigsqcup_{i=0}^{\infty} \mathcal{L}_\lambda((1, \lambda_1 - i), (m, \lambda_1 - \ell - i)) \xrightarrow{\cong} \mathring{\mathcal{L}}_\lambda.$$

Proof. For a lattice path $\mathfrak{p} \in \mathcal{L}_\lambda((1, \lambda_1 - i), (m, \lambda_1 - \ell - i))$, it is clear that $\pi(\mathfrak{p}) \in \mathring{\mathcal{L}}_\lambda$. The inverse map is given by $\mathfrak{p} \mapsto \pi^{-1}(\mathfrak{p}) \cap \boldsymbol{\lambda}$. □

Now, we return to the special case where $\lambda = ((\ell + 1)^m)$. In this case, we have

$$\mathcal{L}_\lambda(u, v) = \mathcal{L}(u, v) \tag{7.1}$$

for all $u, v \in \boldsymbol{\lambda}$.

Proposition 7.4. *Let $\ell, m \in \mathbb{Z}_{\geqq 1}$ and $\lambda = ((\ell+1)^m) \in \mathcal{P}_{m,\ell}$.*

(1) *Let $k \in [0, \ell]$ and let $\nu^{(k)} = (\ell^{m-1}, k) \in \mathcal{P}_{m,\ell}$. Then the correspondence $\mathfrak{p} \mapsto \lambda \setminus \mathfrak{p}$ gives a bijection*

$$\mathcal{L}((1, \ell+1), (m, k+1)) \xrightarrow{\cong} \mathcal{E}_\lambda(\nu^{(k)}).$$

(2) *Let $\mu = (\ell^{m-1}, 0) \in \mathcal{P}_{m,\ell}$. Then the correspondence $\mathfrak{p} \mapsto \mathring{\lambda} \setminus \mathfrak{p}$ gives a bijection*

$$\mathring{\mathcal{L}}_\lambda \xrightarrow{\cong} \mathcal{E}_{\mathring{\lambda}}(\mathring{\mu}).$$

Proof. (1) We write $\mathcal{L} = \mathcal{L}((1, \ell+1), (m, k+1))$ and $\nu = \nu^{(k)}$ in this proof.

Put $\psi(\mathfrak{p}) = \lambda \setminus \mathfrak{p}$. We will prove $\psi(\mathfrak{p})$ is contained in $\mathcal{E}_\lambda(\nu)$ by induction on

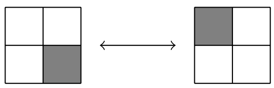
$$r(\mathfrak{p}) := \#\{x \in \boldsymbol{\lambda} \mid x \ngeq y \text{ for any } y \in \mathfrak{p}\},$$

which is the number of cells in $\boldsymbol{\lambda}$ located to the right of $\mathfrak{p}$. If $r(\mathfrak{p}) = 0$, then $\psi(\mathfrak{p}) = \nu \in \mathcal{E}_\lambda(\nu)$.

Let $\mathfrak{p} \in \mathcal{L}$ with $r(\mathfrak{p}) \geqq 1$ and suppose that $\psi(\mathfrak{q}) \in \mathcal{E}_\lambda(\nu)$ for any $\mathfrak{q} \in \mathcal{L}$ such that $r(\mathfrak{q}) < r(\mathfrak{p})$. There exists $y = (a, b) \in \lambda \setminus \mathfrak{p}$ such that

$$(a, b-1), (a-1, b), x = (a-1, b-1) \in \mathfrak{p}.$$

Now $\mathfrak{p}' := (\mathfrak{p} \setminus \{x\}) \cup \{y\}$ is a lattice path with $r(\mathfrak{p}') = r(\mathfrak{p}) - 1$ and hence $\psi(\mathfrak{p}') \in \mathcal{E}_\lambda(\nu)$. Moreover, $\psi(\mathfrak{p})$ is obtained from $\psi(\mathfrak{p}')$ by applying the elementary excitation at x. This implies $\psi(\mathfrak{p}) \in \mathcal{E}_\lambda(\nu)$.



