

**LINEAR STABILITY OF RADially SYMMETRIC
EQUILIBRIUM SOLUTIONS TO THE SINGULAR LIMIT
PROBLEM OF THREE-COMPONENT
ACTIVATOR-INHIBITOR MODEL**

TAKUYA KOJIMA AND YOSHIHITO OSHITA

ABSTRACT. We show linear stability or instability for radially symmetric equilibrium solutions to the system of interface equation and two parabolic equations arising in the singular limit of three-component activator-inhibitor models.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

We are interested in the system of equations

$$(1.1) \quad V_{\Gamma(t)} = W(v_1, v_2) - (N - 1)\alpha H \quad \text{on } \Gamma(t), \quad t > 0,$$

$$(1.2) \quad \theta_1 \frac{\partial v_1}{\partial t} = \Delta v_1 + G_1^+(v_1)\chi_{\Omega^+(t)} + G_1^-(v_1)\chi_{\Omega^-(t)} \quad \text{in } \mathbb{R}^N, \quad t > 0,$$

$$(1.3) \quad \theta_2 \frac{\partial v_2}{\partial t} = \Delta v_2 + G_2^+(v_2)\chi_{\Omega^+(t)} + G_2^-(v_2)\chi_{\Omega^-(t)} \quad \text{in } \mathbb{R}^N, \quad t > 0.$$

Here $\Omega^+(t) \subset \mathbb{R}^N$ is a bounded domain, $\Gamma(t) = \partial\Omega^+(t)$ is an embedded surface called an interface, $\Omega^-(t) = \mathbb{R}^N \setminus \overline{\Omega^+(t)}$, H is the mean curvature at each point of $\Gamma(t)$, and $V_{\Gamma(t)}$ is the normal velocity of $\Gamma(t)$ in the direction of $\Omega^-(t)$. Furthermore, θ_1 and θ_2 are nonnegative constants, α is a positive constant, and χ_A denotes the characteristic function of a subset $A \subset \mathbb{R}^N$. Throughout this paper, we assume that $N \geq 2$. We make the following assumptions on G_j^\pm and W .

(G) $G_j^\pm \in C^1(\mathbb{R})$, $\frac{dG_j^\pm}{dv_j}(v_j) < 0$, and there exist $\underline{v}_j, \bar{v}_j$ such that $G_j^+(\bar{v}_j) = 0$, $G_j^-(\underline{v}_j) = 0$, where $-\infty < \underline{v}_j < \bar{v}_j < \infty$, for each $j = 1, 2$.

(W) $W \in C^1(\mathbb{R}^2)$, $W_{v_1}(v_1, v_2) < 0$, and $W_{v_2}(v_1, v_2) < 0$.

A typical example satisfying the assumptions (G) and (W) is $G_j^\pm(v_j) = \pm 1 - v_j$, and $W(v_1, v_2) = -(av_1 + bv_2 + c)$, where a, b, c are constants with $a, b > 0$.

This problem (1.1), (1.2) and (1.3) can be derived formally by taking the singular limit of the following three-component activator-inhibitor model (or

Mathematics Subject Classification. Primary 35B35; Secondary 35K57.

Key words and phrases. singular limit problem, equilibrium solutions, linear stability.

propagator-controller model):

$$\begin{cases} \frac{1}{\alpha} \frac{\partial u}{\partial t} = \Delta u + \frac{1}{\varepsilon^2} \left(u - u^3 + \frac{\sqrt{2}\varepsilon}{3\alpha} W(v_1, v_2) \right), \\ \theta_1 \frac{\partial v_1}{\partial t} = \Delta v_1 + f_1(u, v_1), \\ \theta_2 \frac{\partial v_2}{\partial t} = \Delta v_2 + f_2(u, v_2). \end{cases}$$

Here $f_j(u, v_j)$ is a function that is monotonically decreasing in v_j , and monotonically increasing in u , θ_1 and θ_2 are nonnegative constants, ε is a small parameter, and α is a given constant. When ε is sufficiently small, the phase domains $\{u \sim 1\}$ and $\{u \sim -1\}$ are formed, and the thin layered region appear between them. The internal transition layer has a width of order ε . The discontinuity surface, which is often called the sharp interface, appears in the limit $\varepsilon \rightarrow 0$. The evolution of the interface is governed by not only the inhibitors v_1 and v_2 but also its mean curvature.

Heijster and Sandstede [9] studied travelling spots that bifurcate from radially symmetric stationary spots of three-component FitzHugh–Nagumo system

$$(1.4) \quad \begin{cases} \frac{\partial u}{\partial t} = \varepsilon^2 \Delta u + u - u^3 - \varepsilon(av + bw + c), \\ \theta_1 \frac{\partial v}{\partial t} = \Delta v + u - v, \\ \theta_2 \frac{\partial w}{\partial t} = d^2 \Delta w + u - w. \end{cases}$$

It is suggested that the supercritical drift bifurcation does not occur in two-component FitzHugh–Nagumo system. The existence and stability of planar radially symmetric spots of (1.4) was studied in [8]. Taniguchi [7] studied the linear stability of spherical interfaces in an equilibrium ball in a two-phase boundary problem. Internal layered patterns and sharp interfaces arising in reaction-diffusion systems including two-component or three-component FitzHugh–Nagumo type have been studied extensively in recent years (see [1, 3, 4, 5, 6, 10] and references therein).

Radially symmetric equilibrium solutions. Denote by $\Gamma(R) = \{x \in \mathbb{R}^N : |x| = R\}$ the radially symmetric interface. To consider the radially symmetric stationary solutions to (1.1), (1.2) and (1.3), we define the following functions. For each $j = 1, 2$ and $R > 0$, let $V_j(r, R)$ be the unique

solution to

$$(1.5) \quad \begin{cases} -\Delta_r v = G_j^+(v(r))\chi_{\{r < R\}} + G_j^-(v(r))\chi_{\{r > R\}}, & 0 < r < \infty \\ v_r(0, R) = 0, \quad v(+\infty, R) = \underline{v}_j \end{cases}$$

where

$$\Delta_r := \partial_r^2 + \frac{N-1}{r} \partial_r$$

and $r = |x|$. It is known that for each $j = 1, 2$ the solution $V_j(r, R)$ satisfies $\frac{\partial V_j}{\partial r}(r, R) < 0$ for all $r > 0$. See [2]. We then define the functions $Z_j(R) := V_j(R, R)$ for $j = 1, 2$. Then define $h(R) := W(Z_1(R), Z_2(R))$ and

$$U(R) := h(R) - \frac{(N-1)\alpha}{R}.$$

We see that $U(R_0) = 0$ if and only if $(\Gamma(R_0), V_1(r, R_0), V_2(r, R_0))$ is a radially symmetric equilibrium solution to (1.1)–(1.3).

