Math. J. Okayama Univ. **63** (2021), 133–151

RECTANGULAR HALL-LITTLEWOOD SYMMETRIC FUNCTIONS AND A SPECIFIC SPIN CHARACTER

Kazuya aokage

Abstract. We derive the Schur function identities coming from the tensor products of the spin representations of the symmetric group S_n. We deal with the tensor products of the basic spin representation $V^{(n)}$ and any spin representation V^{λ} ($\lambda \in SP(n)$). The characteristic map of the tensor product $\zeta_n \otimes \zeta_\lambda$ is described by Stembridge[4] for the case of odd n. We consider the case n is even.

1. Introduction

The aim of this paper is to prove a some identities between the Hall-Littlewood symmetric functions and the Schur functions. The Hall-Littlewood symmetric function $P_\lambda(x;t)$ was defined by Philip Hall by using the Hall algebra which comes from group theory. The Hall-Littlewood symmetric function associated to the partition λ of n is defined by

$$P_\lambda(x;t) = \prod_{i\geq 0}\prod_{j=1}^{m_i} \frac{1-t}{1-t^j} \sum_{w\in S_n} w(x_1^{\lambda_1}\cdots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j}). \tag{1.1}$$

When $t=-1$, the Hall-Littlewood symmetric function coincides with that introduced by Schur in the theory of projective representations of the symmetric group.

A partition λ is any non-increasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ containing only finitely many non-zero terms. The number of parts is the length of λ, denoted by $\ell(\lambda)$. And let $P(n)$ be the set of all partitions of n, $SP(n)$ be the set of partitions of n into distinct parts, and let $OP(n)$ be the set of partitions of n with odd parts. Furthermore the set of hook partition is denoted by the following.

Definition 1.1.

$$HP(n) = \{(k, 1^{n-k}) \in P(n) \ : 1 \leq k \leq n\},$$

$$HOP(n) = \{(k, 1^{n-k}) \in P(n) \ : \ 1 \leq k \leq n, k : odd\}.$$

Mathematics Subject Classification. 20C30, 20C25, 05E05.

Key words and phrases. symmetric group, symmetric function, projective representation.

In this paper, we deal with symmetric functions of variables $x = (x_1, x_2, ...)$. Let $p_r(x) = x_1^r + x_2^r + \dots$ be the power sum symmetric function for $r \geq 1$. The Schur function is defined as follows :

$$s_\lambda(x) = \sum_{\lambda \in P(n)} \chi_\lambda(\rho) z_\rho^{-1} p_\rho(x).$$

Here the integer $\chi_\lambda(\rho)$ is the irreducible character of λ of the symmetric group S_n, evaluated at the conjugacy class ρ, and z_ρ denotes the order of centralizer of an element in the conjugacy class ρ.

Theorem 1.2 is our main results.

Theorem 1.2.
If n is even, then
(i)

$$\sum_{\mu \in HP(n) \setminus HOP(n)} s_\mu(x) = \sum_{\ell(\lambda) \leq 2} (-1)^{\lambda_2} P_\lambda(x; -1)$$

(ii)

$$\sum_{\mu \in HOP(n)} s_\mu(x) = \sum_{\ell(\lambda) = 2} (-1)^{\lambda_2 + 1} P_\lambda(x; -1).$$

Theorem 1.2 arises from the second inner tensor product of the basic spin representation for the Schur covering groups $\widetilde{S_n}$ and $\widetilde{S_n'}$[1]. When n is odd, irreducible decomposition for the second tensor product of the Schur covering groups $\widetilde{S_n}$ has been obtained by Stembridge[4]. He showed that the characteristic map $\mathrm{ch}(\zeta_n \otimes \zeta_\lambda)$ equals the Hall-Littlewood symmetric function :

$$\mathrm{ch}(\zeta_n \otimes \zeta_\lambda) = P_\lambda(x, -1),$$

where ζ_λ is the irreducible spin character of the group $\widetilde{S_n}$. From the basic properties of the spin representation of the covering groups $\widetilde{S_n}$ and $\widetilde{S_n'}$, we notice that, if n is even and $\lambda \neq (n)$,

$$\mathrm{ch}(\zeta_n \otimes \zeta_\lambda) = \mathrm{ch}(\phi_n \otimes \phi_\lambda) = P_\lambda(x, -1).$$

where ϕ_λ is the irreducible spin character of $\widetilde{S_n'}$. In [1], we deals with irreducible decomposition for the second tensor product of $\widetilde{S_n}$ and $\widetilde{S_n'}$ when n is even and $\lambda = (n)$.

Theorem 1.3 ([1]). *If $n = 2k$ is even, then*

$$V^{(n)\bigotimes 2} \simeq \begin{cases} \bigoplus_{\lambda \in HP(n) \setminus HOP(n)} S^\lambda & (k : \mathit{even}) \\ \bigoplus_{\lambda \in HOP(n)} S^\lambda & (k : \mathit{odd}), \end{cases}$$

$$W^{(n)\bigotimes 2} \simeq \begin{cases} \bigoplus_{\lambda\in HOP(n)} S^{\lambda} & (k : even) \\ \bigoplus_{\lambda\in HP(n)\setminus HOP(n)} S^{\lambda} & (k : odd). \end{cases}$$

Our aim of the present paper is to determine the image of the characteristic map for Theorem 1.3. The paper is organized as follows. In Section 2, we quickly review the second tensor product of the basic spin representations. Section 3 is devoted to the proof of Theorem 1.2.

Our argument asserts that Hall-littlewood function for rectangular partition with length 2 includes spin character $\zeta_n(n)$(see Cor 3.8).

2. Quick review of the second tensor product for the basic spin representations.

We present some results described by Schur[3]. Let G be a finite group, and V be a vector space over $\mathbb{C}$. A mapping $\rho : G \to GL(V)$ is called a projective representstion of G over $\mathbb{C}$, if there exists a mapping $\alpha : G\times G \to \mathbb{C}^{\times}$ such that the following properties hold :

$$(1).\ \ \rho(1_G) = \mathrm{id}_V, \qquad (2).\ \ \rho(x)\rho(y) = \alpha(x,y)\rho(xy).$$

Simply, it is called an α-representation. When $\alpha(x,y) = 1$, for any $x,y \in G$, it is the linear representations of G.

Schur showed that each projective representations of G are linearized by a linear representation of $\widetilde{G}$. These groups are called Schur covering groups of G.

