REMARK ON A PAPER
BY IZADI AND BAGHALAGHDAM
ABOUT CUBES AND FIFTH POWERS SUMS

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Abstract. In this paper, we refine the method introduced by Izadi and Baghalaghdam to search integer solutions to the Diophantine equation

\[ X_1^5 + X_2^5 + X_3^5 = Y_1^3 + Y_2^3 + Y_3^3. \]

We show that the Diophantine equation has infinitely many positive solutions.

1. Introduction

In [2], Izadi and Baghalaghdam consider the Diophantine equation:

\[ a(X_1'^5 + X_2'^5) + \sum_{i=0}^{n} a_i X_1^i = b(Y_1'^3 + Y_2'^3) + \sum_{i=0}^{m} b_i Y_i^3. \]

where \( n, m \in \mathbb{N} \cup \{0\} \), \( a, b \neq 0 \), \( a_i, b_i \) are fixed arbitrary rational numbers. They use theory of elliptic curves to find nontrivial integer solutions to (1). In particular, they discuss the equation:

\[ X_1^5 + X_2^5 + X_3^5 = Y_1^3 + Y_2^3 + Y_3^3 \]

and obtain integer solutions, for example:

\[ 8^5 + 6^5 + 14^5 = (-110)^3 + 124^3 + 14^3, \]

\[ 128122^5 + (-79524)^5 + 48598^5 = 359227580^3 + (-251874598)^3 + 107352982^3. \]

However, no positive solutions are presented in their paper [2]. In this paper, we refine their method to find positive solutions to (2).

Consider the Diophantine equation (2). Let:

\[ \begin{cases} X_1 = t + x_1, & X_2 = t - x_1, & X_3 = \alpha t, \\
Y_1 = t + v, & Y_2 = t - v, & Y_3 = \beta t. \end{cases} \]

Then we get a quartic curve:

\[ C : v^2 = \frac{2 + \alpha^5}{6} t^4 + \frac{20x_1^2 - 2 - \beta^3}{6} t^2 + \frac{5x_1^4}{3} \]

with parameters \( x_1, \alpha, \beta \in \mathbb{Q} \). If we get a rational point \((t, v)\) on \( C \), we can compute a rational solution to (2) (see [2]).

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Once we obtain rational solutions to (2), we can obtain integer solutions by multiplying an appropriate value to \(X_i, Y_i\). In the same way, in order to obtain solutions in positive integers, it suffices to search positive rational solutions to equation (2).

2. **Additional Requirements for Positive Solutions**

Suppose that a positive rational solution \((X_i, Y_i)_{1 \leq i \leq 3}\) to (2) is obtained from a given point \((t, v)\) on the quartic \(C\).

**Proposition 2.1.** Let \(\alpha, \beta, x_1 \in \mathbb{Q}\) and

\[
F(t) = \frac{2 + \alpha^5}{6} t^4 + \frac{20x_1^2 - 2 - \beta^3}{6} t^2 + \frac{5x_1^4}{3}.
\]

A rational point \((t, v)\) on the curve \(C : v^2 = F(t)\) in (4) produces a positive rational solution to (2) by (3) if and only if

\[
\alpha, \beta > 0, \quad 0 \leq F(t) < t^2, \quad t > |x_1|
\]

hold.

**Proof.** If \(X_i\) and \(Y_i\) are positive in the solution in the form (3), we have \(t = (X_1 + X_2)/2 > 0, \alpha = 2X_3/(X_1 + X_2) > 0\) and \(\beta = 2Y_3/(Y_1 + Y_2) > 0\). For \((t, v) \in C\), one has that \(0 \leq v^2 = F(t) < v^2 + Y_1Y_2 = t^2\). It follows from \(x_1^2 < x_1^2 + X_1X_2 = t^2\) that \(t > |x_1|\) for \(t > 0\). Conversely, suppose the inequalities in (5) hold. Then the given point \((t, v)\) on \(C\) satisfies \(v^2 = F(t) < t^2\). This and (5) immediately imply \(X_i, Y_i > 0\) in (3). \(\square\)

**Proposition 2.2.** Under the same assumption as Proposition 2.1, let

\[
a = \frac{2 + \alpha^5}{6}, \quad b = \frac{20x_1^2 - 2 - \beta^3}{6}, \quad c = \frac{5}{3} x_1^4.
\]

Then \(a, b, c\) satisfy \(b^2 - 4ac > 0\) and \(b < 0\) if and only if there exists a real number \(t\) such that \(F(t) < t^2\).