Next, we construct an inverse map. We define $\varphi(D) = \boldsymbol{\lambda} \setminus D$ and will prove that $\varphi(D)$ is contained in $\mathcal{L}$ for all $D \in \mathcal{E}_\lambda(\nu)$. For $D = \nu$, we have $\varphi(D) = \lambda/\nu \in \mathcal{L}$. It is easy to see that $\varphi(D_y) \in \mathcal{L}$ for any $\varphi(D)$ and any D-active cell y. Hence $\varphi(D) \in \mathcal{L}$ for any $D \in \mathcal{E}_\lambda(\nu)$. This completes the proof of (1).

The statement (2) is proved by a parallel argument. □

Combining (7.1), Lemma 7.3 (2) and Proposition 7.4, we have the following:

Corollary 7.5. *Let $\ell, m \in \mathbb{Z}_{\geqq 1}$ and $\lambda = ((\ell+1)^m), \mu = (\ell^{m-1}, 0) \in \mathcal{P}_{m,\ell}$. Then the map $\mathfrak{p} \mapsto \mathring{\lambda} \setminus \pi(\mathfrak{p})$ gives a bijection*

$$\bigsqcup_{i=0}^{\infty} \mathcal{L}((1, \ell+1-i), (m, 1-i)) \xrightarrow{\cong} \mathcal{E}_{\mathring{\lambda}}(\mathring{\mu}).$$

For $\ell, m \in \mathbb{Z}_{\geqq 1}$ and $x = (a, b) \in \mathbb{Z}^2$, define

$$h_{m,\ell}(x) = \ell + m - a - b + 2. \tag{7.2}$$

Remark that

$$h_{m,\ell}(x) = h_\lambda(x)$$

for $\lambda = ((\ell+1)^m)$ and $x \in \lambda$. For $s \in \mathbb{Z}_{\geqq 1}$, define

$$F_{(\ell,m;s)} = \sum_{\mathfrak{p} \in \mathcal{L}((1,\ell+1),(m,2))} \prod_{x \in \mathfrak{p}} \frac{1}{h_{m,\ell}(x) + s - 1}. \tag{7.3}$$

Lemma 7.6. *Let $\ell, m, s \in \mathbb{Z}_{\geqq 1}$. Then*

$$F_{(\ell,m;s)} = \frac{(s-1)!}{(\ell+m+s-2)!} \binom{\ell+m-2}{m-1}. \tag{7.4}$$

Proof. We proceed by induction on s. If $s = 1$, then it follows from Proposition 7.4 (1) with $k = 1$ and Theorem 5.4 that

$$F_{(\ell,m;1)} = \sum_{D \in \mathcal{E}_\lambda(\nu^{(1)})} \prod_{x \in \lambda \setminus D} \frac{1}{h_\lambda(x)} = \frac{1}{(\ell+m-1)!} \binom{\ell+m-2}{m-1},$$

where $\lambda = ((\ell+1)^m)$ and $\nu^{(1)} = (\ell^{m-1}, 1)$, and hence (7.4) holds for all $\ell, m \in \mathbb{Z}_{\geqq 1}$.

Take $s \geqq 1$ and suppose that (7.4) holds for all $\ell, m \in \mathbb{Z}_{\geqq 1}$. Via the bijection

$$\mathcal{L}((1,\ell+2),(m,2)) \cong \mathcal{L}((2,\ell+2),(m,2)) \sqcup \mathcal{L}((1,\ell+1),(m,2)),$$

we have

$$F_{(\ell+1,m;s)} = \frac{1}{h_{m,\ell+1}(1,\ell+2) + s - 1} (F_{(\ell+1,m-1;s)} + F_{(\ell,m;s+1)}).$$

Using induction hypothesis, we have

$$\begin{aligned}
F_{(\ell,m;s+1)} &= (h_{m,\ell+1}(1,\ell+2) + s - 1) F_{(\ell+1,m;s)} - F_{(\ell+1,m-1;s)} \\
&= (m+s-1) \cdot \frac{(s-1)!\binom{\ell+m-1}{m-1}}{(\ell+m+s-1)!} - \frac{(s-1)!\binom{\ell+m-2}{m-2}}{(\ell+m+s-2)!} \\
&= \frac{(s-1)!\binom{\ell+m-2}{m-2}}{(\ell+m+s-2)!} \left(\frac{(m+s-1)(\ell+m-1)}{(\ell+m+s-1)(m-1)} - 1 \right) \\
&= \frac{(s-1)!\binom{\ell+m-2}{m-2}}{(\ell+m+s-2)!} \cdot \frac{\ell s}{(\ell+m+s-1)(m-1)} \\
&= \frac{s!}{(\ell+m+s-1)!} \binom{\ell+m-2}{m-1}.
\end{aligned}$$