First we recall two groups. Let $\widetilde{S_n}$ be the group generated by elements $t_1, t_2, \cdots, t_{n-1}, z$ subject to the relations :

- $z^2 = 1,$
- $t_j^2 = z \quad (1 \le j \le n-1),$
- $t_{j+1}t_jt_{j+1} = t_jt_{j+1}t_j \quad (1 \le j \le n-2),$
- $t_it_j = zt_jt_i \quad (|i-j| > 1).$

Let $\widetilde{S_n'}$ be the group generated by elements $s_1, s_2, \cdots, s_{n-1}, z$ subject to the relations :

- $z^2 = 1,$
- $s_j^2 = 1 \quad (1 \le j \le n-1),$
- $s_{j+1}s_js_{j+1} = s_js_{j+1}s_j \quad (1 \le j \le n-2),$
- $s_is_j = zs_js_i \quad (|i-j| > 1).$

These groups are non-isomorphic for $n \neq 6$. It is known that $\widetilde{S_n}$ and $\widetilde{S_n'}$ are equivalent to Schur covering groups of S_n $(n \ge 4)$. They are the only Schur covering groups of S_n.

We consider tensor products of the projective representations. Let $\rho_1 : G \longrightarrow GL(V), \alpha$-representation, and $\rho_2 : G \longrightarrow GL(W), \beta$-representation. Then the map $\rho_1 \otimes \rho_2 : G \longrightarrow GL(V \otimes W)$ defined by $\alpha\beta(x,y) = \alpha(x,y)\beta(x,y)$ is $\alpha\beta$-representation. In the case of S_n, it is known that scalars $\alpha(x,y)$ are either $\alpha(x,y) = 1$ $(\forall\, x, y \in S_n)$ or $\alpha(x,y) = \pm 1$ $(\forall\, x, y \in S_n)$. The projective representations with non-trivial scalar maps α are called the spin representations. Therefore, the tensor products in even (respectively, odd) numbers for spin representations are linear representations of S_n (respectively, spin representations). Especially, we deal with the basic spin representations which are the smallest faithful spin representations. The character values of the basic spin representations for the groups $\widetilde{S_n}$ are given by the following theorem.

Theorem 2.1 ([3]).
(1) If n is odd, we have

$$\zeta_n(\lambda) = \begin{cases} 2^{\frac{\ell(\lambda)-1}{2}} & if\ \lambda \in OP(n) \\ 0 & otherwise, \end{cases}$$

(2) If $n = 2k$ is even, we have

$$\zeta_n^{\pm}(\lambda) = \begin{cases} 2^{\frac{\ell(\lambda)-2}{2}} & if\ \lambda \in OP(n) \\ \pm i^k \sqrt{k} & if\ \lambda = (n) \\ 0 & otherwise. \end{cases}$$

The basic spin character of $\widetilde{S_n'}$ is given by the following theorem.

Theorem 2.2 ([1]).
(1) If n is odd, we have

$$\phi_n(\lambda) = \begin{cases} i^{n-\ell(\lambda)} 2^{\frac{\ell(\lambda)-1}{2}} & if\ \ \lambda \in OP(n) \\ 0 & \ otherwise, \end{cases}$$

(2) If $n = 2k$ is even, we have

$$\phi_n^{\pm}(\lambda) = \begin{cases} i^{n-\ell(\lambda)} 2^{\frac{\ell(\lambda)-2}{2}} & if\ \ \lambda \in OP(n) \\ \pm i^{k-1} \sqrt{k} & if\ \ \lambda = (n) \\ 0 & \ otherwise. \end{cases}$$

In case n is even, from the Theorem 2.1 and 2.2, the Schur covering groups $\widetilde{S_n}$ and $\widetilde{S_n'}$ have two basic spin representations respectively. We write $V^{(n)^+}$, $(W^{(n)^+})$ for the representation space with character ζ_n^+, (ϕ_n^+). We apply a similar definition to the minus case. Clearly,

$$V^{(n)^{+\otimes 2}} \simeq V^{(n)^{-\otimes 2}}, W^{(n)^{+\otimes 2}} \simeq W^{(n)^{-\otimes 2}} \quad as\ \mathbb{C}[S_n] - modules.$$

Simply, we write $V^{(n)^{\otimes 2}}$, $(W^{(n)^{\otimes 2}})$.

Theorem 1.3 implies the irreducible decomposition as $\mathbb{C}[S_n]$-molules. Here we write Specht module corresponding to partition λ as S^λ, and write its character as χ_λ.

Let us now briefly explain what this characteristic map ch is. Let R_n be identified with the $\mathbb{Q}$ span of irreducible characters χ_λ for all partitions $\lambda \in P(n)$. We put

$$R = \bigoplus_{n \geq 0} R_n.$$

We define the characteristic map ch : $R \longrightarrow \Lambda = \mathbb{Q}[p_r(x), r \geq 1]$ as follows : $\mathrm{ch}(\chi_\lambda) = \sum_{\mu \in P(n)} z_\mu^{-1} \chi_\lambda(\mu) p_\mu(x)$. Then the map ch gives isomorphism of these graded algebras. It is well known as Frobenius formula that $\mathrm{ch}(\chi_\lambda) = s_\lambda(x)$. We will find $\mathrm{ch}({\zeta_n}^{\otimes 2})$ and $\mathrm{ch}({\phi_n}^{\otimes 2})$ by calculating the right side of the Theorem 1.3.

3. Schur identity

The Kostka-Foulkes polynomial $K_{\lambda\mu}(t)$ is defined by

$$s_\lambda(x) \quad = \sum_{\mu \in P(n)} K_{\lambda\mu}(t) P_\mu(x;t).$$

It is known that the Kostka-Foukes polynomial $K_{\lambda\mu}(t)$ satisfies the properties :

$$(1).\ K_{\lambda\mu}(t) = 0, \quad unless\ \lambda \geq \mu. \qquad (2).\ K_{\lambda\lambda}(t) = 1. \tag{3.1}$$

For example, when $n = 4$ and $n = 5$, the matrices $(K_{\lambda\mu}(-1))_{\lambda,\mu \in P(n)}$ are given below[5].

$n = 4$

	4	31	2^2	21^2	1^4
4	1	-1	1	-1	1
31	0	1	-1	0	-1
2^2	0	0	1	-1	2
21^2	0	0	0	1	-1
1^4	0	0	0	0	1

,

$n = 5$

	5	41	32	31^2	2^21	21^3	1^5
5	1	-1	1	-1	1	1	1
41	0	1	-1	0	0	-1	0
32	0	0	1	-1	0	1	1
31^2	0	0	0	1	-1	-1	-2
2^21	0	0	0	0	1	0	1
21^3	0	0	0	0	0	1	0
1^5	0	0	0	0	0	0	1

.

For $\lambda \in P(n)$, we write

$$P_\lambda(x;-1) = \sum_{\mu \in P(n)} a_{\lambda\mu} s_\mu(x),$$

where $(a_{\lambda\mu})_{\lambda,\mu \in P(n)} := (K_{\lambda\mu}(-1))^{-1}_{\lambda,\mu \in P(n)}$. For $\lambda \in SP(n)$, The coefficients $a_{\lambda\mu}$ are called the Stembridge coefficients, and are written as $g_{\lambda\mu}$.

