Math. J. Okayama Univ. **64** (2022), 31–45

THE BEST CONSTANT OF THE DISCRETE SOBOLEV INEQUALITIES ON THE COMPLETE BIPARTITE GRAPH

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Abstract. We have the best constants of three kinds of discrete Sobolev inequalities on the complete bipartite graph with $2N$ vertices, that is, $K_{N,N}$. We introduce a discrete Laplacian $\boldsymbol{A}$ on $K_{N,N}$. $\boldsymbol{A}$ is a $2N \times 2N$ real symmetric positive-semidefinite matrix whose eigenvector corresponding to zero eigenvalue is $\mathbf{1} = {}^t(1, 1, \cdots, 1) \in \mathbf{C}^{2N}$. A discrete heat kernel, a Green's matrix and a pseudo Green's matrix play important roles in giving the best constants.

1. Discrete Laplacian

For any fixed $N = 1, 2, 3, \cdots$, we set the indices of vertices on the complete bipartite graph $K_{N,N}$ as Figure 1.

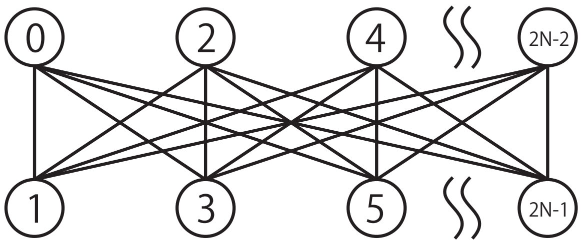

Figure 1. Complete bipartite graph $K_{N,N}$.

We introduce the edge set

$$e = \Big\{ (2i, 2j+1) \,|\, 0 \le i, j \le N-1 \Big\},$$

where the vertices $2i$ and $2j+1$ are connected to an edge. The discrete Laplacian $\boldsymbol{A}$ is defined as

$$\boldsymbol{A} = \Bigg(\; a(i,j) \; \Bigg)_{0 \le i,j \le 2N-1}, \qquad a(i,j) = \begin{cases} N & (i = j) \\ -1 & ((i,j) \in e) \\ 0 & (\text{otherwise}) \end{cases}.$$

Mathematics Subject Classification. Primary 46E39; Secondary 35K08.

Key words and phrases. Discrete Sobolev inequality, Discrete Laplacian, Green's matrix, Reproducing relation.

$\boldsymbol{A}$ is rewritten as

$$(1.1)\qquad \boldsymbol{A} = \left(N\delta(i-j) - \sum_{k=0}^{N-1} \delta(i-j+2k+1) \right)_{0\le i,j\le 2N-1},$$

where the delta function

$$(1.2)\qquad \delta(i) = \begin{cases} 1 & (\mathrm{Mod}(i,2N)=0) \\ 0 & (\mathrm{Mod}(i,2N)\neq 0) \end{cases} \qquad (i\in\mathbf{Z}).$$

Here, we show the concrete form of $\boldsymbol{A} = \boldsymbol{A}_N$ as

$$\boldsymbol{A}_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad \boldsymbol{A}_2 = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix},$$

$$\boldsymbol{A}_3 = \begin{pmatrix} 3 & -1 & 0 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ -1 & 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & 0 & -1 & 3 \end{pmatrix}.$$

$\boldsymbol{A}$ is a $2N\times 2N$ real symmetric positive-semidefinite matrix which has an eigenvalue 0 and whose eigenvector is $\mathbf{1} = {}^t(1,1,\cdots,1)\in\mathbf{C}^{2N}$. We introduce the following three matrices

$$(1.3)\qquad \text{Discrete heat kernel}: \boldsymbol{H}(t) = \exp(-t\boldsymbol{A}),$$

$$(1.4)\qquad \text{Green's matrix}: \boldsymbol{G}(a) = (\boldsymbol{A}+a\boldsymbol{I})^{-1} = \int_0^\infty e^{-at}\boldsymbol{H}(t)dt,$$

$$(1.5)\qquad \text{Pseudo Green's matrix}: \boldsymbol{G}_* = \lim_{a\to+0}\left(\boldsymbol{G}(a) - a^{-1}\boldsymbol{E}_0\right),$$

where a is a positive number and

$$\boldsymbol{E}_0 = (2N)^{-1}\mathbf{1}{}^t\mathbf{1} = \frac{1}{2N}\left(\quad 1 \quad \right)_{0\le i,j\le 2N-1}$$

is a projection matrix to the eigenspace corresponding to the eigenvalue 0 of $\boldsymbol{A}$. $\boldsymbol{G}_*$ satisfies

$$\boldsymbol{A}\boldsymbol{G}_* = \boldsymbol{G}_*\boldsymbol{A} = \boldsymbol{I} - \boldsymbol{E}_0, \qquad \boldsymbol{G}_*\boldsymbol{E}_0 = \boldsymbol{E}_0\boldsymbol{G}_* = \boldsymbol{O}.$$

Here, $\boldsymbol{I}$ is the $2N\times 2N$ identity matrix and $\boldsymbol{O}$ is the $2N\times 2N$ zero matrix. Thus, $\boldsymbol{G}(a)$ is an inverse matrix of $\boldsymbol{A}+a\boldsymbol{I}$ and $\boldsymbol{G}_*$ is a Penrose-Moore generalized inverse matrix of $\boldsymbol{A}$.

This paper is composed of five sections. In section 2, we show Theorem 2.1$\sim$2.3 corresponding to $\boldsymbol{H}(t)$, $\boldsymbol{G}(a)$ and $\boldsymbol{G}_*$. In section 3, we prepare some basic matrices and explain the difference equations. In section 4, we present a reproducing relation. Section 5 is devoted to the proof of Theorem 2.1∼2.3.