This completes the induction step. □

For $\ell, m, s, t \in \mathbb{Z}_{\geqq 1}$ and $x \in \mathbb{Z}^2$, we define

$$h^{s,t}_{m,\ell}(x) = \ell + m - a - b + (d-c)t + s + 1, \tag{7.5}$$

where $x = (a + cm, b - d\ell)$ with $1 \leqq a \leqq m,\ 2 \leqq b \leqq \ell + 1$ and $c, d \geqq 1$. Remark that for $s = 1$ and $t = \ell + m$, the number $h^{s,t}_{m,\ell}(x)$ gives a cylindric hook length:

$$h^{1,\ell+m}_{m,\ell}(x) = h_{\mathring{\lambda}}(\pi(x)) \tag{7.6}$$

for $\lambda = ((\ell + 1)^m)$ and $x \in \boldsymbol{\lambda}$. (See Definition 5.3.)

Note also that for any $s, t, a \in \mathbb{Z}_{\geqq 1}$, we have

$$\lim_{b \to -\infty} h^{s,t}_{m,\ell}(a, b) = +\infty. \tag{7.7}$$

Define

$$F_{(\ell,m;s,t)} = \sum_{i=0}^{\infty} \sum_{\mathfrak{p} \in \mathcal{L}((1,\ell+1-i),(m,1-i))} \prod_{x \in \mathfrak{p}} \frac{1}{h^{s,t}_{m,\ell}(x)}.$$

Lemma 7.7. *Let $\ell, m, s, t \in \mathbb{Z}_{\geqq 1}$. Then*

$$F_{(\ell,m;s,t)} = \frac{1}{t - m + 1} \left(F_{(\ell,m;s)} + F_{(\ell,m-1;s+1,t)} - F_{(\ell,m-1;s,t)} \right). \tag{7.8}$$

Proof. For $i \geqq 0$, put

$$d_i = \ell + 1 - i.$$

Fix ℓ, m, s, t and write

$$h(x) = h^{s,t}_{m,\ell}(x)$$

for a while. We have

$$F_{(\ell,m;s,t)} = \sum_{i=0}^{\infty} \sum_{0 \leqq k_1 \leqq \cdots \leqq k_{m-1} \leqq \ell} \frac{1}{\prod_{r=0}^{k_1} h(1, d_i - r) \prod_{r=k_1}^{k_2} h(2, d_i - r) \cdots \prod_{r=k_{m-1}}^{\ell} h(m, d_i - r)}$$

(See Figure 8.). Combining with

$$h(m, 1 - i) - h(1, \ell + 1 - i) = t - m + 1,$$

we have

$$F_{(\ell,m;s,t)} = \frac{1}{t - m + 1} \sum_{i=0}^{\infty}$$

$$\times \sum_{0 \leqq k_1 \leqq \cdots \leqq k_{m-1} \leqq \ell} \left(\frac{1}{\prod_{r=0}^{k_1} h(1, d_i - r) \prod_{r=k_1}^{k_2} h(2, d_i - r) \cdots \prod_{r=k_{m-1}}^{\ell-1} h(m, d_i - r)} \right.$$

$$- \frac{1}{\prod_{r=1}^{k_1} h(1,d_i-r) \prod_{r=k_1}^{k_2} h(2,d_i-r) \cdots \prod_{r=k_{m-1}}^{\ell} h(m,d_i-r)} \Bigg)$$

