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EXISTENCE AND STABILITY OF STATIONARY SOLUTIONS TO THE ALLEN–CAHN EQUATION DISCRETIZED IN SPACE AND TIME

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Abstract. The existence and stability of the Allen–Cahn equation discretized in space and time are studied in a finite spatial interval. If a parameter is less than or equals to a critical value, the zero solution is the only stationary solution. If the parameter is larger than the critical value, one has a positive stationary solution and this positive stationary solution is asymptotically stable.

1. Introduction

In this paper, we consider the following initial value problem for a difference equation given by

$$(1)\quad \begin{cases} \dfrac{u_j^{n+1}-u_j^n}{\tau} = \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} + f(u_j^n), \ 1 \le j \le N-1, \, n \ge 0, \\ u_0^n = u_N^n = 0, \quad n > 0 \end{cases}$$

for a given initial value $\{u_j^0\}_{1\le j\le N-1}$ satisfying $u_j^0 \ge 0$ for all $1 \le j \le N-1$, and $f \in C^1[0,\infty)$ is a function $f(u) = u(\mu - g(u))$. Now N is a positive integer. Here we assume that g is strictly monotone increasing in $u > 0$ and satisfies $g(0) = g'(0) = 0$. We also assume that there exists a real number $m > 0$ such that $g(m) = \mu$. In other words, f satisfies the following conditions

(A1) $f(u)/u$ is strictly monotone decreasing in $u > 0$,
(A2) $f(0) = 0$, $f'(0) = \mu$ for a constant $\mu > 0$,
(A3) $f(m) = 0$ for a constant $m > 0$.

Note that constant states $u_j^n \equiv 0$ and $u_j^n \equiv m$ satisfy the difference equation. We also consider initial value problem for a difference equation given by

$$(2)\quad \begin{cases} \dfrac{du_j}{dt}(t) = \dfrac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{h^2} + f(u_j(t)), \\ \hfill 1 \le j \le N-1, \, t > 0, \\ u_0(t) = u_N(t) = 0, \quad t > 0, \\ u_j(0) = u_j^0, \quad 0 \le j \le N. \end{cases}$$

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This difference equation is obtained from the space discretization of the Allen–Cahn equation. Interesting questions about traveling fronts, for instance, are about the existence of traveling waves, their monotonicity for space, stability and its convergence rate to a traveling wave. About these problems, for the continuous Allen–Cahn model, we refer the reader to [1, 8, 12, 15, 3, 14], for example. The lattice system (2) arises in chemical reaction theory [7, 9, 11] and biology [2, 10]. A similar model appears for example in [6] in material science and in [13] in image processing. More precisely, [7] introduced coupled Nagumo equations and [9] considered cellular automaton models. The authors in [11] use computers to find propagation failure phenomenon of traveling wave. The lattice system on $\mathbb{Z}$ when zero is a solution for these systems is discussed in [2] and they focused on conditions forcing non-convergence to zero of solutions as time approaches infinity. For a lattice system, propagation and its failure are considered in [10] .

Let $\{v_j\}_{1\le j\le N-1}$ be the stationary solution of the problem for (2). In other words, it is a solution of the difference equation

$$\begin{cases} \dfrac{v_{j+1}-2v_j+v_{j-1}}{h^2}+v_j(\mu-g(v_j))=0, \quad 1\le j\le N-1,\\ v_0=v_N=0. \end{cases} \tag{3}$$

The main assertion in this paper is as follows.

Theorem 1. *Let $h=1/N$ for $N\in\mathbb{N}$ and let $\{u_j^n\}$ be the solution of (1) with the initial value $\{u_j^0\}_{1\le j\le N-1}$ with $u_j^0\ge 0 \quad (1\le j\le N-1)$. Let $K=1+\max_{1\le s\le m}|f'(s)|$ and $\theta=\tau/h^2$. Assume that $\tau>0$ is small enough to satisfy*

$$0<2e^{K\tau}\theta\le\frac{1}{2}$$

and

$$\frac{e^{K\tau}-1}{\tau}+e^{K\tau}\min_{0\le s\le m}f'(s)>0.$$

If $\mu\le\frac{4}{h^2}\sin^2\left(\frac{\pi h}{2}\right)$, $v_j=0$ $(0\le j\le N)$ is the only stationary solution of (1) with $v_j\ge 0 \quad (1\le j\le N-1)$. Moreover, one has $\lim_{n\to\infty}\max_{1\le j\le N-1}|u_j^n|=0$. If $\mu>\frac{4}{h^2}\sin^2\left(\frac{\pi h}{2}\right)$, there exists a unique stationary solution $\{v_j\}_{1\le j\le N-1}$ with $v_j>0$ $(1\le j\le N-1)$. Assume $0<u_j^0<m$ $(1\le j\le N-1)$, then one has $\lim_{n\to\infty}\max_{1\le j\le N-1}|u_j^n-v_j|=0$.

