Math. J. Okayama Univ. **64** (2022), 1–11

A NOTE ON TORSION POINTS ON AMPLE DIVISORS ON ABELIAN VARIETIES

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Abstract. In the present paper, we consider torsion points on ample divisors on abelian varieties. We prove that, for each integer $n \geq 2$, an effective divisor of level n on an abelian variety does not contain the subgroup of n-torsion points. Moreover, we also discuss an application of this result to the study of the p-rank of cyclic coverings of curves in positive characteristic.

Introduction

In the present paper, we consider torsion points on ample divisors on abelian varieties. The main result of the present paper is as follows [cf. Corollary 1.8, (i)].

Theorem A. *Let k be an algebraically closed field, A an abelian variety over k, D an effective divisor on A, and $n \geq 2$ an integer invertible in k. Suppose that the effective divisor D is* **of level n**, *i.e., that there exists an effective divisor D_1 on A such that D_1 gives rise to a* **principal polarization** *on A, and, moreover, D is* **linearly equivalent** *to nD_1 [cf. Definition 1.3, (ii); also Remark 1.3.1]. Then the subgroup of* **n-torsion points** *of A is* **not contained** *in* Supp(D).

Here, let us recall that *R. Auffarth*, *G. P. Pirola*, and *R. S. Manni* proved that if D is an effective divisor on an abelian variety of dimension $g \geq 1$ over the *field of complex numbers* that gives rise to a *principal polarization* on the abelian variety, then, for each integer $n \geq 3$, the set of n-torsion (respectively, 2-torsion) points on Supp(D) is *of cardinality* $\leq n^{2g} - (g+1)n^g$ $(< n^{2g})$ (respectively, $\leq 2^{2g} - 2^{g-1}g - 2^g$ $(< 2^{2g})$) [cf. [1], Theorem 1.1]. Theorem A may be regarded as a *partial generalization* of this result [cf. Remark 1.8.1].

In §2 of the present paper, we apply Theorem A and Raynaud's theory of theta divisors [cf. [5]] to obtain an *application* to the study of the *p-rank* of cyclic coverings of curves in positive characteristic. One consequence of our application is as follows [cf. Theorem 2.7, (i)].

Theorem B. *Let p be an* **odd** *prime number, k an algebraically closed field of characteristic p, and X a projective smooth connected curve over k of*

Mathematics Subject Classification. Primary 14K12; Secondary 14H30.
Key words and phrases. abelian variety, torsion point, curve, p-rank.

genus ≥ 2*. Then there exist a positive integer* n *such that* $p-1 \in n\mathbb{Z}$ *and a finite étale* **cyclic** *covering of* X **of degree** $\boldsymbol{n}$ *whose Jacobian variety is* **of positive** $\boldsymbol{p}$**-rank**.

Here, let us recall that *M. Raynaud* proved that, in the situation of Theorem B, the étale fundamental group of X is *not pro-prime-to-p* [cf. [5], Corollaire 4.3.2]. In §2 of the present paper, we also derive a *refinement* of this result from Theorem B [cf. Remark 2.8.1, (iii)].

1. Torsion Points on Ample Divisors on Abelian Varieties

In the present §1, we discuss *torsion points* on ample divisors on abelian varieties and prove the main result of the present paper [cf. Corollary 1.8 below]. In the present §1, let g be a positive integer, k an algebraically closed field,

$$A$$

an *abelian variety* over k of dimension g, n a positive integer, and

$$\mathcal{L}$$

an *ample* invertible sheaf on A *of separable type* [cf. [2], p.289].

Definition 1.1. We shall write $A[n] \subseteq A$ for the closed subgroup scheme of A obtained by forming the kernel of the endomorphism of A given by multiplication by n.

