Math. J. Okayama Univ. **63** (2021), 167–173

ON p_g-IDEALS

Tony J. Puthenpurakal

Abstract. Let $(A, \mathfrak{m})$ be an excellent normal domain of dimension two. We define an $\mathfrak{m}$-primary ideal I to be a p_g-ideal if the Rees algebra $A[It]$ is a Cohen-Macaulay normal domain. If A has infinite residue field then it follows from a result of Rees that the product of two p_g ideals is p_g. When A contains an algebraically closed field $k \cong A/\mathfrak{m}$ then Okuma, Watanabe and Yoshida proved that A has p_g-ideals and furthermore product of two p_g-ideals is a p_g ideal. In this article we show that if A is an excellent normal domain of dimension two containing a field $k \cong A/\mathfrak{m}$ of characteristic zero then also A has p_g-ideals.

1. introduction

Zariski's theory of integrally closed ideals in a two dimensional regular local ring, $(A, \mathfrak{m})$, has been very influential; see [5, Chapter 14] for a modern exposition. In particular product of two $\mathfrak{m}$-primary integrally closed ideals is integrally closed. If the residue field of A is infinite then every $\mathfrak{m}$-primary integrally closed ideal I is stable i.e., for any minimal reduction Q of I we have $I^2 = QI$. In particular the Rees algebra $\mathcal{R}(I) = A[It]$ is a Cohen-Macaulay normal domain (this also holds if $A/\mathfrak{m}$ is finite). Later Lipman proved that if $(A, \mathfrak{m})$ is a two dimensional rational singularity then analogous results holds, see [6]. However we cannot significantly weaken the hypotheses on A. In fact Cutkosky [1] proved that if $(A, \mathfrak{m})$ is an excellent normal local domain of dimension two such that $A/\mathfrak{m}$ is algebraically closed and if for any $\mathfrak{m}$-primary integrally closed ideal I we have I^2 is integrally closed then A is a rational singularity.

Assume $(A, \mathfrak{m})$ is an excellent normal domain of dimension two containing an algebraically closed field $k \cong A/\mathfrak{m}$. For such rings Okuma, Watanabe and Yoshida in [8] introduced (using geometric techniques) the notion of p_g-ideals as follows: let I be an $\mathfrak{m}$-primary ideal in A. The I has a resolution $f \colon X \to Spec(A)$ with $I\mathcal{O}_X$ invertible. Then $I\mathcal{O}_X = \mathcal{O}_X(-Z)$ for some anti-nef cycle Z. We denote I by I_Z. It can be shown that $\ell_A(H^1(X, \mathcal{O}_X(-Z)) \leq p_g(A)$ where $p_g(A) = \ell_A(H^1(X, \mathcal{O}_X)$ is the geometric genus of A and Z is an anti-nef cycle such that $\mathcal{O}_X(-Z)$ has no fixed component. An integrally closed $\mathfrak{m}$-primary ideal I with $\ell_A(H^1(X, \mathcal{O}_X(-Z)) = p_g(A)$ is called a p_g-ideal. If I, J are two $\mathfrak{m}$-primary p_g ideals then IJ is a p_g-ideal. Furthermore I is

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Mathematics Subject Classification. Primary 13A30, 13B22; Secondary 13A50, 14B05.
Key words and phrases. p_g-ideal, normal Rees rings, Cohen-Macaulay rings, stable ideals.

stable and so the Rees algebra $\mathcal{R}(I)$ is a Cohen-Macaulay normal domain. They also proved that if A is also a rational singularity then any $\mathfrak{m}$-primary integrally closed ideal is a p_g-ideal. In a later paper [9] they showed that if $\mathcal{R}(I)$ is a Cohen-Macaulay normal domain then I is a p_g-ideal. Motivated by this result we make the following definition:

Definition 1.1. Let $(A, \mathfrak{m})$ be a normal domain of dimension two. An $\mathfrak{m}$-primary ideal I is said to be p_g-ideal in A if the Rees algebra $\mathcal{R}(I) = A[It]$ is a normal Cohen-Macaulay domain.

