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REMARK ON A PAPER BY IZADI AND BAGHALAGHDAM ABOUT CUBES AND FIFTH POWERS SUMS

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Abstract. In this paper, we refine the method introduced by Izadi and Baghalaghdam to search integer solutions to the Diophantine equation $X_1^5+X_2^5+X_3^5=Y_1^3+Y_2^3+Y_3^3$. We show that the Diophantine equation has infinitely many positive solutions.

1. Introduction

In [2], Izadi and Baghalaghdam consider the Diophantine equation:

$$a(X_1'^5+X_2'^5)+\sum_{i=0}^{n} a_iX_i^5=b(Y_1'^3+Y_2'^3)+\sum_{i=0}^{m} b_iY_i^3 \tag{1}$$

where $n,m\in\mathbb{N}\cup\{0\},\ a,b\neq 0,\ a_i,b_i$ are fixed arbitrary rational numbers. They use theory of elliptic curves to find nontrivial integer solutions to (1). In particular, they discuss the equation:

$$X_1^5+X_2^5+X_3^5=Y_1^3+Y_2^3+Y_3^3 \tag{2}$$

and obtain integer solutions, for example:

$$8^5+6^5+14^5=(-110)^3+124^3+14^3,$$

$$128122^5+(-79524)^5+48598^5=359227580^3+(-251874598)^3+107352982^3.$$

However, no positive solutions are presented in their paper [2]. In this paper, we refine their method to find positive solutions to (2).

Consider the Diophantine equation (2). Let:

$$\begin{cases} X_1=t+x_1, & X_2=t-x_1, & X_3=\alpha t,\\ Y_1=t+v, & Y_2=t-v, & Y_3=\beta t.\end{cases} \tag{3}$$

Then we get a quartic curve:

$$C: v^2=\frac{2+\alpha^5}{6}t^4+\frac{20x_1^2-2-\beta^3}{6}t^2+\frac{5x_1^4}{3} \tag{4}$$

with parameters $x_1,\ \alpha,\ \beta\in\mathbb{Q}$. If we get a rational point (t,v) on C, we can compute a rational solution to (2) (see [2]).

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Once we obtain rational solutions to (2), we can obtain integer solutions by multiplying an appropriate value to X_i, Y_i. In the same way, in order to obtain solutions in positive integers, it suffices to search positive rational solutions to equation (2).

2. Additional Requirements for Positive Solutions

Suppose that a positive rational solution $(X_i, Y_i)_{1\leq i\leq 3}$ to (2) is obtained from a given point (t, v) on the quartic C.

Proposition 2.1. Let $\alpha, \beta, x_1 \in \mathbb{Q}$ and

$$F(t) = \frac{2+\alpha^5}{6}t^4 + \frac{20x_1^2 - 2 - \beta^3}{6}t^2 + \frac{5x_1^4}{3}.$$

A rational point (t, v) on the curve $C : v^2 = F(t)$ in (4) produces a positive rational solution to (2) by (3) if and only if

$$\alpha, \beta > 0, \quad 0 \leq F(t) < t^2, \quad t > |x_1| \tag{5}$$

hold.

Proof. If X_i and Y_i are positive in the solution in the form (3), we have $t = (X_1 + X_2)/2 > 0$, $\alpha = 2X_3/(X_1 + X_2) > 0$ and $\beta = 2Y_3/(Y_1 + Y_2) > 0$. For $(t, v) \in C$, one has that $0 \leq v^2 = F(t) < v^2 + Y_1Y_2 = t^2$. It follows from $x_1^2 < x_1^2 + X_1X_2 = t^2$ that $t > |x_1|$ for $t > 0$. Conversely, suppose the inequalities in (5) hold. Then the given point (t, v) on C satisfies $v^2 = F(t) < t^2$. This and (5) immediately imply $X_i, Y_i > 0$ in (3). $\square$

Proposition 2.2. Under the same assumption as Proposition 2.1, let

$$a = \frac{2+\alpha^5}{6}, \; b = \frac{20x_1^2 - 8 - \beta^3}{6}, \; c = \frac{5}{3}x_1^4. \tag{6}$$

Then a, b, c satisfy $b^2 - 4ac > 0$ and $b < 0$ if and only if there exists a real number t such that $F(t) < t^2$.

Proof. Let $\tilde{F}(t) = F(t) - t^2$. Since $\tilde{F}(0) = 5x_1^4/3 \geq 0$, and in this case $a > 0$, it is easy to see that the following conditions are equivalent to each others:

(i) There exists a real number t such that $F(t) < t^2$.

(ii) The equation $\tilde{F}(t) = 0$ has four distinct solutions.

(iii) The quadratic equation $ax^2+bx+c = 0$ has two distinct non-negative solutions.

