ANALYTIC EXTENSION OF EXCEPTIONAL
CONSTANT MEAN CURVATURE ONE CATENOIDS
IN DE SITTER 3-SPACE

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Abstract. Catenoids in de Sitter 3-space $S^3_1$ belong to a certain class of
space-like constant mean curvature one surfaces. In a previous work, the
authors classified such catenoids, and found that two different classes of
countably many exceptional elliptic catenoids are not realized as closed
subsets in $S^3_1$. Here we show that such exceptional catenoids have closed
analytic extensions in $S^3_1$ with interesting properties.

1. Introduction.

We denote by $S^3_1$ the de Sitter 3-space, which is a simply-connected
Lorentzian 3-manifold with constant sectional curvature 1. Let $R^4_1$ be the
Lorentz-Minkowski 4-space with the metric $\langle \cdot, \cdot \rangle$ of signature $(-+++)$. Then
$$ S^3_1 = \{ X \in R^4_1 ; \langle X, X \rangle = 1 \} $$
with metric induced from $R^4_1$. We identify $R^4_1$ with the $2 \times 2$ Hermitian
matrices $\text{Herm}(2)$ by
$$ (t, x, y, z) \mapsto \begin{pmatrix} t + z & x + iy \\ x - iy & t - z \end{pmatrix}, $$
where $i = \sqrt{-1}$. Then $S^3_1$ is represented as
$$ S^3_1 = \{ X \in \text{Herm}(2) ; \det X = -1 \} = \{ ae_3 a^* ; a \in \text{SL}(2, \mathbb{C}) \}, $$
where $a^* := \overline{a}^\top$ is the conjugate transpose of $a$, and
$$ e_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. $$

To draw surfaces in $S^3_1$, we use the stereographic hollow ball model given
in [4] as follows:

\begin{enumerate}
\item \( \Pi : S^3_1 \ni (t, x, y, z) \mapsto \frac{1}{\delta} (x, y, z) \in R^3 \)
\end{enumerate}
$$ \delta := t + \sqrt{t^2 + x^2 + y^2 + z^2} = t + \sqrt{2t^2 + 1} \). $$

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This projection $\Pi$ is the composition of central projection of $S^3_1$ to the unit sphere $S^3$ centered at the origin in $\mathbb{R}^4$ and usual stereographic projection of $S^3$ into $\mathbb{R}^3$ from $(0,0,0,-1)$. The image of $\Pi$ is the set

$$D^3 := \left\{ \xi \in \mathbb{R}^3 : \sqrt{2} - 1 < \left| \xi \right| < \sqrt{2} + 1 \right\},$$

where $|\xi| := \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$ for $\xi = (\xi_1, \xi_2, \xi_3)$.

In [1], the authors classified all catenoids in $S^3_1$ (i.e. weakly complete constant mean curvature one surfaces in $S^3_1$ of genus zero with two regular ends whose hyperbolic Gauss map is of degree one). There are three types of catenoids:

- elliptic catenoids,
- the parabolic catenoid, and
- hyperbolic catenoids.

Parabolic catenoids have only one congruence class, whose secondary Gauss map is given by

$$g = \frac{1 + \log z}{-1 + \log z},$$

and they are rotationally symmetric surfaces with one cone-like singular point and two embedded ends. On the other hand, the secondary Gauss maps of hyperbolic catenoids are of the form

$$g = \frac{g_0 - i}{g_0 + 1}, \quad g_0 := \exp((m + i\tau) \log z) = z^m + i\tau,$$

where $m$ is a non-negative integer, and $\tau$ is a non-zero real number. When $m \neq 0$ (resp. $m = 0$), hyperbolic catenoids admit only cuspidal edge singularities (resp. cone-like singular points), see [1, Page 36]. Recently, in a joint work with Seong-Deog Yang, the authors [2] proved that all hyperbolic catenoids do not admit any analytic extension.

On the other hand, there are many subclasses of elliptic catenoids, whose secondary Gauss maps $g$ are given by

(i) $g = z^\alpha$ ($0 < \alpha < 1$),
(ii) $g = z^\alpha$ ($\alpha > 1$),
(iii) $g = z^m + c$ ($m = 2, 3, \ldots$) with $c \in (0, \infty) \setminus \{1\}$,
(iv) $g = z^m + 1$ ($m = 2, 3, \ldots$),
(v) $g = (z^m - 1)/(z^m + 1)$ ($m = 2, 3, \ldots$).