We note that if I is a p_g-ideal then all powers of I are integrally closed. Furthermore if the residue field of A is infinite then I is stable; see [2, Theorem 1]. From the definition it does not follow that if I, J are p_g-ideals then the product IJ is also a p_g ideal. However if A is also analytically unramified with an infinite residue field then by a result of Rees; product of two p_g ideals is p_g, see [10, 2.6]. Also we do not know that whether every normal domain of dimension two has a p_g ideal.

In this paper we first give a considerably simpler proof of Rees's result in a special case.

Theorem 1.2. *Let $(A, \mathfrak{m})$ be an excellent two dimensional normal domain containing a perfect field $k \cong A/\mathfrak{m}$. If I, J are p_g-ideals in A then IJ is also a p_g-ideal in A.*

Our main result is regarding existence of p_g-ideals. We prove

Theorem 1.3. *Let $(A, \mathfrak{m})$ be an excellent two dimensional normal domain containing a field $k \cong A/\mathfrak{m}$ of characteristic zero. Then there exists p_g ideals in A.*

See Remark 4.1 to see the reason why our technique fails in positive characteristic.

We now describe in brief the contents of this paper. In section two we discuss some preliminary results that we need. In section three we describe a construction and prove Theorem 1.2. In the next section we prove Theorem 1.3.

2. PRELIMINARIES

In this section we prove the following preliminary result that we need. Parts of it are already known.

Lemma 2.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring containing a perfect field $k \cong A/\mathfrak{m}$. Let ℓ be a finite extension of k. Set $B = A \otimes_k \ell$. Then we have the following*

(1) *B is a finite flat A-module.*

(2) *B is a Noetherian ring.*
(3) *B is local with maximal ideal* $\mathfrak{m}B$ *and residue field isomorphic to* ℓ.
(4) *B contains* ℓ.
(5) *A is Cohen-Macaulay (Gorenstein, regular) if and only if B is Cohen-Macaulay (Gorenstein, regular).*
(6) *If A is excellent then so is B.*
(7) *If A is normal then so is B.*
(8) *If A is excellent normal and I is an integrally closed ideal in A then IB is an integrally closed ideal in B.*
(9) *If* ℓ *is a Galois extension of k with Galois group G then G acts on B (via* $\sigma(a \otimes t) = a \otimes \sigma(t)$*). Furthermore if* $|G|$ *is invertible in k then* $B^G = A$.

Proof. As k is perfect we have that ℓ is a separable extension of k. So by primitive element theorem we have that $\ell = k(\alpha)$. Let $f(x)$ be the minimal poynomial of α. Then $B \cong A[X]/(f(x))$. We now prove our assertions.

(1) This is clear.
(2) This follows from (1).
(3) Let $\mathfrak{n}$ be a maximal ideal in B. Then as B is finite over A we get $\mathfrak{n} \cap A = \mathfrak{m}$. So $\mathfrak{n}$ contains $\mathfrak{m}B$. Notice $B/\mathfrak{m}B \cong k[x]/(f(x)) \cong \ell$. So $\mathfrak{m}B$ is a maximal ideal in B. It follows $\mathfrak{n} = \mathfrak{m}B$. The result follows.
(4) This is clear.
(5) The extension $A \to B$ is flat with fiber $F \cong \ell$. The result follows from Corollary to Theorem 23.3, Theorem 23.4 and Theorem 23.7 in the text [7].
(6) As A is excellent so is $A[X]$. As B is a quotient of $A[X]$ we get that B is also excellent.
(7) As A is normal it satisfies R_1 and S_2. Let P be a prime ideal in A and let $\kappa(P)$ be the residue field of A_P. Then note that $B \otimes_A \kappa(P) = \ell \otimes_k \kappa(P)$ is a finite direct product of fields and so is regular. The result now follows from Theorem 23.9 in [7].