Lemma 1.2. *The following four conditions are equivalent:*

(1) *There exist an ample invertible sheaf* $\mathcal{L}_1$ *on* A **of degree one** *[cf.* [2]*, p.*289*,* (III)] *and an* **isomorphism** $\mathcal{L} \xrightarrow{\sim} \mathcal{L}_1^{\otimes n}$*.*
(2) *The invertible sheaf* $\mathcal{L}$ *is* **of degree** $\boldsymbol{n^g}$*, and, moreover, there exist an ample invertible sheaf* $\mathcal{L}_1$ *on* A *and an* **isomorphism** $\mathcal{L} \xrightarrow{\sim} \mathcal{L}_1^{\otimes n}$*.*
(3) *The* **equality** $H(\mathcal{L}) = A[n]$ *[cf.* [2]*, p.*288*, Definition] holds, and, moreover, there exist an ample invertible sheaf* $\mathcal{L}_1$ *on* A *and an* **isomorphism** $\mathcal{L} \xrightarrow{\sim} \mathcal{L}_1^{\otimes n}$*.*
(4) *The* **equality** $H(\mathcal{L}) = A[n]$ *holds.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from [2], p.289, (II). Moreover, the equivalence (3) $\Leftrightarrow$ (4) follows from [3], p.214, Theorem 3. Next, since the group scheme $A[n]$ is *of degree* n^{2g} over k [cf. [3], p.60, Proposition, (1)], the implication (3) $\Rightarrow$ (2) follows from [2], p.289, (IV).

Finally, we verify the implication (2) $\Rightarrow$ (3). Suppose that condition (2) is satisfied. Then since $\mathcal{L}$ is *isomorphic* to $\mathcal{L}_1^{\otimes n}$ [cf. condition (2)], the homomorphisms $\Lambda(\mathcal{L})$, $\Lambda(\mathcal{L}_1)\colon A \to A^\wedge$ [cf. [2], p.289, (IV)] satisfy the equality $\Lambda(\mathcal{L}) = n\cdot\Lambda(\mathcal{L}_1)$. Thus, it follows that $A[n] \subseteq \mathrm{Ker}(n\cdot\Lambda(\mathcal{L}_1)) = \mathrm{Ker}(\Lambda(\mathcal{L})) =$

$H(\mathcal{L})$. On the other hand, since $\mathcal{L}$ is *of degree* n^g [cf. condition (2)], it follows from [2], p.289, (IV), that the group scheme $H(\mathcal{L})$ is *of degree* n^{2g} over k. Thus, since the group scheme $A[n]$ is *of degree* n^{2g} over k [cf. [3], p.60, Proposition, (1)], the equality $H(\mathcal{L}) = A[n]$, hence also condition (3), holds, as desired. This completes the proof of the implication (2) $\Rightarrow$ (3), hence also of Lemma 1.2. □

Definition 1.3.

(i) We shall say that the ample invertible sheaf $\mathcal{L}$ of separable type is *of level* n if $\mathcal{L}$ satisfies the four conditions [i.e., with respect to the fixed "n"] in the statement of Lemma 1.2.

(ii) We shall say that an effective divisor D on A is *of level* n if the invertible sheaf $\mathcal{O}_A(D)$ is [ample, of separable type, and] of level n.

Remark 1.3.1. Let $\mathcal{M}$ be an invertible sheaf on A. Then it is immediate that $\mathcal{M}$ gives rise to a *principal polarization* on A if and only if $\mathcal{M}$ is [ample, of separable type, and] *of level one*.

Lemma 1.4. *Let* $\mathcal{M}$ *be an invertible sheaf on* A **algebraically equivalent** *to* $\mathcal{L}$. *Then the following hold:*

(i) *There exists a closed point* $a \in A$ *of* A *such that* $\mathcal{L}$ *is* **isomorphic** *to* $T_a^*\mathcal{M}$ *[cf. [2], p.288, Definition].*

(ii) *The invertible sheaf* $\mathcal{M}$ *is* **ample** *and* **of separable type**.

(iii) *Suppose that* $\mathcal{M}$ *is* **of level** $\boldsymbol{n}$ *[cf. (ii)]. Then* $\mathcal{L}$ *is* **of level** $\boldsymbol{n}$.