The linearized eigenvalue problem. Let $\Phi_n(\xi)$, $\xi \in S^{N-1}$ be any spherically harmonic function of degree n . Then

$$(1.6) \quad -\Delta_{S^{N-1}} \Phi_n = \kappa_n \Phi_n \quad \text{on } S^{N-1},$$

where $\Delta_{S^{N-1}}$ denotes the Laplace–Beltrami operator on S^{N-1} and $\kappa_n = n(n + N - 2)$, $n = 0, 1, 2, \dots$. Our linearized eigenvalue problem around the radially symmetric equilibriums is the following:

$$(1.7) \quad \begin{cases} \lambda_n = -\sum_{j=1}^2 P_j(R_0) [\partial_r V_j(R_0, R_0) + z_{j,n}(R_0)] + \frac{\alpha(N-1-\kappa_n)}{R_0^2}, \\ \left(-\Delta_r + \frac{\kappa_n}{r^2} + g_j(r, R_0) + \theta_j \lambda_n \right) z_{j,n}(r) = Q_j(R_0) \delta_{R_0}(r) \quad (j = 1, 2) \end{cases}$$

for each $n = 0, 1, \dots$, where $\delta_{R_0}(r)$ denotes the Dirac delta function concentrated at $r = R_0$, and

$$\begin{aligned} P_j(R) &:= -\frac{\partial W}{\partial v_j}(V_1(R, R), V_2(R, R)) > 0, \\ Q_j(R) &:= G_j^+(V_j(R, R)) - G_j^-(V_j(R, R)) > 0, \\ g_j(r, R) &:= -\frac{dG_j^+}{dv_j}(V_j(r, R))\chi_{\{r < R\}} - \frac{dG_j^-}{dv_j}(V_j(r, R))\chi_{\{r > R\}} > 0 \end{aligned}$$

for each $j = 1, 2$ and $R > 0$. When $(\lambda_n, z_{1,n}, z_{2,n})$ solves (1.7), we call λ_n an eigenvalue of mode n . We can approximate the solutions near $(\Gamma(R_0))$,

$V_1(r, R_0), V_2(r, R_0)$) as in

$$\begin{aligned} \Gamma(t) &= \{[R_0 + \eta\rho(\xi)e^{\lambda t}]\xi + O(\eta^2) : \xi \in S^{N-1}\}, \\ (1.8) \quad v_1(x, t) &= V_1(r, R_0) + \eta w_1(x)e^{\lambda t} + O(\eta^2), \\ v_2(x, t) &= V_2(r, R_0) + \eta w_2(x)e^{\lambda t} + O(\eta^2) \end{aligned}$$

with a small parameter η , $\lambda = \lambda_n$ and

$$(\rho(\xi), w_1(x), w_2(x)) = (\Phi_n(\xi), z_{1,n}(r)\Phi_n(\xi), z_{2,n}(r)\Phi_n(\xi)).$$

See Appendix A for the derivation of this eigenvalue problem.

To state our main results, we define a function

$$(1.9) \quad f(R) = -\frac{(N+1)h(R)}{(N-1)R} + \sum_{j=1}^2 P_j(R)Q_j(R)[\phi_{j,1}(R, R) - \phi_{j,2}(R, R)],$$

where $\phi_{j,1}(r, R)$ ($j = 1, 2$) is the unique solution to the equation

$$(1.10) \quad \begin{cases} (-\Delta_r + \frac{N-1}{r^2} + g_j(r, R))\phi = \delta_R, \\ \phi(\infty, R) = 0, \quad \phi_r(0, R) = 0, \end{cases}$$

for $R > 0$, and $\phi_{j,2}(r, R)$ ($j = 1, 2$) is the unique solution to the equation

$$(1.11) \quad \begin{cases} (-\Delta_r + \frac{2N}{r^2} + g_j(r, R))\phi = \delta_R, \\ \phi(\infty, R) = 0, \quad \phi_r(0, R) = 0, \end{cases}$$

for $R > 0$. Let $\phi_{j,0}(r)$ ($j = 1, 2$) be the unique solution to the equation

$$(1.12) \quad \begin{cases} (-\Delta_r + g_j(r, R_0))\phi = \delta_{R_0}, \\ \phi(\infty, R_0) = 0, \quad \phi_r(0, R_0) = 0, \end{cases}$$

where $R_0 > 0$ is a solution to $U(R_0) = 0$. Our main result gives criteria for the stability of equilibrium solutions.

Theorem 1.1. *Suppose that $R_0 > 0$ satisfies $U(R_0) = 0$. Then (1.1)–(1.3) has an equilibrium solution $(\Gamma(R_0), V_1(r, R_0), V_2(r, R_0))$. Suppose also that $\theta_j \geq 0$ ($j = 1, 2$) satisfies*

$$(1.13) \quad \frac{1}{R_0^{N-1}} \sum_{j=1}^2 P_j(R_0)Q_j(R_0) \left(\int_0^\infty r^{N-1} |\phi_{j,0}|^2 dr \right) \theta_j < 1.$$

Then we have the following:

- (1) *If $U'(R_0) < 0$ and $f(R_0) < 0$, then the equilibrium solution $(\Gamma(R_0), V_1(r, R_0), V_2(r, R_0))$ to (1.1)–(1.3) is linearly stable.*
- (2) *If either $U'(R_0) > 0$ or $f(R_0) > 0$, then the equilibrium solution $(\Gamma(R_0), V_1(r, R_0), V_2(r, R_0))$ to (1.1)–(1.3) is linearly unstable.*

This paper is organized as follows. In Section 2, we prove Theorem 1.1 by using some of the results in [2] and [7]. In Section 3, we give an example of both stable and unstable radially symmetric equilibrium solutions.

2. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1. We regard a radially symmetric function as a function of $r = |x|$. We define

$$C_{0,\text{rad}}^\infty(\mathbb{R}^N) := \{u \in C_0^\infty(\mathbb{R}^N) \mid u \text{ is a radially symmetric function}\}.$$

Let L_{rad}^2 be the completion of $C_{0,\text{rad}}^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^2 := \int_0^\infty r^{N-1}|u|^2 dr.$$

For each $\kappa \geq 0$, let $H_{\text{rad},\kappa}^1$ be the completion of $C_{0,\text{rad}}^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{H_{\text{rad},\kappa}^1}^2 := \int_0^\infty r^{N-1}(|u_r|^2 + |u|^2) + \kappa r^{N-3}|u|^2 dr.$$

We regard $v \in L_{\text{rad}}^2$ as an element of $(H_{\text{rad},\kappa}^1)'$ such that

$$\langle u, v \rangle = \int_0^\infty r^{N-1}u(r)\overline{v(r)} dr$$

for $u \in H_{\text{rad},\kappa}^1$. For $j = 1, 2$, let $L_j(\kappa, \lambda)$ be a linear operator from $H_{\text{rad},\kappa}^1$ to $(H_{\text{rad},\kappa}^1)'$ such that

$$(2.1) \quad \langle u, L_j v \rangle = \int_0^\infty \left[r^{N-1} \frac{du}{dr} \cdot \frac{d\bar{v}}{dr} + r^{N-3} \kappa u \bar{v} + r^{N-1} (g_j + \theta_j \bar{\lambda}) u \bar{v} \right] dr$$

for all $u, v \in H_{\text{rad},\kappa}^1$. For smooth v , we have

$$(2.2) \quad L_j(\kappa, \lambda)v = -\Delta_r v + \left(\frac{\kappa}{r^2} + g_j(r, R) + \theta_j \lambda \right) v.$$

For $j = 1, 2$, let $u_j(\cdot, \kappa, \lambda)$ be the unique solution to the equation

$$(2.3) \quad L_j(\kappa, \lambda)u_j = \delta_{R_0}, \quad u_j(\cdot, \kappa, \lambda) \in H_{\text{rad},\kappa}^1$$

for $\kappa \geq 0$ and $\text{Re } \lambda \geq 0$. Then for $j = 1, 2$, we have $\phi_{j,0}(r) = u_j(r, 0, 0)$, $\phi_{j,1}(r, R_0) = u_j(r, N - 1, 0)$, and $\phi_{j,2}(r, R_0) = u_j(r, 2N, 0)$.

Let $R_0 > 0$ be a number such that $U(R_0) = 0$, and $(\Gamma(R_0), V_1(r, R_0), V_2(r, R_0))$ be the associated equilibrium solution. Assume that $(\lambda_n, z_{1,n}(r))$,

$z_{2,n}(r)$ solves (1.7). Since $u_j(r, \kappa, \lambda)$ ($j = 1, 2$) satisfy the equation (2.3), we have $z_{j,n}(r) = Q_j u_j(r, \kappa_n, \lambda_n)$ for $j = 1, 2$. Hence, we obtain

$$-\sum_{j=1}^2 (P_j \partial_r V_j(R_0, R_0) + P_j Q_j u_j(R_0, \kappa_n, \lambda_n)) + \frac{\alpha(N-1-\kappa_n)}{R_0^2} - \lambda_n = 0.$$

Now we define

(2.4)

$$F(\kappa, \lambda) := -\sum_{j=1}^2 (P_j \partial_r V_j(R_0, R_0) + P_j Q_j u_j(R_0, \kappa, \lambda)) + \frac{\alpha(N-1-\kappa)}{R_0^2} - \lambda,$$

and

$$(2.5) \quad E(\kappa) := -\sum_{j=1}^2 (P_j \partial_r V_j(R_0, R_0) + P_j Q_j u_j(R_0, \kappa, 0)) + \frac{\alpha(N-1-\kappa)}{R_0^2}.$$

We have the following:

Lemma 1. *For every $\kappa > 0$, there holds $E''(\kappa) < 0$.*

Proof. From (2.5), we get

$$E'(\kappa) = -\frac{\alpha}{R_0^2} - \sum_{j=1}^2 P_j(R_0) Q_j(R_0) \frac{\partial u_j}{\partial \kappa}(R_0, \kappa, 0).$$

Therefore we have

$$(2.6) \quad E''(\kappa) = -\sum_{j=1}^2 P_j(R_0) Q_j(R_0) \frac{\partial^2 u_j}{\partial \kappa^2}(R_0, \kappa, 0).$$

Notice that

$$\begin{aligned} R_0^{N-1} \frac{\partial^2 u_j}{\partial \kappa^2}(R_0, \kappa, 0) &= \left\langle \frac{\partial^2 u_j}{\partial \kappa^2}(\cdot, \kappa, 0), \delta_{R_0} \right\rangle \\ &= \left\langle \frac{\partial^2 u_j}{\partial \kappa^2}(\cdot, \kappa, 0), L_j(\kappa, 0) u_j(\cdot, \kappa, 0) \right\rangle \\ &= \left\langle L_j(\kappa, 0) \frac{\partial^2 u_j}{\partial \kappa^2}(\cdot, \kappa, 0), u_j(\cdot, \kappa, 0) \right\rangle \end{aligned}$$

for $j = 1, 2$. Using

$$L_j(\kappa, 0) \left(\frac{\partial^2 u_j}{\partial \kappa^2}(\cdot, \kappa, 0) \right) = -\frac{2}{r^2} \frac{\partial u_j}{\partial \kappa}(\cdot, \kappa, 0)$$

and

$$L_j(\kappa, 0) \left(\frac{\partial u_j}{\partial \kappa}(\cdot, \kappa, 0) \right) = -\frac{1}{r^2} u_j(\cdot, \kappa, 0),$$

we compute

$$\begin{aligned} & \left\langle L_j(\kappa, 0) \frac{\partial^2 u_j}{\partial \kappa^2}(\cdot, \kappa, 0), u_j(\cdot, \kappa, 0) \right\rangle \\ &= 2 \left\langle \frac{\partial u_j}{\partial \kappa}(\cdot, \kappa, 0), -\frac{1}{r^2} u_j(\cdot, \kappa, 0) \right\rangle \\ &= 2 \left\langle \frac{\partial u_j}{\partial \kappa}(\cdot, \kappa, 0), L_j(\kappa, 0) \frac{\partial u_j}{\partial \kappa}(\cdot, \kappa, 0) \right\rangle > 0 \end{aligned}$$

for $j = 1, 2$. It follows from (2.6) that $E''(\kappa) < 0$ as desired. \square

Lemma 2. *For all $\kappa \geq 0$ and $\operatorname{Re} \lambda \geq 0$, there holds*

$$(2.7) \quad \|u_j(\cdot, \kappa, \lambda)\|^2 \leq \|u_j(\cdot, 0, 0)\|^2.$$

Proof. Assume that $\lambda = \lambda^R + i\lambda^I$ is an eigenvalue of (1.7) with $\lambda^R \geq 0$. Differentiating (2.3) with respect to λ^R , we have

$$-\left(\frac{\partial u_j}{\partial \lambda^R}\right)'' - \frac{N-1}{r} \left(\frac{\partial u_j}{\partial \lambda^R}\right)' + \frac{\kappa}{r^2} \frac{\partial u_j}{\partial \lambda^R} + g_j(r) \frac{\partial u_j}{\partial \lambda^R} + \theta_j u_j + \theta_j \lambda^R \frac{\partial u_j}{\partial \lambda^R} = 0.$$

