Math. J. Okayama Univ. **64** (2022), 63–73

NOTE ON TOTALLY ODD MULTIPLE ZETA VALUES

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Abstract. A partial answer to a conjecture about the rank of the matrix $C_{N,r}$ introduced by Francis Brown in the study of totally odd multiple zeta values is given.

1. Introduction

In this paper, we give an upper bound of the rank of the matrix $C_{N,r}$ introduced by Brown [3] in the study of totally odd multiple zeta values. We first recall the uneven part of the motivic Broadhurst-Kreimer conjecture from [3] and then state our main result.

In [2, Definition 2.1], motivic multiple zeta values $\zeta^{\mathfrak{m}}(n_1,\ldots,n_r)$ are defined as elements of the free comodule $\mathcal{H}=\mathcal{A}^{\mathcal{MT}}\otimes_{\mathbb{Q}}\mathbb{Q}[f_2]$ over $\mathcal{A}^{\mathcal{MT}}$, where $\mathcal{A}^{\mathcal{MT}}$ denotes the graded affine ring of the unipotent radical of the Tannaka group of mixed Tate motives over $\mathbb{Z}$. The theory of motivic multiple zeta values plays an important role in the study of multiple zeta values defined by

$$\zeta(n_1,\ldots,n_r)=\sum_{0<k_1<\cdots<k_r}\frac{1}{k_1^{n_1}\cdots k_r^{n_r}}.$$

We call $n_1+\cdots+n_r$ the weight and r the depth. It is shown by Brown [2, Theorem 1.1] that the $\mathbb{Q}$-algebra $\mathcal{H}$ is generated by motivic multiple zeta values of the form $\zeta^{\mathfrak{m}}(n_1,\ldots,n_r)$, $n_1,\ldots,n_r\in\{2,3\}$.

Let $\mathfrak{D}_r\mathcal{H}$ be the $\mathbb{Q}$-vector subspace of $\mathcal{H}$ spanned by all motivic multiple zeta values of depth $\le r$. This gives rise to increasing filtrations $\mathfrak{D}_\bullet$ on the algebra $\mathcal{H}$ and the quotient algebra $\mathcal{A}=\mathcal{H}/\zeta^m(2)\mathcal{H}$, called the depth filtrations. We define the depth-graded motivic multiple zeta value $\zeta^{\mathfrak{m}}_{\mathfrak{D}}(n_1,\ldots,n_r)$ to be the image of $\zeta^{\mathfrak{m}}(n_1,\ldots,n_r)$ in $\mathrm{gr}^{\mathfrak{D}}_r\mathcal{H}$. Denote by

$$\zeta^{\mathfrak{a}}_{\mathfrak{D}}(n_1,\ldots,n_r)$$

the image of $\zeta^{\mathfrak{m}}_{\mathfrak{D}}(n_1,\ldots,n_r)$ in $\mathrm{gr}^{\mathfrak{D}}_r\mathcal{A}$. Let $\mathrm{gr}^{\mathfrak{D}}_r\mathcal{A}^{odd}_N$ be the $\mathbb{Q}$-vector space spanned by totally odd depth-graded motivic multiple zeta values of weight N and depth r

$$\zeta^{\mathfrak{a}}_{\mathfrak{D}}(2n_1+1,\ldots,2n_r+1),\ 2n_1+\cdots+2n_r=N-r.$$

The uneven part of the motivic Broadhurst-Kreimer conjecture [3, Conjecture 6] states that the generating function of the dimension of the space

Mathematics Subject Classification. Primary 11F32, Secondary 11F67.

Key words and phrases. Multiple zeta values, Period polynomials.

$\mathrm{gr}_r^{\mathfrak{D}}\mathcal{A}_N^{odd}$ is given by

$$(1.1)\qquad 1+\sum_{N,r>0}\dim \mathrm{gr}_r^{\mathfrak{D}}\mathcal{A}_N^{odd}x^Ny^r \overset{?}{=} \frac{1}{1-\mathbb{O}(x)y+\mathbb{S}(x)y^2},$$

where

$$\mathbb{O}(x)=\frac{x^3}{1-x^2}=x^3+x^5+\cdots,\quad \mathbb{S}(x)=\frac{x^{12}}{(1-x^4)(1-x^6)}=x^{12}+x^{16}+\cdots.$$

This conjecture is at present far from being solved (see also [8, 9]).