We also get the results for a semi-discrete equation (2).

Proposition 1. *Let $h=1/N$ for $N\in\mathbb{N}$ and let $\{u_j(t)\}$ be the solution of (2) with the initial value $\{u_j^0\}_{1\le j\le N-1}$ with $u_j^0\ge 0 \quad (1\le j\le N-1)$.*

If $\mu \le \frac{4}{h^2}\sin^2\left(\frac{\pi h}{2}\right)$*, the zero solution* $v_j = 0$ $(1 \le j \le N-1)$ *is the only solution of* (3) *with* $v_j \ge 0$ $(1 \le j \le N-1)$*. Moreover, one has* $\lim_{t\to\infty}\max_{1\le j\le N-1}|u_j(t)| = 0$*. If* $\mu > \frac{4}{h^2}\sin^2\left(\frac{\pi h}{2}\right)$*, there exists a unique positive stationary solution* $\{v_j\}_{1\le j\le N-1}$ *with* $v_j > 0$ $(1 \le j \le N-1)$*. Assume* $0 < u_j^0 < m$ $(1 \le j \le N-1)$*, then one has* $\lim_{t\to\infty}\max_{1\le j\le N-1}|u_j(t) - v_j| = 0$*.*

An analogous result for the continuous model can be found in [16], this is the discrete version of their claim.

The remainder of this paper is organized as follows. Section 2 is devoted to discuss the comparison principles of (1). We discuss the relation between the problems (1) and (2) in Section 3, which is useful in the proof of the following sections. In Section 4, we establish the comparison principles for the problem (2). We recall the fundamental eigenvalue problem for the discrete Laplacian in Section 5. We show Theorem 1 in Section 6. The proof of Theorem 1 is given in Section 7.

2. COMPARISON PRINCIPLES FOR THE SPACE AND TIME DISCRETE MODEL

In this section, we consider the following initial value problem for a difference equation (1). A basic comparison principle for the problem (1) is the following proposition. See Proposition 2.1 of [5] for related work.

Proposition 2. *Assume that*

$$0 < 2e^{K_1\tau}\theta \le \frac{1}{2}, \quad \text{where} \quad \theta = \frac{\tau}{h^2}, \tag{4}$$

$K_1 = 1 + \sup_{1\le j\le N, n\ge 0}|g_j^n|$ *and*

$$g_j^n = \int_0^1 f'(\vartheta u_j^n + (1-\vartheta)v_j^n)\,d\vartheta.$$

Moreover, suppose

$$\frac{e^{K_1\tau}-1}{\tau} + e^{K_1\tau}g_j^n > 0 \tag{5}$$

for all $n \ge 0$ *and* $1 \le j \le N$*. Let* $\{v_j^n\}$ *and* $\{u_j^n\}$ *satisfy*

$$\frac{u_j^{n+1}-u_j^n}{\tau} \ge \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} + f(u_j^n), \quad 1 \le j \le N-1$$

and

$$\frac{v_j^{n+1}-v_j^n}{\tau} \le \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{h^2} + f(v_j^n), \quad 1 \le j \le N-1$$

for $n = 0, 1, 2\cdots$. *Assume that*

$$0 \le v_j^0 \le u_j^0 \le m \quad \text{for all} \quad j \in \mathbb{Z}. \tag{6}$$

Then

$$0 \le v_j^n \le u_j^n \le m \quad \text{for all} \quad j \in \mathbb{Z}, \quad n = 0, 1, 2, \cdots . \tag{7}$$

We remark that the condition (5) is automatically satisfied when $\tau \to 0$, because this condition can be reduced to $K_1 + g_j^n > 0$. By a simple calculation, we can check that $w_j^n = u_j^n - v_j^n$ satisfies

$$\frac{w_j^{n+1} - w_j^n}{\tau} \ge \frac{w_{j+1}^n - 2w_j^n + w_{j-1}^n}{h^2} + g_j^n w_j^n.$$

Hence Proposition 2 can be reduced to the following lemma.