Example 3.1. If $\lambda = (k^2)$, we have

(1). $P_{2^2}(x;-1) = -s_{1^4}(x) + s_{21^2}(x) + s_{2^2}(x)$

(2). $P_{3^2}(x;-1) = s_{1^6}(x) - s_{21^4}(x) + s_{2^3}(x) + s_{31^3}(x) + s_{321} + s_{3^2}(x)$

(3). $P_{4^2}(x;-1) = -s_{1^8}(x) + s_{21^6}(x) + s_{2^4}(x) - s_{31^5}(x) + s_{32^21}(x)$
$-s_{3^22}(x) + s_{41^4}(x) + s_{421^2}(x) + s_{431}(x) + s_{4^2}(x).$

To prepare a proof of the theorem 1.2, we need to calculate the summation for all the hook partitions :

$$\sum_{\lambda \in HP(n)} K_{\lambda\mu}(-1).$$

The following theorem is due to Bryan and Jing [2].

Theorem 3.2 ([2])**.** *For $\lambda = (n-k, 1^k)$, $\mu \leqslant \lambda$,*

$$K_{\lambda\mu}(t) = t^{n(\mu)-k\ell(\mu)+\frac{k(k+1)}{2}} \begin{bmatrix} \ell(\mu)-1 \\ k \end{bmatrix},$$

where

$$n(\mu) := \sum_{i=1}^{\ell(\mu)} (i-1)\mu_i.$$

The symbol $\left[\quad\right]$ is q-binomial form at $q = t$. The algebraic formula for the Kostka-Foukes polynomials is given by use of relations between vertex operators realizing Hall-Littlewood symmetric functions and Schur functions. From Theorem 3.2, we have

$$\begin{aligned} \sum_{\lambda \in HP(n)} K_{\lambda\mu}(t) &= \sum_{\lambda \in HP(n)} t^{n(\mu)-k\ell(\mu)+\frac{k(k+1)}{2}} \begin{bmatrix} \ell(\mu)-1 \\ k \end{bmatrix} \\ &= t^{n(\mu)} \sum_{k=0}^{\ell(\mu)-1} t^{-k\ell(\mu)+\frac{k(k+1)}{2}} \begin{bmatrix} \ell(\mu)-1 \\ k \end{bmatrix}. \end{aligned}$$

Here we recall q-binomial theorem :

$$\prod_{j=1}^{N} (1 + zq^j) = \sum_{k=0}^{N} q^{\frac{k(k+1)}{2}} \begin{bmatrix} N \\ k \end{bmatrix} z^k. \tag{3.2}$$

Here if $N < k$, we put $\begin{bmatrix} N \\ k \end{bmatrix} = 0$.

Put $z = t^{-\ell(\mu)}$ and $q = t$ in (3.2). Immediately we have

$$\prod_{j=1}^{N}(1 + t^{-\ell(\mu)}t^j) = \sum_{k=0}^{N} t^{\frac{k(k+1)}{2}} t^{-\ell(\mu)k} \begin{bmatrix} N \\ k \end{bmatrix}.$$

Hence

$$\sum_{\lambda \in HP(n)} K_{\lambda\mu}(t) \quad = \quad t^{n(\mu)} \prod_{j=1}^{\ell(\mu)-1} (1 + t^{-\ell(\mu)+j}).$$

Under the specialization $t = -1$, when $\ell(\mu) \geq 2$, we have

$$\sum_{\lambda \in HP(n)} K_{\lambda\mu}(-1) \quad = \quad (-1)^{n(\mu)} \prod_{j=1}^{\ell(\mu)-1} (1 + (-1)^{-\ell(\mu)+j}) = 0.$$

Clearly, if $\ell(\mu) = 1$, we have $\mu = (n)$. From (3.1), we have the following.

Proposition 3.3.

$$\sum_{\lambda \in HP(n)} K_{\lambda\mu}(-1) = \begin{cases} 1 & if \quad \ell(\mu) = 1 \\ 0 & otherwise. \end{cases}$$

The left side of Theorem 1.2 is

$$\begin{aligned} \sum_{\lambda \in HP(n)\setminus HOP(n)} s_\lambda(x) \quad &= \quad \sum_{\lambda \in HP(n)\setminus HOP(n)} \sum_{\mu \in P(n)} K_{\lambda\mu}(-1) P_\mu(x; -1) \\ &= \quad \sum_{\mu \in P(n)} \sum_{\lambda \in HP(n)\setminus HOP(n)} K_{\lambda\mu}(-1) P_\mu(x; -1), \\ \sum_{\lambda \in HOP(n)} s_\lambda(x) \quad &= \quad \sum_{\lambda \in HOP(n)} \sum_{\mu \in P(n)} K_{\lambda\mu}(-1) P_\mu(x; -1) \\ &= \quad \sum_{\mu \in P(n)} \sum_{\lambda \in HOP(n)} K_{\lambda\mu}(-1) P_\mu(x; -1). \end{aligned}$$

Therefore, we have only to calculate

$$\sum_{\lambda \in HP(n)\setminus HOP(n)} K_{\lambda\mu}(t) \;\; and \;\; \sum_{\lambda \in HOP(n)} K_{\lambda\mu}(t).$$

We divide the set of hook partitions into two sets.

$$\sum_{\lambda \in HP(n)} K_{\lambda\mu}(t) = \sum_{\lambda \in HP(n)\setminus HOP(n)} K_{\lambda\mu}(t) + \sum_{\lambda \in HOP(n)} K_{\lambda\mu}(t). \tag{3.3}$$

For each sum of the right side of (3.3), we have the following results.

$$\sum_{\lambda\in HP(n)\backslash HOP(n)} K_{\lambda\mu}(t)=\begin{cases}(n: even)\\ t^{n(\mu)}\sum_{k\geq 0,\,even}^{n-2} t^{-k\ell(\mu)+\frac{k(k+1)}{2}}\begin{bmatrix}\ell(\mu)-1\\ k\end{bmatrix}\\ (n: odd)\\ t^{n(\mu)}\sum_{k\geq 1,\,odd}^{n-2} t^{-k\ell(\mu)+\frac{k(k+1)}{2}}\begin{bmatrix}\ell(\mu)-1\\ k\end{bmatrix}.\end{cases}$$

Likewise

$$\sum_{\lambda\in HOP(n)} K_{\lambda\mu}(t)=\begin{cases}(n: even)\\ t^{n(\mu)}\sum_{k\geq 1,\,odd}^{n-1} t^{-k\ell(\mu)+\frac{k(k+1)}{2}}\begin{bmatrix}\ell(\mu)-1\\ k\end{bmatrix}\\ (n: odd)\\ t^{n(\mu)}\sum_{k\geq 0,\,even}^{n-1} t^{-k\ell(\mu)+\frac{k(k+1)}{2}}\begin{bmatrix}\ell(\mu)-1\\ k\end{bmatrix}.\end{cases}$$