In order to study (1.1), Brown introduces the square matrix $C_{N,r}$ (see (2.4) for the clear-cut definition), whose rows and columns are indexed by $\boldsymbol{m}=(m_1,\ldots,m_r)$ and $\boldsymbol{n}=(n_1,\ldots,n_r)$ in the set $\mathbb{I}_{N,r}$ of totally odd indices of weight N and depth r, and whose coefficients are given by

$$\partial_{m_2}\cdots\partial_{m_2}\partial_{m_1}\zeta_{\mathfrak{D}}^{\mathfrak{m}}(n_1,\ldots,n_r)\in\mathbb{Z},$$

where for odd $m\in\mathbb{Z}_{\geq 3}$ the ∂_m is a well-defined derivation corresponding to the canonical generator of the depth-graded motivic Lie algebra $\mathfrak{d}$ of weight $-m$ and depth 1 (see [4, §2.5] and [3, §4] for the definition of $\mathfrak{d}$). The homology conjecture of $\mathfrak{d}$ [3, Conjecture 5] (see also [5]) leads to the following expectations for the matrix $C_{N,r}$ (see [9, Conjecture 1.1]).

Conjecture 1.1. *i) Rational numbers $\{a_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{I}_{N,r}}$ give rise to a linear relation of the form*

$$\sum_{\boldsymbol{n}\in\mathbb{I}_{N,r}}a_{\boldsymbol{n}}\zeta_{\mathfrak{D}}^{\mathfrak{m}}(\boldsymbol{n})=0,$$

if and only if the column vector ${}^t(a_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{I}_{N,r}}$ is a right annihilator of the matrix $C_{N,r}$. Therefore, the rank of the square matrix $C_{N,r}$ equals the dimension of the $\mathbb{Q}$-vector space spanned by all totally odd depth-graded motivic multiple zeta values of weight N and depth r.
ii) Then the generating function of the rank of the matrix $C_{N,r}$ is given by

$$(1.2)\qquad 1+\sum_{N,r>0}\mathrm{rank}\, C_{N,r}x^Ny^r \overset{?}{=} \frac{1}{1-\mathbb{O}(x)y+\mathbb{S}(x)y^2}.$$

Assuming Conjecture 1.1 i), we see that Conjecture 1.1 ii) is equivalent to the uneven part of the motivic Broadhurst-Kreimer conjecture (1.1). Remark that Conjecture 1.1 i) is true for $r=2,3$, but still open for $r\geq 4$. The first example of linear relations among totally odd depth-graded motivic multiple zeta values is deduced from

$$C_{12,2}=\begin{pmatrix} 0 & 0 & 0 & 1\\ -6 & 0 & 1 & 6\\ -15 & -14 & 15 & 15\\ -27 & -42 & 42 & 28\end{pmatrix}.$$

The space of right annihilators of $C_{12,2}$ is generated by the vector ${}^t(14,75,84,0)$, which gives the well-known relation obtained by Gangl, Kaneko and Zagier [6]:

$$14\zeta(3,9)+75\zeta(5,7)+84\zeta(7,5)\equiv 0 \mod \mathbb{Q}\zeta(12).$$

As for Conjecture 1.1 ii), it was shown by [1] that the equality (1.2) on the coefficient of y^2 holds. We also know by Goncharov [7, Theorem 2.5] that $\operatorname{rank} C_{N,3}$ is bounded by the coefficient of $x^N y^3$ in the Taylor expansion of the right-hand side of (1.2) at $x=y=0$. Remark that one of consequences of the equality (1.2) on the coefficient of y^3, which is still open, is that all multiple zeta values of depth 3 and weight odd are $\mathbb{Q}$-linear combinations of $\zeta(odd, odd, odd)$'s and multiple zeta values of depth ≤ 2.

In this paper, we give a partial answer to Conjecture 1.1 ii).

Theorem 1.2. *We have*

$$1+\sum_{N,r>0}\operatorname{rank} C_{N,r}x^N y^r \leq \frac{1}{1-\mathbb{O}(x)y+\mathbb{S}(x)y^2},$$

where $\sum a_{N,r}x^N y^r \leq \sum b_{N,r}x^N y^r$ *means* $a_{N,r}\leq b_{N,r}$ *for any* N,r.

The principle of our proof is to relate left annihilators of the square matrix $C_{N,r}$ with the restricted even period polynomials.

The contents of the paper are as follows. In Section 2, the matrix $C_{N,r}$ is defined. We recall some properties of the matrix $C_{N,r}$ from [9]. Section 3 is devoted to the proof of Theorem 1.2.