$$= \frac{1}{t-m+1} \sum_{i=0}^{\infty} (A_i + B_i - C_i - D_i).$$

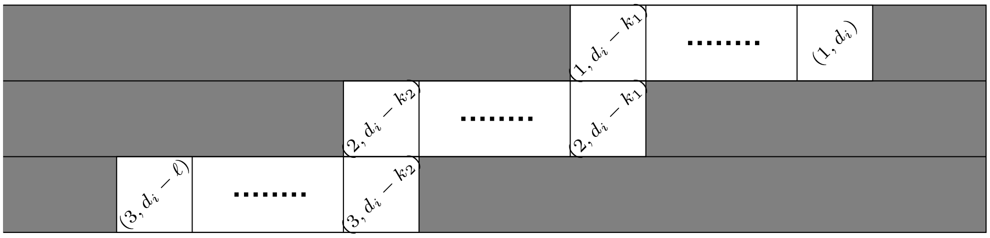


FIGURE 8. An excited diagram for $\mathring{\lambda}/\mathring{\mu}$ with $\lambda = (\ell^3)$, $\mu = ((\ell-1)^2, 0)$.

Here, A_i, B_i, C_i and D_i are

$$A_i = \sum_{0 \leqq k_1 \leqq \cdots \leqq k_{m-1} \leqq \ell-1} \frac{1}{\prod_{r=0}^{k_1} h(1,d_i-r) \prod_{r=k_1}^{k_2} h(2,d_i-r) \cdots \prod_{r=k_{m-1}}^{\ell-1} h(m,d_i-r)},$$

$$B_i = \sum_{0 \leqq k_1 \leqq \cdots \leqq k_{m-2} \leqq \ell} \frac{1}{\prod_{r=0}^{k_1} h(1,d_i-r) \prod_{r=k_1}^{k_2} h(2,d_i-r) \cdots \prod_{r=k_{m-2}}^{\ell} h(m-1,d_i-r)},$$

$$C_i = \sum_{0 \leqq k_2 \leqq \cdots \leqq k_{m-1} \leqq \ell} \frac{1}{\prod_{r=0}^{k_2} h(2,d_i-r) \prod_{r=k_1}^{k_2} h(3,d_i-r) \cdots \prod_{r=k_{m-1}}^{\ell} h(m,d_i-r)},$$

$$D_i = \sum_{1 \leqq k_1 \leqq \cdots \leqq k_{m-1} \leqq \ell} \frac{1}{\prod_{r=1}^{k_1} h(1,d_i-r) \prod_{r=k_1}^{k_2} h(2,d_i-r) \cdots \prod_{r=k_{m-1}}^{\ell} h(m,d_i-r)}$$

$$= A_{i+1}.$$

Now we have

$$\sum_{i=0}^{\infty} B_i = \sum_{i=0}^{\infty} \sum_{\mathfrak{p} \in \mathcal{L}((1,\ell+1-i),(m-1,1-i))} \prod_{x \in \mathfrak{p}} \frac{1}{h(x)}$$

$$= \sum_{i=0}^{\infty} \sum_{\mathfrak{p} \in \mathcal{L}((1,\ell+1-i),(m-1,1-i))} \prod_{x \in \mathfrak{p}} \frac{1}{h_{m-1,\ell}^{s+1,t}(x)} = F_{(\ell,m-1;s+1,t)},$$

$$\sum_{i=0}^{\infty} C_i = \sum_{i=0}^{\infty} \sum_{\mathfrak{p} \in \mathcal{L}((2,\ell+1-i),(m,1-i))} \prod_{x \in \mathfrak{p}} \frac{1}{h(x)} = F_{(\ell,m-1;s,t)}.$$

By using (7.7), we have

$$\lim_{i\to\infty} A_i = 0,$$

and hence

$$\begin{aligned}\sum_{i=0}^{\infty}(A_i - D_i) &= \sum_{i=0}^{\infty}(A_i - A_{i+1}) = A_0 \\ &= \sum_{i=0}^{\infty} \sum_{\mathfrak{p}\in\mathcal{L}((1,\ell+1),(m,2))} \prod_{x\in\mathfrak{p}} \frac{1}{h(x)} = F_{(\ell,m;s)}.\end{aligned}$$

Therefore,

$$F_{(\ell,m;s,t)} = \frac{1}{t-m+1}\left(F_{(\ell,m;s)} + F_{(\ell,m-1;s+1,t)} - F_{(\ell,m-1;s,t)}\right).$$