2. Discrete Sobolev inequality

In this section, we state the best constants of three kinds of discrete Sobolev inequalities on $K_{N,N}$. For $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{C}^{2N}$, we introduce sesquilinear forms

$$(\boldsymbol{u},\, \boldsymbol{v}) = \boldsymbol{v}^* \boldsymbol{u}, \qquad \|\, \boldsymbol{u} \,\|^2 = (\boldsymbol{u}, \boldsymbol{u}),$$
$$(\boldsymbol{u},\, \boldsymbol{v})_H = ((\boldsymbol{A} + a\boldsymbol{I})\boldsymbol{u},\, \boldsymbol{v}) = \boldsymbol{v}^*(\boldsymbol{A} + a\boldsymbol{I})\boldsymbol{u}, \qquad \|\, \boldsymbol{u} \,\|_H^2 = (\boldsymbol{u}, \boldsymbol{u})_H,$$

where $\boldsymbol{u}^*$ denotes $\boldsymbol{u}^* = {}^t\overline{\boldsymbol{u}}$. For $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{C}_0^{2N} := \{\, \boldsymbol{u} \,|\, \boldsymbol{u} \in \mathbf{C}^{2N} \text{ and } {}^t\mathbf{1}\boldsymbol{u} = 0 \,\}$, we introduce a sesquilinear form

$$(\boldsymbol{u},\, \boldsymbol{v})_A = (\boldsymbol{A}\boldsymbol{u},\, \boldsymbol{v}) = \boldsymbol{v}^* \boldsymbol{A}\boldsymbol{u}, \qquad \|\, \boldsymbol{u} \,\|_A^2 = (\boldsymbol{u}, \boldsymbol{u})_A.$$

$(\cdot, \cdot)_H$ and $(\cdot, \cdot)_A$ are proved to be an inner product afterwards. We rewrite $\|\, \boldsymbol{u} \,\|_H^2$ and $\|\, \boldsymbol{u} \,\|_A^2$ as

$$\|\, \boldsymbol{u} \,\|_H^2 = \|\, \boldsymbol{u} \,\|_A^2 + a \sum_{i=0}^{2N-1} |\, u(i) \,|^2, \qquad \|\, \boldsymbol{u} \,\|_A^2 = \sum_{(i,j)\in e} |\, u(i) - u(j) \,|^2.$$

The concrete forms of $\|\, \boldsymbol{u} \,\|_A^2 = \|\, \boldsymbol{u} \,\|_{AN}^2$ are as

$$\|\, \boldsymbol{u} \,\|_{A1}^2 = |\, u(0) - u(1) \,|^2,$$

$$\|\, \boldsymbol{u} \,\|_{A2}^2 =$$
$$|\, u(0) - u(1) \,|^2 + |\, u(0) - u(3) \,|^2 + |\, u(2) - u(1) \,|^2 + |\, u(2) - u(3) \,|^2,$$

$$\|\, \boldsymbol{u} \,\|_{A3}^2 =$$
$$|\, u(0) - u(1) \,|^2 + |\, u(0) - u(3) \,|^2 + |\, u(0) - u(5) \,|^2 +$$
$$|\, u(2) - u(1) \,|^2 + |\, u(2) - u(3) \,|^2 + |\, u(2) - u(5) \,|^2 +$$
$$|\, u(4) - u(1) \,|^2 + |\, u(4) - u(3) \,|^2 + |\, u(4) - u(5) \,|^2.$$

To describe theorems, for any j $(0 \le j \le 2N-1)$ fixed, we use the $2N$-dimensional vector

$$\boldsymbol{\delta}_j = {}^t(\cdots, \delta(i-j), \cdots)_{0 \le i \le 2N-1},$$

where $\delta(i)$ is defined in (1.2).

Theorem 2.1. *For any* $\boldsymbol{u} = {}^t(u(0), u(1), \cdots, u(2N-1)) \in \mathbf{C}_0^{2N}$*, there exists a positive constant* C *which is independent of* $\boldsymbol{u}$*, such that the discrete Sobolev inequality*

$$\left(\max_{0\le j\le 2N-1} |\,u(j)\,|\right)^2 \le C \,\|\,\boldsymbol{u}\,\|_A^2 \tag{2.1}$$

holds. Among such C*, for any* j_0 $(0 \le j_0 \le 2N-1)$*, the best constant is*

$$C_0 = \max_{0\le j\le 2N-1} {}^t\boldsymbol{\delta}_j \boldsymbol{G}_* \boldsymbol{\delta}_j = {}^t\boldsymbol{\delta}_{j_0} \boldsymbol{G}_* \boldsymbol{\delta}_{j_0} = \frac{4N-3}{4N^2}.$$

If we replace C *by* C_0 *in* (2.1)*, the equality holds if and only if* $\boldsymbol{u}$ *is parallel to*

$$\boldsymbol{G}_*\boldsymbol{\delta}_{j_0} = \frac{1}{4N^2}\left(4N\delta(i-j_0) - 2 - (-1)^{i-j_0}\right)_{0\le i\le 2N-1}.$$

Theorem 2.2. *For any* $\boldsymbol{u} = {}^t(u(0), u(1), \cdots, u(2N-1)) \in \mathbf{C}^{2N}$*, there exists a positive constant* C *which is independent of* $\boldsymbol{u}$*, such that the discrete Sobolev inequality*

$$\left(\max_{0\le j\le 2N-1} |\,u(j)\,|\right)^2 \le C \,\|\,\boldsymbol{u}\,\|_H^2 \tag{2.2}$$

holds. Among such C*, for any* j_0 $(0 \le j_0 \le 2N-1)$*, the best constant is*

$$C_0(a) = \max_{0\le j\le 2N-1} {}^t\boldsymbol{\delta}_j \boldsymbol{G}(a) \boldsymbol{\delta}_j = {}^t\boldsymbol{\delta}_{j_0} \boldsymbol{G}(a) \boldsymbol{\delta}_{j_0} = \frac{N+2Na+a^2}{a(N+a)(2N+a)}.$$

If we replace C *by* $C_0(a)$ *in* (2.2)*, the equality holds if and only if* $\boldsymbol{u}$ *is parallel to*

$$\boldsymbol{G}(a)\boldsymbol{\delta}_{j_0} = \frac{1}{N+a}\left(\delta(i-j_0) + \frac{1}{2a} - \frac{(-1)^{i-j_0}}{2(2N+a)}\right)_{0\le i\le 2N-1}.$$