Proof. First, we verify assertion (i). Let us first observe that the homomorphism $A(k) \to \mathrm{Pic}^0(A)$ determined by $\Lambda(\mathcal{L})$ is *surjective* [cf. [2], p.289, (IV)]. Thus, there exists a closed point $a \in A$ of A such that $\mathcal{M} \otimes_{\mathcal{O}_A} \mathcal{L}^{-1}$ is *isomorphic* to $T_{-a}^*\mathcal{L} \otimes_{\mathcal{O}_A} \mathcal{L}^{-1}$. Thus, we conclude that $\mathcal{L}$ is *isomorphic* to $T_a^*\mathcal{M}$, as desired. This completes the proof of assertion (i). Assertions (ii), (iii) follow from assertion (i). This completes the proof of Lemma 1.4. □

Lemma 1.5. *Suppose that* $\mathcal{L}$ *is* **of level** $\boldsymbol{n}$. *Then the following two conditions are equivalent:*

(1) *The* **inequality** $n > 1$ *holds.*

(2) *The invertible sheaf* $\mathcal{L}$ *is* **generated by global sections**.

Proof. The implication (1) $\Rightarrow$ (2) follows immediately from [3], pp.57-58, Application 1, (iii). Next, to verify the implication (2) $\Rightarrow$ (1), assume that condition (2) is *satisfied*, but that condition (1) is *not satisfied* [i.e., that $n = 1$]. Then it follows from [2], p.289, (II), that $\Gamma(A, \mathcal{L})$ is *of dimension one*. Thus, since $\mathcal{L}$ is *generated by global sections* [cf. condition (2)], the invertible sheaf $\mathcal{L}$ is *trivial*. In particular, since [we have assumed that] $\mathcal{L}$ is *ample*, we conclude that $g = 0$. Thus, since [we have assumed that]

$g > 0$, we obtain a *contradiction*, as desired. This completes the proof of the implication (2) $\Rightarrow$ (1), hence also of Lemma 1.5. $\square$

One main technical observation of the present paper is as follows.

Lemma 1.6. *Let D be an effective divisor on A obtained by forming the* **zero locus** *of a nonzero global section of the invertible sheaf $\mathcal{L}$. Write $H(D) \subseteq H(\mathcal{L})$ for the subgroup of $H(\mathcal{L})$ consisting of $a \in A$ such that $T_a^* D = D$. Let $H \subseteq H(\mathcal{L})$ be a subgroup of $H(\mathcal{L})$ such that $H + H(D)$ ($\stackrel{\text{def}}{=} \{ h + h_d \in H(\mathcal{L}) \,|\, h \in H, h_d \in H(D) \}$) $= H(\mathcal{L})$. Suppose that the inclusion*

$$H \subseteq \mathrm{Supp}(D)$$

holds. Then the subset $H \subseteq A$ of A is **contained** *in the* **base locus** *of the [complete linear system associated to the] invertible sheaf $\mathcal{L}$.*

Proof. Let $s \in \Gamma(A, \mathcal{L})$ be a nonzero global section of $\mathcal{L}$ whose zero locus is given by D.

Here, let us recall the exact sequence

$$0 \longrightarrow k^\times \longrightarrow \mathcal{G}(\mathcal{L}) \longrightarrow H(\mathcal{L}) \longrightarrow 0$$

in [2], p.290, concerning the *theta group* $\mathcal{G}(\mathcal{L})$ associated to $\mathcal{L}$. It follows from the definition of $\mathcal{G}(\mathcal{L})$ that there exists a natural action of $\mathcal{G}(\mathcal{L})$ on the linear space $\Gamma(A, \mathcal{L})$ over k, which restricts to the natural action of the subgroup $k^\times \subseteq \mathcal{G}(\mathcal{L})$ on $\Gamma(A, \mathcal{L})$ [cf. [2], p.295, Definition]. In particular,

(a) for each $a \in H(\mathcal{L})$, if $\widetilde{a} \in \mathcal{G}(\mathcal{L})$ is a lifting of $a \in H(\mathcal{L})$, then the zero locus of the nonzero global section $\widetilde{a} \cdot s \in \Gamma(A, \mathcal{L})$ is given by $T_{-a}^* D$.

