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DIFFERENTIAL OPERATORS ON MODULAR FORMS ASSOCIATED TO JACOBI FORMS

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Abstract. Given Jacobi forms, we determine associated Jacobi-like forms, whose coefficients are quasimodular forms. We then use these quasimodular forms to construct differential operators on modular forms, which are expressed in terms of the Fourier coefficients of the given Jacobi forms.

1. Introduction

Jacobi forms were introduced by Eichler and Zagier in [4], and they play an important role in number theory. Jacobi-like forms are formal power series with coefficients in the ring of holomorphic functions on the Poincaré upper half plane $\mathcal{H}$ which are invariant under a certain action of a discrete subgroup of $SL(2, \mathbb{R})$, and they generalize Jacobi forms in some sense (cf. [3], [8]). On the other hand, quasimodular forms, which were introduced by Kaneko and Zagier in [5], generalize modular forms, and they are closely linked to Jacobi-like forms. More specifically, each coefficient of a Jacobi-like form is a quasimodular form, and there is a lifting map from quasimodular forms to Jacobi-like forms so that the lifting of a quasimodular form has this form as one of its coefficients (cf. [1], [6]).

Although the derivative of a modular form is not a modular form in general, there are a number of ways of constructing differential operators on modular forms. For example, the Serre operator can be considered by using the Eisenstein series E_2, and Rankin-Cohen brackets determine certain types of differential operators on modular forms. The goal of this paper is to construct differential operators on modular forms associated a Jacobi form by using a method of constructing such operators from quasimodular forms introduced in our earlier paper [7]. To be more specific, we first determine Jacobi-like forms corresponding to Jacobi forms, so that their coefficients are quasimodular forms. We then use these quasimodular forms to construct differential operators on modular forms by using the above-mentioned method. These operators are expressed in terms of the Fourier coefficients of the given Jacobi forms.

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2. Jacobi forms and Jacobi-like forms

Let $\mathcal{H}$ be the Poincaré upper half plane on which the group $SL(2, \mathbb{R})$ acts as usual by linear fractional transformations. Thus we may write

$$\gamma z = \frac{az + b}{cz + d}$$

for all $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$. For the same z and γ, we set

$$\mathfrak{J}(\gamma, z) = cz + d, \quad \mathfrak{K}(\gamma, z) = \frac{c}{cz + d}. \tag{2.1}$$

Then the resulting maps $\mathfrak{J}, \mathfrak{K} : SL(2, \mathbb{R}) \times \mathcal{H} \to \mathbb{C}$ satisfy

$$\mathfrak{J}(\gamma\gamma', z) = \mathfrak{J}(\gamma, \gamma' z)\mathfrak{J}(\gamma', z), \quad \mathfrak{K}(\gamma, \gamma' z) = \mathfrak{J}(\gamma', z)^2 (\mathfrak{K}(\gamma\gamma', z) - \mathfrak{K}(\gamma', z))$$

for all $\gamma, \gamma' \in SL(2, \mathbb{R})$ and $z \in \mathcal{H}$.

We now consider a subgroup Γ of $SL(2, \mathbb{Z})$ of finite index and choose nonnegative integers ν and μ. Then Jacobi forms for Γ introduced by Eichler and Zagier [4] can be described as follows.

Definition 2.1. A holomorphic function $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ is a *Jacobi form of weight* ν *and index* μ *for* Γ if it satisfies the following conditions:

(i) If $\mathfrak{J}$ and $\mathfrak{K}$ are as in (2.1), the transformation formula

$$\phi(\gamma z, \mathfrak{J}(\gamma, z)^{-1} w) = \mathfrak{J}(\gamma, z)^{\nu} e^{2\pi i \mu \mathfrak{K}(\gamma, z) w^2} \phi(z, w) \tag{2.2}$$

holds for all $\gamma \in \Gamma$ and $(z, w) \in \mathcal{H} \times \mathbb{C}$.

(ii) If $(p, q) \in \mathbb{Z}^2$, the relation

$$\phi(z, w + pz + q) = e^{2\pi i \mu(-p^2 z - 2pw)} \phi(z, w) \tag{2.3}$$

holds for all $(z, w) \in \mathcal{H} \times \mathbb{C}$.

(iii) The function ϕ has a Fourier development of the form

$$\phi(z, w) = \sum_{n=0}^{\infty} \sum_{r^2 \leq 4\mu n/\ell} C_{\phi}(n, r) e^{2\pi i n z/\ell} e^{2\pi i r w} \tag{2.4}$$

for all $(z, w) \in \mathcal{H} \times \mathbb{C}$, where ℓ is the least positive integer such that $\phi(z + \ell, w) = \phi(z, w)$.