Except for the two cases (iv) and (v), all elliptic catenoids are closed subsets of $S^3_1$, since the singular sets of catenoids of type (i)–(iii) are compact. In this paper, we call the catenoids in the class (iv) (resp. (v)) exceptional catenoids of type I (resp. exceptional catenoids of type II) and we study these two classes.
For each $m = 2, 3, \ldots$, we set

\begin{equation}
F^I_m := z^{-\frac{m+1}{2}} \frac{(m+1)z}{m-1} \frac{z((m-1)z^m - m - 1)}{(m+1)z^m - m + 1}
\end{equation}

and

\begin{equation}
F^\|_m := z^{-\frac{m+1}{2}} \frac{z((1-m)z^m + m + 1)}{-(m+1)z^m - m - 1} \frac{z((m-1)z^m + m + 1)}{(m+1)z^m + m - 1}.
\end{equation}

The maps $f^I_m : C \setminus \{0\} \to S^3_1$ defined by

\begin{equation}
f^I_m := F^I_m e_3 (F^I_m)^* \quad (J = I, \|)
\end{equation}

give the exceptional catenoids. These expressions are obtained by shifting $m$ to $m-1$ in [1, Prop. 4.9]. We will show that the image of each $f^I_m$ ($J = I, \|$) has an analytic extension $C^J_m$ which is a closed set in $S^3_1$.

A subset $\mathcal{A}$ of a manifold $M^n$ is called almost embedded (resp. almost immersed) if there is a discrete subset $D$ of $\mathcal{A}$ such that $\mathcal{A} \setminus D$ is the image of an embedding (resp. an immersion) of a manifold into $M^n$. For example (cf. [1]),

- catenoids of class (iii) are not almost immersed,
- catenoids of class (i) are almost immersed, but not almost embedded,
- catenoids of class (ii) are almost embedded.

In Section 2, we investigate the geometric properties of $f^I_m$, and show that the image of each $f^I_m$ has an analytic extension whose image is immersed outside of a compact set. See Figures 1, 2, 3, where $f^I_3, C^I_2$ and $C^I_3$ are drawn in the stereographic hollow ball model (1). In Section 3, we show that each $C^\|_m$ can be realized as a warped product of a certain trochoid and hyperbola. In particular, $C^\|_2$ and $C^\|_3$ are almost embedded, and $C^\|_m$ ($m \geq 4$) are almost immersed (cf. Section 3). See Figures 4, 5, where $f^\|_2, C^\|_2$ and $C^\|_3$ are drawn in the stereographic hollow ball model (1) as well.
It is well-known that the de Sitter space $S_1^3$ can be compactified by including two spheres $\partial_{\pm}S_1^3$. These two sets $\partial_{\pm}S_1^3$ are called the ideal boundaries. In the stereographic hollow ball model, the relations

$$
\partial_{\pm}S_1^3 = \{ \xi \in \mathbb{R}^3 ; |\xi| = \sqrt{2} \mp 1 \}
$$

Figure 2. The set $C^1_2$ (left) and halves of it (center and right).

Figure 3. The set $C^1_3$ (left) and halves of it (center and right).

Figure 4. The set $C^1_2$ (left) and the image of $f^1_2$ (right).
The set $C^I_J$ and half of it.

Figure 5. The set $C^I_J$ and half of it.

hold (cf. (2)). If a subset $A$ of $S^3_1$ is closed, then each element of the set

$$\overline{\Pi(A)} \cap \partial D^3 (\subset \partial_- S^3_1 \cup \partial_+ S^3_1)$$

is called an endpoint, where $\overline{\Pi(A)}$ is the closure of $\Pi(A)$ in $\mathbb{R}^3$. Then the set $\overline{\Pi(C^I_J)} \cap \partial_+ S^3_1$ consists of one (resp. two) point(s) if $J = I$ and $m$ is odd (resp. if $J = I$ and $m$ is even, or $J = \Xi$). On the other hand, $\overline{\Pi(C^I_J)} \cap \partial_- S^3_1$ always consists of two points, that is, the number of the endpoints of $C^I_J$ ($J = I, \Xi$) is three or four (cf. Theorems 4 and 6). This is a remarkable phenomenon, since other elliptic catenoids in $S^3_1$ do not have any analytic extensions and have exactly two endpoints.