Lemma 1. *Let* $\{g_j^n\}$ *satisfy* $\sup_{n\ge0,1\le j\le N-1} |g_j^n| < \infty$ *and let* $\{w_j^n\}$ *satisfy*

$$\begin{cases} \frac{w_j^{n+1}-w_j^n}{\tau} \ge \frac{w_{j+1}^n-2w_j^n+w_{j-1}^n}{h^2} + g_j^n w_j^n, & 1 \le j \le N-1,\ n \ge 0, \\ w_j^0 \ge 0 \quad 1 \le j \le N-1. \end{cases}$$

Assume that (4) *and* (5)*, where* $K_1 = 1 + \sup_{1\le j\le N, n\ge0} |g_j^n|$*. Then one has* $w_j^n \ge 0$ *for all* $1 \le j \le N-1$ *and* $n \ge 0$.

Proof. Define $W_j^n := e^{K_1 n\tau} w_j^n$. Then by a calculation, we can check that

$$\frac{W_j^{n+1} - W_j^n}{\tau} = e^{K_1\tau} \frac{W_{j+1}^n - 2W_j^n + W_{j-1}^n}{h^2} + e^{K_1\tau} g_j^n W_j^n + \frac{e^{K_1\tau} - 1}{\tau} W_j^n.$$

By solving this equation, we get

$$W_j^{n+1} = \theta e^{K_1\tau} W_{j+1}^n + (1 - 2e^{K_1\tau}\theta) W_j^n + \theta e^{K_1\tau} W_{j-1}^n + \tau\Big(\frac{e^{K_1\tau} - 1}{\tau} + e^{K_1\tau} g_j^n\Big) W_j^n.$$

The right hand side is non-negative, hence by the induction argument, we conclude that W_j^n for all $n \ge 0$ and $0 \le j \le N$. $\square$

Next we shall prove that the monotonicity of solutions in time is guaranteed.

Lemma 2. *Suppose the same condition as Proposition 2 for the functions*

$$\bar{g}_j^n = \int_0^1 f'(\phi u_j^{n+1} + (1 - \phi) u_j^n)\, d\phi.$$

instead of $\{g_j^n\}$*. Assume*

$$0 \le u_j^0 \le m \quad \text{for all} \quad j \in \mathbb{Z}.$$

and

$$u_j^1 \ge u_j^0 \quad \text{for all} \quad j \in \mathbb{Z}.$$

Then

$$u_j^{n+1} \geq u_j^n \quad \textit{for all} \quad j \in \mathbb{Z}, \quad n = 0, 1, 2, \cdots . \tag{8}$$

Proof. We define $\bar{w}_j^n = \frac{u_j^{n+1} - u_j^n}{\tau}$, then the lemma follows from Lemma 1. □

3. Relations between the two discrete Allen–Cahn equations

In this section we recall the standard Euler method to estimate difference between the solution of the (1) and that of (2). Let $T > 0$ be positive constant and consider an initial value problem for an ODE system

$$\boldsymbol{y}'(t) = \boldsymbol{F}(\boldsymbol{y}(t)), \quad 0 < t \leq T, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0,$$

where $\boldsymbol{y}(t) = (y_1(t), \ldots, y_{N-1}(t)) \in \mathbb{R}^{N-1}$ is a vector valued functions and $\boldsymbol{F} : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ is a locally Lipschitz vector valued map with constant L. More precisely, for any positive constant $\rho > 0$ there exists $L > 0$ such that

$$\|\boldsymbol{F}(\boldsymbol{z}) - \boldsymbol{F}(\boldsymbol{y})\| \leq L\|\boldsymbol{z} - \boldsymbol{y}\| \qquad \text{if} \qquad \|\boldsymbol{z} - \boldsymbol{y}_0\|, \|\boldsymbol{y} - \boldsymbol{y}_0\| \leq \rho, \tag{9}$$

where $\|\cdot\|$ is the standard Euclidean norm. We also define

$$M = \sup_{\|\boldsymbol{y} - \boldsymbol{y}_0\| \leq \rho} \|\boldsymbol{F}(\boldsymbol{y})\| < \infty. \tag{10}$$

We also choose small $T > 0$ such that $MT \leq \rho$. We consider time variables

$$t_n = n\tau, \qquad 0 \leq n \leq \left[\frac{T}{\tau}\right].$$

Here $[T/\tau]$ is the largest integer that is less than or equals T/τ. The Euler method is a scheme for obtaining an approximated value $\boldsymbol{Y}^{n+1}$ for $\boldsymbol{y}(t_{n+1})$ using only the approximation $\{\boldsymbol{Y}^n\}_{0 \leq n \leq [T/\tau]}$ for $\boldsymbol{y}(t_n)$ and the vector function $\mathbf{F}$, namely

$$\begin{cases} \boldsymbol{Y}^{n+1} = \boldsymbol{Y}^n + \tau \boldsymbol{F}(\boldsymbol{Y}^n), \quad 0 \leq n \leq \left[\frac{T}{\tau}\right] \\ \boldsymbol{Y}^0 = \boldsymbol{y}^0. \end{cases} \tag{11}$$

We define the global truncation error at step n by

$$\boldsymbol{r}^n = \boldsymbol{Y}^n - \boldsymbol{y}(t_n). \tag{12}$$

$\{\boldsymbol{Y}^n\}_{0 \leq n \leq [T/\tau]}$ is called the Euler approximation.