Theorem 2.3. *For any* $\boldsymbol{u}(t) = {}^t(u(0,t), u(1,t), \cdots, u(2N-1,t)) \in \mathbf{C}^{2N}$*, there exists a positive constant* C *which is independent of* $\boldsymbol{u}(t)$*, such that the discrete Sobolev-type inequality*

$$\left(\sup_{\substack{0\le j\le 2N-1\\ -\infty<s<\infty}} |\,u(j,s)\,|\right)^2 \le C \int_{-\infty}^{\infty} \left\| \left(\frac{d}{dt} + \boldsymbol{A} + a\boldsymbol{I}\right)\boldsymbol{u}(t)\right\|^2 dt \tag{2.3}$$

holds. Among such C*, the best constant is*

$$C_1(a) = \frac{1}{2}C_0(a),$$

where $C_0(a)$ is given in Theorem 2.2. If we replace C by $C_1(a)$ in (2.3), for any j_0 $(0 \le j_0 \le 2N-1)$, the equality holds if and only if $\boldsymbol{u}(t)$ is parallel to

$$\int_{|t|}^{\infty} \frac{1}{2} e^{-a\sigma} \boldsymbol{H}(\sigma) \boldsymbol{\delta}_{j_0} d\sigma = \frac{1}{2(N+a)} \Bigg(e^{-(N+a)|t|} \delta(i-j_0) + \frac{N+a}{2Na} e^{-a|t|} - \frac{1}{2N} e^{-(N+a)|t|} - \frac{(-1)^{i-j_0}}{2N} e^{-(N+a)|t|} + \frac{(-1)^{i-j_0}(N+a)}{2N(2N+a)} e^{-(2N+a)|t|} \Bigg)_{0 \le i \le 2N-1} \tag{2.4}$$

$$(-\infty < t < \infty).$$

In our previous papers, we obtained the best constant of the discrete Sobolev inequalities (2.1) and (2.2) on graphs such as the complete graph K_N [10], N-sided polygons [4, 5, 9], regular polyhedra [2, 7, 8], and truncated regular polyhedra [1, 3, 6].

3. Difference equations

Let us put ω as $\omega = \exp(\sqrt{-1}\,\pi/N)$ which satisfies $\omega^{2N} = 1$ and put normalized orthogonal vectors as

$$\boldsymbol{\varphi}_k = \frac{1}{\sqrt{2N}} {}^t(\,\cdots, \omega^{ik},\, \cdots)_{0 \le i \le 2N-1} \qquad (0 \le k \le 2N-1),$$

which satisfies $\boldsymbol{\varphi}_l^* \boldsymbol{\varphi}_k = \delta(k-l)$. Hereafter, we introduce some $2N \times 2N$ matrices. $\boldsymbol{Q}$ is defined as

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{\varphi}_0 & \cdots & \boldsymbol{\varphi}_{2N-1} \end{pmatrix} = \frac{1}{\sqrt{2N}} \begin{pmatrix} \omega^{ij} \end{pmatrix}_{0 \le i,j \le 2N-1}.$$

$\boldsymbol{E}_k$ are orthogonal projection matrices defined as

$$\boldsymbol{E}_k = \boldsymbol{\varphi}_k \boldsymbol{\varphi}_k^* = \frac{1}{2N} \begin{pmatrix} \omega^{(i-j)k} \end{pmatrix}_{0 \le i,j \le 2N-1} \qquad (0 \le k \le 2N-1),$$

which satisfy $\boldsymbol{E}_k \boldsymbol{E}_l = \delta(k-l)\boldsymbol{E}_k$ and $\boldsymbol{E}_k^* = \boldsymbol{E}_k$. Using $\boldsymbol{E}_k$, we have the spectral decomposition of $\boldsymbol{I}$ as

$$\boldsymbol{I} = \boldsymbol{Q}\boldsymbol{Q}^* = \sum_{k=0}^{2N-1} \boldsymbol{\varphi}_k \boldsymbol{\varphi}_k^* = \sum_{k=0}^{2N-1} \boldsymbol{E}_k. \tag{3.1}$$

$\boldsymbol{L}$ is a rotate-left matrix defined as

$$\boldsymbol{L} = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix} = \Big(\ \delta(i-j+1) \ \Big)_{0 \le i,j \le 2N-1},$$

which satisfies $\boldsymbol{L}^* = {}^t\boldsymbol{L} = \boldsymbol{L}^{-1} = \boldsymbol{L}^{2N-1}$ and

$$\boldsymbol{L}^k = \Big(\ \delta(i-j+k) \ \Big)_{0 \le i,j \le 2N-1} \qquad (0 \le k \le 2N-1), \qquad \boldsymbol{L}^{2N} = \boldsymbol{I}.$$

Thus $\boldsymbol{L}$ is a unitary matrix. $\boldsymbol{L}$ has eigenvalues ω^k $(0 \le k \le 2N-1)$ corresponding to the normalized orthogonal eigenvectors $\boldsymbol{\varphi}_k$ $(0 \le k \le 2N-1)$. So $\boldsymbol{L}$ is diagonalized by the matrix $\boldsymbol{Q}$ as

$$\boldsymbol{L} = \boldsymbol{Q}\widehat{\boldsymbol{L}}\boldsymbol{Q}^*, \qquad \widehat{\boldsymbol{L}} = \Big(\ \omega^i \delta(i-j) \ \Big)_{0 \le i,j \le 2N-1}.$$

Using $\boldsymbol{E}_k$, we have the spectral decomposition of $\boldsymbol{L}$ as

$$\boldsymbol{L} = \boldsymbol{Q}\widehat{\boldsymbol{L}}\boldsymbol{Q}^* = \sum_{k=0}^{2N-1} \omega_k \boldsymbol{\varphi}_k \boldsymbol{\varphi}_k^* = \sum_{k=0}^{2N-1} \omega^k \boldsymbol{E}_k. \tag{3.2}$$

Using (3.1) and (3.2), we rewrite $\boldsymbol{A}$ given in (1.1) as

$$\boldsymbol{A} = N\boldsymbol{I} - \sum_{k=0}^{2N-1} \boldsymbol{L}^{2k+1} = N\boldsymbol{Q}\boldsymbol{Q}^* - \sum_{k=0}^{N-1} \boldsymbol{Q}\widehat{\boldsymbol{L}}^{2k+1}\boldsymbol{Q}^* = \boldsymbol{Q}\widehat{\boldsymbol{A}}\boldsymbol{Q}^*,$$

where

$$\widehat{\boldsymbol{A}} = N\boldsymbol{I} - \sum_{k=0}^{N-1} \widehat{\boldsymbol{L}}^{2k+1} = \Big(\ \lambda_i \delta(i-j) \ \Big)_{0 \le i,j \le 2N-1},$$

$$\lambda_i = N - \sum_{k=0}^{N-1} \omega^{i(2k+1)} = \begin{cases} 0 & (i=0) \\ N & (i \ne 0 \text{ and } i \ne N) \\ 2N & (i=N) \end{cases}.$$