Now we apply the specialization $t=-1$, and write down :

$$\sum_{\lambda\in HP(n)\backslash HOP(n)} K_{\lambda\mu}(-1)$$

$$=\begin{cases}(n: even)\\ (-1)^{n(\mu)}\{\begin{bmatrix}\ell(\mu)-1\\ 0\end{bmatrix}-\begin{bmatrix}\ell(\mu)-1\\ 2\end{bmatrix}+\begin{bmatrix}\ell(\mu)-1\\ 4\end{bmatrix}+\dots\\ \qquad\qquad +(-1)^{\frac{(n-2)(n-1)}{2}}\begin{bmatrix}\ell(\mu)-1\\ n-2\end{bmatrix}\}\\ (n: odd)\\ (-1)^{n(\mu)+\ell(\mu)}\{-\begin{bmatrix}\ell(\mu)-1\\ 1\end{bmatrix}+\begin{bmatrix}\ell(\mu)-1\\ 3\end{bmatrix}-\begin{bmatrix}\ell(\mu)-1\\ 5\end{bmatrix}+\dots\\ \qquad\qquad +(-1)^{\frac{(n-2)(n-1)}{2}}\begin{bmatrix}\ell(\mu)-1\\ n-2\end{bmatrix}\}.\end{cases}$$

$$\sum_{\lambda\in HOP(n)} K_{\lambda\mu}(-1)$$
$$= \begin{cases} (n : even) \\ (-1)^{n(\mu)+\ell(\mu)}\{- \begin{bmatrix}\ell(\mu)-1\\1\end{bmatrix} + \begin{bmatrix}\ell(\mu)-1\\3\end{bmatrix} - \begin{bmatrix}\ell(\mu)-1\\5\end{bmatrix} + \dots \\ \qquad\qquad\qquad\qquad + (-1)^{\frac{(n-1)n}{2}} \begin{bmatrix}\ell(\mu)-1\\n-1\end{bmatrix}\} \\ (n : odd) \\ (-1)^{n(\mu)}\{ \begin{bmatrix}\ell(\mu)-1\\0\end{bmatrix} - \begin{bmatrix}\ell(\mu)-1\\2\end{bmatrix} + \begin{bmatrix}\ell(\mu)-1\\4\end{bmatrix} + \dots \\ \qquad\qquad\qquad\qquad + (-1)^{\frac{(n-1)n}{2}} \begin{bmatrix}\ell(\mu)-1\\n-1\end{bmatrix}\}. \end{cases}$$

For the partitions μ such that $\ell(\mu) \le 2$, we show the following.

Proposition 3.4.
(1) If $\ell(\mu) = 1$ (i.e. $\mu = (n)$), we have

$$\sum_{\lambda\in HP(n)\setminus HOP(n)} K_{\lambda\mu}(-1) = \begin{cases} 1 & if \quad n : even \\ 0 & if \quad n : odd, \end{cases}$$

$$\sum_{\lambda\in HOP(n)} K_{\lambda\mu}(-1) = \begin{cases} 0 & if \quad n : even \\ 1 & if \quad n : odd. \end{cases}$$

(2) If $\ell(\mu) = 2$ and $\mu = (n-i, i)$, $(i \ge 1)$, we have

$$\sum_{\lambda\in HP(n)\setminus HOP(n)} K_{\lambda\mu}(-1) = \begin{cases} (-1)^i & if \quad n : even \\ (-1)^{i+1} & if \quad n : odd, \end{cases}$$

$$\sum_{\lambda\in HOP(n)} K_{\lambda\mu}(-1) = \begin{cases} (-1)^{i+1} & if \quad n : even \\ (-1)^i & if \quad n : odd. \end{cases}$$

Proof. (1). If $\mu = (n)$, from (3.1) we have

$$\sum_{\lambda\in HOP(n)} K_{\lambda\mu}(-1) = \begin{cases} 0 & if \quad n : even \\ 1 & if \quad n : odd. \end{cases}$$

Hence, Proposition 3.3 and (3.3) assert the required result.

(2). By assumption, we have

$$n(\mu) = i.$$

When n is even, we have

$$\sum_{\lambda \in HP(n) \setminus HOP(n)} K_{\lambda\mu}(-1) = (-1)^i \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (-1)^i,$$

$$\sum_{\lambda \in HOP(n)} K_{\lambda\mu}(-1) = (-1)^{i+2}(-\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = (-1)^{i+1}.$$

When n is odd, we have

$$\sum_{\lambda \in HP(n) \setminus HOP(n)} K_{\lambda\mu}(-1) = (-1)^{i+2}(-\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = (-1)^{i+1},$$

$$\sum_{\lambda \in HOP(n)} K_{\lambda\mu}(-1) = (-1)^i \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (-1)^i.$$

$\square$

Next we investigate μ such that $\ell(\mu) \geq 3$. Here we recall q-binomial relation

$$\begin{bmatrix} N+k \\ k \end{bmatrix} = \begin{bmatrix} N+k \\ N \end{bmatrix}.$$

Proposition 3.5. *If $\ell(\mu)$ is odd ≥ 3, we have*

$$\sum_{\lambda \in HOP(n)} K_{\lambda\mu}(-1) = \sum_{\lambda \in HP(n) \setminus HOP(n)} K_{\lambda\mu}(-1) = 0.$$

Proof. Let n be even. (i). If $\ell(\mu) = 2m+1$ and m is even, we have

$$\begin{aligned}
&\sum_{\lambda \in HOP(n)} K_{\lambda\mu}(-1) \\
= \;& (-1)^{n(\mu)} \{ \begin{bmatrix} \ell(\mu)-1 \\ 1 \end{bmatrix} - \begin{bmatrix} \ell(\mu)-1 \\ 3 \end{bmatrix} + \begin{bmatrix} \ell(\mu)-1 \\ 5 \end{bmatrix} + \dots - \begin{bmatrix} \ell(\mu)-1 \\ \ell(\mu)-2 \end{bmatrix} \} \\
= \;& (-1)^{n(\mu)} \{ \begin{bmatrix} \ell(\mu)-1 \\ 1 \end{bmatrix} - \begin{bmatrix} \ell(\mu)-1 \\ 3 \end{bmatrix} + \begin{bmatrix} \ell(\mu)-1 \\ 5 \end{bmatrix} + \dots - \begin{bmatrix} \ell(\mu)-1 \\ 1 \end{bmatrix} \} \\
= \;& (-1)^{n(\mu)} \{ \begin{bmatrix} \ell(\mu)-1 \\ 1 \end{bmatrix} - \begin{bmatrix} \ell(\mu)-1 \\ 1 \end{bmatrix} + \begin{bmatrix} \ell(\mu)-1 \\ 3 \end{bmatrix} - \begin{bmatrix} \ell(\mu)-1 \\ 3 \end{bmatrix} + \dots \\
&\qquad + \begin{bmatrix} \ell(\mu)-1 \\ \frac{\ell(\mu)-3}{2} \end{bmatrix} - \begin{bmatrix} \ell(\mu)-1 \\ \frac{\ell(\mu)-3}{2} \end{bmatrix} \} \\
= \;& 0.
\end{aligned}$$