**Proof.** Let \(\tilde{F}(t) = F(t) - t^2\). Since \(\tilde{F}(0) = 5x_1^4/3 \geq 0\), and in this case \(a > 0\), it is easy to see that the following conditions are equivalent to each others:

(i) There exists a real number \(t\) such that \(F(t) < t^2\).

(ii) The equation \(\tilde{F}(t) = 0\) has four distinct solutions.

(iii) The quadratic equation \(ax^2 + bx + c = 0\) has two distinct non-negative solutions.

(iv) The discriminant \(D = b^2 - 4ac\) of the quadratic function \(f(x) = ax^2 + bx + c\) is positive, and the axis of the quadratic function \(-b/2a\) is positive, and \(f(0) \geq 0\).

The condition (iv) holds if and only if “\(b^2 - 4ac > 0\) and \(b < 0\)”, since \(a > 0\) and \(f(0) = c = 5x_1^4/3 \geq 0\). \(\square\)
3. **Example for** $X_1^5 + X_2^5 + X_3^5 = Y_1^3 + Y_2^3 + Y_3^3$

Let us first search parameters $(x_1, \alpha, \beta)$ such that

$$0 < \alpha, \beta, b < b^2 - 4ac$$

with $a, b, c$ given by (6) and such that the quartic curve $C$ of (4) has at least one rational point. Note that these are necessary to satisfy conditions of Proposition 2.1, 2.2. Then, the curve $C$ is birationally equivalent to an elliptic curve $E$ over $\mathbb{Q}$. If $E$ has positive rank, then $C$ has infinitely many rational points.

Let $(x_1, \alpha, \beta) = (2, 1, 16)$. Then the quartic:

$$C : v^2 = \frac{1}{2} t^4 - \frac{2009}{3} t^2 + \frac{80}{3},$$

has a rational point $(t, v) = (44, 760)$. By $T = t - 44$, we transform $C$ into

$$C' : v^2 = \frac{1}{2} T^4 + 88T^3 + \frac{15415}{3}T^2 + \frac{333412}{3}T + 760^2$$

which is birationally equivalent over $\mathbb{Q}$ to the cubic elliptic curve (see [5, Theorem 2.17], [2]):

$$E : y^2 + \frac{41789}{285}xy + 133760y = x^3 - \frac{76876021}{324900}x^2 - 1155200x + \frac{2460032672}{9},$$

where:

$$T = \frac{2 \cdot 760(x + \frac{15415}{3}) - \frac{334312^2}{2 \cdot 3^2 \cdot 760}}{y}, \quad v = -760 + \frac{T(Tx - \frac{334312}{3})}{2 \cdot 760}.$$

Using the Sage software [3], we find that the cubic curve $E$ is an elliptic curve which has rank 2 and the generators of $E$ are:

$$P_1 = \left( -\frac{1802189}{1521}, \frac{5513659679}{417430} \right), \quad P_2 = \left( -\frac{351379}{363}, \frac{47356344241}{2276010} \right).$$