Proposition 3. *Let $\{\boldsymbol{Y}^n\}_{0 \leq n \leq [T/\tau]}$ be given by (11). Define $L > 0$ and $M > 0$ by (9) and (10), respectively. Suppose that $MT < \rho$, then $\boldsymbol{r}^n$ satisfies $\|\boldsymbol{r}^n\| \leq \frac{M\tau}{2} e^{TL}$ for $0 \leq n \leq [T/\tau]$.*

Proof. First we shall show that

$$|\boldsymbol{Y}^n - \boldsymbol{y}^0| \le \rho \quad 0 \le n \le \left[\frac{T}{\tau}\right]. \tag{13}$$

For the case $n = 0$ is trivial. Assume that it is holds true until $n-1$. Then

$$\boldsymbol{Y}^n = \sum_{i=1}^{n}(\boldsymbol{Y}^i - \boldsymbol{Y}^{i-1}) + \boldsymbol{Y}^0 = \tau\sum_{i=1}^{n}\boldsymbol{F}(\boldsymbol{Y}^{i-1}) + \boldsymbol{y}_0$$

and

$$\|\boldsymbol{Y}^n - \boldsymbol{y}^0\| \le \tau\sum_{i=1}^{n}\|\boldsymbol{F}(\boldsymbol{Y}^{i-1})\| \le Mt_n \le MT \le \rho.$$

Thus (13) holds for all $0 \le n \le \frac{T}{\tau}$. Then one has

$$\boldsymbol{y}(t_{n+1}) - \boldsymbol{y}(t_n) = \int_{t_n}^{t_{n+1}} \boldsymbol{y}'(t)dt = \int_{t_n}^{t_{n+1}} \boldsymbol{F}(\boldsymbol{y}(t))dt = \tau\int_0^1 \boldsymbol{F}(\boldsymbol{y}(t_n + \tau s))ds.$$

Combining this with (11), we conclude

$$\boldsymbol{r}^{n+1} = \boldsymbol{r}^n - \tau\int_0^1 (\boldsymbol{F}(\boldsymbol{y}(t_n + \tau s)) - \boldsymbol{F}(\boldsymbol{Y}^j))\, ds. \tag{14}$$

Here one has

$$\|\boldsymbol{y}(t_n + \tau s) - \boldsymbol{y}(t_n)\| = \left\|\int_{t_n}^{t_n+\tau s} \boldsymbol{y}'(\sigma)d\sigma\right\| = \left\|\int_{t_n}^{t_n+\tau s} \boldsymbol{F}(\boldsymbol{y}(\sigma))d\sigma\right\| \le M\tau s.$$

Combining this with (14), one has

$$\|\boldsymbol{r}^{n+1} - \boldsymbol{r}^n + \boldsymbol{F}(\boldsymbol{y}(t_n))\tau - \boldsymbol{F}(\boldsymbol{Y}^n)\tau\| \le \tau\int_0^1 LM\tau s\, ds = \frac{1}{2}LM\tau^2.$$

Here we apply an inequality

$$|\boldsymbol{F}(\boldsymbol{y}(t_n)) - \boldsymbol{F}(\boldsymbol{Y}^n)| \le L\|\boldsymbol{y}(t_n) - \boldsymbol{Y}^n\| \le L\|\boldsymbol{r}^n\|$$

together with the triangle inequality to conclude that

$$\|\boldsymbol{r}^{n+1}\| \le \|\boldsymbol{r}^n\| + \|\boldsymbol{F}(\boldsymbol{y}(t_n)) - \boldsymbol{F}(\boldsymbol{Y}^n)\|\tau + \frac{1}{2}LM\tau^2 \le (1+\tau L)\|\boldsymbol{r}^n\| + \frac{1}{2}LM\tau^2$$

for all $0 \le n \le K$. By the induction argument starting from $\|\boldsymbol{r}^0\| = 0$, this inequality yields

$$\|\boldsymbol{r}^n\| \le \frac{LM\tau^2}{2}\sum_{k=1}^{n-1}(1+\tau L)^k = \frac{M\tau}{2}\{(1+\tau L)^n - 1\} \le \frac{M\tau}{2}e^{TL}.$$

We complete the proof. □

4. COMPARISON PRINCIPLES FOR THE SPACE DISCRETE MODEL

First we prove the comparison principle for the discrete reaction-diffusion equation. See Lemma 1 of [2] and Lemma 3.4 of [4] for related work.