Now let us fix a subset

$$\widetilde{H} \subseteq \mathcal{G}(\mathcal{L})$$

of $\mathcal{G}(\mathcal{L})$ such that the composite $\widetilde{H} \hookrightarrow \mathcal{G}(\mathcal{L}) \twoheadrightarrow H(\mathcal{L})$ determines a *bijection* $\widetilde{H} \xrightarrow{\sim} H$. Then since [we have assumed that] the inclusion $H \subseteq \mathrm{Supp}(D)$ holds, it follows from (a) that,

(b) for every $\widetilde{a} \in \widetilde{H}$, the subset $H \subseteq A$ [i.e., the subset "$T_{-a}^* H$" of A — where we write a for the image of $\widetilde{a} \in \widetilde{H}$ in H] is *contained* in the zero locus of the nonzero global section $\widetilde{a} \cdot s \in \Gamma(A, \mathcal{L})$.

Next, let us observe that it follows immediately from (a), together with our assumption that $H + H(D) = H(\mathcal{L})$, that

(c) the linear subspace of $\Gamma(A, \mathcal{L})$ generated by the $\mathcal{G}(\mathcal{L})$-orbit of $s \in \Gamma(A, \mathcal{L})$ *coincides* with the linear subspace of $\Gamma(A, \mathcal{L})$ generated by the subset $\{\widetilde{a} \cdot s\}_{\widetilde{a} \in \widetilde{H}} \subseteq \Gamma(A, \mathcal{L})$.

On the other hand, it follows from [2], p.297, Theorem 2, that the action of $\mathcal{G}(\mathcal{L})$ on $\Gamma(A, \mathcal{L})$ is *irreducible*. Thus, we conclude from (c) that

(d) the subset $\{\widetilde{a} \cdot s\}_{\widetilde{a} \in \widetilde{H}} \subseteq \Gamma(A, \mathcal{L})$ *generates* the linear space $\Gamma(A, \mathcal{L})$.

Thus, it follows from (b) and (d) that the subset $H \subseteq A$ is contained in the *base locus* of the invertible sheaf $\mathcal{L}$, as desired. This completes the proof of Lemma 1.6. □

Theorem 1.7. *Let k be an algebraically closed field, A an abelian variety over k, and D an effective divisor on A. Suppose that the invertible sheaf $\mathcal{O}_A(D)$ is* **ample**, **of separable type** [*cf.* [2], *p.*289], *and* **generated by global sections**. *Then the following hold:*

(i) *Recall the closed subgroup scheme $H(\mathcal{O}_A(D)) \subseteq A$ of A defined in* [2], *p.*288, *Definition. Then $H(\mathcal{O}_A(D))$ is* **not contained** *in* $\mathrm{Supp}(D)$.

(ii) *Write* $\deg(D)$ *for the degree of the ample invertible sheaf $\mathcal{O}_A(D)$* [*cf.* [2], *p.*289, (III)]. *Then $A[\deg(D)]$* [*cf. Definition* 1.1] *is* **not contained** *in* $\mathrm{Supp}(D)$.

Proof. Assertion (i) follows from Lemma 1.6. Assertion (ii) follows from assertion (i), together with the inclusion $H(\mathcal{O}_A(D)) \subseteq A[\deg(D)]$ [cf. [2], p.289, (IV); [2], p.293, Theorem 1; also the first Definition in [2], p.294]. □

The main result of the present paper is as follows.

Corollary 1.8. *Let k be an algebraically closed field, A an abelian variety over k, D an effective divisor on A, and n a positive integer invertible in k. Suppose that the effective divisor D is* **of level n** [*cf. Definition* 1.3, (ii)]. *Then the following hold:*

(i) *Suppose that $n \geq 2$. Then $A[n]$ is* **not contained** *in* $\mathrm{Supp}(D)$.

(ii) *Suppose that $n = 1$. Then, for each integer $m \geq 2$ invertible in k, $A[m]$ is* **not contained** *in* $\mathrm{Supp}(D)$.

Proof. Let us recall from condition (4) of Lemma 1.2 that the equality $H(\mathcal{O}_A(D)) = A[n]$ holds. Thus, assertion (i) follows from Lemma 1.5 and Theorem 1.7, (i).

