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SYMBOLIC POWERS OF MONOMIAL IDEALS

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Abstract. Let K be a field and consider the standard grading on $A = K[X_1, \ldots, X_d]$. Let I, J be monomial ideals in A. Let $I_n(J) = (I^n \colon J^\infty)$ be the n^{th} symbolic power of I with respect to J. It is easy to see that the function $f_J^I(n) = e_0(I_n(J)/I^n)$ is of quasi-polynomial type, say of period g and degree c. For $n \gg 0$ say

$$f_J^I(n) = a_c(n)n^c + a_{c-1}(n)n^{c-1} + \text{lower terms},$$

where for $i = 0, \ldots, c$, $a_i \colon \mathbb{N} \to \mathbb{Q}$ are periodic functions of period g and $a_c \neq 0$. In [4, 2.4] we (together with Herzog and Verma) proved that $\dim I_n(J)/I^n$ is constant for $n \gg 0$ and $a_c(-)$ is a constant. In this paper we prove that if I is generated by some elements of the same degree and height $I \geq 2$ then $a_{c-1}(-)$ is also a constant.

1. introduction

Let us recall that a function $f \colon \mathbb{N} \to \mathbb{Z}$ is called of quasi-polynomial of period g and degree c if

$$f(n) = a_c(n)n^c + a_{c-1}(n)n^{c-1} + \text{lower terms}$$

where for $i = 0, \ldots, c$, $a_i \colon \mathbb{N} \to \mathbb{Q}$ are periodic functions of period g and $a_c \neq 0$. Many interesting functions are of quasi-polynomial type (i.e., it coincides with a quasi-polynomial for $n \gg 0$). For instance the Hilbert function of a graded module over a not-necessarily standard graded K-algebra (here K is a field) is of quasi-polynomial type; see [2, 4.4.3].

Let f be of quasi-polynomial type. Following Erhart we let the *grade* of f denote the smallest integer $\delta \geq -1$ such that $a_j(-)$ is constant for all $j > \delta$. Erhart had conjectured that if P is a d-dimensional rational polytope in $\mathbb{R}^m$ such that for some δ the affine span of every δ-dimensional face of P contains a point with integer coordinates then the grade of the Erhart quasi-polynomial of P is $< \delta$. This conjecture was proved by independently by McMullen [5] and Stanley [6, Theorem 2.8]. For a purely algebraic proof see [1, Theorem 5].

1.1. *Our interest in quasi-polynomials arise in the following context (see* [4]*): Let $A = K[X_1, \ldots, X_d]$ be a standard graded polynomial ring over a field K. If M is a graded A-module then let $e_0(M)$ denote its multiplicity. Let I, J be monomial ideals. Let $I_n(J) = I^n \colon J^\infty$ be the n^{th} symbolic power*

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of I with respect to J. Then by [3, 3.2] *we have that $\bigoplus_{n\geq 0} I_n(J)$ is a finitely generated A-algebra. From this it is easy to see that $f_J^I(n) = e_0(I_n(J)/I^n)$ is of quasi-polynomial type, say of period g and degree c. We write*

$$f_J^I(n) = a_c(n)n^c + a_{c-1}(n)n^{c-1} + \textit{lower terms.}$$

In [4, 2.4] *we proved that $\dim I_n(J)/I^n$ is constant for $n \gg 0$ and $a_c(-)$ is a positive constant. In this short paper we prove*

Theorem 1.2. *[with hypotheses as in 1.1] Further assume that I is generated by some elements of the same degree and* height $I \geq 2$*. Then $a_{c-1}(-)$ is also a constant.*

2. Preliminaries

Let K be a field and let $R = \bigoplus_{m\geq 0} R_m$ be a standard graded, finitely generated $K = R_0$-algebra. We do not assume R is a polynomial ring. Let $M = \bigoplus_{m\geq 0} M_m$ be a graded R-module. Its Hilbert series is $H_M(z) = \sum_{m\geq 0} \dim_K M_m z^m$. It is well-known that $H_M(z) = h_M(z)/(1-z)^{\dim M}$ where $h_M(z) \in \mathbb{Z}[z]$. The number $e_0(M) = h_M(1)$ is called the multiplicity *of M.*

2.1. *Let I be a homogeneous ideal in R. An element $x \in I$ is called I-*superficial *if there exists n_0 such that $(I^{n+1} \colon x) \cap I^{n_0} = I^n$ for all $n \geq n_0$. Superficial elements exist when K is infinite. If* grade$(I) > 0$ *then by using Artin-Rees lemma it can be shown that $(I^{n+1} \colon x) = I^n$ for all $n \gg 0$. If $I = (u_1, \ldots, u_l)$ and K is infinite then a general linear sum of the u_i is an I-superficial element. Thus I-superficial elements need not be homogeneous even if I is. However if I is generated by some elements of the same degree then it is clear that there exists homogeneous I-superficial elements.*

The following result can be shown in the same way as in the proof of Proposition 2.3 and Theorem 2.4 in [4].