2. Preliminaries

2.1. **Notations.** We call a tuple of positive integers $\boldsymbol{n}=(n_1,\dots,n_r)$ an index. We define the weight $\mathrm{wt}(\boldsymbol{n})$ and the depth $\mathrm{dep}(\boldsymbol{n})$ of an index $\boldsymbol{n}=(n_1,\dots,n_r)$ by $\mathrm{wt}(\boldsymbol{n})=n_1+\cdots+n_r$ and $\mathrm{dep}(\boldsymbol{n})=r$, respectively. For an index $\boldsymbol{n}=(n_1,\dots,n_r)$, write

$$\boldsymbol{x}^{\boldsymbol{n}-1}=x_1^{n_1-1}\cdots x_r^{n_r-1}.$$

Let $\mathbb{I}_{N,r}$ be the set of totally odd indices of weight N and depth r:

$$\mathbb{I}_{N,r}=\{\boldsymbol{n}=(n_1,\dots,n_r)\in\mathbb{N}^r \mid \mathrm{wt}(\boldsymbol{n})=N, n_1,\dots,n_r\geq 3:\text{odd}\}.$$

2.2. **Linearized Ihara action.** We denote by $\underline{\circ}$ the linearised Ihara action ([9, §3.1]), which is the dual to the depth-graded Ihara coaction on $\mathrm{gr}^{\mathfrak{D}}\mathcal{H}$

([3, §3.1]). It is given by the formula

(2.1)
$$f \,\underline{\circ}\, g(x_1,\dots,x_{r+s}) = \sum_{i=0}^{s} f(x_{i+1}-x_i,\dots,x_{i+r}-x_i)g(x_1,\dots,x_i,x_{i+r+1},\dots,x_{r+s})$$
$$+(-1)^{\deg(f)+r}\sum_{i=1}^{s} f(x_{i+r-1}-x_{i+r},\dots,x_i-x_{i+r})g(x_1,\dots,x_{i-1},x_{i+r},\dots,x_{r+s})$$

for homogeneous polynomials $f(x_1,\dots,x_r)$ and $g(x_1,\dots,x_s)$, where $x_0=0$.

Denote by $\sigma_r^{(i)}$ $(1\le i\le r-1)$ the following change of variables:

$$\begin{aligned} f(x_1,\dots,x_r)\big|\sigma_r^{(i)} =& f(x_{i+1}-x_i,x_1,\dots,x_i,x_{i+2},\dots,x_r)\\ &- f(x_{i+1}-x_i,x_1,\dots,x_{i-1},x_{i+1},\dots,x_r).\end{aligned}$$

We regard $\sigma_r^{(i)}$ as an element of the group ring $\mathbb{Z}[\mathrm{GL}_r(\mathbb{Z})]$. Write $\sigma_r = \sum_{i=1}^{r-1}\sigma_r^{(i)}$. By (2.1) we have

$$x_1^{m_1-1}\,\underline{\circ}\,(x_1^{m_2-1}\cdots x_{r-1}^{m_r-1}) = \boldsymbol{x}^{\boldsymbol{m}-1}\big|(1+\sigma_r) \tag{2.2}$$

for any $\boldsymbol{m}\in\mathbb{N}^r$, where $f\big|(1+\sigma_r^{(i)}+\sigma_r^{(j)})$ means $f+f\big|\sigma_r^{(i)}+f\big|\sigma_r^{(j)}$.

2.3. **Matrices.** Following [9, Eq. (3.5) and Definition 2.3], we now define the matrices $E_{N,r}^{(q)}$ and $C_{N,r}$ (see [3, Eq. (11.2)] for the original).

For indices $\boldsymbol{m}=(m_1,\dots,m_r)$ and $\boldsymbol{n}=(n_1,\dots,n_r)$, let us define $\delta\binom{\boldsymbol{m}}{\boldsymbol{n}}$ as the Kronecker delta given by

$$\delta\binom{\boldsymbol{m}}{\boldsymbol{n}} = \begin{cases} 1 & \text{if } m_i=n_i \text{ for all } i\in\{1,\dots,r\}\\ 0 & \text{otherwise}\end{cases}.$$

For indices $\boldsymbol{m}$ and $\boldsymbol{n}$ of depth $r\ge 2$, we define the integer $e\binom{\boldsymbol{m}}{\boldsymbol{n}}$ by

$$\boldsymbol{x}^{\boldsymbol{m}-1}\big|(1+\sigma_r) = \sum_{\substack{\boldsymbol{n}\in\mathbb{N}^r\\ \mathrm{wt}(\boldsymbol{n})=\mathrm{wt}(\boldsymbol{m})}} e\binom{\boldsymbol{m}}{\boldsymbol{n}}\boldsymbol{x}^{\boldsymbol{n}-1}.$$

We set $e\binom{m_1}{n_1}=\delta\binom{m_1}{n_1}$. For the explicit formula of the integer $e\binom{\boldsymbol{m}}{\boldsymbol{n}}$ we refer the reader to [9, Lemma 3.1]. We define the integer $c\binom{\boldsymbol{m}}{\boldsymbol{n}}$ by

$$x_1^{m_1-1}\,\underline{\circ}\,(\cdots\,\underline{\circ}\,(x_1^{m_{r-1}-1}\,\underline{\circ}\,x_1^{m_r-1})\cdots) = \sum_{\substack{\boldsymbol{n}\in\mathbb{N}^r\\ \mathrm{wt}(\boldsymbol{n})=\mathrm{wt}(\boldsymbol{m})}} c\binom{\boldsymbol{m}}{\boldsymbol{n}}\boldsymbol{x}^{\boldsymbol{n}-1}$$

for each index $\boldsymbol{m}=(m_1,\dots,m_r)$ with $r\ge 2$. We also let $c\binom{m_1}{n_1}=\delta\binom{m_1}{n_1}$.