Next, we verify assertion (ii). Let $m \geq 2$ be an integer invertible in k. Then since D is *of level one*, it is immediate that mD is *of level m*. Thus, since $\mathrm{Supp}(mD) = \mathrm{Supp}(D)$, it follows from assertion (i) that $A[m]$ is *not contained* in $\mathrm{Supp}(D)$, as desired. This completes the proof of assertion (ii), hence also of Corollary 1.8. □

Remark 1.8.1. *R. Auffarth*, *G. P. Pirola*, and *R. S. Manni* proved that, in the situation of Corollary 1.8, if, moreover, k is the *field of complex numbers*, and $n = 1$ [i.e., the divisor D gives rise to a *principal polarization* on A — cf. Remark 1.3.1], then, for each integer $m \geq 3$, the set $A[m] \cap \mathrm{Supp}(D)$

(respectively, $A[2] \cap \mathrm{Supp}(D)$) is *of cardinality* $\leq n^{2g} - (g+1)n^g$ $(< n^{2g})$ (respectively, $\leq 2^{2g} - 2^{g-1}g - 2^g$ $(< 2^{2g})$) — where we write g for the dimension of A [cf. [1], Theorem 1.1]. Corollary 1.8 may be regarded as a *partial generalization* of this result.

2. Application: p-Rank of Cyclic Coverings of Curves

In the present §2, we apply the main result of the present paper and Raynaud's theory of theta divisors [cf. [5]] to obtain an *application* to the study of the *p-rank* of cyclic coverings of curves in positive characteristic [cf. Theorem 2.7 below]. In the present §2, let p be a prime number, k an algebraically closed field of characteristic p, $g \geq 2$ an integer,

$$X$$

a *projective smooth connected curve* over k of genus g, $n \geq 2$ an integer invertible in k, and

$$\mathcal{L}$$

an invertible sheaf on X *of order* n.

Definition 2.1. We shall write X^F for the projective smooth connected curve over k obtained by forming the base-change of X by the absolute Frobenius endomorphism of k, $\mathcal{L}^F$ for the invertible sheaf on X^F obtained by forming the base-change of $\mathcal{L}$ by the absolute Frobenius endomorphism of k, and $\Phi \colon X \to X^F$ for the relative Frobenius morphism associated to X over k.

Remark 2.1.1. Let us recall that we have a natural isomorphism of invertible sheaves on X

$$\mathcal{L}^{\otimes p} \xrightarrow{\sim} \Phi^* \mathcal{L}^F$$

given by, for each local section l of $\mathcal{L}$, mapping $l^{\otimes p}$ to $\Phi^{-1} l^F$ — where we write l^F for the local section of $\mathcal{L}^F$ obtained by forming the base-change of the local section l by the absolute Frobenius endomorphism of k. Let us identify $\mathcal{L}^{\otimes p}$ with $\Phi^* \mathcal{L}^F$ by means of this isomorphism.

Definition 2.2.

(i) Let i be an element of $\{1, \ldots, n\}$. Then we shall write

$$\gamma_{\mathcal{L},i} \colon H^1(X^F, (\mathcal{L}^F)^{\otimes i}) \longrightarrow H^1(X, \mathcal{L}^{\otimes pi})$$

for the k-linear homomorphism obtained by applying "$H^1(X^F, (-) \otimes_{\mathcal{O}_{X^F}} (\mathcal{L}^F)^{\otimes i})$" to the homomorphism $\mathcal{O}_{X^F} \to \Phi_* \mathcal{O}_X$ determined by Φ [cf. also Remark 2.1.1].

(ii) We shall say that the invertible sheaf $\mathcal{L}$ is *new-ordinary* if, for every element $i \in \{1, \ldots, n-1\}$ with $n\mathbb{Z} + i\mathbb{Z} = \mathbb{Z}$, the homomorphism $\gamma_{\mathcal{L},i}$ of (i) is an isomorphism.

Remark 2.2.1.

(i) One verifies immediately from the theory of finite étale cyclic coverings and generalized Hasse-Witt invariants [cf., e.g., [6], §2.1, or [7], pp.73-74] that

the existence of a *new-ordinary* invertible sheaf on X *of order* n

is *equivalent* to

the existence of a *new-ordinary* finite étale cyclic covering of X *of degree* n, i.e., a finite étale cyclic covering of X *of degree* n that *has a new ordinary part* in the sense of [6], Définition 2.1.1,

which thus *implies*

the existence of a finite étale cyclic covering of X *of degree* n whose Jacobian variety is *of p-rank* $\geq (g-1) \cdot \sharp(\mathbb{Z}/n\mathbb{Z})^{\times}$ (> 0).