From (3.3), we have

$$\sum_{\lambda \in HP(n) \setminus HOP(n)} K_{\lambda\mu}(-1) = 0.$$

(ii). If $\ell(\mu) = 2m+1$ and m is odd, we have

$$\begin{aligned}
&\sum_{\lambda \in HP(n) \setminus HOP(n)} K_{\lambda\mu}(-1) \\
= \;& (-1)^{n(\mu)} \{ \begin{bmatrix} \ell(\mu)-1 \\ 0 \end{bmatrix} - \begin{bmatrix} \ell(\mu)-1 \\ 2 \end{bmatrix} + \begin{bmatrix} \ell(\mu)-1 \\ 4 \end{bmatrix} + \dots - \begin{bmatrix} \ell(\mu)-1 \\ \ell(\mu)-1 \end{bmatrix} \} \\
= \;& (-1)^{n(\mu)} \{ \begin{bmatrix} \ell(\mu)-1 \\ 0 \end{bmatrix} - \begin{bmatrix} \ell(\mu)-1 \\ 2 \end{bmatrix} + \begin{bmatrix} \ell(\mu)-1 \\ 4 \end{bmatrix} + \dots - \begin{bmatrix} \ell(\mu)-1 \\ 0 \end{bmatrix} \} \\
= \;& (-1)^{n(\mu)} \{ \begin{bmatrix} \ell(\mu)-1 \\ 0 \end{bmatrix} - \begin{bmatrix} \ell(\mu)-1 \\ 0 \end{bmatrix} + \begin{bmatrix} \ell(\mu)-1 \\ 2 \end{bmatrix} - \begin{bmatrix} \ell(\mu)-1 \\ 2 \end{bmatrix} + \dots \\
&\qquad + \begin{bmatrix} \ell(\mu)-1 \\ \frac{\ell(\mu)-3}{2} \end{bmatrix} - \begin{bmatrix} \ell(\mu)-1 \\ \frac{\ell(\mu)-3}{2} \end{bmatrix} \} \\
= \;& 0.
\end{aligned}$$

Likewise from (3.3), it follows that

$$\sum_{\lambda \in HOP(n)} K_{\lambda\mu}(-1) = 0.$$

When n is odd, it suffices to exchange $\sum_{\lambda \in HOP(n)}$ and $\sum_{\lambda \in HP(n) \setminus HOP(n)}$. □

Finally, we show the following.

Proposition 3.6.
If $\ell(\mu)$ is even ≥ 4, we have

$$\sum_{\lambda\in HOP(n)} K_{\lambda\mu}(-1) = \sum_{\lambda\in HP(n)\backslash HOP(n)} K_{\lambda\mu}(-1) = 0.$$

Proof.
Let n be even. (i). If $\ell(\mu)=2k$ and k is even, we have

$$\begin{aligned} &\sum_{\lambda\in HP(n)\backslash HOP(n)} K_{\lambda\mu}(-1) \\ = \ & (-1)^{n(\mu)}(\begin{bmatrix}\ell(\mu)-1\\0\end{bmatrix} - \begin{bmatrix}\ell(\mu)-1\\2\end{bmatrix} + \begin{bmatrix}\ell(\mu)-1\\4\end{bmatrix} + \dots - \begin{bmatrix}\ell(\mu)-1\\\ell(\mu)-2\end{bmatrix}) \\ = \ & (-1)^{n(\mu)}(\begin{bmatrix}\ell(\mu)-1\\0\end{bmatrix} - \begin{bmatrix}\ell(\mu)-1\\2\end{bmatrix} + \begin{bmatrix}\ell(\mu)-1\\4\end{bmatrix} + \dots - \begin{bmatrix}\ell(\mu)-1\\1\end{bmatrix}) \\ = \ & (-1)^{n(\mu)}(\begin{bmatrix}\ell(\mu)-1\\0\end{bmatrix} - \begin{bmatrix}\ell(\mu)-1\\1\end{bmatrix} - \begin{bmatrix}\ell(\mu)-1\\2\end{bmatrix} + \begin{bmatrix}\ell(\mu)-1\\3\end{bmatrix} + \dots \\ & \qquad + \begin{bmatrix}\ell(\mu)-1\\\frac{\ell(\mu)-2}{2}\end{bmatrix}). \end{aligned}$$

Also

$$\begin{aligned} &\sum_{\lambda\in HOP(n)} K_{\lambda\mu}(-1) \\ = \ & (-1)^{n(\mu)}(-\begin{bmatrix}\ell(\mu)-1\\1\end{bmatrix} + \begin{bmatrix}\ell(\mu)-1\\3\end{bmatrix} - \begin{bmatrix}\ell(\mu)-1\\5\end{bmatrix} + \dots + \begin{bmatrix}\ell(\mu)-1\\\ell(\mu)-1\end{bmatrix}) \\ = \ & (-1)^{n(\mu)}(-\begin{bmatrix}\ell(\mu)-1\\1\end{bmatrix} + \begin{bmatrix}\ell(\mu)-1\\3\end{bmatrix} - \begin{bmatrix}\ell(\mu)-1\\5\end{bmatrix} + \dots + \begin{bmatrix}\ell(\mu)-1\\0\end{bmatrix}) \\ = \ & (-1)^{n(\mu)}(\begin{bmatrix}\ell(\mu)-1\\0\end{bmatrix} - \begin{bmatrix}\ell(\mu)-1\\1\end{bmatrix} - \begin{bmatrix}\ell(\mu)-1\\2\end{bmatrix} + \begin{bmatrix}\ell(\mu)-1\\3\end{bmatrix} + \dots \\ & \qquad + \begin{bmatrix}\ell(\mu)-1\\\frac{\ell(\mu)-2}{2}\end{bmatrix}). \end{aligned}$$

Hence

$$\begin{aligned} \sum_{\lambda\in HP(n)} K_{\lambda\mu}(-1) = 2(-1)^{n(\mu)}(&\begin{bmatrix}\ell(\mu)-1\\0\end{bmatrix} - \begin{bmatrix}\ell(\mu)-1\\1\end{bmatrix} - \begin{bmatrix}\ell(\mu)-1\\2\end{bmatrix} \\ & + \begin{bmatrix}\ell(\mu)-1\\3\end{bmatrix} + \dots + \begin{bmatrix}\ell(\mu)-1\\\frac{\ell(\mu)-2}{2}\end{bmatrix}). \end{aligned}$$