(iv) The discriminant $D = b^2 - 4ac$ of the quadratic function $f(x) = ax^2 + bx + c$ is positive, and the axis of the quadratic function $-b/2a$ is positive, and $f(0) \geq 0$.

The condition (iv) holds if and only if "$b^2 - 4ac > 0$ and $b < 0$", since $a > 0$ and $f(0) = c = 5x_1^4/3 \geq 0$. $\square$

3. Example for $X_1^5 + X_2^5 + X_3^5 = Y_1^3 + Y_2^3 + Y_3^3$

Let us first search parameters (x_1, α, β) such that

$$0 < \alpha, \beta, \quad b < 0 < b^2 - 4ac$$

with a, b, c given by (6) and such that the quartic curve C of (4) has at least one rational point. Note that these are necessary to satisfy conditions of Proposition 2.1, 2.2. Then, the curve C is birationally equivalent to an elliptic curve E over $\mathbb{Q}$. If E has positive rank, then C has infinitely many rational points.

Let $(x_1, \alpha, \beta) = (2, 1, 16)$. Then the quartic:

$$C : v^2 = \frac{1}{2}t^4 - \frac{2009}{3}t^2 + \frac{80}{3},$$

has a rational point $(t, v) = (44, 760)$. By $T = t - 44$, we transform C into

$$C' : v^2 = \frac{1}{2}T^4 + 88T^3 + \frac{15415}{3}T^2 + \frac{334312}{3}T + 760^2$$

which is birationally equivalent over $\mathbb{Q}$ to the cubic elliptic curve (see [5, Theorem 2.17], [2]):

$$E : y^2 + \frac{41789}{285}xy + 133760y = x^3 - \frac{76876021}{324900}x^2 - 1155200x + \frac{2460032672}{9},$$

where:

$$T = \frac{2 \cdot 760(x + \frac{15415}{3}) - \frac{334312^2}{2 \cdot 3^2 \cdot 760}}{y}, \quad v = -760 + \frac{T(Tx - \frac{334312}{3})}{2 \cdot 760}.$$

Using the Sage software [3], we find that the cubic curve E is an elliptic curve which has rank 2 and the generators of E are:

$$P_1 = \left(-\frac{1802189}{1521}, \frac{5513659679}{417430}\right), \quad P_2 = \left(-\frac{351379}{363}, \frac{47356344241}{2276010}\right).$$

We now consider the subset

$$C_0 = \left\{(t, v) \in C \mid 0 \le F(t) < t^2\right\} \subset C$$

whose points satisfy another condition (5) of Proposition 2.1. The two quartic equations:

$$F(t) = \frac{1}{2}t^4 - \frac{2009}{3}t^2 + \frac{80}{3} = 0, \quad \tilde{F}(t) = \frac{1}{2}t^4 - \frac{2009}{3}t^2 + \frac{80}{3} - t^2 = 0$$

have respectively solutions:

$$t = \pm\frac{1}{3}\sqrt{6027 \pm 3\sqrt{4035601}}, \quad t = \pm\frac{2}{3}\sqrt{1509 \pm 3\sqrt{252979}}.$$

Let us take larger solutions as:

$$\begin{aligned} a_1 &= \frac{1}{3}\sqrt{6027+3\sqrt{4035601}} \simeq 36.59635926... \\ a_2 &= \frac{2}{3}\sqrt{1509+3\sqrt{252979}} \simeq 36.62367500... \end{aligned}$$

If a point (t_0, v_0) on C satisfies $a_1 \leq t_0 \leq a_2$, then (t_0, v_0) lies on C_0. We now make use of the composition law of points on the elliptic curve E. Since E has positive rank, we can test infinitely many rational points of E till finding a point (t_0, v_0) on C_0. We find that the rational point

$$Q = 2P_1 - P_2 = \left(\tfrac{3048453811921111829037}{58470412871306667}, -\tfrac{4767546475726965161322288395890039}{46528437561782035616437457770}\right)$$

on E corresponds to

$$(t_0, v_0) = \left(\tfrac{1708156198441559091562 04}{4664941095250009917983}, -\tfrac{6907408840626256639198729252916998776830290 96}{21761675422152362106175457381859866386788289}\right)$$

on C_0, and creates a positive rational solution:

$$\begin{aligned} X_1 &= \tfrac{180145502034655928992170}{4664941095250009917983} \simeq 38.61688676... \\ X_2 &= \tfrac{161485737653655889320238}{4664941095250009917983} \simeq 34.61688676... \\ X_3 &= \tfrac{170815619844155909156204}{4664941095250009917983} \simeq 36.61688676... \\ Y_1 &= \tfrac{106103920658980331397442614601687483092587436}{21761675422152362106175457381859866386788289} \simeq 4.875723886... \\ Y_2 &= \tfrac{1487585688784231659237188465185087238458645628}{21761675422152362106175457381859866386788289} \simeq 68.35804964... \\ Y_3 &= \tfrac{2733049917506494546499264}{4664941095250009917983} \simeq 585.8701882... \end{aligned}$$

Next we shall prove that the Diophantine equation (2) has infinitely many positive solutions. The real locus of elliptic curve $E(\mathbb{R})$ can be regarded as a compact topological subspace of complex projective variety E.