Definition 2.1. *For positive integers N, r, q with $1 \le q \le r$, we define the $|\mathbb{I}_{N,r}| \times |\mathbb{I}_{N,r}|$ matrices $E^{(q)}_{N,r}$ and $C_{N,r}$ by*

$$E^{(q)}_{N,r} = \left(\delta\binom{m_1,\dots,m_{r-q}}{n_1,\dots,n_{r-q}} \cdot e\binom{m_{r-q+1},\dots,m_r}{n_{r-q+1},\dots,n_r}\right)_{\substack{(m_1,\dots,m_r)\in\mathbb{I}_{N,r}\\(n_1,\dots,n_r)\in\mathbb{I}_{N,r}}} \tag{2.3}$$

and

$$C_{N,r} = \left(c\binom{\boldsymbol{m}}{\boldsymbol{n}}\right)_{\substack{\boldsymbol{m}\in\mathbb{I}_{N,r}\\\boldsymbol{n}\in\mathbb{I}_{N,r}}}, \tag{2.4}$$

where rows and columns are indexed by $\boldsymbol{m}$ and $\boldsymbol{n}$ in the set $\mathbb{I}_{N,r}$. It is understood that the matrices $E^{(q)}_{N,r}$ and $C_{N,r}$ are an empty matrix when $|\mathbb{I}_{N,r}| = 0$ (i.e. $\operatorname{rank} C_{N,r} = 0$ *and* $\ker C_{N,r} = \{0\}$*).*

The matrix $E^{(1)}_{N,r}$ is the identity matrix when $|\mathbb{I}_{N,r}| \neq 0$. One can write the matrix $C_{N,r}$ in the form

$$C_{N,r} = E^{(1)}_{N,r} E^{(2)}_{N,r} \cdot \cdots \cdot E^{(r-1)}_{N,r} \cdot E^{(r)}_{N,r} \tag{2.5}$$

for positive integers N, r (see [9, Proposition 3.3]).

2.4. **Linear maps.** For positive integers $N, r \ge 1$, let $\mathbf{V}_{N,r}$ denote the $|\mathbb{I}_{N,r}|$-dimensional vector space over $\mathbb{Q}$ of row vectors $(a_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{I}_{N,r}}$ indexed by totally odd indices $\boldsymbol{n} \in \mathbb{I}_{N,r}$ with rational coefficients:

$$\mathbf{V}_{N,r} = \{(a_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{I}_{N,r}} \mid a_{\boldsymbol{n}} \in \mathbb{Q}\}.$$

If $|\mathbb{I}_{N,r}| = 0$, we set $\mathbf{V}_{N,r} = \{0\}$. The matrix $C_{N,r}$ can be viewed as the linear map on $\mathbf{V}_{N,r}$ in the following manner (see also [9, §2.2]):

$$\begin{aligned} C_{N,r} : \mathbf{V}_{N,r} &\longrightarrow \mathbf{V}_{N,r} \\ v = (a_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{I}_{N,r}} &\longmapsto v \cdot C_{N,r} = \Big(\sum_{\boldsymbol{m}\in\mathbb{I}_{N,r}} a_{\boldsymbol{m}} c\binom{\boldsymbol{m}}{\boldsymbol{n}}\Big)_{\boldsymbol{n}\in\mathbb{I}_{N,r}}. \end{aligned}$$

For any subspace $\mathbf{W}$ of $\mathbf{V}_{N,r}$, we denote the image of $\mathbf{W}$ under the map $C_{N,r}$ by

$$\mathbf{W}C_{N,r} = \{v \cdot C_{N,r} \mid v \in \mathbf{W}\} \subset \mathbf{V}_{N,r}.$$

The $\mathbb{Q}$-vector subspace of $\mathbf{V}_{N,r}$ of left annihilators of the matrix $C_{N,r}$ is denoted by

$$\ker C_{N,r} = \{v \in \mathbf{V}_{N,r} \mid v \cdot C_{N,r} = 0\}.$$

We also apply the above convention to the matrices $E^{(q)}_{N,r}$, $q = 1, \dots, r$.