This implies that

$$(2.8) \quad L_j \left(\frac{\partial u_j}{\partial \lambda^R} \right) = -\theta_j u_j \quad (j = 1, 2).$$

Similarly we have

$$(2.9) \quad L_j \left(\frac{\partial u_j}{\partial \lambda^I} \right) = -i\theta_j u_j \quad (j = 1, 2).$$

Furthermore, we differentiate (2.3) with respect to κ , we obtain

$$(2.10) \quad L_j \left(\frac{\partial u_j}{\partial \kappa} \right) = -\frac{u_j}{r^2} \quad (j = 1, 2).$$

We show that for all $\lambda^I \neq 0$, there holds $\|u_j(\cdot, \kappa, \lambda^R + i\lambda^I)\|^2 < \|u_j(\cdot, \kappa, \lambda^R)\|^2$ ($j = 1, 2$). It follows from (2.9) that

$$\begin{aligned} \frac{\partial}{\partial \lambda^I} \|u_j(\cdot, \kappa, \lambda^R + i\lambda^I)\|^2 &= 2 \operatorname{Re} \left\langle u_j, \frac{\partial u_j}{\partial \lambda^I} \right\rangle \\ &= 2 \operatorname{Re} \left\langle \frac{i}{\theta_j} L_j \left(\frac{\partial u_j}{\partial \lambda^I} \right), \frac{\partial u_j}{\partial \lambda^I} \right\rangle \\ &= -2\lambda^I \left\| \frac{\partial u_j}{\partial \lambda^I} \right\|^2 \end{aligned}$$

for $j = 1, 2$. Therefore, $\|u_j(\cdot, \kappa, \lambda^R + i\lambda^I)\|^2 < \|u_j(\cdot, \kappa, \lambda^R)\|^2$ ($j = 1, 2$) for all $\lambda^I \neq 0$.

We consider the case of $\lambda^I = 0$. Since $R_0^{N-1}u_j(R_0, \kappa, \lambda^R) = \langle u_j, L_j u_j \rangle$, we find that $u_j(R_0, \kappa, \lambda^R) > 0$ holds for $j = 1, 2$. By $u_j(\infty, \kappa, \lambda^R) = 0$ and the maximum principle, we see that $u_j(r, \kappa, \lambda^R) > 0$ ($j = 1, 2$) for all $r > 0$. It therefore follows from (2.8) and (2.10) that

$$L_j \left(\frac{\partial u_j}{\partial \lambda^R} \right) < 0, \quad L_j \left(\frac{\partial u_j}{\partial \kappa} \right) < 0$$

for $j = 1, 2$. By

$$\frac{\partial u_j}{\partial \lambda^R}(\infty, \kappa, \lambda^R) = \frac{\partial u_j}{\partial \kappa}(\infty, \kappa, \lambda^R) = 0$$

and the maximum principle, we obtain that

$$\frac{\partial u_j}{\partial \lambda^R}(r, \kappa, \lambda^R) < 0, \quad \frac{\partial u_j}{\partial \kappa}(r, \kappa, \lambda^R) < 0$$

for all $r > 0$ and $j = 1, 2$. Thus it follows from

$$\begin{aligned} \frac{\partial}{\partial \lambda^R} \|u_j(\cdot, \kappa, \lambda^R)\|^2 &= 2 \left\langle u_j, \frac{\partial u_j}{\partial \lambda^R} \right\rangle, \\ \frac{\partial}{\partial \kappa} \|u_j(\cdot, \kappa, \lambda^R)\|^2 &= 2 \left\langle u_j, \frac{\partial u_j}{\partial \kappa} \right\rangle, \end{aligned}$$

that $\frac{\partial}{\partial \lambda^R} \|u_j(\cdot, \kappa, \lambda^R)\|^2 < 0$ and $\frac{\partial}{\partial \kappa} \|u_j(\cdot, \kappa, \lambda^R)\|^2 < 0$ for $j = 1, 2$. We conclude that $\|u_j(\cdot, \kappa, \lambda)\|^2 \leq \|u_j(\cdot, 0, 0)\|^2$ ($j = 1, 2$) for all $\kappa \geq 0$ and $\text{Re } \lambda \geq 0$. This completes the proof. \square

We consider the equation

$$(2.11) \quad F(\kappa, \lambda) = 0, \quad \text{Re } \lambda \geq 0$$

for each $\kappa \geq 0$.

Lemma 3. *Assume that (1.13). Then any solution λ of (2.11) with a nonnegative real part must be real.*

Proof. For $j = 1, 2$, we write $u_j(r, \kappa, \lambda) = u_j^R + iu_j^I$ where both u_j^R and u_j^I are real. We can calculate $u_j(R_0, \kappa, \lambda)$ ($j = 1, 2$) as

$$(2.12) \quad R_0^{N-1}u_j(R_0, \kappa, \lambda) = \langle u_j, \delta_{R_0} \rangle = \langle u_j, L_j u_j \rangle.$$

Taking the imaginary part, we have $R_0^{N-1}u_j^I(R_0, \kappa, \lambda) = -\lambda^I \theta_j \|u_j(\cdot, \kappa, \lambda)\|^2$ for $j = 1, 2$. We obtain from $\text{Im } F(\kappa, \lambda) = 0$ that

$$\lambda^I \left[\frac{1}{R_0^{N-1}} \sum_{j=1}^2 P_j Q_j \theta_j \|u_j(\cdot, \kappa, \lambda)\|^2 - 1 \right] = 0.$$

It follows from (2.7) and (1.13) that

$$\frac{1}{R_0^{N-1}} \sum_{j=1}^2 P_j Q_j \theta_j \|u_j(\cdot, \kappa, \lambda)\|^2 - 1 \leq \frac{1}{R_0^{N-1}} \sum_{j=1}^2 P_j Q_j \theta_j \|u_j(\cdot, 0, 0)\|^2 - 1 < 0.$$