Hence $\boldsymbol{A}$ has eigenvalues λ_k $(0 \le k \le 2N-1)$ corresponding to the normalized orthogonal eigenvectors $\boldsymbol{\varphi}_k$ $(0 \le k \le 2N-1)$. Then, the Jordan canonical form of $\boldsymbol{A}$ is given as

$$\boldsymbol{A} = \boldsymbol{Q}\widehat{\boldsymbol{A}}\boldsymbol{Q}^*, \qquad \widehat{\boldsymbol{A}} = \Big(\ \lambda_i \delta(i-j) \ \Big)_{0 \le i,j \le 2N-1}.$$

Using $\boldsymbol{E}_k$, we have the spectral decomposition of $\boldsymbol{A}$ as

$$\boldsymbol{A} = \boldsymbol{Q}\widehat{\boldsymbol{A}}\boldsymbol{Q}^* = \sum_{k=0}^{2N-1} \lambda_k \boldsymbol{\varphi}_k \boldsymbol{\varphi}_k^* = \sum_{k=0}^{2N-1} \lambda_k \boldsymbol{E}_k = N(\boldsymbol{I} - \boldsymbol{E}_0 + \boldsymbol{E}_N). \tag{3.3}$$

First, we explain three difference equations concerning the discrete heat kernel (1.3), the Green's matrix (1.4) and the pseudo Green's matrix (1.5).

Proposition 3.1. *For any* $\boldsymbol{f}(t) \in \mathbf{C}^{2N}$, *the discrete heat equation*

$$\left(\frac{d}{dt} + \boldsymbol{A} + a\boldsymbol{I}\right)\boldsymbol{u} = \boldsymbol{f}(t) \qquad (-\infty < t < \infty) \tag{3.4}$$

has the unique solution given as

$$\boldsymbol{u}(t) = \int_{-\infty}^{\infty} \boldsymbol{H}_*(t-s)\boldsymbol{f}(s)ds \qquad (-\infty < t < \infty), \tag{3.5}$$

$$\boldsymbol{H}_*(t) = Y(t)e^{-at}\boldsymbol{H}(t) \qquad (-\infty < t < \infty), \tag{3.6}$$

where $Y(t) = 1\ (0 \le t < \infty),\ 0\ (-\infty < t < 0)$ *is the Heaviside step function and* $\boldsymbol{H}(t)$ *is the discrete heat kernel expressed as*

$$\boldsymbol{H}(t) = \exp(-\boldsymbol{A}t) = e^{-Nt}\boldsymbol{I} + \left(1 - e^{-Nt}\right)\boldsymbol{E}_0 - \left(e^{-Nt} - e^{-2Nt}\right)\boldsymbol{E}_N = \left(e^{-Nt}\delta(i-j) + \frac{1}{2N}\left(1 - e^{-Nt}\right) - \frac{(-1)^{i-j}}{2N}\left(e^{-Nt} - e^{-2Nt}\right)\right)_{0 \le i,j \le 2N-1}. \tag{3.7}$$

The concrete forms of $\boldsymbol{H}(t) = \boldsymbol{H}_N(t)\ (N = 1,2,3)$ are as follows:

$$\boldsymbol{H}_1(t) = \begin{pmatrix} h_0 & h_1 \\ h_1 & h_0 \end{pmatrix}, \quad \begin{pmatrix} h_0 \\ h_1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 + e^{-2t} \\ 1 - e^{-2t} \end{pmatrix},$$

$$\boldsymbol{H}_2(t) = \begin{pmatrix} h_0 & h_1 & h_2 & h_1 \\ h_1 & h_0 & h_1 & h_2 \\ h_2 & h_1 & h_0 & h_1 \\ h_1 & h_2 & h_1 & h_0 \end{pmatrix}, \quad \begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} (1 + e^{-2t})^2 \\ 1 - e^{-4t} \\ (1 - e^{-2t})^2 \end{pmatrix},$$

$$\boldsymbol{H}_3(t) = \begin{pmatrix} h_0 & h_1 & h_2 & h_1 & h_2 & h_1 \\ h_1 & h_0 & h_1 & h_2 & h_1 & h_2 \\ h_2 & h_1 & h_0 & h_1 & h_2 & h_1 \\ h_1 & h_2 & h_1 & h_0 & h_1 & h_2 \\ h_2 & h_1 & h_2 & h_1 & h_0 & h_1 \\ h_1 & h_2 & h_1 & h_2 & h_1 & h_0 \end{pmatrix}, \quad \begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 + 4e^{-3t} + e^{-6t} \\ 1 - e^{-6t} \\ (1 - e^{-3t})^2 \end{pmatrix}.$$

Proof of Proposition 3.1 Using the Fourier transform

$$\boldsymbol{u}(t) \quad \overset{\widehat{\ }}{\longrightarrow} \quad \widehat{\boldsymbol{u}}(\omega) = \int_{-\infty}^{\infty} e^{-\sqrt{-1}\omega t}\boldsymbol{u}(t)\,dt,$$

we transform (3.4) into

$$\left(\sqrt{-1}\,\omega\boldsymbol{I} + \boldsymbol{A} + a\boldsymbol{I}\right)\widehat{\boldsymbol{u}}(\omega) = \widehat{\boldsymbol{f}}(\omega) \qquad (-\infty < \omega < \infty).$$

Solving this relation, we have $\widehat{\boldsymbol{u}}(\omega) = \widehat{\boldsymbol{H}}_*(\omega)\,\widehat{\boldsymbol{f}}(\omega)$, where

$$\widehat{\boldsymbol{H}}_*(\omega) = \left(\sqrt{-1}\,\omega\boldsymbol{I} + \boldsymbol{A} + a\boldsymbol{I}\right)^{-1} = \int_{-\infty}^{\infty} e^{-\sqrt{-1}\omega t} Y(t) e^{-at} \boldsymbol{H}(t)\,dt.$$

Using the inverse Fourier transform, we have (3.5) and (3.6). From

$$\boldsymbol{H}(t) = \exp(-\boldsymbol{A}t) = \boldsymbol{Q}\exp(-\widehat{\boldsymbol{A}}t)\boldsymbol{Q}^* = \sum_{k=0}^{2N-1} e^{-\lambda_k t}\boldsymbol{E}_k =$$
$$\boldsymbol{E}_0 + e^{-Nt}\left(\boldsymbol{E}_1 + \cdots + \boldsymbol{E}_{N-1} + \boldsymbol{E}_{N+1} + \cdots + \boldsymbol{E}_{2N-1}\right) + e^{-2Nt}\boldsymbol{E}_N =$$
$$\boldsymbol{E}_0 + e^{-Nt}\left(\boldsymbol{I} - \boldsymbol{E}_0 - \boldsymbol{E}_N\right) + e^{-2Nt}\boldsymbol{E}_N,$$

we have (3.7). It should be noted that $\boldsymbol{H}_*(t)$ satisfies

$$\left(\frac{d}{dt} + \boldsymbol{A} + a\boldsymbol{I}\right)\boldsymbol{H}_*(t) = \boldsymbol{O},$$
$$\boldsymbol{H}_*(t-s)\Big|_{s=t-0} - \boldsymbol{H}_*(t-s)\Big|_{s=t+0} = \boldsymbol{I} \qquad (-\infty < t < \infty).$$