We denote by $J_{\nu,\mu}(\Gamma)$ the space of Jacobi forms of weight ν and index μ for Γ.

Remark 2.2. Given a Jacobi form $\phi \in J_{\nu,\mu}(\Gamma)$, we set $\phi^{-}(z, w) = \phi(z, -w)$. Then for $\gamma \in \Gamma$, $(z, w) \in \mathcal{H} \times \mathbb{C}$ and $(p, q) \in \mathbb{Z}^2$, using (2.2) and(2.3), we have

$$\begin{aligned}\phi^-(\gamma z, \mathfrak{J}(\gamma,z)^{-1}w) &= \phi(\gamma z, \mathfrak{J}(\gamma,z)^{-1}(-w))\\ &= \mathfrak{J}(\gamma,z)^{\nu} e^{2\pi i\mu\mathfrak{K}(\gamma,z)w^2}\phi(z,-w)\\ &= \mathfrak{J}(\gamma,z)^{\nu} e^{2\pi i\mu\mathfrak{K}(\gamma,z)w^2}\phi^-(z,w),\end{aligned}$$

$$\begin{aligned}\phi^-(z, w+pz+q) &= \phi(z,(-w)+(-p)z+(-q))\\ &= e^{2\pi i\mu(-p^2z-2pw)}\phi(z,-w)\\ &= e^{2\pi i\mu(-p^2z-2pw)}\phi^-(z,w).\end{aligned}$$

Thus it follows that ϕ^- is also a Jacobi form belonging to $J_{\nu,\mu}(\Gamma)$.

Let $\mathcal{F}$ be the ring of holomorphic functions $f : \mathcal{H} \to \mathbb{C}$ that are polynomially bounded, meaning that there is a constant $N > 0$ such that

$$f(z) = O((1+|z|^2)/y^N)$$

for $z \in \mathcal{H}$ with $y = \mathrm{Im}(z)$ as $y \to \infty$ and $y \to 0$ (see e.g. [2, §5.1]). We denote by $\mathcal{F}[[X]]$ the complex algebra of formal power series in X with coefficients in $\mathcal{F}$, so that its elements can be written in the form

$$\Phi(z,X) = \sum_{k=0}^{\infty} \phi_k(z)X^k$$

with $\phi_k \in \mathcal{F}$ for each $k \geq 0$. Given an integer λ, we can consider right actions $|_\lambda$ and $|_\lambda^J$ of $SL(2,\mathbb{R})$ on $\mathcal{F}$ and $\mathcal{F}[[X]]$, respectively, defined by

$$(f \mid_\lambda \gamma)(z) = \mathfrak{J}(\gamma,z)^{-\lambda} f(z)$$

$$(\Phi \mid_\lambda^J \gamma)(z,X) = \mathfrak{J}(\gamma,z)^{-\lambda} e^{-\mathfrak{K}(\gamma,z)X}\Phi(\gamma z, \mathfrak{J}(\gamma,z)^{-2}X), \tag{2.5}$$

for $f \in \mathcal{F}$, $\Phi(z,X) \in \mathcal{F}[[X]]$, $\gamma \in SL(2,\mathbb{R})$ and $z \in \mathcal{H}$.

Definition 2.3. (i) A holomorphic function $f \in \mathcal{F}$ is a *modular form of weight* λ *for* Γ if it satisfies

$$f \mid_\lambda \gamma = f$$

for all $\gamma \in \Gamma$.

(ii) A *Jacobi-like form of weight* λ *for* Γ is a formal power series $\Phi(z,X) \in \mathcal{F}[[X]]$ satisfying

$$(\Phi \mid_\lambda^J \gamma)(z,X) = \Phi(z,X) \tag{2.6}$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$.

We denote by $M_\lambda(\Gamma)$ and $\mathcal{J}_\lambda(\Gamma)$ the spaces of modular forms and Jacobi-like forms, respectively, of weight λ for Γ.

We now fix a Jacobi form $\phi \in J_{\nu,\mu}(\Gamma)$, which has a Fourier development as in (2.4), and consider the associated formal power series $F_\phi(z, X) \in \mathcal{F}[[X]]$ defined by

$$F_\phi(z, X) = \sum_{k=0}^{\infty} \xi_k^\phi(z) X^k, \tag{2.7}$$

where

$$\xi_k^\phi(z) = \frac{(2\pi i)^k}{(2k)!\mu^k} \sum_{n=0}^{\infty} \sum_{r^2 \leq 4\mu n/\ell} r^{2k} C_\phi(n, r) e^{2\pi i n z/\ell} \tag{2.8}$$

for $k \geq 0$.