Lemma 3. *Let $g_j(t)$ be functions satisfying $\sup_{0\le j\le N, 0\le t\le T} g_j(t) < \infty$. Suppose that functions $w_j(t)$ satisfy*

$$\begin{cases} \frac{d}{dt}w_j \ge \frac{w_{j+1}-2w_j+w_{j-1}}{h^2} + g_j(t)w_j, & 1\le j\le N-1, t>0, \\ w_0 = w_N = 0, \quad w_j(0)\ge 0, & 1\le j\le N-1. \end{cases}$$

Then $w_j(t)\ge 0$ for all $(j,t)\in\{0,1,,\dots,N\}\times(0,T)$.

Proof. We discrete the time $\theta=\tau/h^2\in(0,1/2)$ and $t_n = n\tau$ for some small $\tau>0$ and denote the approximate value of $w_j(t_n)$ by w_j^n. Choosing $\tau>0$ sufficiently close to 0, we can assume without loss of generality (4) and (5) holds. From Lemma 1, we conclude that $w_j^n\ge 0$ for all $1\le j\le N-1$ and $n\ge 0$. Finally, by taking a limit $\tau\to 0$ and by applying Proposition 3, we conclude that $w_j(t)\ge 0$ for all $t\ge 0$. □

Proposition 4. *Let $T>0$ and suppose that real-value functions $u_j, v_j : [0,T)\to\mathbb{R}$ are differentiable in $t\in(0,T)$ for each $j\in\{1,2,\dots N-1\}$ and satisfy*

(15)
$$\frac{d}{dt}u_j - \frac{u_{j+1}-2u_j+u_{j-1}}{h^2} - f(u_j) \ge \frac{d}{dt}v_j - \frac{v_{j+1}-2v_j+v_{j-1}}{h^2} - f(v_j), \quad 1\le j\le N-1, t\in(0,T),$$

(16)
$$u_0 = u_N = 0 = v_0 = v_N = 0, \qquad u_j(0)\ge v_j(0), \quad 1\le j\le N-1.$$

Then $u_j(t)\ge v_j(t)$ for all $(j,t)\in\{0,1,\dots N\}\times(0,T)$.

Proof. Let us define $w_j(t) = u_j(t)-v_j(t)$ for all $1\le j\le N-1$ and $t\in(0,T)$. Then w_j satisfies

$$\frac{d}{dt}w_j - \frac{w_{j+1}-2w_j+w_{j-1}}{h^2} \ge f(u_j(t)) - f(v_j(t)).$$

Hence we obtain

$$\frac{d}{dt}w_j(t) \ge \frac{w_{j+1}(t)-2w_j(t)+w_{j-1}(t)}{h^2} + g_j(t)w_j(t),$$

where

$$g_j(t) = \int_0^1 f'(\vartheta u_j(t) + (1-\vartheta)v_j(t))\, d\vartheta.$$

Set $K = 1+\sup_{0\le s\le m}|f'(s)|$, and apply Lemma 3 to get the desired result. □

Next we establish the strong comparison principle. See Lemma 3.5 of [4] for related work.

Proposition 5. *Let* $T > 0$ *and suppose that real-value functions* $u_j(t), v_j(t) : [0,T) \to \mathbb{R}$ *are differentiable in* $t \in (0,T)$ *for each* $j \in \{1, 2, \dots N-1\}$ *and satisfy* (15)-(16). *Moreover there exists* $1 \leq J \leq N-1$ *such that* $u_J(0) > v_J(0)$. *Then* $u_j(t) > v_j(t)$ *for all* $(j,t) \in \{1, \dots N-1\} \times (0,T)$.

Proof. We define the function $w_j(t)$ as in the proof of Proposition 4. Set $K = 1 + \sup_{0 \leq s \leq m} |f'(s)|$, and define $w_j(t) := e^{-Kt} W_j(t)$, then we have

$$\frac{d}{dt} W_j(t) \geq \frac{W_{j+1}(t) - 2W_j(t) + W_{j-1}(t)}{h^2} + \{K + g_j(t)\} W_j(t). \tag{17}$$