This completes the proof of Proposition 3.1. ■

Proposition 3.2. *For any $\boldsymbol{f} \in \mathbf{C}^{2N}$, the difference equation $(\boldsymbol{A}+a\boldsymbol{I})\boldsymbol{u} = \boldsymbol{f}$ has the unique solution given as $\boldsymbol{u} = \boldsymbol{G}(a)\boldsymbol{f}$, where $\boldsymbol{G}(a)$ is the Green's matrix expressed as*

$$\text{(3.8)} \quad \boldsymbol{G}(a) = (\boldsymbol{A} + a\boldsymbol{I})^{-1} = \frac{1}{N+a}\left(\boldsymbol{I} + \frac{N}{a}\boldsymbol{E}_0 - \frac{N}{2N+a}\boldsymbol{E}_N\right) =$$
$$\frac{1}{N+a}\left(\delta(i-j) + \frac{1}{2a} - \frac{(-1)^{i-j}}{2(2N+a)}\right)_{0\le i,j\le 2N-1}.$$

The concrete forms of $\boldsymbol{G}(a) = \boldsymbol{G}_N(a)$ $(N = 1, 2, 3)$ are as follows:

$$\boldsymbol{G}_1(a) = \begin{pmatrix} g_0 & g_1 \\ g_1 & g_0 \end{pmatrix},$$
$$\begin{pmatrix} g_0 \\ g_1 \end{pmatrix} = \frac{1}{a(a+2)}\begin{pmatrix} a+1 \\ 1 \end{pmatrix},$$
$$\boldsymbol{G}_2(a) = \begin{pmatrix} g_0 & g_1 & g_2 & g_1 \\ g_1 & g_0 & g_1 & g_2 \\ g_2 & g_1 & g_0 & g_1 \\ g_1 & g_2 & g_1 & g_0 \end{pmatrix},$$
$$\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix} = \frac{1}{a(a+2)(a+4)}\begin{pmatrix} a^2+4a+2 \\ a+2 \\ 2 \end{pmatrix},$$

$$\boldsymbol{G}_3(a) = \begin{pmatrix} g_0 & g_1 & g_2 & g_1 & g_2 & g_1 \\ g_1 & g_0 & g_1 & g_2 & g_1 & g_2 \\ g_2 & g_1 & g_0 & g_1 & g_2 & g_1 \\ g_1 & g_2 & g_1 & g_0 & g_1 & g_2 \\ g_2 & g_1 & g_2 & g_1 & g_0 & g_1 \\ g_1 & g_2 & g_1 & g_2 & g_1 & g_0 \end{pmatrix},$$

$$\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix} = \frac{1}{a(a+3)(a+6)} \begin{pmatrix} a^2+6a+3 \\ a+3 \\ 3 \end{pmatrix}.$$

Proof of Proposition 3.2 Using (3.1) and (3.3), we have

$$\sum_{k=0}^{2N-1} \boldsymbol{E}_k \boldsymbol{f} = \boldsymbol{I}\boldsymbol{f} = \boldsymbol{f} = (\boldsymbol{A} + a\boldsymbol{I})\boldsymbol{u} = \sum_{k=0}^{2N-1} (\lambda_k + a)\boldsymbol{E}_k \boldsymbol{u}.$$

Multiplying $\boldsymbol{E}_l$ on both sides of the above relation from the left and using the relation $\boldsymbol{E}_k\boldsymbol{E}_l = \delta(k-l)\boldsymbol{E}_k$, we obtain $\boldsymbol{E}_l\boldsymbol{u} = (\lambda_l + a)^{-1}\boldsymbol{E}_l\boldsymbol{f}$. Then, we see that

$$\boldsymbol{u} = \boldsymbol{I}\boldsymbol{u} = \sum_{l=0}^{2N-1} \boldsymbol{E}_l\boldsymbol{u} = \sum_{l=0}^{2N-1} \frac{1}{\lambda_l + a}\boldsymbol{E}_l\boldsymbol{f} = \boldsymbol{G}(a)\boldsymbol{f},$$

$$\begin{aligned} \boldsymbol{G}(a) &= \sum_{l=0}^{2N-1} \frac{1}{\lambda_l + a}\boldsymbol{E}_l = \\ &\frac{1}{a}\boldsymbol{E}_0 + \frac{1}{2N+a}\boldsymbol{E}_N + \\ &\frac{1}{N+a}\left(\boldsymbol{E}_1 + \cdots + \boldsymbol{E}_{N-1} + \boldsymbol{E}_{N+1} + \cdots + \boldsymbol{E}_{2N-1}\right) = \\ &\frac{1}{a}\boldsymbol{E}_0 + \frac{1}{2N+a}\boldsymbol{E}_N + \frac{1}{N+a}\left(\boldsymbol{I} - \boldsymbol{E}_0 - \boldsymbol{E}_N\right). \end{aligned}$$

This completes the proof of Proposition 3.2. ■

Proposition 3.3. *For any $\boldsymbol{f} \in \mathbf{C}^{2N}$ with the solvability condition ${}^t\mathbf{1}\boldsymbol{f} = 0$, the difference equation $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$ with the orthogonality condition ${}^t\mathbf{1}\boldsymbol{u} = 0$ has the unique solution given as $\boldsymbol{u} = \boldsymbol{G}_*\boldsymbol{f}$, where $\boldsymbol{G}_*$ is the pseudo Green's matrix expressed as*

$$(3.9)\quad \begin{aligned} \boldsymbol{G}_* &= \lim_{a\to+0}\left(\boldsymbol{G}(a) - a^{-1}\boldsymbol{E}_0\right) = \frac{1}{N}\left(\boldsymbol{I} - \boldsymbol{E}_0 - \frac{1}{2}\boldsymbol{E}_N\right) = \\ &\frac{1}{4N^2}\left(4N\delta(i-j) - 2 - (-1)^{i-j} \right)_{0\le i,j\le 2N-1}. \end{aligned}$$