2. Exceptional catenoids of type I.

In this section, we show that $f^I_m$ has an analytic extension. For each integer $m \geq 2$, we set

$$f^I_m(r, \theta) = (x_0(r, \theta), x_1(r, \theta), x_2(r, \theta), x_3(r, \theta)),$$

with $z = re^{i\theta}$ ($r > 0$, $\theta \in S^1 := \mathbb{R}/2\pi \mathbb{Z}$). Then

$$x_0 \pm x_3 = \frac{m^2 - 1}{4m} r^{m+1} \left( 2 \cos m \theta - \frac{m \pm 1}{m \pm 1} r^m \right),$$

$$x_1 + ix_2 = \frac{(m - 1)^2}{4m} e^{i(m+1)\theta} + \frac{(m + 1)^2}{4m} e^{-i(m-1)\theta} - e^{i\theta} \frac{m^2 - 1}{4m} r^m.$$  

We know that $f^I_m(r, \theta)$ has self-intersections, since it contains swallowtail singularities (cf. Proposition A.1 in Appendix A). The limit curve

$$\gamma_m(\theta) := \lim_{r \to 0} (x_1, x_2)$$

gives a closed regular planar curve.

A hypo-trochoid is a roulette traced by a point attached to a disk of radius $r_c$ rolling along the inside of a fixed circle of radius $r_m$, where the point is
a distance $d$ from the center of the interior circle. The parametrization of a hypo-trochoid is given by

\[
\begin{align*}
x(s) &= (r_c - r_m) \cos s + d \cos \left( \frac{r_c - r_m}{r_m} s \right), \\
y(s) &= (r_c - r_m) \sin s - d \sin \left( \frac{r_c - r_m}{r_m} s \right).
\end{align*}
\]

We prove the following:

**Proposition 1.** The plane curve $\gamma_m(\theta)$ has the following properties:

(a) $\gamma_m(\theta + \pi) = (-1)^{m+1} \gamma_m(\theta)$ for $\theta \in \mathbb{R}$,

(b) the image of $\gamma_m$ is a convex curve if $m = 2, 3$,

(c) $\gamma_m$ is a hypo-trochoid with (cf. Figure 6)

\[
\begin{align*}
r_c &= \frac{m - 1}{2}, \\
r_m &= \frac{m^2 - 1}{4m}, \\
d &= \frac{(m + 1)^2}{4m}.
\end{align*}
\]

**Proof.** The first two assertions follow immediately. The last assertion follows from the expressions

\[
\begin{align*}
x_1 &= \frac{(m - 1)^2 \cos(m + 1)\theta + (m + 1)^2 \cos(m - 1)\theta}{4m}, \\
x_2 &= \frac{(m - 1)^2 \sin(m + 1)\theta - (m + 1)^2 \sin(m - 1)\theta}{4m}.
\end{align*}
\]

We set $\Omega := \Omega^+ \cup \Omega^-$, where

\[
\Omega^\pm := \{(r, \theta) \in \mathbb{R} \times S^1 ; \pm r > 0\} \quad (S^1 := \mathbb{R}/2\pi \mathbb{Z}).
\]

The expressions (6) and (7) are meaningful for $r < 0$ as well, and $f_m^1$ can be extended to $\Omega$. We denote this extension by $\tilde{f}_m^1 : \Omega \to S^3_1$. If $m$ is odd, then

\[
\tilde{f}_m^1(-r, \theta + \pi) = \tilde{f}_m^1(r, \theta).
\]
Figure 7. The domains of $\tilde{f}_m^1$ and their singular sets.