This implies $\lambda^I = 0$. Therefore the eigenvalue λ must be real. This completes the proof. \square

Lemma 4. *For all $\kappa \geq 0$ and $\lambda \in \mathbb{R}$ with $\lambda \geq 0$, there holds*

$$F_\lambda(\kappa, \lambda) \leq \frac{1}{R_0^{N-1}} \sum_{j=1}^2 P_j Q_j \theta_j \|u_j(\cdot, 0, 0)\|^2 - 1.$$

Proof. Let $\kappa \geq 0$ and $\lambda \geq 0$. From (2.1) and (2.8), we get

$$R_0^{N-1} \frac{\partial u_j}{\partial \lambda}(R_0, \kappa, \lambda) = -\theta_j \|u_j\|^2$$

for $j = 1, 2$. It then follows that

$$\begin{aligned} F_\lambda(\kappa, \lambda) &= -1 - \sum_{j=1}^2 P_j(R_0) Q_j(R_0) \frac{\partial u_j}{\partial \lambda}(R_0, \kappa, \lambda) \\ &= -1 + \frac{1}{R_0^{N-1}} \sum_{j=1}^2 P_j Q_j \theta_j \|u_j(\cdot, \kappa, \lambda)\|^2 \\ &\leq \frac{1}{R_0^{N-1}} \sum_{j=1}^2 P_j Q_j \theta_j \|u_j(\cdot, 0, 0)\|^2 - 1 \end{aligned}$$

as desired. \square

Proposition 5. *Assume that (1.13) holds. Then*

- (1) (2.11) has a unique solution $\lambda > 0$ if $E(\kappa) > 0$.
- (2) (2.11) has a unique solution $\lambda = 0$ if $E(\kappa) = 0$.
- (3) (2.11) has no solution if $E(\kappa) < 0$.

Proof. Note that by Lemma 3, any solution λ of (2.11) with a nonnegative real part is real. By Lemma 4, we have

$$(2.13) \quad F(\kappa, \lambda) \leq F(\kappa, 0) - A\lambda = E(\kappa) - A\lambda$$

for $\lambda \geq 0$ with

$$A := 1 - \frac{1}{R_0^{N-1}} \sum_{j=1}^2 P_j Q_j \theta_j \|u_j(\cdot, 0, 0)\|^2 > 0.$$

The claims (2) and (3) follow from (2.13).

(1) Let $E(\kappa) > 0$. Then it follows from (2.13) that $F(\kappa, 0) > 0 > F(\kappa, \lambda)$ if $\lambda > E(\kappa)/A$. Therefore by the monotonicity of $F(\kappa, \cdot)$ on $[0, \infty)$, there exists a unique $\lambda_* > 0$ such that $F(\kappa, \lambda_*) = 0$. This completes the proof of (1). \square

In order to study the stability of $(\Gamma(R_0), V_1(r, R_0), V_2(r, R_0))$, we need to determine the sign of $E(0)$ and $E(2N)$.

Lemma 6. *Assume $U(R_0) = 0$. Then there holds*

$$E(0) = U'(R_0).$$

Proof. Differentiating $U(R)$ with respect to R , we get

$$(2.14) \quad U'(R) = \frac{\partial W}{\partial v_1} \cdot \frac{dV_1}{dR}(R, R) + \frac{\partial W}{\partial v_2} \cdot \frac{dV_2}{dR}(R, R) + \frac{(N-1)\alpha}{R^2}.$$

Differentiating (1.5) with respect to R , we have

$$(2.15) \quad (-\Delta_r + g_j(r, R)) \frac{\partial V_j}{\partial R} = [G_j^+(V_j(r, R)) - G_j^-(V_j(r, R))] \delta_R(r)$$

for $j = 1, 2$. Substituting $R = R_0$ into (2.15), we obtain

$$(-\Delta_r + g_j(R_0, R_0)) \frac{\partial V_j}{\partial R}(R_0, R_0) = Q_j(R_0) \delta_{R_0},$$

and thus

$$(2.16) \quad \frac{\partial V_j}{\partial R}(R_0, R_0) = Q_j(R_0) u_j(R_0, 0, 0)$$

for $j = 1, 2$. For each $j = 1, 2$ and all $R > 0$, we have

$$\frac{dV_j}{dR}(R, R) = \frac{\partial V_j}{\partial r}(R, R) + \frac{\partial V_j}{\partial R}(R, R).$$

Therefore, substituting $R = R_0$ into (2.14) and using the equations (2.15) and (2.16), we obtain

$$U'(R_0) = - \sum_{j=1}^2 (P_j \partial_r V_j(R_0, R_0) + P_j Q_j u_j(R_0, 0, 0)) + \frac{(N-1)\alpha}{R_0^2}.$$

By the definition of $E(\kappa)$, we obtain the desired relation. This completes the proof. \square

Lemma 7. *Assume $U(R_0) = 0$. Then*

$$E(2N) = f(R_0).$$

Proof. Differentiating (1.5) with respect to r , we have

$$L_j(N - 1, 0) \left(\frac{\partial V_j}{\partial r} \right) = -Q_j(R) \delta_R$$

for $j = 1, 2$. Thus we find that

$$(2.17) \quad -\frac{\partial V_j}{\partial r}(r, R) = Q_j(R) \phi_{j,1}(r, R)$$

for $j = 1, 2$. From $U(R_0) = 0$, we get

$$\alpha = \frac{R_0 h(R_0)}{N - 1}.$$

Therefore, by using the definition of $E(\kappa)$, we obtain $E(2N) = f(R_0)$, where f is defined as in (1.9). This completes the proof. \square

Completion of Proof of Theorem 1.1.

Case 1: Assume that $U'(R_0) > 0$. Then this means that $E(0) > 0$ by Lemma 6. Hence there exists a positive eigenvalue $\lambda_0 > 0$ of mode 0 by Proposition 5 (1).

Case 2: Assume that $f(R_0) > 0$. Then $E(2N) > 0$ by Lemma 7. By Proposition 5 (1), we see that there exists a positive eigenvalue $\lambda_2 > 0$ of mode 2.