The concrete forms of $\boldsymbol{G}_* = \boldsymbol{G}_{*N}$ $(N = 1, 2, 3)$ are as follows:

$$\boldsymbol{G}_{*1} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad \boldsymbol{G}_{*2} = \frac{1}{16}\begin{pmatrix} 5 & -1 & -3 & -1 \\ -1 & 5 & -1 & -3 \\ -3 & -1 & 5 & -1 \\ -1 & -3 & -1 & 5 \end{pmatrix},$$

$$\boldsymbol{G}_{*3} = \frac{1}{36}\begin{pmatrix} 9 & -1 & -3 & -1 & -3 & -1 \\ -1 & 9 & -1 & -3 & -1 & -3 \\ -3 & -1 & 9 & -1 & -3 & -1 \\ -1 & -3 & -1 & 9 & -1 & -3 \\ -3 & -1 & -3 & -1 & 9 & -1 \\ -1 & -3 & -1 & -3 & -1 & 9 \end{pmatrix}.$$

Proof of Proposition 3.3 Using (3.1), (3.3) and $\boldsymbol{E}_0\boldsymbol{f} = N^{-1}\boldsymbol{1}^t\boldsymbol{1}\boldsymbol{f} = \boldsymbol{0}$, where $\boldsymbol{0}$ is zero vector, we have

$$\sum_{k=1}^{2N-1} \boldsymbol{E}_k\boldsymbol{f} = \sum_{k=0}^{2N-1} \boldsymbol{E}_k\boldsymbol{f} = \boldsymbol{I}\boldsymbol{f} = \boldsymbol{f} = \boldsymbol{A}\boldsymbol{u} = \sum_{k=1}^{2N-1} \lambda_k \boldsymbol{E}_k\boldsymbol{u}.$$

Multiplying $\boldsymbol{E}_l$ on both sides of the above relation from the left and using the relation $\boldsymbol{E}_k\boldsymbol{E}_l = \delta(k-l)\boldsymbol{E}_k$, we obtain $\boldsymbol{E}_l\boldsymbol{u} = \lambda_l^{-1}\boldsymbol{E}_l\boldsymbol{f}$ $(1 \le l \le 2N-1)$. Then, using $\boldsymbol{E}_0\boldsymbol{u} = (2N)^{-1}\boldsymbol{1}^t\boldsymbol{1}\boldsymbol{u} = \boldsymbol{0}$, we see that

$$\boldsymbol{u} = \boldsymbol{I}\boldsymbol{u} = \sum_{l=0}^{N-1} \boldsymbol{E}_l\boldsymbol{u} = \sum_{l=1}^{N-1} \boldsymbol{E}_l\boldsymbol{u} = \sum_{l=1}^{N-1} \lambda_l^{-1}\boldsymbol{E}_l\boldsymbol{f} = \boldsymbol{G}_*\boldsymbol{f},$$

where

$$\boldsymbol{G}_* = \sum_{l=1}^{N-1} \lambda_l^{-1}\boldsymbol{E}_l =$$
$$\frac{1}{N}\left(\boldsymbol{E}_1 + \cdots + \boldsymbol{E}_{N-1} + \boldsymbol{E}_{N+1} + \cdots + \boldsymbol{E}_{2N-1}\right) + \frac{1}{2N}\boldsymbol{E}_N =$$
$$\frac{1}{N}\left(\boldsymbol{I} - \boldsymbol{E}_0 - \boldsymbol{E}_N\right) + \frac{1}{2N}\boldsymbol{E}_N = \frac{1}{N}\left(\boldsymbol{I} - \boldsymbol{E}_0 - \frac{1}{2}\boldsymbol{E}_N\right).$$

On the otherhand, taking the limit as $a \to +0$ on both sides of

$$\boldsymbol{G}(a) - a^{-1}\boldsymbol{E}_0 = \frac{1}{N+a}\left(\boldsymbol{I} - \boldsymbol{E}_0 - \frac{N}{2N+a}\boldsymbol{E}_N\right),$$

we have the same $\boldsymbol{G}_*$. This completes the proof of Proposition 3.3. ■

Next, we show that the diagonalvalues of $\boldsymbol{G}_*$ and $\boldsymbol{G}(a)$ are equal to the best constants of the discrete Sobolev inequalities (2.1) and (2.2), respectively. The most important fact is that the diagonal elements of $\boldsymbol{G}_*$ and

$\boldsymbol{G}(a)$ are the same. Using the diagonalvalues of $\boldsymbol{G}(a)$, we have the square of L^2 norm of $\|\boldsymbol{H}_*(t)\boldsymbol{\delta}_j\|$.

Lemma 3.1. *For any fixed j $(0 \le j \le 2N-1)$, we have the following relations:*

$$ {}^t\boldsymbol{\delta}_j \boldsymbol{G}_* \boldsymbol{\delta}_j = \frac{4N-3}{4N^2}, \tag{3.10} $$

$$ {}^t\boldsymbol{\delta}_j \boldsymbol{G}(a) \boldsymbol{\delta}_j = \frac{N+2Na+a^2}{a(N+a)(2N+a)}, \tag{3.11} $$

$$ \int_{-\infty}^{\infty} \|\boldsymbol{H}_*(t)\boldsymbol{\delta}_j\|^2 \, dt = \frac{1}{2} {}^t\boldsymbol{\delta}_j \boldsymbol{G}(a) \boldsymbol{\delta}_j. \tag{3.12} $$

Proof of Lemma 3.1 (3.10) and (3.11) follows from (3.9) and (3.8), respectively. Noting ${}^t\boldsymbol{H}(t) = \boldsymbol{H}(t)$, $(\boldsymbol{H}(t))^2 = \boldsymbol{H}(2t)$ and (3.6), we have (3.12) as

$$ \int_{-\infty}^{\infty} \|\boldsymbol{H}_*(t)\boldsymbol{\delta}_j\|^2 \, dt = \int_{-\infty}^{\infty} {}^t\Big(\boldsymbol{H}_*(t)\boldsymbol{\delta}_j\Big)\Big(\boldsymbol{H}_*(t)\boldsymbol{\delta}_j\Big) dt = $$
$$ \int_{-\infty}^{\infty} {}^t\boldsymbol{\delta}_j \boldsymbol{H}_*(2t)\boldsymbol{\delta}_j dt = \frac{1}{2} {}^t\boldsymbol{\delta}_j \int_{-\infty}^{\infty} \boldsymbol{H}_*(\tau) d\tau \, \boldsymbol{\delta}_j = $$
$$ \frac{1}{2} {}^t\boldsymbol{\delta}_j \int_0^{\infty} e^{-a\tau} \boldsymbol{H}(\tau) d\tau \, \boldsymbol{\delta}_j = \frac{1}{2} {}^t\boldsymbol{\delta}_j \boldsymbol{G}(a) \boldsymbol{\delta}_j. $$

This completes the proof of Lemma 3.1. ■

4. Reproducing relation

We show that $\boldsymbol{G}(a)$ and $\boldsymbol{G}_*$ are a reproducing matrix for the inner products $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_A$, respectively.