We now consider the subset

$$C_0 = \{(t, v) \in C \mid 0 \leq F(t) < t^2 \} \subset C$$

whose points satisfy another condition (5) of Proposition 2.1. The two quartic equations:

$$F(t) = \frac{1}{2} t^4 - \frac{2009}{3} t^2 + \frac{80}{3} = 0, \quad \tilde{F}(t) = \frac{1}{2} t^4 - \frac{2009}{3} t^2 + \frac{80}{3} - t^2 = 0$$

have respectively solutions:

$$t = \pm \frac{1}{3} \sqrt{6027 \pm 3 \sqrt{4035601}}, \quad t = \pm \frac{2}{3} \sqrt{1509 \pm 3 \sqrt{252979}}.$$
Let us take larger solutions as:
$$a_1 = \frac{1}{3} \sqrt[3]{6027 + 3\sqrt[3]{4035601}} \simeq 36.59635926...$$
$$a_2 = \frac{2}{3} \sqrt[3]{1509 + 3\sqrt[3]{252979}} \simeq 36.62367500...$$

If a point \((t_0, v_0)\) on \(C\) satisfies \(a_1 \leq t_0 \leq a_2\), then \((t_0, v_0)\) lies on \(C_0\). We now make use of the composition law of points on the elliptic curve \(E\). Since \(E\) has positive rank, we can test infinitely many rational points of \(E\) till finding a point \((t_0, v_0)\) on \(C_0\). We find that the rational point
\[Q = 2P_1 - P_2 = \left(\frac{30484538119211829037}{584704128713906667}, -\frac{4767546475726965161322288395890039}{4652843756178203561643745770}\right)\]
on \(E\) corresponds to
\[(t_0, v_0) = \left(\frac{170815619844155909156204}{466494109525000917983}, -\frac{60974088406262566391987292529169987763029096}{21761675422152362106175457381559866386788289}\right)\]
on \(C_0\), and creates a positive rational solution:

$$X_1 = \frac{180145502034655928992170}{466494109525000917983} \simeq 38.61688676...$$
$$X_2 = \frac{161485737655365588930283}{466494109525000917983} \simeq 34.61688676...$$
$$X_3 = \frac{170815619844155909156204}{466494109525000917983} \simeq 36.61688676...$$

Next we shall prove that the Diophantine equation (2) has infinitely many positive solutions. The real locus of elliptic curve \(E(\mathbb{R})\) can be regarded as a compact topological subspace of complex projective variety \(E\).

**Lemma 3.1.** If the rank of elliptic curve \(E\) over \(\mathbb{Q}\) is positive, every point of \(E(\mathbb{Q})\) is an accumulation point in \(E(\mathbb{R})\).

**Proof.** Since \(E(\mathbb{R})\) is a compact topological group, and \(E(\mathbb{Q})\) is an infinite subgroup of \(E(\mathbb{R})\), there is at least one accumulation point of \(E(\mathbb{Q})\) in \(E(\mathbb{R})\). The group operations are homeomorphisms from \(E(\mathbb{R})\) to itself. Therefore all points of \(E(\mathbb{Q})\) are accumulation points of \(E(\mathbb{R})\). \(\square\)

**Theorem 3.2.** The Diophantine equation (2) has infinitely many positive solutions.

**Proof.** The part of \(C_0\) has one rational point \((t_0, v_0)\) which corresponds to the above point \(Q\). By Lemma 3.1, the point \(Q\) is an accumulation point of \(E(\mathbb{Q})\) in \(E(\mathbb{R})\), and \((t_0, v_0)\) is that of \(C(\mathbb{Q})\) in \(C(\mathbb{R})\). Thus the part of \(C_0\) includes infinitely many rational points. Since \(2 = |x_1| < a_1 = 36.59635926...\), they correspond to positive rational solutions to (2). \(\square\)
4. Example for $X_1^5 + X_2^5 = Y_1^3 + Y_2^3 + Y_3^3$