From Proposition 3.3

$$\sum_{\lambda\in HP(n)\setminus HOP(n)} K_{\lambda\mu}(-1) = \sum_{\lambda\in HOP(n)} K_{\lambda\mu}(-1) = 0.$$

(i). If $\ell(\mu)=2k$ and k is odd, we have

$$\begin{aligned}
&\sum_{\lambda\in HP(n)\setminus HOP(n)} K_{\lambda\mu}(-1)\\
=\;& (-1)^{n(\mu)}\left(\begin{bmatrix}\ell(\mu)-1\\0\end{bmatrix}-\begin{bmatrix}\ell(\mu)-1\\2\end{bmatrix}+\begin{bmatrix}\ell(\mu)-1\\4\end{bmatrix}+\dots+\begin{bmatrix}\ell(\mu)-1\\\ell(\mu)-2\end{bmatrix}\right)\\
=\;& (-1)^{n(\mu)}\left(\begin{bmatrix}\ell(\mu)-1\\0\end{bmatrix}-\begin{bmatrix}\ell(\mu)-1\\2\end{bmatrix}+\begin{bmatrix}\ell(\mu)-1\\4\end{bmatrix}+\dots+\begin{bmatrix}\ell(\mu)-1\\1\end{bmatrix}\right)\\
=\;& (-1)^{n(\mu)}\left(\begin{bmatrix}\ell(\mu)-1\\0\end{bmatrix}+\begin{bmatrix}\ell(\mu)-1\\1\end{bmatrix}-\begin{bmatrix}\ell(\mu)-1\\2\end{bmatrix}-\begin{bmatrix}\ell(\mu)-1\\3\end{bmatrix}+\dots\right.\\
&\left.+\begin{bmatrix}\ell(\mu)-1\\\frac{\ell(\mu)-2}{2}\end{bmatrix}\right).
\end{aligned}$$

Also

$$\begin{aligned}
&\sum_{\lambda\in HOP(n)} K_{\lambda\mu}(-1)\\
=\;& (-1)^{n(\mu)}\left(-\begin{bmatrix}\ell(\mu)-1\\1\end{bmatrix}+\begin{bmatrix}\ell(\mu)-1\\3\end{bmatrix}-\begin{bmatrix}\ell(\mu)-1\\5\end{bmatrix}+\dots-\begin{bmatrix}\ell(\mu)-1\\\ell(\mu)-1\end{bmatrix}\right)\\
=\;& (-1)^{n(\mu)}\left(-\begin{bmatrix}\ell(\mu)-1\\1\end{bmatrix}+\begin{bmatrix}\ell(\mu)-1\\3\end{bmatrix}-\begin{bmatrix}\ell(\mu)-1\\5\end{bmatrix}+\dots-\begin{bmatrix}\ell(\mu)-1\\0\end{bmatrix}\right)\\
=\;& (-1)^{n(\mu)}\left(-\begin{bmatrix}\ell(\mu)-1\\0\end{bmatrix}-\begin{bmatrix}\ell(\mu)-1\\1\end{bmatrix}+\begin{bmatrix}\ell(\mu)-1\\2\end{bmatrix}+\begin{bmatrix}\ell(\mu)-1\\3\end{bmatrix}+\dots\right.\\
&\left.-\begin{bmatrix}\ell(\mu)-1\\\frac{\ell(\mu)-2}{2}\end{bmatrix}\right).
\end{aligned}$$

Hence

$$\sum_{\lambda\in HP(n)\setminus HOP(n)} K_{\lambda\mu}(-1) = -\sum_{\lambda\in HOP(n)} K_{\lambda\mu}(-1).$$

Also by multiplying both sides of (3.2) by z we have

$$z\prod_{j=1}^{N}(1+zq^j)=\sum_{k=0}^{N} q^{\frac{k(k+1)}{2}}\begin{bmatrix}N\\k\end{bmatrix} z^{k+1}.$$

If we put $q=-1$, $z=-1$, and $N=\ell(\mu)-1$, we have

$$\begin{aligned}
0 &= -\prod_{j=1}^{\ell(\mu)-1}(1+(-1)^{j+1}) = \sum_{k=0}^{\ell(\mu)-1}(-1)^{\frac{(k+1)(k+2)}{2}}\begin{bmatrix}\ell(\mu)-1\\ k\end{bmatrix}\\
&= \begin{bmatrix}\ell(\mu)-1\\ 0\end{bmatrix}+\begin{bmatrix}\ell(\mu)-1\\ 1\end{bmatrix}-\begin{bmatrix}\ell(\mu)-1\\ 2\end{bmatrix}-\begin{bmatrix}\ell(\mu)-1\\ 3\end{bmatrix}\\
&\quad+\begin{bmatrix}\ell(\mu)-1\\ 4\end{bmatrix}+\begin{bmatrix}\ell(\mu)-1\\ 5\end{bmatrix}+\ldots+\begin{bmatrix}\ell(\mu)-1\\ \ell(\mu)-2\end{bmatrix}+\begin{bmatrix}\ell(\mu)-1\\ \ell(\mu)-1\end{bmatrix}\\
&= \begin{bmatrix}\ell(\mu)-1\\ 0\end{bmatrix}+\begin{bmatrix}\ell(\mu)-1\\ 1\end{bmatrix}-\begin{bmatrix}\ell(\mu)-1\\ 2\end{bmatrix}-\begin{bmatrix}\ell(\mu)-1\\ 3\end{bmatrix}\\
&\quad+\begin{bmatrix}\ell(\mu)-1\\ 4\end{bmatrix}+\begin{bmatrix}\ell(\mu)-1\\ 5\end{bmatrix}+\ldots+\begin{bmatrix}\ell(\mu)-1\\ 1\end{bmatrix}+\begin{bmatrix}\ell(\mu)-1\\ 0\end{bmatrix}\\
&= 2(\begin{bmatrix}\ell(\mu)-1\\ 0\end{bmatrix}+\begin{bmatrix}\ell(\mu)-1\\ 1\end{bmatrix}-\begin{bmatrix}\ell(\mu)-1\\ 2\end{bmatrix}-\begin{bmatrix}\ell(\mu)-1\\ 3\end{bmatrix}+\ldots+\begin{bmatrix}\ell(\mu)-1\\ \frac{\ell(\mu)-2}{2}\end{bmatrix}).
\end{aligned}$$

Hence

$$\sum_{\lambda\in HP(n)\backslash HOP(n)} K_{\lambda\mu}(-1)=0 \quad and \quad \sum_{\lambda\in HOP(n)} K_{\lambda\mu}(-1)=0.$$