(8) As A is normal then so is $A[t]$. Let $\mathcal{R} = A[It]$ be the Rees algebra of A with respect to I. Set $\overline{\mathcal{R}} = \bigoplus_{n \geq 0} \overline{I^n}$ be the Rees-ring of the integral closure filtration of I. As A is normal and excellent it follows that the completion $\widehat{A}$ is also normal. In particular it is reduced. So $\overline{\mathcal{R}}$ is a finite extension of $\mathcal{R}$. We note that $\overline{\mathcal{R}}$ is the integral closure of $\mathcal{R}$ in $A[t]$. So $\overline{\mathcal{R}}$ is normal. By graded version of (7) we get that $\overline{\mathcal{R}} \otimes_k \ell$ is normal. We note that $\mathcal{R} \otimes_k \ell = \bigoplus_{n \geq 0} I^n B$ and $\overline{\mathcal{R}} \otimes_k \ell = \bigoplus_{n \geq 0} \overline{I^n} B$. We have graded inclusions

$$\mathcal{R} \otimes_k \ell \subseteq \overline{\mathcal{R}} \otimes_k \ell \subseteq B[t].$$

We note that as B is normal we get $B[t]$ is normal. Also $\overline{\mathcal{R}} \otimes_k \ell$ is a finite extension of $\mathcal{R} \otimes_k \ell$. It follows that $\overline{\mathcal{R}} \otimes_k \ell$ is the integral closure of $\mathcal{R} \otimes_k \ell$

in $B[t]$. In particular we have $\overline{I^n}B = \overline{I^nB}$ for $n \geq 1$. So for $n = 1$ we get $IB = \overline{I}B = \overline{IB}$. Thus IB is integrally closed in B.

(9) It is clear that G acts on B (via the action described) and $A \subseteq B^G$. Now assume $|G|$ is invertible in k. Let ρ^ℓ_k, ρ^B_A be the corresponding Reynolds operators. Let $\xi = \sum_{i=1}^r a_i \otimes t_i \in B^G$. Then note

$$\xi = \rho^B_A(\xi) = \sum_{i=1}^r \left(a_i \otimes \rho^\ell_k(t_i)\right) = \left(\sum_{i=1}^r a_i \rho^\ell_k(t_i)\right) \otimes 1 \in A.$$

□

3. A construction and proof of Theorem 1.2

Throughout this section $(A, \mathfrak{m})$ is a Noetherian local ring containing a perfect field $k \cong A/\mathfrak{m}$. *Also throughout we assume* $\dim A = 2$. Fix an algebraic closure $\overline{k}$ of k. We investigate properties of $A \otimes_k \overline{k}$. Some of the results here are already known. However some of our applications regarding p_g-ideals is new and crucial to prove Theorem 1.2 and Theorem 1.3.

3.1. Let

$$\mathcal{C}_k = \{E \mid E \text{ is a finite extension of } k \text{ in } \overline{k}\}.$$

We note that $\mathcal{C}_k$ is a directed system of fields with $\lim_{E \in \mathcal{C}_k} E = \overline{k}$. For $E \in \mathcal{C}_k$ set $A^E = A \otimes_k E$. Then by 2.1 A^E is a finite flat extension of A. Also A^E is local with maximal ideal $\mathfrak{m}^E = \mathfrak{m}A^E$. Clearly $\{A^E\}_{E \in \mathcal{C}_k}$ forms a directed system of local rings and we have $\lim_{E \in \mathcal{C}_k} A^E = A \otimes_k \overline{k}$. By [4, Chap. 0. (10.3.13)] it follows that $A \otimes_k \overline{k}$ is a Noetherian local ring (say with maximal ideal $\mathfrak{m}^{\overline{k}}$). Note that we may consider A^E as a subring of $A \otimes_k \overline{k}$. We have

$$A \otimes_k \overline{k} = \bigcup_{E \in \mathcal{C}_k} A^E \quad \text{and} \quad \mathfrak{m}^{\overline{k}} = \bigcup_{E \in \mathcal{C}_k} \mathfrak{m}^E.$$

It follows that $\mathfrak{m}(A \otimes_k \overline{k}) = \mathfrak{m}^{\overline{k}}$. It is also clear that $A \otimes_k \overline{k}$ contains $\overline{k}$ and its residue field is isomorphic to $\overline{k}$. The extension $A \to A \otimes_k \overline{k}$ is flat with fiber $\cong \overline{k}$. In particular $\dim A \otimes_k \overline{k}$ is two.

3.2. Let $F \in \mathcal{C}_k$. Set

$$\mathcal{C}_F = \{E \mid E \in \mathcal{C}_k, E \supseteq F\}.$$

Then $\mathcal{C}_F$ is cofinal in $\mathcal{C}_k$. So we have $\lim_{E \in \mathcal{C}_F} A^E = A \otimes_k \overline{k}$. Also note that if $E \in \mathcal{C}_F$ then

$$A^E = A \otimes_k E = A \otimes_k F \otimes_F E = A^F \otimes_F E.$$

It also follows that $\mathfrak{m}^E = \mathfrak{m}^F A^E$.