Lemma 4.1. *For any $\boldsymbol{u} \in \mathbf{C}_0^{2N}$ and fixed j $(0 \le j \le 2N-1)$, we have the following reproducing relations:*

$$ u(j) = (\boldsymbol{u}, \, \boldsymbol{G}_*\boldsymbol{\delta}_j)_A. \tag{4.1} $$

$$ {}^t\boldsymbol{\delta}_j \boldsymbol{G}_* \boldsymbol{\delta}_j = \| \, \boldsymbol{G}_*\boldsymbol{\delta}_j \, \|_A^2. \tag{4.2} $$

Proof of Lemma 4.1 Noting $\boldsymbol{G}_*^* = \boldsymbol{G}_*$, we have (4.1) as

$$ (\boldsymbol{u}, \, \boldsymbol{G}_*\boldsymbol{\delta}_j)_A = {}^t\boldsymbol{\delta}_j \boldsymbol{G}_* \boldsymbol{A} \boldsymbol{u} = {}^t\boldsymbol{\delta}_j (\boldsymbol{I} - \boldsymbol{E}_0)\boldsymbol{u} = {}^t\boldsymbol{\delta}_j \boldsymbol{u} - \frac{1}{N} \mathbf{1} {}^t\mathbf{1} \boldsymbol{u} = u(j). $$

Putting $\boldsymbol{u} = \boldsymbol{G}_*\boldsymbol{\delta}_j$ in (4.1), we obtain (4.2). ■

Lemma 4.2. *For any $\boldsymbol{u} \in \mathbf{C}^{2N}$ and fixed j $(0 \le j \le 2N-1)$, we have the following reproducing relations:*

$$ u(j) = (\boldsymbol{u}, \, \boldsymbol{G}(a)\boldsymbol{\delta}_j)_H. \tag{4.3} $$

$$(4.4)\qquad {}^t\boldsymbol{\delta}_j \boldsymbol{G}(a)\boldsymbol{\delta}_j = \|\, \boldsymbol{G}(a)\boldsymbol{\delta}_j \,\|_H^2.$$

The proof of Lemma 4.2 Noting $(\boldsymbol{G}(a))^* = \boldsymbol{G}(a)$, we have (4.3) as

$$(\boldsymbol{u},\, \boldsymbol{G}(a)\boldsymbol{\delta}_j)_H = {}^t\boldsymbol{\delta}_j \boldsymbol{G}(a)(\boldsymbol{A} + a\boldsymbol{I})\boldsymbol{u} = {}^t\boldsymbol{\delta}_j \boldsymbol{I}\boldsymbol{u} = u(j).$$

Putting $\boldsymbol{u} = \boldsymbol{G}(a)\boldsymbol{\delta}_j$ in (4.3), we obtain (4.4). ■

5. Proof of Theorems

This section is devoted to the proof of main theorems.

Proof of Theorem 2.1 For any $\boldsymbol{u} \in \mathbf{C}_0^{2N}$, applying the Schwarz inequality to (4.1) and using (4.2), we have

$$|\, u(j) \,|^2 \le \|\, \boldsymbol{u} \,\|_A^2 \|\, \boldsymbol{G}_*\boldsymbol{\delta}_j \,\|_A^2 = {}^t\boldsymbol{\delta}_j \boldsymbol{G}_*\boldsymbol{\delta}_j \|\, \boldsymbol{u} \,\|_A^2.$$

Taking the maximum with respect to j on both sides, we obtain the discrete Sobolev inequality

$$(5.1)\qquad \left(\max_{0 \le j \le 2N-1} |\, u(j) \,| \right)^2 \le C_0 \,\|\, \boldsymbol{u} \,\|_A^2,$$

where for any j_0 $(0 \le j_0 \le 2N-1)$, we put

$$C_0 = \max_{0 \le j \le 2N-1} {}^t\boldsymbol{\delta}_j \boldsymbol{G}_*\boldsymbol{\delta}_j = {}^t\boldsymbol{\delta}_{j_0} \boldsymbol{G}_*\boldsymbol{\delta}_{j_0}.$$

From the above inequality (5.1), $\|\, \boldsymbol{u} \,\|_A^2 = 0$ holds if and only if $\boldsymbol{u} = \mathbf{0}$. This shows that the sesquilinear form $(\boldsymbol{u}, \boldsymbol{v})_A$ is an inner product of vector space $\mathbf{C}_0^N$. If we take $\boldsymbol{u} = \boldsymbol{G}_*\boldsymbol{\delta}_{j_0}$ in (5.1), then we have

$$\left(\max_{0 \le j \le 2N-1} |\, {}^t\boldsymbol{\delta}_j \boldsymbol{G}_*\boldsymbol{\delta}_{j_0} \,| \right)^2 \le C_0 \,\|\, \boldsymbol{G}_*\boldsymbol{\delta}_{j_0} \,\|_A^2 = (C_0)^2.$$

Combining this with the trivial inequality

$$(C_0)^2 = |\, {}^t\boldsymbol{\delta}_{j_0} \boldsymbol{G}_*\boldsymbol{\delta}_{j_0} \,|^2 \le \left(\max_{0 \le j \le 2N-1} |\, {}^t\boldsymbol{\delta}_j \boldsymbol{G}_*\boldsymbol{\delta}_{j_0} \,| \right)^2,$$

we have

$$\left(\max_{0 \le j \le 2N-1} |\, {}^t\boldsymbol{\delta}_j \boldsymbol{G}_*\boldsymbol{\delta}_{j_0} \,| \right)^2 = C_0 \,\|\, \boldsymbol{G}_*\boldsymbol{\delta}_{j_0} \,\|_A^2.$$

This shows that C_0 is the best constant of (5.1) and the equality holds for any column of $\boldsymbol{G}_*$. The concrete form of C_0 is given in (3.10). This completes the proof of Theorem 2.1. ■