Proposition 2.4. *The formal power series $F_\phi(z, X)$ in X given by (2.7) is a Jacobi-like form belonging to $\mathcal{J}_\nu(\Gamma)$.*

Proof. From (2.4) we see that the power series expansion of the Jacobi form $\phi(z, w)$ in w can be written as

$$\phi(z, w) = \sum_{k=0}^{\infty} \beta_k^\phi(z) w^k,$$

where

$$\beta_k^\phi(z) = \sum_{n=0}^{\infty} \sum_{r^2 \leq 4\mu n/\ell} \frac{(2\pi i r)^k}{k!} C_\phi(n, r) e^{2\pi i n z/\ell} \tag{2.9}$$

for $k \geq 0$. We consider the corresponding even series given by

$$\widehat{\phi}(z, w) = \frac{\phi(z, w) + \phi(z, -w)}{2} = \sum_{k=0}^{\infty} \beta_{2k}^\phi(z) w^{2k}. \tag{2.10}$$

On the other hand, if we set

$$X = 2\pi i \mu w^2,$$

from (2.7), (2.8), (2.9) and (2.10) we obtain

$$
\begin{aligned}
(2.11)\qquad F_\phi(z,X) &= \sum_{k=0}^{\infty} \frac{(2\pi i)^k}{(2k)!\mu^k} \sum_{n=0}^{\infty} \sum_{r^2\le 4\mu n/\ell} r^{2k} C_\phi(n,r) e^{2\pi i n z/\ell} X^k \\
&= \sum_{k=0}^{\infty} \frac{1}{(2\pi i\mu)^k} \sum_{n=0}^{\infty} \sum_{r^2\le 4\mu n/\ell} \frac{(2\pi i r)^{2k}}{(2k)!} C_\phi(n,r) e^{2\pi i n z/\ell} X^k \\
&= \sum_{k=0}^{\infty} \beta_{2k}^{\phi}(z) \left(\frac{X}{2\pi i\mu}\right)^k = \sum_{k=0}^{\infty} \beta_{2k}^{\phi}(z) w^{2k} \\
&= \widehat{\phi}(z,w).
\end{aligned}
$$

Since ϕ is a Jacobi form of weight ν and index μ, by Remark 2.2 the same is true for $\widehat{\phi}$, and therefore by (2.2) it satisfies

$$\widehat{\phi}(\gamma z, \mathfrak{J}(\gamma,z)^{-1} w) = \mathfrak{J}(\gamma,z)^{\nu} e^{2\pi i\mu \mathfrak{K}(\gamma,z) w^2} \widehat{\phi}(z,w)$$

for all $\gamma \in \Gamma$ and $(z,w) \in \mathcal{H} \times \mathbb{C}$. From this and (2.11)we see that

$$
\begin{aligned}
F_\phi(\gamma z, \mathfrak{J}(\gamma,z)^{-2} X) &= \widehat{\phi}\left(\gamma z, \mathfrak{J}(\gamma,z)^{-1} \sqrt{\frac{X}{2\pi i\mu}}\right) \\
&= \mathfrak{J}(\gamma,z)^{\nu} e^{\mathfrak{K}(\gamma,z)X} \widehat{\phi}\left(z, \sqrt{\frac{X}{2\pi i\mu}}\right) \\
&= \mathfrak{J}(\gamma,z)^{\nu} e^{\mathfrak{K}(\gamma,z)X} F_\phi(z,X);
\end{aligned}
$$

hence $F_\phi(z,X)$ belongs to $\mathcal{J}_\nu(\Gamma)$. □

3. Quasimodular forms and differential operators

Let Γ and $\mathcal{F}$ be as in Section 2, and let m and λ be integers with $\lambda \geq 2m \geq 0$.

Definition 3.1. An element $f \in \mathcal{F}$ is a *quasimodular form for* Γ *of weight* λ *and depth at most* m if there are functions $f_0, \dots, f_m \in \mathcal{F}$ satisfying

$$(3.1)\qquad (f \mid_\lambda \gamma)(z) = \sum_{r=0}^{m} f_r(z) \mathfrak{K}(\gamma,z)^r$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$, where $\mathfrak{K}(\gamma,z)$ is as in (2.1). We denote by $QM_\lambda^m(\Gamma)$ the space of such quasimodular forms.

Remark 3.2. As was pointed out by Zagier in [9, Section 5.3] and [10, Section 2], the condition for quasimodular forms given in Definition 3.1 was suggested by Werner Nahm.