The following result is definitely known to experts. We give a proof for the convenience of the reader.

Lemma 3.1. *If A is excellent then so is $A \otimes_k \overline{k}$*

Proof. In the directed system $\{A^E\}_{E \in \mathcal{C}_k}$ each map $A^F \to A^E$ (when $F \subseteq E$) is etale as $A^E = A^F \otimes_F E$ and E is seperable over F. So by a result of [3, 5.3] it follows that $A \otimes_k \overline{k}$ is excellent. $\square$

We now show the main properties of $A \otimes_k \overline{k}$ that we need

Theorem 3.2. *(with hypotheses as above) Set $T = A \otimes_k \overline{k}$ and $\mathfrak{n} = \mathfrak{m}^{\overline{k}}$. We have*

(1) *A is Cohen-Macaulay (Gorenstein, regular) if and only if T is Cohen-Macaulay (Gorenstein, regular).*
(2) *If A is a normal domain if and only if T is a normal domain.*
(3) *Assume A is an excellent normal domain. Then we have*
   (a) *I is integrally closed in A if and only if IT is integrally closed in T*
   (b) *I is a p_g ideal in A if and only if IT is a p_g ideal in T.*

Proof. (1) The extension $A \to T$ is flat local with fiber ring $\overline{k}$. The result follows from Corollary to Theorem 23.3, Theorem 23.4 and Theorem 23.7 in the text [7].

(2) If A is normal then so is A^E for every $E \in \mathcal{C}_k$. In particular $T = \bigcup_{E \in \mathcal{C}_k} A^E$ is a domain. If R is a domain let $K(R)$ denote the fraction field of R. Clearly $K(T) = \bigcup_{E \in \mathcal{C}_k} K(A^E)$. Let $\xi = a/b \in K(T)$ be integral over T. Then ξ satisfies a monic polynomial $h(x) \in T[x]$. Choose $E \in \mathcal{C}_k$ such that a, b and all coefficients of h are in A^E. Then $\xi \in K(A^E)$ is integral over A^E. As A^E is normal we have $\xi \in A^E$. So $\xi \in T$. Thus T is normal. Conversely if T is normal then as the extension $A \to T$ is flat we get by Corollary to Theorem 23.7 in [7] we get that A is normal.

(3)(a) If IT is integrally closed in T then $IT \cap A$ is integrally closed in A. But $A \to T$ is faithfully flat. So $IT \cap A = I$ (note we did not use excellence of A to prove this). Conversely assume I is integrally closed in A. As A is excellent and normal, by Lemma 2.1(8), we have that IA^E is integrally closed in A^E for every $E \in \mathcal{C}_k$. Let $\xi \in T$ be integral over IT. Say we have an equation

$$\xi^n + a_1\xi^{n-1} + \cdots a_{n-1}\xi + a_n = 0$$

with $a_i \in (IT)^i = I^iT$. We may choose $F \in \mathcal{C}_k$ such that $\xi \in A^T$ and $a_i \in I^iA^F$. So ξ is integral over IA^F. But IA^F is integrally closed. Therefore $\xi \in IA^F$. So $\xi \in IT$. Thus IT is integrally closed.

(3)(b) Let $\mathcal{R}(I), \mathcal{R}(IT)$ be the Rees Algebra of I and IT respectively. Notice $\mathcal{R}(IT) = \mathcal{R}(I) \otimes_A T = \mathcal{R}(I) \otimes_k \overline{k}$. The rings $\mathcal{R}(I)$ and $\mathcal{R}(IT)$ are

$*$-local. Furthermore the extension $\mathcal{R}(I) \to \mathcal{R}(IT)$ is flat with fiber $\overline{k}$. So by graded analog of (1) we get that $\mathcal{R}(I)$ is Cohen-Macaulay if and only if $\mathcal{R}(IT)$ is Cohen-Macaulay.