Lemma 3.1. If the rank of elliptic curve E over $\mathbb{Q}$ is positive, every point of $E(\mathbb{Q})$ is an accumulation point in $E(\mathbb{R})$.

Proof. Since $E(\mathbb{R})$ is a compact topological group, and $E(\mathbb{Q})$ is an infinite subgroup of $E(\mathbb{R})$, there is at least one accumulation point of $E(\mathbb{Q})$ in $E(\mathbb{R})$. The group operations are homeomorphisms from $E(\mathbb{R})$ to itself. Therefore all points of $E(\mathbb{Q})$ are accumulation points of $E(\mathbb{R})$. $\square$

Theorem 3.2. The Diophantine equation (2) has infinitely many positive solutions.

Proof. The part of C_0 has one rational point (t_0, v_0) which corresponds to the above point Q. By Lemma 3.1, the point Q is an accumulation point of $E(\mathbb{Q})$ in $E(\mathbb{R})$, and (t_0, v_0) is that of $C(\mathbb{Q})$ in $C(\mathbb{R})$. Thus the part of C_0 includes infinitely many rational points. Since $2 = |x_1| < a_1 = 36.59635926...$, they correspond to positive rational solutions to (2). $\square$

4. Example for $X_1^5 + X_2^5 = Y_1^3 + Y_2^3 + Y_3^3$

Let $\alpha = 0$. Then (2) gives another Diophantine equation:

$$X_1^5 + X_2^5 = Y_1^3 + Y_2^3 + Y_3^3. \tag{7}$$

In the same way, we can obtain a rational or positive rational solutions of it. For example, let $x_1 = 10,\ \beta = 18$. Then the quartic curve:

$$C : v^2 = \frac{1}{3}t^4 - 639t^2 + \frac{50000}{3}$$

has a rational point $(t, v) = (-5, 30)$ and can be regarded as an elliptic curve over $\mathbb{Q}$ that has rank 2. It is birationally equivalent to:

$$E : y^2 + \frac{1867}{9}xy - 400y = x^3 - \frac{3676525}{324}x^2 - 1200x + \frac{367652500}{27}.$$

From this, we can compute positive rational solutions to (7). For example, there is a point $Q = (x_0, y_0)$ on E with

$$x_0 = \frac{9233921838917810856046138588468998730}{712268521667621224056167067666475947}$$

corresponding to (t_0, v_0) on C with

$$t_0 = \frac{7869911761727476320751662986237524106650}{180965667579279848488380712753242417827}$$

which creates the following solution to (7):

$$X_1 = \frac{96795684375202748056354701137699482849 20}{180965667579279848488380712753242417827},$$

$$X_2 = \frac{6060255085934677835867855858705099928380}{180965667579279848488380712753242417827},$$

$$Y_1 = \frac{210257939758607749685886980412651199398809460130710098627025850356764517703 5000}{32748572842414417658282657731373155447687070419319181813277645661864847401929},$$

$$Y_2 = \frac{74578827391600073826502709521328510557987014359519664434475473364684324146 4100}{32748572842414417658282657731373155447687070419319181813277645661864847401929},$$

$$Y_3 = \frac{14165841171109457377352993375227543391970 0}{180965667579279848488380712753242417827}.$$

The case of $\beta = 0$ will be discussed briefly in 5.2 below.

5. Parameters (x_1, α, β) from Trivial Solutions

5.1. There are several trivial solutions; for example:

$$1^5 + 1^5 + 1^5 = 1^3 + 1^3 + 1^3.$$

We call solutions to (2) which consist of 0, ± 1 trivial. We are going to check some of them to search integer (or positive) solutions.

A solution to (2) may decide parameter. For example, when $X_i = Y_i = 1$ $(i = 1, 2, 3)$, we get $(x_1, \alpha, \beta) = (0, 1, 1)$. Then:

$$C : v^2 = \frac{1}{2}t^4 - \frac{1}{2}t^2$$

has a singular point $(t, v) = (0, 0)$ and can be parametrized by one parameter. Let us divide both sides of C by t^4 and substitute s, w for $1/t$, v/t^2 respectively. Then:

$$C' : w^2 = \frac{1}{2} - \frac{1}{2}s^2$$

has a rational point $(s, w) = (1, 0)$. Hence we can parametrize rational points on C' and integer solutions to (2). That is to say we have:

$$\begin{aligned}&\left(\frac{2k^2+1}{2k^2-1}\right)^5 + \left(\frac{2k^2+1}{2k^2-1}\right)^5 + \left(\frac{2k^2+1}{2k^2-1}\right)^5 \\ &= \left(\frac{4k^4-4k^3-2k-1}{(2k^2-1)^2}\right)^3 + \left(\frac{4k^4+4k^3+2k-1}{(2k^2-1)^2}\right)^3 + \left(\frac{2k^2+1}{2k^2-1}\right)^3\end{aligned}$$

where $k \in \mathbb{Q}$. We can see that large enough k give positive solutions to (2). For example:

$$\left(\frac{9}{7}\right)^5 + \left(\frac{9}{7}\right)^5 + \left(\frac{9}{7}\right)^5 = \left(\frac{27}{49}\right)^3 + \left(\frac{99}{49}\right)^3 + \left(\frac{9}{7}\right)^3$$

where $k = 2$. Since $X_1 = X_2 = X_3 = Y_3$, this solution also gives positive solution to another Diophantine equation $3X^5 = Y_1^3 + Y_2^3 + X^3$. Moreover it satisfies $X_1 + X_2 + X_3 = Y_1 + Y_2 + Y_3$ because $\alpha = \beta$.

5.2. From another trivial solution:

$$1^5 + 0^5 + 0^5 = 1^3 + 0^3 + 0^3,$$

we can derive parameters $(x_1, \alpha, \beta) = (\frac{1}{2}, 0, 0)$. Then:

$$C : v^2 = \frac{1}{3}t^4 + \frac{1}{2}t^2 + \frac{5}{48}$$

is an elliptic curve defined over $\mathbb{Q}$ with rational point $(t, v) = (\frac{1}{2}, \frac{1}{2})$. It is birationally equivalent to:

$$E : y^2 + \frac{4}{3}xy + \frac{2}{3}y = x^3 + \frac{5}{9}x^2 - \frac{1}{3}x - \frac{5}{27}$$

over $\mathbb{Q}$ and has rank 1. Hence we can apply the method of Section 3 to compute positive solutions to

$$X_1^5 + X_2^5 = Y_1^3 + Y_2^3 \tag{8}$$

as a special case of (2) with $X_1, X_2, Y_1, Y_2 > 0$, $X_3 = Y_3 = 0$ (where $\alpha = \beta = 0$ in (3)). For example, a point

$$Q = \left(\tfrac{10017045137918654785}{165672066306928896}, \tfrac{29224609136538294659462738431}{67433225470590933809197056}\right)$$

on E corresponding to the point

$$(t_0, v_0) = \left(\tfrac{2806052350871126431439}{4379016004568066987998}, \tfrac{57979267831620055028079719147866926110822 09}{95878905841316384399486679714185599380240 02}\right)$$

on C creates the positive solution to (8):

$$X_1 = \frac{2497780176577579962719}{2189508002284033493999},\ X_2 = \frac{3082721742935464687 20}{2189508002284033493999},$$
$$Y_1 = \frac{597090043011113067437970036067559605125838 5}{4793945292065819219974333985709279969012001},$$
$$Y_2 = \frac{1729736469491251715717284458889034401761 76}{4793945292065819219974333985709279969012001}.$$

5.3. There exists one more parameter with $\beta = 0$, $(x_1, \alpha, \beta) = (0, 0, 0)$, which is derived from the trivial solution:

$$1^5 + 1^5 + 0^5 = 1^3 + 1^3 + 0^3.$$

Then the rational points on:

$$C : v^2 = \frac{1}{3}t^4 - \frac{1}{3}t^2$$

can be parametrized. Thus we have:

$$\left(\frac{3k^2+1}{3k^2-1}\right)^5 + \left(\frac{3k^2+1}{3k^2-1}\right)^5 = \left(\frac{9k^4-6k^3-2k-1}{(3k^2-1)^2}\right)^3 + \left(\frac{9k^4+6k^3+2k-1}{(3k^2-1)^2}\right)^3,$$

where $k \in \mathbb{Q}$. For example, substituting 2 for k, we have:

$$\left(\frac{13}{11}\right)^5 + \left(\frac{13}{11}\right)^5 + 0^5 = \left(\frac{91}{121}\right)^3 + \left(\frac{195}{121}\right)^3 + 0^3.$$

The solutions which are obtained in these ways give solutions to another Diophantine equation $2X^5 = Y_1^3 + Y_2^3$.

5.4. It is not simple to find parameters (x_1, α, β) that produce elliptic curves for non-trivial solutions $(X_i, Y_i)_{1 \leq i \leq 3}$. In particular, the author could not find a good parameter for $\beta = 0,\ \alpha \neq 0$:

Question 5.1. Find (a good method for) positive solutions to:

$$X_1^5 + X_2^5 + X_3^5 = Y_1^3 + Y_2^3.$$

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