Case 3: Assume that $U'(R_0) < 0$ and $f(R_0) < 0$, then we have $E(0) < 0$ and $E(2N) < 0$. Note that we have

$$(2.18) \quad u_j(r, N - 1, 0) = \phi_{j,1}(r, R_0)$$

for $j = 1, 2$. Substituting $\kappa = N - 1$ into (2.5) and using the equations (2.17) and (2.18), we get $E(N - 1) = 0$. Combining this fact and $E''(\kappa) < 0$, we see that $E(\kappa) \leq E(2N) < 0$ for all $\kappa \in [2N, \infty)$. Therefore $E(\kappa_n) < 0$ for all $n \neq 1$. By Proposition 5 (3), we see that $\text{Re } \lambda_n < 0$ for all $n \neq 1$. Moreover by Proposition 5 (2), there exists no eigenvalue λ_1 of mode 1 such that $\text{Re } \lambda_1 \geq 0$ and $\lambda_1 \neq 0$.

This completes the proof of Theorem 1.1.

3. AN EXAMPLE

In this section, we present an example to illustrate the existence and the stability of equilibrium solutions. If θ_1 and θ_2 are sufficiently small, the stability of equilibria is determined by the eigenvalues λ_0 and λ_2 .

Example 1. *Let $N = 3$. Consider the following problem:*

$$(3.1) \quad \begin{aligned} V_{\Gamma(t)} &= -kv_1 - (1-k)v_2 - 2\alpha H \quad \text{on } \Gamma(t), \quad t > 0, \\ \theta_1 \frac{\partial v_1}{\partial t} &= \Delta v_1 + (1 - b^2 v_1 + c)\chi_{\Omega^+} + (-1 - b^2 v_1 + c)\chi_{\Omega^-} \quad \text{in } \mathbb{R}^3, \quad t > 0, \\ \theta_2 \frac{\partial v_2}{\partial t} &= \Delta v_2 + (1 - b^2 v_2 + c)\chi_{\Omega^+} + (-1 - b^2 v_2 + c)\chi_{\Omega^-} \quad \text{in } \mathbb{R}^3, \quad t > 0. \end{aligned}$$

Here, $b \in (0, \infty)$, $k \in (0, 1)$, $c = 1 - 2e^{-2} \approx 0.72933$ are constants, and $\alpha > 0$ is a parameter. Assume that θ_1 and θ_2 satisfy

$$(3.2) \quad \theta_1 \geq 0, \quad \theta_2 \geq 0, \quad k\theta_1 + (1-k)\theta_2 \leq 2b^3.$$

Let $G_j^\pm(v_j) = \pm 1 - b^2 v_j + c$ ($j = 1, 2$) and $W(v_1, v_2) = -kv_1 - (1-k)v_2$. Then G_j^\pm and W satisfy all the assumptions (G) and (W) in Section 1 with $\underline{v}_j = -b^{-2}(1-c) < 0$, $\bar{v}_j = b^{-2}(1+c) > 0$. We use the same notations $h(R)$, $U(R)$, $P_j(R)$, $Q_j(R)$, $V_j(r, R)$, $g_j(r, R)$, $\phi_{j,1}(r, R)$, and $\phi_{j,2}(r, R)$ as in Section 1.

The radially symmetric stationary problem of (3.1) such that $v_j(x)$ has a finite limit as $|x| \rightarrow \infty$ is given by

$$(3.3) \quad h(R) = \frac{2\alpha}{R},$$

$$(3.4) \quad \left(-\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} \right) v_j = (1 - b^2 v_j + c)\chi_{\{r < R\}} + (-1 - b^2 v_j + c)\chi_{\{r > R\}},$$

$$(3.5) \quad v_j'(0) = 0,$$

$$(3.6) \quad \lim_{r \rightarrow \infty} v_j(r) = -b^{-2}(1-c),$$

where $h(R) = -kv_1(R) - (1-k)v_2(R)$ and $j = 1, 2$. The explicit solution $v_j(r) = V_j(r, R)$ of (3.4)–(3.6) is

$$v_j(r) = \begin{cases} b^{-2}[1 + c - 2(1 + bR)e^{-bR}(br)^{-1} \sinh(br)] & \text{if } r < R, \\ 2b^{-2}[bR \cosh(bR) - \sinh(bR)](br)^{-1} e^{-br} - b^{-2}(1-c) & \text{if } r > R \end{cases}$$

for $j = 1, 2$. Therefore we get

$$(3.7) \quad h(R) = b^{-2} \left[\frac{1}{bR} - \frac{e^{-2bR}}{bR} - e^{-2bR} - c \right].$$

We find that $h'(R) < 0$ for $R > 0$, and $h(b^{-1}) = 0$. Hence $h(R) > 0$ for $R \in (0, b^{-1})$ and $h(R) < 0$ for $R \in (b^{-1}, \infty)$.

We consider the equation (3.3), that is, $U(R) = 0$. Now $U(R) = 0$ if and only if

$$\alpha = \frac{R}{2}b^{-2} \left[\frac{1}{bR} - \frac{e^{-2bR}}{bR} - e^{-2bR} - c \right] =: F_0(R).$$

Then we see that $F_0''(R) < 0$ for $R > 0$, $\lim_{R \rightarrow 0^+} F_0(R) = 0$, and $F_0(b^{-1}) = 0$. Hence there exists a unique $R_* \in (0, b^{-1})$ such that $F_0'(R_*) = 0$. We have $F_0'(R) > 0$ for $R \in (0, R_*)$, and $F_0'(R) < 0$ for $R \in (R_*, \infty)$. Let $\alpha_1 = F_0(R_*) > 0$. Then we have the following:

- $U(R) = 0$ has two solutions $R = R_1(\alpha), R_2(\alpha)$ for each $\alpha \in (0, \alpha_1)$, where $0 < R_1(\alpha) < R_2(\alpha)$, $R_1(\alpha)$ is monotonically increasing, $R_2(\alpha)$ is monotonically decreasing in $(0, \alpha_1)$, $\lim_{\alpha \rightarrow 0^+} R_2(\alpha) = b^{-1}$, and $\lim_{\alpha \rightarrow \alpha_1^-} R_2(\alpha) = \lim_{\alpha \rightarrow \alpha_1^-} R_1(\alpha) = R_*$. Moreover $U'(R_1(\alpha)) > 0$ and $U'(R_2(\alpha)) < 0$ for $\alpha \in (0, \alpha_1)$.
- $U(R) = 0$ has exactly one solution $R = R_*$, and $U'(R_*) = 0$ for $\alpha = \alpha_1$.
- $U(R) = 0$ has no solution for each $\alpha \in (\alpha_1, \infty)$.