First assume I is a p_g ideal in A. Then I^n is integrally closed in A for all $n \geq 1$. By 3(a) we get that I^nT is integrally closed in T for all $n \geq 1$. Also as T is normal we get $T[t]$ is normal. As $\mathcal{R}(IT)$ is integrally closed in $T[t]$ we get that it is a normal domain. Also as $\mathcal{R}(I)$ is Cohen-Macaulay, as discussed earlier we get that $\mathcal{R}(IT)$ is Cohen-Macaulay. So IT is a p_g ideal in T.

Conversely assume that IT is a p_g ideal in T. Then $(IT)^n = I^nT$ is integrally closed for all $n \geq 1$. By 3(a) we get that I^n is integrally closed for all $n \geq 1$. As A is normal, as argued before we get that $\mathcal{R}(I)$ is normal. Also as $\mathcal{R}(IT)$ is Cohen-Macaulay, as discussed earlier we get that $\mathcal{R}(I)$ is Cohen-Macaulay. So I is a p_g ideal in A. $\square$

We now give

Proof of Theorem 1.2. Set $T = A \otimes_k \overline{k}$. Let $\mathfrak{n}$ be the maximal ideal of T. We note that T is an excellent normal domain containing $\overline{k} \cong T/\mathfrak{n}$ (see 3.1, 3.1 and 3.2(2)). Let I, J be two p_g ideals in A. Then by 3.23(b) we get that IT, JT are p_g ideals in T. By [8, 3.5] we get that $(IT)(JT) = IJT$ is a p_g ideal in T. So again by 3.23(b) we get that IJ is a p_g-ideal in A. $\square$

4. PROOF OF THEOREM 1.3

In this section we give

Proof of Theorem 1.3. Set $T = A \otimes_k \overline{k}$. Let $\mathfrak{n}$ be the maximal ideal of T. We note that T is an excellent normal domain containing $\overline{k} \cong T/\mathfrak{n}$ (see 3.1, 3.1 and 3.2(2)). By [8, 4.1] there exists a p_g ideal J in T. By 3.1 we have $T = \bigcup_{E \in \mathcal{C}_k} A^E$. So there exists $F \in \mathcal{C}_k$ which contains a set of minimal generators of J. We may further assume (by enlarging) that F is Galois over k. Thus there exists ideal W in A^F with $WT = J$. By 3.2(3)(b) we get that W is a p_g ideal in A^F. Let G be the Galois group of F over k. Then G acts on A^F (via $\sigma(a \otimes f) = a \otimes \sigma(f)$). As k has characteristic zero we have by 2.1(9) that $(A^F)^G = A$. We also note that we have a natural G action on $A^F[t]$ (fixing t) and clearly its invariant ring is $A[t]$. Let $\sigma \in G$. It's action on $A^F[t]$ induces an isomorphism of between the Rees algebra's $\mathcal{R}(W)$ and $\mathcal{R}(\sigma(W))$. So $\sigma(W)$ is a p_g ideal in A^F. By Theorem 1.2 we get that $K = \prod_{\sigma \in G} \sigma(W)$ is a p_g ideal in A^F. Note K is G-invariant. So the G action of $A^F[t]$ restricts to a G-action on $\mathcal{R}(K)$. As characteristic k is zero we get that $V = \mathcal{R}(K)^G$ is a Cohen-Macaulay normal subring of $A[t]$. Set $V = \bigoplus_{n \geq 0} V_n$. Note $V_0 = A$ and $V_n = K^n \cap A$ are integrally closed

$\mathfrak{m}$-primary ideals of A. Note V is not necessarily standard graded. However it is well-known that a Veronese subring $V^{<l>}$ of V is standard graded. Note $V^{<l>}$ is a Cohen-Macaulay normal domain. Observe that $V^{<l>} = \mathcal{R}(V_l)$. Thus V_l is a p_g-ideal in A. $\square$

Remark 4.1. *Our proof of Theorem 1.3 would go through in positive characteristic would go through if we knew order of G is invertible in k. However we have no control on G. So our proof does not extend in this case.*

Acknowledgements

I thank Keiichi Watanabe and Ken-ichi Yoshida for many fruitful discussions regarding this paper.

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Department of Mathematics, IIT Bombay, Powai, Mumbai 400 076
e-mail address: tputhen@math.iitb.ac.in

(*Received August 21, 2019*)
(*Accepted September 2, 2020*)