2.5. **Tensor product.** For convenience we view $\mathbf{V}_{N,r}\otimes_{\mathbb{Q}}\mathbf{V}_{M,s}$ as a subspace of $\mathbf{V}_{N+M,r+s}$ in the following manner. For two row vectors $(a_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{I}_{N,r}} \in \mathbf{V}_{N,r}$ and $(b_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{I}_{M,s}} \in \mathbf{V}_{M,s}$, the coefficient $c_{n_1,\dots,n_{r+s}}$ of the row vector

$$(c_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{I}_{N+M,r+s}} = (a_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{I}_{N,r}} \otimes (b_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{I}_{M,s}} \in \mathbf{V}_{N+M,r+s}$$

is defined by

$$c_{n_1,\dots,n_{r+s}} = \begin{cases} a_{n_1,\dots,n_r} b_{n_{r+1},\dots,n_{r+s}} \\ \qquad \text{if } (n_1,\dots,n_r)\in\mathbb{I}_{N,r} \text{ and } (n_{r+1},\dots,n_{r+s})\in\mathbb{I}_{M,s} \\ 0 \qquad \text{otherwise} \end{cases}$$

for each $(n_1,\dots,n_{r+s})\in\mathbb{I}_{N+M,r+s}$. Note that for $\boldsymbol{n}\in\mathbb{I}_{N+M,r+s}$ the above $c_{\boldsymbol{n}}$ can be obtained from the coefficient of $\boldsymbol{x}^{\boldsymbol{n}-1}$ in $f(x_1,\dots,x_r)g(x_{r+1},\dots,x_{r+s})$, where we write $f(x_1,\dots,x_r) = \sum_{\boldsymbol{n}\in\mathbb{I}_{N,r}} a_{\boldsymbol{n}}\boldsymbol{x}^{\boldsymbol{n}-1}$ and $g(x_1,\dots,x_s) = \sum_{\boldsymbol{n}\in\mathbb{I}_{M,s}} b_{\boldsymbol{n}}\boldsymbol{x}^{\boldsymbol{n}-1}$. With this notation, we let

$$\mathbf{V}_{N,r}\otimes\mathbf{V}_{M,s} = \{v\otimes w \mid v\in\mathbf{V}_{N,r}, w\in\mathbf{V}_{M,s}\},$$

which is a $\mathbb{Q}$-subvector space of $\mathbf{V}_{N+M,r+s}$. We remark that since $\mathbf{V}_{N,r} = \bigoplus_{N_1+N_2=N}\left(\mathbf{V}_{N_1,r-q}\otimes\mathbf{V}_{N_2,q}\right)$ holds for any $0<q<r$, by definition (2.3) we have

$$\mathbf{V}_{N,r}E_{N,r}^{(q)} = \bigoplus_{N_1+N_2=N}\left(\mathbf{V}_{N_1,r-q}\otimes\mathbf{V}_{N_2,q}E_{N_2,q}^{(q)}\right). \tag{2.6}$$

2.6. **Restricted even period polynomials.** Let

$$\mathbf{W}_{N,2} = \ker E_{N,2}^{(2)} = \{v\in\mathbf{V}_{N,2} \mid v\cdot E_{N,2}^{(2)} = 0\}.$$

It was shown by [1] (see also [9, Proposition 3.4]) that the row vector $(a_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{I}_{N,2}} \in \mathbf{V}_{N,2}$ is an element in $\mathbf{W}_{N,2}$ if and only if the polynomial $f(x_1,x_2) = \sum_{\boldsymbol{m}\in\mathbb{I}_{N,2}} a_{\boldsymbol{m}}\boldsymbol{x}^{\boldsymbol{m}-1}$ satisfies

$$0 = f(x_1,x_2) - f(x_2-x_1,x_2) + f(x_2-x_1,x_1) = f(x_1,x_2)\big|(1+\sigma_2^{(1)}).$$

Thus, the space $\mathbf{W}_{N,2}$ is isomorphic to the $\mathbb{Q}$-vector space of restricted even period polynomials of degree $N-2$ (see [6, §5]). We therefore find that

$$\sum_{N>0}\dim\mathbf{W}_{N,2}x^N = \mathbb{S}(x). \tag{2.7}$$

For positive integers N,r,p with $2\le p\le r-2$, we consider the subspace

$$\mathbf{W}_{N,r}^{(p)} = \bigoplus_{N_1+N_2+N_3=N}\left(\mathbf{V}_{N_1,p-1}\otimes\mathbf{W}_{N_2,2}\otimes\mathbf{V}_{N_3,r-p-1}\right)$$

of $\mathbf{V}_{N,r} = \bigoplus_{N_1+N_2+N_3=N} \left(\mathbf{V}_{N_1,p-1} \otimes \mathbf{V}_{N_2,2} \otimes \mathbf{V}_{N_3,r-p-1}\right)$. We also consider

$$\mathbf{W}_{N,r}^{(1)} = \bigoplus_{N_1+N_2=N} (\mathbf{W}_{N_1,2} \otimes \mathbf{V}_{N_2,r-2}), \ \mathbf{W}_{N,r}^{(r-1)} = \bigoplus_{N_1+N_2=N} (\mathbf{V}_{N_1,r-2} \otimes \mathbf{W}_{N_2,2}).$$