By Proposition 4, all we need to prove is that the solution of (17) starting from the initial data

$$W_J(0) > 0, \quad W_j(0) = 0, \quad \text{for} \quad j \neq J$$

satisfies

$$W_j(t) > 0 \quad \text{for all} \quad 1 \leq j \leq N-1.$$

If $t_1 \in (0,T)$ is sufficiently small then $W_J(t_1) > 0$. Moreover, we have

$$W_{J-1}(t_1) > 0, \quad W_{J+1}(t_1) > 0$$

since the right hand side of (17) is positive at time $t = 0$ on $j = J \pm 1$. Proposition 4 implies that we can assume that $W_j(t_1) = 0$ for $|j - J| \geq 2$ without loss of generality. Then by a similar argument again, if $t_2 \in (0,T)$ is sufficiently small, we get

$$W_J(t + t_2) > 0 \quad \text{for all} \quad J-2 \leq j \leq J+2.$$

Continuing the same argument, we have

$$W_j(t) > 0, \qquad \text{for all} \quad 0 \leq j \leq N.$$

The proof is complete. □

We give a result which guarantee the monotonicity of solutions in time.

Proposition 6. *Let* $T > 0$ *and suppose that real-value functions* $u_j(t) : [0,T) \to \mathbb{R}$ *are differentiable in* $t \in (0,T)$ *for each* $j \in \{1, 2, \dots N-1\}$ *and satisfy*

$$\frac{u_{j+1}(0) - 2u_j(0) + u_{j-1}(0)}{h^2} - f(u_j(0)) \geq 0, \quad 1 \leq j \leq N-1, \quad u_0 = u_N = 0,$$

Then $\frac{d}{dt} u_j(t) \geq 0$ *for all* $(j,t) \in \{0, 1, \dots N\} \times (0,T)$.

Proof. The function $U_j(t) = \frac{d}{dt}u_j(t)$ satisfies

$$\frac{dU_j}{dt} = \frac{U_{j+1}(t) - U_j(t) + U_{j-1}(t)}{h^2} + f'(u_j(t))U_j(t).$$

Hence we can apply Proposition 4, since f' is smooth, and we conclude that $U_j(t) \geq 0$ for all $t \geq 0$ and $1 \leq j \leq N$. ☐

5. Eigenvalues and eigenfunctions

Let us introduce notations

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{N-2} \\ v_{N-1} \end{pmatrix}, \quad A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

and

$$\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_{N-1}^2}.$$

We also denote the standard inner products of two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ by $(\boldsymbol{v}, \boldsymbol{w})$. For the discrete Laplacian on a line it is well known that

$$\boldsymbol{p}_i = \begin{pmatrix} \sin\theta_i \\ \sin 2\theta_i \\ \vdots \\ \sin(N-2)\theta_i \\ \sin(N-1)\theta_i \end{pmatrix}, \quad \text{where} \quad \theta_i = \frac{i\pi}{N}$$

for $1 \leq i \leq N-1$ and

$$\lambda_i = \frac{4}{h^2}\sin^2\left(\frac{\theta_i}{2}\right). \tag{18}$$

These values are characterized by the min-max principle, and the next property about the eigenvalue problem of the discrete Laplacian is useful in the following argument.

Lemma 4. *There exists a vector* $\hat{\boldsymbol{p}}_1$ *whose components are all positive such that*

$$A\hat{\boldsymbol{p}}_1 = \lambda_1\hat{\boldsymbol{p}}_1 \tag{19}$$

and $\|\hat{\boldsymbol{p}}_1\| = 1$*. Moreover, the first eigenvalue is given by*

$$\lambda_1 = \min_{v \neq 0}\frac{(A\boldsymbol{v}, \boldsymbol{v})}{(\boldsymbol{v}, \boldsymbol{v})} \tag{20}$$

and the maximum eigenvalue is represented as

$$\lambda_{N-1} = \max_{v \neq 0} \frac{(A\boldsymbol{v}, \boldsymbol{v})}{(\boldsymbol{v}, \boldsymbol{v})}$$

6. Proof of Proposition 1

We prove the non-existence result for a positive stationary solution. Now we multiply $\boldsymbol{v}$ to both hand sides of (3) to obtain

$$-(A\boldsymbol{v}, \boldsymbol{v}) + \sum_{j=1}^{N-1} v_j^2(\mu - g(v_j)) = 0.$$

By (20), we get

$$-\lambda_1 \|\boldsymbol{v}\|^2 + \sum_{j=1}^{N-1} v_j^2(\mu - g(v_j)) \geq 0.$$

and

$$(\lambda_1 - \mu)\|\boldsymbol{v}\|^2 + \sum_{j=1}^{N-1} v_j^2 g(v_j) \leq 0.$$