Let $\alpha = 0$. Then (2) gives another Diophantine equation:

$$(7) \quad X_1^5 + X_2^5 = Y_1^3 + Y_2^3 + Y_3^3.$$ 

In the same way, we can obtain a rational or positive rational solutions of it. For example, let $x_1 = 10, \beta = 18$. Then the quartic curve:

$$C : v^2 = \frac{1}{3}t^4 - 639t^2 + \frac{50000}{3}$$

has a rational point $(t, v) = (-5, 30)$ and can be regarded as an elliptic curve over $\mathbb{Q}$ that has rank 2. It is birationally equivalent to:

$$E : y^2 + \frac{1867}{9}xy - 400y = x^3 - \frac{3676525}{324}x^2 - 1200x + \frac{367652500}{27}.$$ 

From this, we can compute rational solutions to (7). For example, there is a point $Q = (x_0, y_0)$ on $E$ with

$$x_0 = \frac{923392183891785064613858846898730}{7122085216670212240561670676647947},$$

corresponding to $(t_0, v_0)$ on $C$ with

$$t_0 = \frac{7869911761727476320751662986237524166650}{1809666757927984888380712753242417827},$$

which creates the following solution to (7):

$$X_1 = \frac{9679568437520274005635470113769948284920}{1809666757927984888380712753242417827},$$

$$X_2 = \frac{60602550859346778358675558750999928380}{1809666757927984888380712753242417827},$$

$$Y_1 = \frac{21025792975860774968586980412651199308809460130710098627025850356764517703500}{32748872832411411768528265731344765807041931918181527764566186484701929},$$

$$Y_2 = \frac{745788273916000738265026952132815055798701435951966444344754733664843241464100}{327488728421441765828265731344765807041931918181327764566186484701929},$$

$$Y_3 = \frac{1416584117110945737735299337527543919700}{1809666757927984888380712753242417827}.$$ 

The case of $\beta = 0$ will be discussed briefly in 5.2 below.

5. Parameters $(x_1, \alpha, \beta)$ from Trivial Solutions

5.1. There are several trivial solutions; for example:

$$1^5 + 1^5 + 1^5 = 1^3 + 1^3 + 1^3.$$ 

We call solutions to (2) which consist of 0, $\pm 1$ trivial. We are going to check some of them to search integer (or positive) solutions.

A solution to (2) may decide parameter. For example, when $X_i = Y_i = 1$ $(i = 1, 2, 3)$, we get $(x_1, \alpha, \beta) = (0, 1, 1)$. Then:

$$C : v^2 = \frac{1}{2}t^4 - \frac{1}{2}t^2$$
has a singular point \((t, v) = (0, 0)\) and can be parametrized by one parameter. Let us divide both sides of \(C\) by \(t^4\) and substitute \(s, w\) for \(1/t, v/t^2\) respectively. Then:

\[
C' : w^2 = \frac{1}{2} - \frac{1}{2}s^2
\]

has a rational point \((s, w) = (1, 0)\). Hence we can parametrize rational points on \(C'\) and integer solutions to (2). That is to say we have:

\[
\begin{align*}
&\left(\frac{2k^2 + 1}{2k^2 - 1}\right)^5 + \left(\frac{2k^2 + 1}{2k^2 - 1}\right)^5 + \left(\frac{2k^2 + 1}{2k^2 - 1}\right)^5 \\
= &\left(\frac{4k^4 - 4k^3 - 2k - 1}{(2k^2 - 1)^2}\right)^3 + \left(\frac{4k^4 + 4k^3 + 2k - 1}{(2k^2 - 1)^2}\right)^3 + \left(\frac{2k^2 + 1}{2k^2 - 1}\right)^3
\end{align*}
\]

where \(k \in \mathbb{Q}\). We can see that large enough \(k\) give positive solutions to (2). For example:

\[
\begin{align*}
\left(\frac{9}{7}\right)^5 + \left(\frac{9}{7}\right)^5 + \left(\frac{9}{7}\right)^5 &= \left(\frac{27}{49}\right)^3 + \left(\frac{99}{49}\right)^3 + \left(\frac{9}{7}\right)^3
\end{align*}
\]

where \(k = 2\). Since \(X_1 = X_2 = X_3 = Y_3\), this solution also gives positive solution to another Diophantine equation \(3X^5 = Y_1^3 + Y_2^3 + X^3\). Moreover it satisfies \(X_1 + X_2 + X_3 = Y_1 + Y_2 + Y_3\) because \(\alpha = \beta\).