□

Proposition 7.8. *Let* $\ell, m, s, t \in \mathbb{Z}_{\geqq 1}$. *Then*

$$F_{(\ell,m:s,t)} = \frac{(s-1)!}{(\ell+m+s-2)!t}\binom{\ell+m-2}{m-1}. \tag{7.9}$$

Proof. We proceed by induction on m. If $m = 1$, then putting $d_i = \ell+1-i$, we have

$$\begin{aligned}F_{(\ell,1;s,t)} &= \sum_{i=0} \frac{1}{\prod_{k=0}^{\ell} h_{1,\ell}^{s,t}(1, d_i - k)} \\ &= \frac{1}{t}\sum_{i=0}\left(\frac{1}{\prod_{k=0}^{\ell-1} h_{1,\ell}^{s,t}(1, d_i - k)} - \frac{1}{\prod_{k=1}^{\ell} h_{1,\ell}^{s,t}(1, d_i - k)}\right) \\ &= \frac{1}{t}\frac{1}{\prod_{k=0}^{\ell-1} h_{1,\ell}^{s,t}(1, \ell+1-k)} \quad \text{(by (7.7))} \\ &= \frac{1}{t}\cdot\frac{1}{s(s+1)\cdots(s+\ell-1)}\end{aligned}$$

for any $\ell, s, t \in \mathbb{Z}_{\geqq 1}$. This proves (7.9) when $m = 1$.

Let $m > 1$ and suppose that

$$F_{(\ell,m-1:s,t)} = \frac{(s-1)!}{(\ell+(m-1)+s-2)!t}\binom{\ell+(m-1)-2}{m-2}. \tag{7.10}$$

for all ℓ, s, t. By Lemma 7.7, we have

$$\begin{aligned}F_{(\ell,m;s,t)} &= \frac{1}{t-m+1}\left(F_{(\ell,m;s)} + F_{(\ell,m-1;s+1,t)} - F_{(\ell,m-1;s,t)}\right) \\ &= \frac{1}{t-m+1}\left(\frac{(s-1)!\binom{\ell+m-2}{m-1}}{(\ell+m+s-2)!} + \frac{s!\binom{\ell+m-3}{m-2}}{(\ell+m+s-2)!t} - \frac{(s-1)!\binom{\ell+m-3}{m-2}}{(\ell+m+s-3)!t}\right)\end{aligned}$$

(by Lemma 7.6 and induction hypothesis)

$$= \frac{(s-1)!\binom{\ell+m-3}{m-2}}{(\ell+m+s-3)!(t-m+1)}\left(\frac{\ell+m-2}{(m-1)(\ell+m+s-2)}+\frac{s}{(\ell+m+s-2)t}-\frac{1}{t}\right)$$

$$= \frac{(s-1)!}{(\ell+m+s-3)!(t-m+1)}\binom{\ell+m-3}{m-2}\cdot\frac{(\ell+m-2)(t-m+1)}{(m-1)(\ell+m+s-2)t}$$

$$= \frac{(s-1)!}{(\ell+m+s-2)!t}\binom{\ell+m-2}{m-1}.$$

We have proved Proposition 7.8. □

Finally, by applying Corollary 7.5 and Proposition 7.8 with $s = 1$ and $t = \ell + m = n$, we obtain

$$\begin{aligned} g^{\mathring{\lambda}/\mathring{\mu}} &= n! \sum_{D\in\mathcal{E}_{\mathring{\lambda}}(\mathring{\mu})} \prod_{x\in\mathring{\lambda}\setminus D} \frac{1}{h_{\mathring{\lambda}}(x)} \\ &= n!\sum_{i=0}^{\infty}\sum_{\mathfrak{p}\in\mathcal{L}((1,n-m+1-i),(m,1-i))}\prod_{x\in\mathfrak{p}}\frac{1}{h_{m,\ell}^{1,\ell+m}(x)} \\ &= n!\cdot F_{(\ell,m;1,\ell+m)} \\ &= n!\cdot\frac{1}{(\ell+m-1)!(\ell+m)}\binom{\ell+m-2}{m-1} \\ &= \binom{\ell+m-2}{m-1}. \end{aligned}$$

This completes the proof of Theorem 5.9.

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