Hence we conclude that $v_j = 0$ for all $1 \leq j \leq N-1$ provided that

$$\frac{4}{h^2} \sin^2\left(\frac{\pi h}{2}\right) = \lambda_1 \geq \mu.$$

Here we apply (18). Now we prove the convergence to the zero vector from any solution of (2). Let us multiply the equation (2) by $u_j(t)$ and summing for $1 \leq j \leq N-1$, we obtain a Lyapunov functional:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(t)\|^2 &= \sum_{j=1}^{N-1} u_j(t)u_j'(t) \\ &= -(A\boldsymbol{u}(t), \boldsymbol{u}(t)) + \mu\|\boldsymbol{u}(t)\|^2 - \sum_{j=1}^{N-1} u_j(t)^2 g(u_j(t)) \\ &\leq -(\lambda_1 - \mu)\|\boldsymbol{u}(t)\|^2 - \sum_{j=1}^{N-1} u_j(t)^2 g(u_j(t)).\end{aligned}$$

Note that the right-hand side is equal or less than zero for all $t \geq 0$, hence the trajectory is bounded for all $t \geq 0$. From a general theory of dynamical system, Lyapunov functional is constant on the omega limit set, that means the right-hand side must be zero on the omega limit set. Since each term of the right hand side is nonpositive and g is strict increasing and $g(0) = 0$, the omega limit set consists of only zero sequence $\{v_j\}_{1 \leq j \leq N-1} = \{0\}_{1 \leq j \leq N-1}$.

This means that the omega limit set of any solution consists of this single point set, hence any solution $\{u_j(t)\}_{1\le j\le N-1}$ converges to zero as $t\to\infty$.

Next, we prove the unique existence of a positive stationary solution for the case

$$\mu > \frac{4}{h^2}\sin^2\left(\frac{\pi h}{2}\right) = \lambda_1. \tag{21}$$

In order to prove it, we shall construct a supersolution and a subsolution and consider time evolution from those initial data. Let us introduce a supersolution, which is given by a constant vector

$$\bar{\boldsymbol{v}} = \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_{N-2} \\ \bar{v}_{N-1} \end{pmatrix} = \begin{pmatrix} m \\ m \\ \vdots \\ m \\ m \end{pmatrix}, \tag{22}$$

where $m>0$ is a real number satisfying $g(m)=\mu$. By a calculation, it is easy to check

$$\frac{\bar{v}_{j+1}-2\bar{v}_j+\bar{v}_{j-1}}{h^2} + \bar{v}_j(\mu - g(\bar{v}_j)) = \begin{cases} 0, & 2\le j\le N-2, \\ -m/h^2<0, & j=1, N-1. \end{cases} \tag{23}$$

Thus the above constant vector is a supersolution. Next we shall introduce a subsolution

$$\underline{\boldsymbol{v}} = \begin{pmatrix} \underline{v}_1 \\ \underline{v}_2 \\ \vdots \\ \underline{v}_{N-2} \\ \underline{v}_{N-1} \end{pmatrix} = \varepsilon \boldsymbol{p}_1 = \varepsilon \begin{pmatrix} \sin\theta_1 \\ \sin 2\theta_1 \\ \vdots \\ \sin(N-2)\theta_1 \\ \sin(N-1)\theta_1 \end{pmatrix}, \tag{24}$$

where $\varepsilon>0$ is sufficiently small to be determined later. Then the assumption $g(0)=0$ together with the continuity of g, $g(\underline{v}_j)\le\mu$ for all $1\le j\le N-1$ if $\varepsilon\in(0,\mu)$ is sufficiently small. On the other hand, (19) yields $A\boldsymbol{p}_1=\lambda_1\boldsymbol{p}_1$. Thus all we need to check is

$$-\lambda_1 p_1^j + p_1^j(\mu - g(\varepsilon p_1^j)) \ge 0$$

for all $1\le j\le N-1$, where p_1^j is the j-th component of the eigenvector $\boldsymbol{p}_1$. Recall that $g(0)=g'(0)=0$, thus by taking $\varepsilon\in(0,\mu)$ sufficiently small, we get the desired inequality

$$-\lambda_1 + \mu - g(\varepsilon p_1^j) \ge 0.$$

thus $\underline{\boldsymbol{v}}$ is a subsolution. We denote the solution of this problem (2) starting from the initial vector $\boldsymbol{u}^0 = \{u_j^0\}_{0\le j\le N}$ by $\boldsymbol{u}(t;\boldsymbol{u}^0)$. Define

$$\bar{\boldsymbol{u}}(t) = \boldsymbol{u}(t;\bar{\boldsymbol{v}}), \quad \underline{\boldsymbol{u}}(t) = \boldsymbol{u}(t;\underline{\boldsymbol{v}}).$$