(ii) Suppose that $p - 1 \in n\mathbb{Z}$. Then each trivialization ι of $\mathcal{L}^{\otimes n}$ determines an isomorphism of invertible sheaves on X

$$\iota^{(p-1)/n} \colon \mathcal{L}^{\otimes p} \xrightarrow{\sim} \mathcal{L}.$$

Thus, the homomorphism $\gamma_{\mathcal{L},i}$ may be "identified", i.e., by means of $\iota^{(p-1)/n}$, with the homomorphism

$$H^1(X^F, (\mathcal{L}^F)^{\otimes i}) \longrightarrow H^1(X, \mathcal{L}^{\otimes i}).$$

In particular, one verifies immediately from the theory of finite étale cyclic coverings and generalized Hasse-Witt invariants [cf., e.g., [6], §2.1, or [7], pp.73-74] that

the existence of an invertible sheaf $\mathcal{M}$ on X *of order* n such that the homomorphism $\gamma_{\mathcal{M},i}$ is an isomorphism for some $i \in \{1, \ldots, n-1\}$

implies

the existence of a finite étale cyclic covering of X *of degree* n whose Jacobian variety is *of p-rank* $\geq \dim_k H^1(X, \mathcal{L}^{\otimes i}) = g - 1$ (> 0).

In the remainder of the present §2, write J^F for the Jacobian variety of X^F and $\mathcal{B}^F$ for the $\mathcal{O}_{X^F}$-module obtained by forming the cokernel of the homomorphism $\mathcal{O}_{X^F} \to \Phi_* \mathcal{O}_X$ determined by Φ. Moreover, let us fix a universal invertible sheaf $\mathcal{P}^F$ on $X^F \times_k J^F$ of degree zero.

Definition 2.3. We shall write

$$\Theta_{\mathcal{B}^F} \subseteq J^F$$

for the closed subscheme of J^F defined by the zeroth Fitting ideal of the coherent $\mathcal{O}_{J^F}$-module

$$\mathbb{R}^1(X^F \times_k J^F \stackrel{\mathrm{pr}_2}{\to} J^F)_*\big(\mathcal{P}^F \otimes_{\mathcal{O}_{X^F \times_k J^F}} (X^F \times_k J^F \stackrel{\mathrm{pr}_1}{\to} X^F)^* \mathcal{B}^F\big)$$

[cf. also [7], Remark 1.1].

Proposition 2.4. *The following hold:*

(i) *The closed subscheme $\Theta_{\mathcal{B}^F} \subseteq J^F$ of J^F forms a [necessarily effective]* **divisor** *on J^F* **of level $p-1$** *[cf. Definition 1.3, (ii)].*

(ii) *Let $x \in J^F$ be a closed point of J^F and $\mathcal{M}^F$ an invertible sheaf on X^F of degree zero whose isomorphism class corresponds to $x \in J^F$. Then the following three conditions are equivalent:*
   (1) *The closed point $x \in J^F$ is* **not contained** *in $\Theta_{\mathcal{B}^F}$.*
   (2) *The* **equality** *$\Gamma(X^F, \mathcal{M}^F \otimes_{\mathcal{O}_{X^F}} \mathcal{B}^F) = \{0\}$ holds.*
   (3) *The* **equality** *$H^1(X^F, \mathcal{M}^F \otimes_{\mathcal{O}_{X^F}} \mathcal{B}^F) = \{0\}$ holds.*

(iii) *The underlying closed subset of the closed subscheme $\Theta_{\mathcal{B}^F} \subseteq J^F$ of J^F is* **stabilized** *by the automorphism of J^F given by multiplication by -1.*

Proof. First, we verify assertion (i). It follows from [5], Théorème 4.1.1, that the closed subscheme $\Theta_{\mathcal{B}^F} \subseteq J^F$ of J^F forms a [necessarily effective] *divisor* on J^F. Moreover, since [it is well-known that] the "classical theta divisor" on J^F gives rise to a *principal polarization* on J^F, it follows from [5], Proposition 1.8.1, (2) [cf. also [5], §4], together with Lemma 1.4, (iii), of the present paper [cf. also Remark 1.3.1 of the present paper], that the divisor determined by $\Theta_{\mathcal{B}^F} \subseteq J^F$ is *of level* $p-1$, as desired. This completes the proof of assertion (i).