For any q satisfying $1 \le q \le r-p-1$, it follows that

$$\mathbf{W}_{N,r}^{(p)} E_{N,r}^{(q)} = \bigoplus_{N_1+N_2+N_3=N} \left(\mathbf{V}_{N_1,p-1} \otimes \mathbf{W}_{N_2,2} \otimes \mathbf{V}_{N_3,r-p-1} E_{N_3,r-p-1}^{(q)}\right).$$

Hence, by (2.6) we have

$$\mathbf{W}_{N,r}^{(p)} E_{N,r}^{(q)} \subset \mathbf{W}_{N,r}^{(p)} \quad (1 \le q \le r-p-1). \tag{2.8}$$

3. Proof of Theorem 1.2

Since $\operatorname{rank} C_{N,r} = |\mathbb{I}_{N,r}| - \dim\ker C_{N,r}$, we give a lower bound of $\dim\ker C_{N,r}$ in order to prove Theorem 1.2. Note that since $\sum_{N>0} |\mathbb{I}_{N,r}| x^N = \mathbb{O}(x)^r$, it suffices to show the inequality

$$\sum_{N,r\ge 2} \dim\ker C_{N,r} x^N y^r \ge \frac{\mathbb{S}(x)y^2}{(1-\mathbb{O}(x)y)(1-\mathbb{O}(x)y+\mathbb{S}(x)y^2)}. \tag{3.1}$$

We begin with the following proposition.

Proposition 3.1. *For any positive integers $N, r \ge 2$, we have*

$$\sum_{p=1}^{r-1} \mathbf{W}_{N,r}^{(p)} \subset \ker C_{N,r}.$$

Proof. For $1 \le p \le r-1$, by (2.5) and (2.8) we have

$$\mathbf{W}_{N,r}^{(p)} C_{N,r} \subset \mathbf{W}_{N,r}^{(p)} E_{N,r}^{(r-p)} E_{N,r}^{(r-p+1)} \cdots E_{N,r}^{(r-1)} E_{N,r}^{(r)}.$$

We now prove the inclusion

$$\mathbf{W}_{N,r}^{(p)} E_{N,r}^{(r-p)} \subset \ker E_{N,r}^{(r-p+1)}, \tag{3.2}$$

from which Proposition 3.1 follows. Since $\mathbf{W}_{N,r}^{(1)} = \bigoplus_{N_1+N_2=N} (\mathbf{W}_{N_1,2} \otimes \mathbf{V}_{N_2,r-2})$, for $2 \le p \le r-1$ one can write $\mathbf{W}_{N,r}^{(p)}$ in the form

$$\mathbf{W}_{N,r}^{(p)} = \bigoplus_{N_1+N_2=N} \left(\mathbf{V}_{N_1,p-1} \otimes \mathbf{W}_{N_2,r-p+1}^{(1)}\right). \tag{3.3}$$

Applying the matrix $E_{N,r}^{(r-p)}$ to the space (3.3) we have

$$\mathbf{W}_{N,r}^{(p)} E_{N,r}^{(r-p)} = \bigoplus_{N_1+N_2=N} \left(\mathbf{V}_{N_1,p-1} \otimes \mathbf{W}_{N_2,r-p+1}^{(1)} E_{N_2,r-p+1}^{(r-p)} \right).$$

Hence, what is left is to show that the inclusion $\mathbf{W}_{N,q}^{(1)} E_{N,q}^{(q-1)} \subset \ker E_{N,q}^{(q)}$ holds for all $N, q \geq 2$. The case $q = 2$ is immediate from the definition. We assume $q = r \geq 3$. Let $(a_{\boldsymbol{n}})_{\boldsymbol{n} \in \mathbb{I}_{N,r}}$ be an element in $\mathbf{W}_{N,r}^{(1)}$. We write $(b_{\boldsymbol{n}})_{\boldsymbol{n} \in \mathbb{I}_{N,r}} = (a_{\boldsymbol{n}})_{\boldsymbol{n} \in \mathbb{I}_{N,r}} E_{N,r}^{(r-1)} E_{N,r}^{(r)}$ and set $p(x_1, \ldots, x_r) = \sum_{\boldsymbol{m} \in \mathbb{I}_{N,r}} a_{\boldsymbol{m}} \boldsymbol{x}^{\boldsymbol{m}-1}$. Using (2.2), one can easily check that for any $\boldsymbol{n} \in \mathbb{I}_{N,r}$ we have $b_{\boldsymbol{n}}$ =coefficient of $\boldsymbol{x}^{\boldsymbol{n}-1}$ in