By Proposition 6, each component of $\bar{\boldsymbol{u}}(t)$ is monotone decreasing in t, and each component of $\underline{\boldsymbol{u}}(t)$ is monotone increasing in t. Thus we can define a limit function

$$\boldsymbol{U} = \lim_{t\to\infty} \bar{\boldsymbol{u}}(t), \quad \boldsymbol{V} = \lim_{t\to\infty} \underline{\boldsymbol{u}}(t).$$

These vectors satisfy

$$\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} + U_j(\mu - g(U_j)) = 0,$$
$$\frac{V_{j+1} - 2V_j + V_{j-1}}{h^2} + V_j(\mu - g(V_j)) = 0.$$

Now we multiply the first equation by V_j and the second equation by U_j, calculate their difference and sum up together for j to get

$$\sum_{j=1}^{N-1} U_j V_j (g(U_j) - g(V_j)) = 0. \tag{25}$$

Here we used the symmetry relation $(A\boldsymbol{U}, \boldsymbol{V}) = (\boldsymbol{U}, A\boldsymbol{V})$ of the discrete Laplacian. By the monotonicity for time, $U_j, V_j > 0$ for all $1 \le j \le N-1$. The comparison principle yields $U_j \ge V_j$ for all $1 \le j \le N-1$, hence $g(U_j) \ge g(V_j)$. Thus $g(U_j) = g(V_j)$ must hold for all $1 \le j \le N-1$ from (25). Since g is strictly monotone increasing for $u > 0$ to conclude that $U_j = V_j$ for all $1 \le j \le N-1$. Proposition 4 yields the convergence result $\lim_{t\to\infty} \max_{1\le j\le N-1} |u_j(t) - v_j| = 0$ for any initial data satisfying the inequalities $0 < u_j^0 < m$ for all $1 \le j \le N-1$.

7. Proof of Theorem 1

The proof about the existence of the stationary problem has already done, since the stationary problem is the same between (1) and (2).

Let us define the solution of (1) starting from the initial vector $\boldsymbol{u}^0 = \{u_j^0\}_{0\le j\le N}$ by $\boldsymbol{u}^n(\boldsymbol{u}^0)$. Define $\bar{\boldsymbol{u}}^n := \boldsymbol{u}^n(\bar{\boldsymbol{v}})$, where $\bar{\boldsymbol{v}}$ is given in (22). Lemma 2 implies that each component of $\bar{\boldsymbol{u}}^n$ is nonnegative and monotone non-increasing for n. Hence we can define

$$\boldsymbol{U} := \lim_{n\to\infty} \bar{\boldsymbol{u}}^n.$$

By taking a limit $n \to \infty$ in (1) and using the continuity of the function f, we obtain

$$\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} + f(U_j) = 0, \quad 1 \le j \le N-1.$$

First we consider the case,

$$\frac{4}{h^2}\sin^2\left(\frac{\pi h}{2}\right) = \lambda_1 \geq \mu.$$

Under this assumption the only nonnegative stationary solution is zero vector, which implies that $U_j = 0$ for all $1 \leq j \leq N-1$. Note that we can check the assumption (4)-(5), and we can apply Proposition 2 and the comparison principle. Hence $0 \leq u_j^n \leq \bar{u}_j^n$ for all $n \geq 0$ and $0 \leq j \leq N$. By taking a limit $n \to 0$, we prove the desired result.

Next we consider the case

$$\frac{4}{h^2}\sin^2\left(\frac{\pi h}{2}\right) = \lambda_1 < \mu$$

and prove the convergence to the positive stationary solution. Define

$$\underline{\boldsymbol{u}}^n := \boldsymbol{u}^n(\underline{\boldsymbol{v}}),$$

where $\underline{\boldsymbol{v}}$ is given in (24). This time by applying Lemma 2, we conclude that each component of $\bar{\boldsymbol{u}}^n$ is monotone non-increasing for n and each component of $\underline{\boldsymbol{u}}^n$ is monotone non-decreasing for n. Also all components of these vectors are bounded from above and below. Hence there exists

$$\boldsymbol{U} = \lim_{n\to\infty} \bar{\boldsymbol{u}}^n, \quad \boldsymbol{V} = \lim_{n\to\infty} \underline{\boldsymbol{u}}^n.$$

These vectors are solutions to the same stationary problem as discussed in Section 6, and its proof is completely the same as that of Section 6. Now we apply Proposition 2 to conclude that the solution u_j^n satisfies $\underline{u}_j^n \leq u_j^n \leq \bar{u}_j^n$ for all $n \geq 0$ and $0 \leq j \leq N$. Finally, by taking a limit $n \to 0$, we can prove the desired result.

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