We note that the functions f_r in (3.1) are uniquely determined by f and that they are in fact quasimodular forms such that

$$f_r \in QM^{m-r}_{\lambda-2r}(\Gamma)$$

for $0 \le r \le m$ (see e.g. [1]). Thus by setting

$$\mathfrak{S}_r(f) = f_r, \tag{3.2}$$

we obtain the complex linear map

$$\mathfrak{S}_r : QM^m_\lambda(\Gamma) \to QM^{m-r}_{\lambda-2r}(\Gamma)$$

for each r.

Quasimodular forms are closely linked to Jacobi-like forms. Indeed, if the formal power series

$$\Psi(z, X) = \sum_{k=0}^{\infty} \psi_k(z) X^k. \tag{3.3}$$

is a Jacobi-like form belonging to $\mathcal{J}_\nu(\Gamma)$, then the coefficient function ψ_m with $m \ge 0$ is a quasimodular form belonging to $QM^m_{\nu+2m}(\Gamma)$. Thus we can consider the complex linear map

$$\Pi_m : \mathcal{J}_\nu(\Gamma) \to QM^m_{\nu+2m}(\Gamma) \tag{3.4}$$

defined by

$$\Pi_m(\Psi(z, X)) = \psi_m$$

for $\Psi(z, X) \in \mathcal{J}_\nu(\Gamma)$ as in (3.3). Furthermore, if $\mathfrak{S}_j : QM^m_{\nu+2m}(\Gamma) \to QM^{m-j}_{\nu+2m-2j}(\Gamma)$ is as in (3.2), using Corollary 3.7 and Proposition 3.10 in [1], we have

$$\mathfrak{S}_j(\psi_m) = \frac{1}{j!} \psi_{m-j} \tag{3.5}$$

for $0 \le j \le m$. The next proposition provides a method of determining a differential operator on modular forms associated to a quasimodular form.

Proposition 3.3. *Given a quasimodular form $\psi \in QM^m_\lambda(\Gamma)$ and nonnegative integers u and σ with $u \le m$, there is a differential operator*

$$\mathcal{A}^\sigma_{\psi,u} : M_\sigma(\Gamma) \to M_{\sigma+\lambda-2m+2u}(\Gamma)$$

on modular forms given by

$$\mathcal{A}^\sigma_{\psi,u} = \sum_{s=0}^{u} \sum_{p=0}^{s} \frac{(-1)^s (m-u+s)!(2u+\sigma+\lambda-2m-s-2)!}{p!(s-p)!} \times (\mathfrak{S}_{m-u+s}\psi)^{(s-p)} \frac{d^p}{dz^p}. \tag{3.6}$$

Proof. See [7]. □

We now consider a Jacobi form $\phi \in J_{\nu,\mu}(\Gamma)$ whose Fourier coefficients $C_\phi(n,r) \in \mathbb{C}$ are as in (2.4), and set

$$K^{\phi,u,\sigma}_{n,p} = \sum_{s=p}^{u} \sum_{r^2 \leq 4\mu n/\ell} \frac{(-1)^s (2\pi i)^{u-p} r^{2u-2s} n^{s-p}}{\mu^{u-s} \ell^{s-p}} \times \frac{(2u+\sigma+\nu-s-2)!}{p!(s-p)!(2u-2s)!} C_\phi(n,r). \tag{3.7}$$

for nonnegative integers k, p, σ and u with $p \leq u$. We then define the associated functions

$$F^{\phi,u,\sigma}_p : \mathcal{H} \to \mathbb{C}$$

by the series

$$F^{\phi,u,\sigma}_p(z) = \sum_{n=0}^{\infty} K^{\phi,u,\sigma}_{n,p} e^{2\pi i n z/\ell}$$

for all $z \in \mathcal{H}$. The differential operator on modular forms associated to ϕ is then described in the next theorem.

Theorem 3.4. *Given a Jacobi form $\phi \in J_{\nu,\mu}(\Gamma)$ and nonnegative integers u and σ, the formula*

$$\mathcal{D}_{\phi,u,\sigma} = \sum_{p=0}^{u} F^{\phi,u,\sigma}_p(z) \frac{d^p}{dz^p}$$

determines a differential operator

$$\mathcal{D}_{\phi,u,\sigma} : M_\sigma(\Gamma) \to M_{\sigma+\nu+2u}(\Gamma)$$

on modular forms for Γ.