Assertion (ii) follows immediately from the definition of the closed subscheme $\Theta_{\mathcal{B}^F} \subseteq J^F$ [cf. also [5], §4; [7], Lemma 1.2]. Finally, we verify assertion (iii). Let us recall from the discussion preceding [5], Théorème 4.1.1, that there exists an isomorphism $\mathcal{B}^F \stackrel{\sim}{\to} \mathcal{H}om_{\mathcal{O}_{X^F}}(\mathcal{B}^F, \Omega^1_{X^F/k})$ of $\mathcal{O}_{X^F}$-modules. Thus, assertion (iii) follows immediately from assertion (ii), together with *Serre duality.* This completes the proof of assertion (iii), hence also of Proposition 2.4. □

Lemma 2.5. *The following hold:*

(i) *Suppose that $p \neq 2$. Then $J^F[p-1]$ [cf. Definition 1.1] is* **not contained** *in $\Theta_{\mathcal{B}^F}$.*

(ii) *Suppose that* $p = 2$. *Then, for each* **odd** *integer* $m \geq 3$, $J^F[m]$ *is* **not contained** *in* $\Theta_{\mathcal{B}^F}$.

Proof. These assertions follow from Corollary 1.8 and Proposition 2.4, (i). □

Lemma 2.6. *The following hold:*

(i) *Let* i *be an element of* $\{1, \ldots, n\}$. *Then it holds that the homomorphism* $\gamma_{\mathcal{L},i}$ *is an* **isomorphism** *if and only if the closed point of* J^F *that corresponds to* $(\mathcal{L}^F)^{\otimes i}$ *is* **not contained** *in* $\Theta_{\mathcal{B}^F} \subseteq J^F$.

(ii) *It holds that the Jacobian variety of* X *is* **ordinary** *if and only if the identity element of* J^F *is* **not contained** *in* $\Theta_{\mathcal{B}^F} \subseteq J^F$.

(iii) *It holds that the invertible sheaf* $\mathcal{L}$ *is* **new-ordinary** *if and only if, for every element* $i \in \{1, \ldots, n-1\}$ *with* $n\mathbb{Z}+i\mathbb{Z} = \mathbb{Z}$, *the closed point of* J^F *that corresponds to* $(\mathcal{L}^F)^{\otimes i}$ *is* **not contained** *in* $\Theta_{\mathcal{B}^F} \subseteq J^F$.

(iv) *Suppose that* $n \in \{2, 3, 4, 6\}$. *Then it holds that the invertible sheaf* $\mathcal{L}$ *is* **new-ordinary** *if and only if there exists an element* $i \in \{1, \ldots, n-1\}$ *such that* $n\mathbb{Z}+i\mathbb{Z} = \mathbb{Z}$, *and, moreover, the closed point of* J^F *that corresponds to* $(\mathcal{L}^F)^{\otimes i}$ *is* **not contained** *in* $\Theta_{\mathcal{B}^F} \subseteq J^F$.

Proof. Assertion (i) follows immediately from Proposition 2.4, (ii), together with the definition of the $\mathcal{O}_{X^F}$-module $\mathcal{B}^F$. Assertions (ii), (iii) follow from assertion (i) [cf. also [6], §2.1]. Finally, we verify assertion (iv). The necessity follows from assertion (iii). The sufficiency follows from Proposition 2.4, (iii), and assertion (iii). This completes the proof of assertion (iv), hence also of Lemma 2.6. □

One interesting application of the main result of the present paper is as follows.

Theorem 2.7. *Let* p *be a prime number,* k *an algebraically closed field of characteristic* p, *and* X *a projective smooth connected curve over* k *of genus* ≥ 2. *Then the following hold:*

(i) *Suppose that* $p \neq 2$. *Then there exist a positive integer* n *such that* $p-1 \in n\mathbb{Z}$ *and a finite étale* **cyclic** *covering of* X **of degree** $\boldsymbol{n}$ *whose Jacobian variety is* **of positive** $\boldsymbol{p}$**-rank**.