Theorem 2.2. *(with setup as above). Let $\mathcal{F} = \{J_n\}_{n\geq 0}$ be a multiplicative filtration of homogeneous ideals with $J_0 = R$ and $I \subseteq J_1$. Further assume that $\mathcal{S}(\mathcal{F}) = \bigoplus_{n\geq 0} J_n$ is a finitely generated R-algebra. Then the function $f_{\mathcal{F}}^I(n) = e_0(J_n/I^n)$ is of quasi-polynomial type, say of degree c and period g. For $n \gg 0$ we write $f_{\mathcal{F}}^I(n) = a_c(n)n^c +$ lower terms. If* grade$(I) > 0$ *then* $\dim(J_n/I^n)$ *is constant for $n \gg 0$ and $a_c(-)$ is a constant.*

2.3. *Let $\mathcal{F} = \{J_n\}_{n\geq 0}$ be a multiplicative filtration of homogeneous ideals with $J_0 = R$ and $I \subseteq J_1$. Let $\mathcal{R} = R[It]$ be the Rees-algebra of I. Then $L^{\mathcal{F}} = \bigoplus_{n\geq 0} R/J_{n+1}$ has a structure of $\mathcal{R}$-module. This can be seen as follows. Note $\mathcal{S}(\mathcal{F}) = \bigoplus_{n\geq 0} J_n$ is a graded $\mathcal{R}$-module. We have a short* *exact sequence*

$$0 \to \mathcal{S}(\mathcal{F}) \to R[t] \to L^{\mathcal{F}}(-1) \to 0.$$

This gives an $\mathcal{R}$*-module structure on* $L^{\mathcal{F}}(-1)$ *and so on* $L^{\mathcal{F}}$*.*

3. Proof of Theorem 1.2

In this section we give

Proof of Theorem 1.2. We may assume that K is an infinite field.
Let $f(n) = e_0(I_n(J)/I^n)$. Then by our earlier result f is of polynomial type say of degree c and period g. Say

$$f(n) = a_c(n)n^c + a_{c-1}(n)n^{c-1} + \text{lower terms} \quad \text{for } n \gg 0,$$

where $a_c(n) = a$ is a positive constant. We have nothing to show if $c = 0$. So assume $c > 0$. We also have $\dim I_n(J)/I^n$ is constant for $n \gg 0$. Let this constant be r.

As I is generated by some elements of the same degree we can choose homogeneous u which is I-superficial. Say $(I^{n+1} \colon u) = I^n$ for $n \geq n_0$.

Claim-1: $(I_{n+1}(J) \colon u) = I_n(J)$ for $n \geq n_0$.
Fix $n \geq n_0$. Let $p \in (I_{n+1}(J) \colon u)$. Then $up \in I_{n+1}(J)$. So $upJ^m \subseteq I^{n+1}$ for some $m > 0$. Then $pJ^m \subseteq (I^{n+1} \colon u) = I^n$. So $p \in I_n(J)$.

Set $\mathcal{F} = \{I_n(J)\}_{n\geq 0}$. Let $L^{\mathcal{F}} = \bigoplus_{n\geq 0} A/I_{n+1}(J)$ and $L^I = \bigoplus_{n\geq 0} A/I^{n+1}$ be given the $\mathcal{R} = A[It]$-module structures as described in 2.3. Set $W = \bigoplus_{n\geq 0} I_{n+1}(J)/I^{n+1}$. Then we have a short exact sequence of $\mathcal{R}$-modules

$$(*) \qquad 0 \to W \to L^I \to L^{\mathcal{F}} \to 0.$$

Let $p = ut$. By claim-1; $\ker(L^{\mathcal{F}}(-1) \xrightarrow{p} L^{\mathcal{F}})$ is concentrated in degrees $\leq n_0$. So we have a short exact sequence

$$(**) \qquad 0 \to \overline{W}_{n\geq n_0+1} \to \overline{L^I}_{n\geq n_0+1} \to \overline{L^{\mathcal{F}}}_{n\geq n_0+1} \to 0;$$

(here $\overline{(-)} = (-)/p(-)$). We note that $\overline{L^{\mathcal{F}}}_n = A/(I_{n+1}(J), u)$ and $\overline{L^I}_n = A/(I^{n+1}, u)$. Consider the standard graded K-algebra $R = A/(u)$. Consider the R-ideal $\overline{I} = I/(u)$ and the Noetherian R-filtration $\overline{\mathcal{F}} = \{\overline{I_n(J)}\}_{n\geq 0}$ where $\overline{I_n(J)} = (I_n(J) + (u))/(u)$. Notice $\operatorname{grade} I = \operatorname{height} I \geq 2$. So $\operatorname{grade} \overline{I} \geq 1$. Set $g(n) = e_0(\overline{I_n(J)}/\overline{I^n})$. By 2.2 we have $\dim \overline{I_n(J)}/\overline{I^n}$ is constant for $n \gg 0$ (say equal to s). Furthermore $g(n)$ is of quasi-polynomial type say of period g' and degree l. Say

$$g(n) = b_l(n)n^l + \text{lower terms}.$$

We also have $b_l(-) = b$ a constant.

By (*) and (**) we have a short exact sequence for all $n \gg 0$

$$0 \to I_{n-1}(J)/I^{n-1} \to I_n(J)/I^n \to \overline{I_n(J)}/\overline{I^n} \to 0.$$

So we have $r \geq s$. If $r > s$ then notice $c = 0$ which is a contradiction. So $r = s$. It follows that $l = c - 1$. As $f(n) - f(n-1) = g(n)$ we get that

$$ac + a_{c-1}(n) - a_{c-1}(n-1) = b \quad \text{for all } n \gg 0.$$

So for $n \gg 0$ we have $a_{c-1}(n) = \alpha n + \beta$ for $n \gg 0$ for some constants α, β; see [2, 4.1.2]. As $a_{c-1}(-)$ is periodic it follows from that $\alpha = 0$ and so $a_{c-1}(-)$ is a constant. The result follows. □

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