$$\Big(\sum_{(m_1,\ldots,m_r) \in \mathbb{I}_{N,r}} a_{m_1,\ldots,m_r} x_1^{m_1-1} h_{m_2,\ldots,m_r}(x_2, \ldots, x_r) \Big) \big| (1 + \sigma_r),$$

where we write $h_{m_2,\ldots,m_r}(x_1, \ldots, x_{r-1}) = x_1^{m_2-1} \,\underline{\circ}\, (x_1^{m_3-1} \cdots x_{r-2}^{m_r-1})$. Since

$$p(x_1, \ldots, x_r) \big| (1 + \sigma_r^{(1)}) = 0, \tag{3.4}$$

we have

$$p(x_1, \ldots, x_r) = -p(x_2, x_1, x_3, \ldots, x_r). \tag{3.5}$$

By (2.2) one can compute

$$\begin{aligned}
&\sum_{(m_1,\ldots,m_r) \in \mathbb{I}_{N,r}} a_{m_1,\ldots,m_r} x_1^{m_1-1} h_{m_2,\ldots,m_r}(x_2, \ldots, x_r) \\
&= p(x_1, \ldots, x_r) + p(x_1, x_3 - x_2, x_2, x_4, \ldots, x_r) - p(x_1, x_3 - x_2, x_3, x_4, \ldots, x_r) \\
&\quad + p(x_1, x_4 - x_3, x_2, x_3, x_5, \ldots, x_r) - p(x_1, x_4 - x_3, x_2, x_4, x_5, \ldots, x_r) \\
&\quad + \cdots \\
&\quad + p(x_1, x_r - x_{r-1}, x_2, \ldots, x_{r-2}, x_{r-1}) - p(x_1, x_r - x_{r-1}, x_2, \ldots, x_{r-2}, x_r) \\
&= p(x_1, \ldots, x_r) \big| (1 - \sigma_r^{(2)} - \sigma_r^{(3)} - \cdots - \sigma_r^{(r-1)}) = -p(x_1, \ldots, x_r) \big| \sigma_r,
\end{aligned}$$

where for the second (resp. the last) equality we have used (3.5) (resp. (3.4)). Then, the statement that $b_{\boldsymbol{n}} = 0$ for all $\boldsymbol{n} \in \mathbb{I}_{N,r}$ follows from the equation (see [9, Eq. (3.31)])

$$p(x_1, \ldots, x_r) \big| \sigma_r \big| (1 + \sigma_r) = 0,$$

which completes the proof of (3.2). □

In what follows, we compute the dimension of the space $\sum_{1\le p\le r-1}\mathbf{W}_{N,r}^{(p)}$ to give a lower bound of $\dim\ker C_{N,r}$. By linear algebra, we see that

$$\dim\left(\sum_{p=1}^{r-1}\mathbf{W}_{N,r}^{(p)}\right)=\sum_{l=1}^{r-1}(-1)^{l+1}\sum_{0<t_1<\cdots<t_l<r}\dim\left(\bigcap_{j=1}^{l}\mathbf{W}_{N,r}^{(t_j)}\right). \tag{3.6}$$

Lemma 3.2. *For any integers $0<t_1<\cdots<t_l<r$ we have*

$$\sum_{N>0}\dim\left(\bigcap_{j=1}^{l}\mathbf{W}_{N,r}^{(t_j)}\right)x^N=\begin{cases}\mathbb{S}(x)^l\mathbb{O}(x)^{r-2l} & \text{if } t_j-t_{j-1}\ge 2\ (2\le j\le l),\\ 0 & \text{otherwise.}\end{cases}$$

Proof. By [5, Proposition 6.4], we have $\mathbf{W}_{N,3}^{(1)}\cap\mathbf{W}_{N,3}^{(2)}=\{0\}$, and hence, the intersection $\bigcap_{j=1}^{l}\mathbf{W}_{N,r}^{(t_j)}$ is trivial if there exists $j\in\{2,\dots,l\}$ such that $t_j-t_{j-1}=1$. Only consider the cases when $t_j-t_{j-1}\ge 2$ for all $2\le j\le l$. In these cases, the intersection $\bigcap_{j=1}^{l}\mathbf{W}_{N,r}^{(t_j)}$ is a subspace of

$$\mathbf{V}_{N,r}=\bigoplus_{N_1+\cdots+N_{l+1}=N}\left(\mathbf{V}_{N_1,t_1-1}\otimes\left(\bigotimes_{j=2}^{l}\mathbf{V}_{N_j,t_j-t_{j-1}}\right)\otimes\mathbf{V}_{N_{l+1},r-t_l+1}\right).$$

By (3.3), we have

$$\bigcap_{j=1}^{l}\mathbf{W}_{N,r}^{(t_j)}=\bigoplus_{\substack{N_1+\cdots+N_{l+1}\\=N}}\left(\mathbf{V}_{N_1,t_1-1}\otimes\left(\bigotimes_{j=2}^{l}\mathbf{W}_{N_j,t_j-t_{j-1}}^{(1)}\right)\otimes\mathbf{W}_{N_{l+1},r-t_l+1}^{(1)}\right).$$

Then the formula is immediate from (2.7). ☐

We are now in a position to prove (3.1).