The same reason as before holds when n is odd. □

Proof of Theorem 1.1.
When n is even, we have the following equation (3.4) :

$$\begin{aligned}
(3.4)\qquad &\sum_{\lambda\in HP(n)\backslash HOP(n)} s_\lambda(x)\\
&= \sum_{\lambda\in HP(n)\backslash HOP(n)}\sum_{\mu\in P(n)} K_{\lambda\mu}(-1)P_\mu(x;-1)\\
&= \sum_{\mu\in P(n)}\sum_{\lambda\in HP(n)\backslash HOP(n)} K_{\lambda\mu}(-1)P_\mu(x;-1)\\
&= \sum_{\lambda\in HP(n)\backslash HOP(n)} K_{\lambda n}(-1)P_n(x;-1)\\
&\quad+\sum_{\ell(\mu)=2}\sum_{\lambda\in HP(n)\backslash HOP(n)} K_{\lambda\mu}(-1)P_\mu(x;-1)\\
&\quad+\sum_{\ell(\mu)\geq 3}\sum_{\lambda\in HP(n)\backslash HOP(n)} K_{\lambda\mu}(-1)P_\mu(x;-1).
\end{aligned}$$

From Proposition 3.4, 3.5 and 3.6, we have

$$\begin{aligned}(3.4) &= P_n(x;-1) + \sum_{\ell(\mu)=2} (-1)^{\mu_i} P_\mu(x;-1) \\ &= \sum_{\ell(\mu)\leq 2} (-1)^{\mu_i} P_\mu(x;-1).\end{aligned}$$

Likewise

$$\begin{aligned}&\sum_{\lambda\in HOP(n)} s_\lambda(x) \\ = &\sum_{\lambda\in HOP(n)} K_{\lambda n}(-1)P_n(x;-1) + \sum_{\ell(\mu)=2}\sum_{\lambda\in HOP(n)} K_{\lambda\mu}(-1)P_\mu(x;-1) \\ &+ \sum_{\ell(\mu)\geq 3}\sum_{\lambda\in HOP(n)} K_{\lambda\mu}(-1)P_\mu(x;-1) \\ = &\sum_{\ell(\mu)=2} (-1)^{\mu_i+1} P_\mu(x;-1).\end{aligned}$$

□

In parallel when n is odd, from Proposition 3.4, 3.5 and 3.6, we have the following.

Remark 3.7.
If n is odd, we have
(i)

$$\sum_{\mu\in HP(n)\backslash HOP(n)} s_\mu(x) = \sum_{\ell(\lambda)=2} (-1)^{\lambda_2+1} P_\lambda(x;-1)$$

(ii)

$$\sum_{\mu\in HOP(n)} s(x) = \sum_{\ell(\lambda)\leq 2} (-1)^{\lambda_2} P_\lambda(x;-1).$$

These formulas do not correspond to the representation of the second tensor.

The symmetric function $Q_\lambda(x;-1)$ is introduced in Schur's paper[3] on projective representations. For $\lambda \in SP(n)$

$$Q_\lambda(x;-1) = \sum_{\rho\in OP(n)} 2^{\frac{\ell(\lambda)+\ell(\rho)+\epsilon(\lambda)}{2}} {z_\rho}^{-1}\zeta_\lambda(\rho)p_\rho(x),$$

$$\epsilon(\lambda) := \begin{cases} 1 & if \quad \ell(\lambda)+\ell(\rho) : odd \\ 0 & if \quad \ell(\lambda)+\ell(\rho) : even. \end{cases}$$

And it is known that $Q_\lambda(x;-1) = 2^{\ell(\lambda)}P_\lambda(x;-1)$, if $\lambda \in SP(n)$. Spin characters with conjagacy class consisting of odd partitions appear in the terms of Schur's $Q_\lambda(x;-1)$ function. In other words, otherwise they do not appear. Corollary 3.8 assert that we can find the spin character $\zeta_n(n)$ in the term of Hall-Littlewood function for a rectangular partition of length 2.

Corollary 3.8. *If $n = 2k$ is even, we have*

$$\zeta_n(n) = \sqrt{\langle P_{k^2}(x;-1), p_n\rangle_{t=0}} = i^k\sqrt{k}.$$

Proof. From Theorem 1.2, we have

$$P_n(x;-1) + 2\sum_{i\geq 1}^{k}(-1)^i P_{(n-i,i)} = \sum_{\lambda\in HP(n)}(-1)^{\mathrm{leg}(\lambda)}s_\lambda(x).$$

By using the Murnaghan-Nakayama's recursion formulas for $\lambda \in HP(n)$, we immediately have

$$\begin{aligned} p_n(x) &= \sum_{\lambda\in P(n)}\chi_\lambda(n)s_\lambda(x) \\ &= \sum_{\lambda\in HP(n)}(-1)^{\mathrm{leg}(\lambda)}s_\lambda(x). \end{aligned}$$

The formula below will gives expansions for the power sum symmetric functions :

$$p_n(x) = P_n(x;-1) + 2\sum_{i\geq 1}^{k}(-1)^i P_{(n-i,i)}(x;-1).$$

By using the Schur's Q-functions, we have

$$p_n(x) = \frac{1}{2}\sum_{i\geq 0}^{k-1}(-1)^i Q_{(n-i,i)}(x;-1) + (-1)^k 2P_{k^2}(x;-1).$$

$$P_{k^2}(x;-1) = \frac{1}{4}\sum_{i\geq 0}^{k-1}(-1)^{i+k+1}Q_{(n-i,i)}(x;-1) + (-1)^k\frac{1}{2}p_n(x).$$

Hence

$$\begin{aligned} P_{k^2}(x;-1) &= \sum_{i\geq 0}^{k-1}(-1)^{i+k+1}\sum_{\rho\in OP(n)}2^{\frac{\ell(\rho)-2}{2}}{z_\rho}^{-1}\zeta_{(n-i,i)}(\rho)p_\rho(x) \\ &\quad +(-1)^k\frac{1}{2}p_n(x). \end{aligned} \tag{3.5}$$

We define inner product on Λ [5]

$$\langle p_\lambda(x), p_\mu(x)\rangle_{t=0} := z_\lambda\delta_{\lambda\mu}.$$

From (3.5), we have

$$\langle P_{k^2}(x;-1), p_n(x)\rangle_{t=0} = (-1)^k \frac{1}{2} z_n = (-1)^k \frac{n}{2} = (-1)^k k.$$

□

Example 3.9.

$$\begin{aligned}
&(1). \quad P_{2^2}(x;-1) = \frac{1}{6}p_{1^4}(x) - \frac{2}{3}p_{31}(x) + \frac{1}{2}p_4(x)\\
&(2). \quad P_{3^2}(x;-1) = \frac{2}{45}p_{1^6}(x) - \frac{2}{9}p_{31^3}(x) + \frac{5}{18}p_{3^2}(x) + \frac{2}{5}p_{51}(x) - \frac{1}{2}p_6(x)\\
&(3). \quad P_{4^2}(x;-1) = \frac{1}{126}p_{1^8}(x) - \frac{2}{45}p_{31^5}(x) + \frac{2}{9}p_{3^21^2}(x) - \frac{2}{5}p_{53}(x)\\
&\qquad\qquad\qquad\qquad - \frac{2}{7}p_{71}(x) + \frac{1}{2}p_8(x).
\end{aligned}$$