(ii) *Suppose that the Jacobian variety of* X *is* **not ordinary**. *Let* n *be an integer such that* $(p, n) \in \{(2, 3), (3, 2)\}$. *Then there exists a* **new-ordinary** *finite étale* **cyclic** *covering of* X **of degree** $\boldsymbol{n}$, *i.e., a finite étale* **cyclic** *covering of* X **of degree** $\boldsymbol{n}$ *that* **has a new ordinary part** *in the sense of* [6], *Définition* 2.1.1.

Proof. Assertion (i) follows immediately — in light of Remark 2.2.1, (ii) — from Lemma 2.5, (i), and Lemma 2.6, (i), (ii). Assertion (ii) follows immediately — in light of Remark 2.2.1, (i) — from Lemma 2.5, (i), (ii), and Lemma 2.6, (ii), (iv). □

Remark 2.7.1. Some results closely related to the content of Theorem 2.7 are as follows: In the situation of Theorem 2.7, suppose that X is *of genus* g (≥ 2). Then:

(i) *M. Raynaud* proved that if, moreover, l is a prime number such that $l + 1 \geq (p-1)3^{g-1}g!$, then there exists a *new-ordinary* finite étale cyclic covering of X *of degree* l [cf. [5], Théorème 4.3.1; also [7], Remark 3.11].

(ii) *S. Nakajima* proved that if, moreover, $(g, p) = (2, 2)$, and the Jacobian variety of X is *not ordinary* [i.e., the curve X is either *of type I* or *of type II* in the sense of [4], §6], then every finite étale cyclic covering of X *of degree three* is *new-ordinary* [i.e., the curve X is 3-*ordinary* in the sense of the discussion at the beginning of [4], §4] [cf. [4], §6].

Corollary 2.8. *Let p be a prime number, k an algebraically closed field of characteristic p, and X a projective smooth connected curve over k of genus ≥ 2. Write $\pi_1(X)$ for the étale fundamental group [for some choice of basepoint] of X,*

$$n_p \stackrel{\text{def}}{=} \begin{cases} p-1 & \text{if } p \neq 2 \\ 3 & \text{if } p = 2, \end{cases}$$

$N \subseteq \pi_1(X)$ for the normal open subgroup of $\pi_1(X)$ obtained by forming the kernel of the natural surjective homomorphism

$$\pi_1(X) \twoheadrightarrow \pi_1(X)^{\text{ab}} \otimes_{\widehat{\mathbb{Z}}} (\mathbb{Z}/n_p\mathbb{Z}),$$

and $Y \to X$ for the finite étale abelian covering that corresponds to the normal open subgroup $N \subseteq \pi_1(X)$. Then the Jacobian variety of Y is **of positive p-rank**. *In particular, the* **maximal pro-p abelian** *quotient of N is* **nontrivial** [*cf. Remark* 2.8.1, (i), *below*].

Proof. This assertion is a formal consequence of Theorem 2.7, (i), (ii). □

Remark 2.8.1.

(i) Let us recall that it is well-known that, in the situation of Corollary 2.8, the maximal pro-p abelian quotient of $\pi_1(X)$ has a natural structure of *finitely generated free $\mathbb{Z}_p$-module whose rank coincides with the p-rank of the Jacobian variety of X*.

(ii) Let G be a profinite group and l a prime number. Then it is immediate that the following three conditions are equivalent:

(1) The profinite group G is *pro-prime-to-l*.

(2) The *maximal pro-l abelian* quotient of every open subgroup of G is *trivial.*

(3) An arbitrary [or, alternatively, some] *pro-l Sylow* subgroup of G is *trivial.*

(iii) *M. Raynaud* proved that, in the situation of Corollary 2.8, the profinite group $\pi_1(X)$ is *not pro-prime-to-p* [cf. [5], Corollaire 4.3.2]. Let us observe that it follows from the observation of (ii) that Corollary 2.8 may be regarded as a *refinement* of this result by Raynaud.

Acknowledgement

This research was supported by JSPS KAKENHI Grant Number 18K03239 and by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

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(*Received November 20, 2019*)
(*Accepted March 16, 2021*)