Proof of Theorem 2.2 We can show Theorem 2.2 in the same way as Theorem 2.1. So we omit the proof of Theorem 2.2. ■

Proof of Theorem 2.3 Replacing t by s in (3.5), we have

$$\boldsymbol{u}(s) = \int_{-\infty}^{\infty} \boldsymbol{H}_*(s-t)\boldsymbol{f}(t)dt,$$

or equivalently

$$u(j,s) = {}^t\boldsymbol{\delta}_j \boldsymbol{u}(s) = \int_{-\infty}^{\infty} {}^t\boldsymbol{\delta}_j \boldsymbol{H}_*(s-t)\boldsymbol{f}(t)dt = \int_{-\infty}^{\infty} {}^t\Big(\boldsymbol{H}_*(s-t)\boldsymbol{\delta}_j\Big)\boldsymbol{f}(t)dt. \tag{5.2}$$

Applying the Schwarz inequality to (5.2), we have

$$|\,u(j,s)\,|^2 \leq \int_{-\infty}^{\infty} \|\boldsymbol{H}_*(s-t)\boldsymbol{\delta}_j\|^2\,dt \int_{-\infty}^{\infty} \|\boldsymbol{f}(t)\|^2\,dt = \int_{-\infty}^{\infty} \|\boldsymbol{H}_*(t)\boldsymbol{\delta}_j\|^2\,dt \int_{-\infty}^{\infty} \left\|\left(\frac{d}{dt}+\boldsymbol{A}+a\boldsymbol{I}\right)\boldsymbol{u}(t)\right\|^2 dt,$$

where we use (3.4). Taking the supremum with respect to j and s, we obtain the discrete Sobolev-type inequality

$$\left(\sup_{\substack{0\leq j\leq 2N-1\\ -\infty<s<\infty}} |\,u(j,s)\,|\right)^2 \leq C_1(a)\int_{-\infty}^{\infty} \left\|\left(\frac{d}{dt}+\boldsymbol{A}+a\boldsymbol{I}\right)\boldsymbol{u}(t)\right\|^2 dt, \tag{5.3}$$

where for any j_0 $(0\leq j_0\leq 2N-1)$, we put

$$C_1(a) = \max_{0\leq j\leq 2N-1}\int_{-\infty}^{\infty} \|\boldsymbol{H}_*(t)\boldsymbol{\delta}_j\|^2\,dt = \int_{-\infty}^{\infty} \|\boldsymbol{H}_*(t)\boldsymbol{\delta}_{j_0}\|^2\,dt.$$

Here, we introduce the vector $\boldsymbol{U}(t)$ defined as

$$\boldsymbol{U}(t) = \int_{-\infty}^{\infty} \boldsymbol{H}_*(t-s)\boldsymbol{H}_*(-s)\boldsymbol{\delta}_{j_0}ds, \tag{5.4}$$

$$U(j,t) = {}^t\boldsymbol{\delta}_j\boldsymbol{U}(t) = \int_{-\infty}^{\infty} {}^t\boldsymbol{\delta}_j\boldsymbol{H}_*(t-s)\boldsymbol{H}_*(-s)\boldsymbol{\delta}_{j_0}ds.$$

Then we have

$$\left(\sup_{\substack{0\leq j\leq 2N-1\\ -\infty<s<\infty}} |\,U(j,s)\,|\right)^2 \leq C_1(a)\int_{-\infty}^{\infty} \left\|\left(\frac{d}{dt}+\boldsymbol{A}+a\boldsymbol{I}\right)\boldsymbol{U}(t)\right\|^2 dt = C_1(a)\int_{-\infty}^{\infty} \|\boldsymbol{H}_*(-t)\boldsymbol{\delta}_{j_0}\|^2\,dt = (C_1(a))^2.$$

Combining this with the trivial inequality

$$(C_1(a))^2 = |\, U(j_0,0)\,|^2 \le \left(\sup_{\substack{0\le j\le 2N-1 \\ -\infty<s<\infty}} |\, U(j,s)\,| \right)^2,$$

we have

$$\left(\sup_{\substack{0\le j\le 2N-1 \\ -\infty<s<\infty}} |\, U(j,s)\,| \right)^2 = C_1(a) \int_{-\infty}^{\infty} \left\| \left(\frac{d}{dt} + \boldsymbol{A} + a\boldsymbol{I} \right) \boldsymbol{U}(t) \right\|^2 dt.$$

This shows that $C_1(a)$ is the best constant of (5.3) and the equality holds for $\boldsymbol{u}(t) = \boldsymbol{U}(t)$. The concrete form of $C_1(a)$ is given in (3.12). From (5.4), we have

$$\text{(5.5)} \quad \boldsymbol{U}(t) = \int_{-\infty}^{\infty} \boldsymbol{H}_*(t-s)\boldsymbol{H}_*(-s)\boldsymbol{\delta}_{j_0} ds =$$
$$\int_{-\infty}^{\infty} Y(t-s)e^{-a(t-s)}\boldsymbol{H}(t-s)Y(-s)e^{-a(-s)}\boldsymbol{H}(-s)\boldsymbol{\delta}_{j_0} ds =$$
$$\int_{-\infty}^{0\wedge t} e^{-a(t-2s)}\boldsymbol{H}(t-2s)\boldsymbol{\delta}_{j_0} ds,$$

where $x \vee y = \max\{x,y\}$ and $x \wedge y = \min\{x,y\}$ satisfies the relation

$$\begin{cases} x\vee y + x\wedge y = x+y \\ x\vee y - x\wedge y = |x-y| \end{cases} \Leftrightarrow \begin{cases} x\vee y = \dfrac{1}{2}(x+y+|x-y|) \\ x\wedge y = \dfrac{1}{2}(x+y-|x-y|) \end{cases}.$$

From this relation, we have

$$0\wedge t = \frac{1}{2}(0+t-|0-t|) = \frac{1}{2}(t-|t|).$$

For (5.5), if we replace $\sigma = t-2s$

$$\begin{array}{c|ccc} s & -\infty & \to & 0\wedge t \\ \hline \sigma & \infty & \to & |t| \end{array} \qquad ds = -\frac{1}{2}d\sigma,$$

then we have (2.4). This completes the proof of Theorem 2.3. ■

Acknowledgment

This research is supported by J. S. P. S. Grant-in-Aid for Scientific Research (C) No.18K03340.

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(*Received February 11, 2020*)
(*Accepted November 2, 2020*)