Proof of Theorem 1.2. It suffices to show

$$\sum_{N,r\ge 2}\dim\left(\sum_{p=1}^{r-1}\mathbf{W}_{N,r}^{(p)}\right)x^Ny^r=\frac{\mathbb{S}(x)y^2}{(1-\mathbb{O}(x)y)(1-\mathbb{O}(x)y+\mathbb{S}(x)y^2)},$$

from which by Proposition 3.1 the inequality (3.1) follows. We write $f(x,y)$ for the left-hand side. Using (3.6) and Lemma 3.2, one computes

$$f(x,y)=\sum_{N>0}\sum_{r\ge 2}\left(\sum_{l=1}^{r-1}(-1)^{l+1}\sum_{\substack{0<t_1<\cdots<t_l<r\\ t_j-t_{j-1}\ge 2}}\dim\left(\bigcap_{j=1}^{l}\mathbf{W}_{N,r}^{(t_j)}\right)\right)x^Ny^r$$

$$= \sum_{r\geq 2}\left(\sum_{l=1}^{r/2}(-1)^{l+1}\mathbb{S}(x)^l\mathbb{O}(x)^{r-2l}\sum_{\substack{0<t_1<\cdots<t_l<r\\ t_j-t_{j-1}\geq 2}}1\right)y^r,$$

where for the last equality we note that $\dim\left(\bigcap_{j=1}^{l}\mathbf{W}_{N,r}^{(t_j)}\right)=0$ if $l>r/2$ (recall [5, Proposition 6.4]). For simplicity of notation, we let $X=\mathbb{O}(x)y, Y=\mathbb{S}(x)y^2$. Then we have

$$\begin{aligned}
f(x,y) &= \sum_{r\geq 2}\left(\sum_{l=1}^{r/2}(-1)^{l+1}Y^lX^{r-2l}\sum_{\substack{0<t_1<\cdots<t_l<r\\ t_j-t_{j-1}\geq 2}}1\right)\\
&= -\sum_{l\geq 1}(-Y)^l\sum_{r\geq l}X^{2r-2l}\sum_{\substack{0<t_1<\cdots<t_l<2r\\ t_j-t_{j-1}\geq 2}}1\\
&\quad -\sum_{l\geq 1}(-Y)^l\sum_{r\geq l}X^{2r+1-2l}\sum_{\substack{0<t_1<\cdots<t_l<2r+1\\ t_j-t_{j-1}\geq 2}}1\\
&= -\sum_{l\geq 1}(-Y)^l\sum_{r\geq 0}X^{r}\sum_{\substack{0<t_1<\cdots<t_l<r+2l\\ t_j-t_{j-1}\geq 2}}1.
\end{aligned}$$

Using $1/(1-X)=\sum_{t\geq 0}X^t$, one can check the identity

$$\sum_{r\geq 0}X^r\sum_{\substack{0<t_1<\cdots<t_l<r+2l\\ t_j-t_{j-1}\geq 2}}1=\frac{1}{(1-X)^{l+1}}$$

by induction on l. Hence,

$$\begin{aligned}
f(x,y) &= -\sum_{l\geq 1}\frac{(-Y)^l}{(1-X)^{l+1}} = -\frac{1}{1-X}\sum_{l\geq 1}\left(\frac{-Y}{1-X}\right)^l\\
&= -\frac{1}{1-X}\frac{\frac{-Y}{1-X}}{1-\frac{-Y}{1-X}} = \frac{Y}{(1-X)(1-X+Y)},
\end{aligned}$$

which completes the proof. □

Acknowledgements

The author would like to express his sincere gratitude to Ding Ma for sharing the idea of the proof of Proposition 3.1 and for encouragement to improve the author's previous work [9]. The author is grateful to Professor Francis Brown and Professor Benjamin Enriquez for answering his questions on [3] and [5, Proposition 6.4]. This work was partially supported by JSPS KAKENHI Grant Number 16H07115. The author thanks Max Planck Institute for Mathematics, where the paper was written, for the invitation and hospitality. Many thanks also go to the referee for valuable comments.

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(*Received March 2, 2020*)
(*Accepted February 18, 2021*)