Next we introduce some relations about the coefficients $a_{k^2\mu}$. First,

$$\begin{aligned}
P_\lambda(x;-1) &= \sum_{\mu\in P(n)} a_{\lambda\mu}s_\mu(x) = \sum_{\mu\in P(n)} a_{\lambda\mu} \sum_{\tau\in P(n)} {z_\tau}^{-1}\chi_\mu(\tau)p_\tau(x)\\
&= \sum_{\tau\in P(n)} (\sum_{\mu\in P(n)} a_{\lambda\mu}\chi_\mu(\tau)){z_\tau}^{-1}p_\tau(x).
\end{aligned}$$

For $\lambda \in SP(n)$,

$$Q_\lambda(x) = 2^{\ell(\lambda)} \sum_{\tau\in P(n)} (\sum_{\mu\in P(n)} a_{\lambda\mu}\chi_\mu(\tau)){z_\tau}^{-1}p_\tau(x).$$

We have

$$\sum_{\mu\in P(n)} a_{\lambda\mu}\chi_\mu(\tau) = \begin{cases} 2^{\frac{-\ell(\lambda)+\ell(\tau)+\epsilon}{2}}\zeta_\lambda(\tau) & if\ \tau \in OP(n)\\ 0 & otherwise.\end{cases}$$

Likewise, for $\lambda = (k^2)$ we have

Corollary 3.10. *If $n = 2k$ is even, we have*

$$\sum_{\mu\in P(n)} a_{k^2\mu}\chi_\mu(\tau) = \begin{cases} 2^{\frac{\ell(\mu)-2}{2}} \sum_{i\geq 0}^{k-1} (-1)^{i+k+1}\zeta_{(n-i,i)}(\tau) & if\ \tau \in OP(n)\\ (-1)^k k & if\ \tau = (n)\\ 0 & otherwise.\end{cases}$$

Second, we calculate the number $K_{k^2\mu}(-1)$.

$$\begin{aligned} P_\lambda(x;-1) &= \sum_{\tau\in P(n)} \langle P_\lambda(x;-1),\, p_\tau(x)\rangle_{t=0} {z_\tau}^{-1} p_\tau(x) \\ &= \sum_{\tau\in P(n)} \langle P_\lambda(x;-1),\, p_\tau(x)\rangle_{t=0} {z_\tau}^{-1} \sum_{\mu\in P(n)} \chi_\mu(\tau) s_\mu(x) \\ &= \sum_{\mu\in P(n)} (\sum_{\tau\in P(n)} \langle P_\lambda(x;-1),\, p_\tau(x)\rangle_{t=0} {z_\tau}^{-1} \chi_\mu(\tau)) s_\mu(x). \end{aligned} \tag{3.6}$$

When $\lambda = (k^2)$, we have

$$\begin{aligned} &\langle P_{k^2}(x;-1),\, p_\tau(x)\rangle_{t=0} \\ = &\sum_{i\geq 0}^{k-1} (-1)^{i+k+1} 2^{\frac{\ell(\tau)-2}{2}} \zeta_{(n-i,i)}(\tau) + (-1)^k \frac{1}{2} \langle p_n(x),\, p_\tau(x)\rangle_{t=0}. \end{aligned}$$

Hence

$$\begin{aligned} &\sum_{\tau\in P(n)} \langle P_{k^2}(x;-1),\, p_\tau(x)\rangle_{t=0} {z_\tau}^{-1} \chi_\mu(\tau) \\ = &\sum_{\tau\in P(n)} (\sum_{i\geq 0}^{k-1} (-1)^{i+k+1} 2^{\frac{\ell(\tau)-2}{2}} \zeta_{(n-i,i)}(\tau) \\ &\qquad + (-1)^k \frac{1}{2} \langle p_n(x),\, p_\tau(x)\rangle_{t=0}) {z_\tau}^{-1} \chi_\mu(\tau) \\ = &\sum_{\tau\in OP(n)} (\sum_{i\geq 0}^{k-1} (-1)^{i+k+1} 2^{\frac{\ell(\tau)-2}{2}} \zeta_{(n-i,i)}(\tau)) z_\tau^{-1} \chi_\mu(\tau) + (-1)^k \frac{1}{2} \chi_\mu(n). \end{aligned}$$

From (3.6), we have

$$\begin{aligned} &P_{k^2}(x;-1) \\ = &\sum_{\mu\in P(n)} (\sum_{\tau\in OP(n)} (\sum_{i\geq 0}^{k-1} (-1)^{i+k+1} 2^{\frac{\ell(\tau)-2}{2}} \zeta_{(n-i,i)}(\tau) z_\tau^{-1} \chi_\mu(\tau)) \\ &\qquad + (-1)^k \frac{1}{2} \chi_\mu(n)) s_\mu(x). \end{aligned}$$

Corollary 3.11. *For $\mu \in P(n)$ and $n = 2k$, we have*

$$K_{k^2\mu}(-1) = (-1)^k \frac{1}{2} \chi_\mu(n) + \sum_{\tau\in OP(n)} \sum_{i\geq 0}^{k-1} (-1)^{i+k+1} 2^{\frac{\ell(\tau)-2}{2}} z_\tau^{-1} \zeta_{(n-i,i)}(\tau) \chi_\mu(\tau).$$

Finally, from example 3.1 we suggest the following conjecture.

Conjecture 3.12. *For $\mu \in HP(n)$,*

$$a_{k^2\mu} = \begin{cases} 0 & if \ \mathrm{leg}(\mu) < k \\ (-1)^{k+\mathrm{leg}(\mu)} & if \ \mathrm{leg}(\mu) \geq k. \end{cases}$$

References

[1] K. Aokage, Tensor square of the basic spin representations of Schur covering groups for the symmetric groups, J. Algebraic Combin. (2020).

[2] T. W. Bryan, N. Jing, An algebraic formula for the Kostka-Foulkes polynomials, arXiv:math/1608.01775v1.

[3] I. Schur, Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Snbstitutionen, J. Reine Angew. Math. 139 (1911) 155-250.

[4] J. R. Stembridge, Shifted tableaux and the projective representations of symmetric groups, Adv. in Math. 74 (1989) 87-134.

[5] I. G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd. ed, Oxford Univ Press, 1995.

Department of Mathematics, National Institute of Technology, Ariake College, Fukuoka, 836-8585 Japan.

e-mail address: aokage@ariake-nct.ac.jp

(*Received June 22, 2019*)
(*Accepted September 2, 2019*)