5.2. From another trivial solution:

\[1^5 + 0^5 + 0^5 = 1^3 + 0^3 + 0^3,\]

we can derive parameters \((x_1, \alpha, \beta) = (\frac{1}{2}, 0, 0)\). Then:

\[
C : v^2 = \frac{1}{3}t^4 + \frac{1}{2}t^2 + \frac{5}{48}
\]

is an elliptic curve defined over \(\mathbb{Q}\) with rational point \((t, v) = (\frac{1}{2}, \frac{1}{2})\). It is birationally equivalent to:

\[E : y^2 + \frac{4}{3}xy + \frac{2}{3}y = x^3 + \frac{5}{9}x^2 - \frac{1}{3}x - \frac{5}{27}\]

over \(\mathbb{Q}\) and has rank 1. Hence we can apply the method of Section 3 to compute positive solutions to

\[(8) \quad X_1^5 + X_2^5 = Y_1^3 + Y_2^3\]

as a special case of (2) with \(X_1, X_2, Y_1, Y_2 > 0, \ X_3 = Y_3 = 0\) (where \(\alpha = \beta = 0\) in (3)). For example, a point

\[
Q = (10017045137918654785, 29224609136538294659462738431) \quad \frac{16567266306928896}{474322254705993809197056}
\]

on \(E\) corresponding to the point

\[(t_0, v_0) = (2806052350871126431439, 5797926783162005502807971914786692611082209) \quad \frac{1379016004568066987998}{9587890584131638439948667971418559938024002}
\]
on $C$ creates the positive solution to (8):

$$X_1 = \frac{2497780176577579902719}{218955905802284033493999}, \quad X_2 = \frac{308272174293546468720}{218955905802284033493999},$$

$$Y_1 = \frac{59709004301113027437990360675596051258385}{4793945292065819219974333985709279969012001}, \quad Y_2 = \frac{172973646949125171571728445888903440176176}{4793945292065819219974333985709279969012001}.$$

5.3. There exists one more parameter with $\beta = 0$, $(x_1, \alpha, \beta) = (0, 0, 0)$, which is derived from the trivial solution:

$$1^5 + 1^5 + 0^5 = 1^3 + 1^3 + 0^3.$$

Then the rational points on:

$$C : v^2 = \frac{1}{3}t^4 - \frac{1}{3}t^2$$

can be parametrized. Thus we have:

$$\left(\frac{3k^2 + 1}{3k^2 - 1}\right)^5 + \left(\frac{3k^2 + 1}{3k^2 - 1}\right)^5 = \left(\frac{9k^4 - 6k^3 - 2k - 1}{(3k^2 - 1)^2}\right)^3 + \left(\frac{9k^4 + 6k^3 + 2k - 1}{(3k^2 - 1)^2}\right)^3,$$

where $k \in \mathbb{Q}$. For example, substituting 2 for $k$, we have:

$$\left(\frac{13}{11}\right)^5 + \left(\frac{13}{11}\right)^5 + 0^5 = \left(\frac{91}{121}\right)^3 + \left(\frac{195}{121}\right)^3 + 0^3.$$

The solutions which are obtained in these ways give solutions to another Diophantine equation $2X_1^5 = Y_1^3 + Y_2^3$.

5.4. It is not simple to find parameters $(x_1, \alpha, \beta)$ that produce elliptic curves for non-trivial solutions $(X_i, Y_i)_{1 \leq i \leq 3}$. In particular, the author could not find a good parameter for $\beta = 0$, $\alpha \neq 0$:

**Question 5.1.** Find (a good method for) positive solutions to:

$$X_1^5 + X_2^5 + X_3^5 = Y_1^3 + Y_2^3.$$

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