Next we consider the linear stability of these equilibriums. Note that we have $P_1(R) = k, P_2(R) = 1 - k, Q_j(R) = 2$, and $g_j(r, R) \equiv b^2$. For $j = 1, 2$, let $u_j(r, R, \kappa_n)$ be the unique solution to

$$\mathcal{L}_j(\kappa_n)u(r) = \delta_R, \quad u \in H_{\text{rad}, \kappa}^1$$

where $R > 0, \kappa_n = n(n + 1)$, and the operator $\mathcal{L}_j(\kappa)$ is defined as in

$$\mathcal{L}_j(\kappa) = -\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{\kappa}{r^2} + b^2.$$

Then for $j = 1, 2, u_j(r, R, \kappa_n)$ can be expressed as

$$u_j(r, R, \kappa_n) = \begin{cases} R\sqrt{R/r}I_{n+\frac{1}{2}}(br)K_{n+\frac{1}{2}}(bR) & \text{if } r < R \\ R\sqrt{R/r}I_{n+\frac{1}{2}}(bR)K_{n+\frac{1}{2}}(br) & \text{if } r > R, \end{cases}$$

where $I_{n+\frac{1}{2}}$ and $K_{n+\frac{1}{2}}$ are the modified Bessel functions of the first kind and the second kind, respectively. Since $\phi_{j,1}(r, R) = u_j(r, R, \kappa_1)$ and $\phi_{j,2}(r, R) = u_j(r, R, \kappa_2)$, we have

$$\begin{aligned} \phi_{j,1}(R, R) &= RI_{\frac{3}{2}}(bR)K_{\frac{3}{2}}(bR), \\ \phi_{j,2}(R, R) &= RI_{\frac{5}{2}}(bR)K_{\frac{5}{2}}(bR) \end{aligned}$$

for $j = 1, 2$. We set $s = bR$. By using the expression (1.9) of $f(R)$, we obtain

$$(3.8) \quad f(R) = 2R(I_{\frac{3}{2}}(s)K_{\frac{3}{2}}(s) - I_{\frac{5}{2}}(s)K_{\frac{5}{2}}(s)) - \frac{2h(R)}{R}.$$

By the elementary computation, we get

$$(3.9) \quad \begin{aligned} & I_{\frac{3}{2}}(s)K_{\frac{3}{2}}(s) - I_{\frac{5}{2}}(s)K_{\frac{5}{2}}(s) \\ &= \frac{1}{2s^5 e^{2s}} [(2s^2 - 9)e^{2s} + 2s^4 + 8s^3 + 16s^2 + 18s + 9]. \end{aligned}$$

Substituting this relation and (3.7) into (3.8), we have

$$f(R) = \frac{R}{s^5 e^{2s}} [(2cs^3 - 9)e^{2s} + 2s^4 + 10s^3 + 18s^2 + 18s + 9].$$

Now we define

$$\begin{aligned} F_1(s) &= (2cs^3 - 9)e^{2s} + 2s^4 + 10s^3 + 18s^2 + 18s + 9, \\ F_2(s) &= s^5 e^{2s} > 0 \end{aligned}$$

for $s > 0$. By the assumption $0 < c < 1$, we find that $F_1(s)$ has a unique zero point $s_0 > 0$ with $F_1(s_0) = 0$, $F_1(s) < 0$ for $s \in (0, s_0)$, and $F_1(s) > 0$ for $s \in (s_0, \infty)$. Since $F_1(1) = 53 - 7e^2 \approx 1.2766 > 0$, we see that $s_0 < 1$. On the other hand, if $\alpha = \alpha_1$, then $R = R_*$ is an equilibrium such that $E(0) = U'(R_*) = 0$. Since $E''(\kappa)$ is negative and $E(N - 1) = 0$, we see that $f(R_*) = E(2N)$ should be negative. Hence $bR_* < s_0$. Therefore there exists a unique $\alpha_2 \in (0, \alpha_1)$ such that $R_2(\alpha_2) = b^{-1}s_0$.

Let $R_0 > 0$ be an equilibrium solution. Note that $\phi_{j,0}(r)$ ($j = 1, 2$) is given by

$$\phi_{j,0}(r) = \begin{cases} R_0 \sqrt{R_0/r} I_{\frac{1}{2}}(br) K_{\frac{1}{2}}(bR_0) & \text{if } r < R_0 \\ R_0 \sqrt{R_0/r} I_{\frac{1}{2}}(bR_0) K_{\frac{1}{2}}(br) & \text{if } r > R_0. \end{cases}$$

Then the condition (1.13) becomes

$$\begin{aligned} & \left(K_{\frac{1}{2}}(bR_0)^2 \int_0^{R_0} r I_{\frac{1}{2}}(br)^2 dr + I_{\frac{1}{2}}(bR_0)^2 \int_{R_0}^{\infty} r K_{\frac{1}{2}}(br)^2 dr \right) \\ & \times (k\theta_1 + (1 - k)\theta_2) < \frac{1}{2R_0}, \end{aligned}$$

that is,

$$k\theta_1 + (1 - k)\theta_2 < \frac{2b^3}{e^{-2bR_0}(e^{2bR_0} - 1 - 2bR_0)}.$$

Therefore under the condition (3.2), we have the following:

- $R_0 = R_2(\alpha)$ is linearly unstable for $\alpha \in (0, \alpha_2)$.

- $R_0 = R_2(\alpha)$ is linearly stable for $\alpha \in (\alpha_2, \alpha_1)$.
- $R_0 = R_1(\alpha)$ is linearly unstable for $\alpha \in (0, \alpha_1)$.

We remark that numerical computations show that

$$\begin{aligned} R_* &\approx 0.508739 \cdot b^{-1}, & \alpha_1 &\approx 0.0417721 \cdot b^{-3}, \\ s_0 &\approx 0.808191, & \alpha_2 &\approx 0.0257134 \cdot b^{-3}. \end{aligned}$$

APPENDIX A. DERIVATION OF THE LINEARIZED EIGENVALUE PROBLEM

To approximate solutions near the stationary solution $(\Gamma(R_0), V_1(r, R_0), V_2(r, R_0))$, set

$$\begin{aligned} \Gamma(t) &= \{[R_0 + \eta\rho(\xi)e^{\lambda t}]\xi + O(\eta^2) : \xi \in S^{N-1}\}, \\ (A.1) \quad v_1(x, t) &= V_1(r, R_0) + \eta w_1(x)e^{\lambda t} + O(\eta^2), \\ v_2(x, t) &= V_2(r, R_0) + \eta w_2(x)e^{\lambda t} + O(\eta^2) \end{aligned}$$

with small parameter η . Here $\lambda \in \mathbb{C}$, while $\rho(\xi)$ and $w_j(x)$ ($j = 1, 2$) are real valued functions on S^{N-1} and \mathbb{R}^N , respectively.