Proof. Let $F_\phi(z,X)$ be the formal power series given by (2.7), which belongs to $\mathcal{J}_\lambda(\Gamma)$ by Proposition 2.4. If Π_m with $m \geq 0$ is the map in (3.4), then we see that the function

$$\xi^\phi_m = \Pi_m(F_\phi(z,X))$$

is a quasimodular form belonging to $QM^m_{\nu+2m}(\Gamma)$, which satisfies

$$\mathfrak{S}_j(\xi^\phi_m) = \frac{1}{j!} \xi^\phi_{m-j}$$

for $0 \leq j \leq m$ by (3.5). From this relation and (2.8) we obtain

$$\mathfrak{S}_j(\xi^\phi_m) = \sum_{n=0}^{\infty} \sum_{r^2 \leq 4\mu n/\ell} \frac{(2\pi i)^{m-j} r^{2m-2j}}{j!(2m-2j)! \mu^{m-j}} C_\phi(n,r) e^{2\pi i n z/\ell}. \tag{3.8}$$

We now apply Proposition 3.3 by using $\lambda = \nu + 2m$ in (3.6) and setting $\mathcal{D}_{\phi,u,\sigma} = \mathcal{A}^{\sigma}_{\xi^{\phi}_m,u}$ to obtain the operator

$$\mathcal{D}_{\phi,u,\sigma} : M_{\sigma}(\Gamma) \to M_{\sigma+\nu+2u}(\Gamma)$$

given by

$$\mathcal{D}_{\phi,u,\sigma} = \sum_{s=0}^{u} \sum_{p=0}^{s} \frac{(-1)^s (m-u+s)!(2u+\sigma+\nu-s-2)!}{p!(s-p)!} \times (\mathfrak{S}_{m-u+s}\xi^{\phi}_m)^{(s-p)} \frac{d^p}{dz^p}.$$

On the other hand, using (3.8), we have

$$(\mathfrak{S}_{m-u+s}\xi^{\phi}_m)(z) = \sum_{n=0}^{\infty} \sum_{r^2 \le 4\mu n/\ell} \frac{(2\pi i)^{u-s} r^{2u-2s}}{(m-u+s)!(2u-2s)!\mu^{u-s}} C_{\phi}(n,r) e^{2\pi i n z/\ell}$$

for $0 \le s \le u \le m$, so that

$$(\mathfrak{S}_{m-u+s}\xi^{\phi}_m)^{(s-p)}(z) = \sum_{n=0}^{\infty} \sum_{r^2 \le 4\mu n/\ell} \frac{(2\pi i)^{u-s} r^{2u-2s}}{(m-u+s)!(2u-2s)!\mu^{u-s}} \times \left(\frac{2\pi i n}{\ell}\right)^{s-p} C_{\phi}(n,r) e^{2\pi i n z/\ell}$$

for $0 \le p \le s$. Thus we obtain

$$\begin{aligned}
\mathcal{D}_{\phi,u,\sigma} &= \sum_{s=0}^{u} \sum_{p=0}^{s} \sum_{n=0}^{\infty} \sum_{r^2 \le 4\mu n/\ell} \frac{(-1)^s (2u+\sigma+\nu-s-2)!}{p!(s-p)!} \\
&\qquad \times \frac{(2\pi i)^{u-p} r^{2u-2s}}{(2u-2s)!\mu^{u-s}} \left(\frac{n}{\ell}\right)^{s-p} C_{\phi}(n,r) e^{2\pi i n z/\ell} \frac{d^p}{dz^p} \\
&= \sum_{s=0}^{u} \sum_{p=0}^{s} \sum_{n=0}^{\infty} \sum_{r^2 \le 4\mu n/\ell} \frac{(-1)^s (2\pi i)^{u-p} (2u+\sigma+\nu-s-2)! r^{2u-2s} n^{s-p}}{p!(s-p)!(2u-2s)!\mu^{u-s}\ell^{s-p}} \\
&\qquad \times C_{\phi}(n,r) e^{2\pi i n z/\ell} \frac{d^p}{dz^p} \\
&= \sum_{p=0}^{u} \sum_{s=p}^{u} \sum_{n=0}^{\infty} \sum_{r^2 \le 4\mu n/\ell} \frac{(-1)^s (2\pi i)^{u-p} (2u+\sigma+\nu-s-2)! r^{2u-2s} n^{s-p}}{p!(s-p)!(2u-2s)!\mu^{u-s}\ell^{s-p}} \\
&\qquad \times C_{\phi}(n,r) e^{2\pi i n z/\ell} \frac{d^p}{dz^p} \\
&= \sum_{p=0}^{u} \sum_{n=0}^{\infty} K^{\phi,u,\sigma}_{n,p} e^{2\pi i n z/\ell} \frac{d^p}{dz^p} = \sum_{p=0}^{u} F^{\phi,u,\sigma}_p(z) \frac{d^p}{dz^p},
\end{aligned}$$

where we used (3.7); hence the proof of the theorem is complete. □

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