By substituting (A.1) into (1.1), (1.2), and (1.3), dividing both sides by $\eta e^{\lambda t}$, and sending η to 0, we obtain

$$\begin{aligned} (A.2) \quad \lambda\rho(\xi) &= -\sum_{j=1}^2 P_j(R_0)[V'_j(R_0, R_0)\rho(\xi) + w_j(R_0\xi)] \\ &\quad + \frac{\alpha}{R_0^2}[(N-1)\rho(\xi) + \Delta_{S^{N-1}}\rho(\xi)]. \end{aligned}$$

$$(A.3) \quad (-\Delta + g_j(R_0, R_0) + \theta_j\lambda)w_j = \rho(\xi)Q_j(R_0)\delta_{R_0} \quad (j = 1, 2).$$

Here $\Delta_{S^{N-1}}$ denotes the Laplace–Beltrami operator on S^{N-1} .

Since the set $\{\Phi_n\}_{n=0}^\infty$ of the spherically harmonic functions is complete for the continuous functions on S^{N-1} , we can expand $\rho(\xi)$, $w_1(x)$, $w_2(x)$ in a Fourier series:

$$(A.4) \quad \rho(\xi) = \sum_{n=0}^\infty \rho_n \Phi_n(\xi), \quad w_j(x) = \sum_{n=0}^\infty w_{j,n}(r) \Phi_n(\xi) \quad (j = 1, 2).$$

Then we have

$$\begin{aligned} \lambda \sum_{n=0}^\infty \rho_n \Phi_n(\xi) &= -\sum_{j=1}^2 \sum_{n=0}^\infty P_j(R_0)[V'_j(R_0, R_0)\rho_n \Phi_n(\xi) + w_{j,n}(R_0)\Phi_n(\xi)] \\ &\quad + \sum_{n=0}^\infty \frac{\alpha(N-1-\kappa_n)}{R_0^2} \rho_n \Phi_n(\xi), \end{aligned}$$

$$\sum_{n=0}^{\infty} \left(-\Delta_r + \frac{\kappa_n}{r^2} + g_j + \theta_j \lambda \right) w_{j,n}(r) \Phi_n(\xi) = \sum_{n=0}^{\infty} \rho_n Q_j(R_0) \Phi_n(\xi) \delta_{R_0}$$

for $j = 1, 2$. Therefore for each n ,

$$\begin{aligned} \lambda \rho_n &= - \sum_{j=1}^2 P_j(R_0) [V_j'(R_0, R_0) \rho_n + w_{j,n}(R_0)] \\ &\quad + \frac{\alpha(N-1-\kappa_n)}{R_0^2} \rho_n, \end{aligned}$$

$$\left(-\Delta_r + \frac{\kappa_n}{r^2} + g_j + \theta_j \lambda \right) w_{j,n}(r) = \rho_n Q_j(R_0) \delta_{R_0} \quad (j = 1, 2).$$

If $\rho_n \neq 0$ for some n , then setting $z_{j,n} = \frac{w_{j,n}}{\rho_n}$ ($j = 1, 2$), $(\lambda, z_{1,n}, z_{2,n})$ solves

$$\begin{aligned} \lambda &= - \sum_{j=1}^2 P_j(R_0) [V_j'(R_0, R_0) + z_{j,n}(R_0)] \\ &\quad + \frac{\alpha(N-1-\kappa_n)}{R_0^2}, \end{aligned}$$

$$\left(-\Delta_r + \frac{\kappa_n}{r^2} + g_j + \theta_j \lambda \right) z_{j,n}(r) = Q_j(R_0) \delta_{R_0} \quad (j = 1, 2).$$

REFERENCES

- [1] X. Chen, *Generation and propagation of interfaces in reaction-diffusion systems*, Trans. Amer. Math. Soc. **334**, 877–913 (1992)
- [2] X. Chen and M. Taniguchi, *Instability of spherical interfaces in a nonlinear free boundary problem*, Advances in Differential Equations **5**, 747–772 (2000)
- [3] P.C. Fife, *Dynamics of internal layers and diffusive interfaces*, CBMS-NSF Regional Conference Series in Applied Mathematics 53, SIAM, Philadelphia, 1988
- [4] D. Hilhorst, Y. Nishiura and M. Mimura, *A free boundary problem arising in some reaction-diffusion systems*, Proc. Roy. Soc. Edinburgh **118A**, 355–378 (1991)
- [5] T. Kajiwara and K. Kurata, *On a variational problem arising from the three-component FitzHugh-Nagumo type reaction-diffusion systems*, Tokyo J. Math. **41**, 131–174 (2018)
- [6] Y. Oshita, *Singular limit problem for some elliptic systems*, SIAM J. Math. Anal. **38**, 1886–1911 (2007)
- [7] M. Taniguchi, *Multiple existence and linear stability of equilibrium balls in a nonlinear free boundary problem*, Quarterly of Applied Mathematics **58**, 283–302 (2000)
- [8] P. van Heijster and B. Sandstede, *Planar radial spots in a three-component FitzHugh-Nagumo system*, Nonlinear Science **21**, 705–745 (2011)
- [9] P. van Heijster and B. Sandstede, *Bifurcations to travelling planar spots in a three-component FitzHugh-Nagumo system*, Physica D **275**, 19–34 (2014)
- [10] P. van Heijster, C-N. Chen, Y. Nishiura, and T. Teramoto, *Pinned solutions in a heterogeneous three-component FitzHugh-Nagumo model*, J. Dynam. Differential Equations **31**, 153–203 (2019)

TAKUYA KOJIMA
GRADUATE SCHOOL OF NATURAL SCIENCE AND TECHNOLOGY
OKAYAMA UNIVERSITY
OKAYAMA 700-8530, JAPAN

e-mail address: puqc9o2q@s.okayama-u.ac.jp

YOSHIHITO OSHITA
DEPARTMENT OF MATHEMATICS, OKAYAMA UNIVERSITY, 3-1-1 TSUSHIMA-NAKA,
OKAYAMA 700-8530, JAPAN

e-mail address: oshita@okayama-u.ac.jp

(Received December 3, 2019)

(Accepted September 18, 2020)