In particular, if $m$ is odd, the image of $f_m^1$ coincides with that of $\tilde{f}_m^1$. On the other hand, if $m$ is even,

$$\tilde{f}_m^1(-r, \theta) = \iota \circ f_m^1(r, \theta),$$

where $\iota$ is the isometric involution given by

$$\iota: S^3_1 \ni (t, x, y, z) \mapsto (-t, x, y, -z) \in S^3_1.$$ 

Thus, if $m$ is even, $f_m^1(\Omega^+)$ and $\tilde{f}_m^1(\Omega^-)$ are congruent, but do not coincide with each other. The singular set of $\tilde{f}_m^1$ is $\Sigma_m := \Sigma_m^+ \cup \Sigma_m^-$, where

$$\Sigma_m^\pm := \{(r, \theta) \in \Omega^\pm : r^m + 2 \cos m\theta = 0\},$$

each of which consists of $m$ components. The image of each component of the singular set is a curve with singularities which is bounded in $S^3_1$, whose endpoints are

$$P_k := (0, \gamma_m(\alpha_k), 0), \quad \alpha_k := \frac{2k + 1}{2m} \pi \quad (k = 0, \ldots, 2m - 1).$$

We denote by $A_m^\pm$ the domain in $\Omega^\pm$ containing a neighborhood of $r = \pm \infty$, and $B_m^\pm := \Omega^\pm \setminus A_m^\pm$. Then we have the expressions

$$A_m^\pm = \{(r, \theta) \in \Omega^\pm : \epsilon_m(r^m + 2 \cos m\theta) > 0\},$$

$$B_m^\pm = \{(r, \theta) \in \Omega^\pm : \epsilon_m(r^m + 2 \cos m\theta) < 0\},$$

where $\epsilon$ is the sign of $r$ (cf. Figure 7). We next consider the light-like lines

$$L_k := \{(t, \gamma_m(\alpha_k), -t) : t \in \mathbb{R}\} \subset S^3_1$$

passing through $P_k$ for $k = 0, 1, \ldots, 2m - 1$, and set

$$C_m^1 := \tilde{f}_m^1(\Omega) \cup L_0 \cup \cdots \cup L_{2m-1}.$$ 

Then $C_m^1$ is the analytic extension of $f_m^1$. In fact,

**Theorem 2.** For each integer $m \geq 2$, 

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**Diagram:**

- Figure 7 shows the domains of $\tilde{f}_m^1$ and their singular sets.
- The domains are labeled with $m = 2$ and $m = 3$.
- The singular sets are depicted with shaded regions.
- The light-like lines $L_k$ are indicated, passing through points $P_k$.
- The analytic extension $C_m^1$ is illustrated as the union of domains and light-like lines.
(i) $C_{m}^{I}$ is a closed set of $S_{1}^{3}$. In particular, if $m$ is odd, then $C_{m}^{I}$ is the closure of the image of $f_{m}^{I}$. On the other hand, if $m$ is even, then the closure of the image of $f_{m}^{I}$ is just half of $C_{m}^{I}$. The other half can be obtained by the isometric involution $i$ of $S_{1}^{3}$ given in (10).

(ii) Moreover, $C_{m}^{I}$ is analytically immersed outside the compact set consisting of the image of $\Sigma_{m}$, and the points \{(0, \gamma_{m}(\alpha_{k}), 0); k = 0, \ldots, 2m - 1\}.

Proof. By (6), $x_{0}(r, \theta)$ diverges for $r \to \pm \infty$. Take a sequence $\{\zeta_{j} = (r_{j}, \theta_{j})\}_{j=1,2,\ldots}$ on $\Omega$ such that $\lim_{j \to \infty} r_{j} = 0$. Taking a subsequence if necessary, we may assume $\{\zeta_{j}\}$ is included in $\Omega^{+}$ or $\Omega^{-}$, and $\lim_{j \to \infty} \theta_{j} = \beta$. If $\cos m\beta \neq 0$, (6) implies that $\lim_{j \to \infty} x_{0}(r_{j}, \theta_{j})$ diverges. On the other hand, if $\cos m\beta = 0$, that is, $\beta = \alpha_{k}$ for some $k$, then $\lim_{j \to \infty} (x_{0}(\zeta_{j}) + x_{3}(\zeta_{j}))$ tends to 0, that is, $f_{m}^{I}(\zeta_{j})$ is asymptotic to the line $L_{k}$. Conversely, for each point $Q_{k,t} := (t, \gamma_{m}(\alpha_{k}), -t) \in L_{k}$, we set

$$
\zeta_{j} := \left(\frac{1}{j}, \frac{1}{m} \cos^{-1} \frac{4mt}{j(m^{2} - 1)}\right) \quad (j = 1, 2, \ldots),
$$

where $\cos^{-1}$ is the inverse function of $\cos$ as a map

$$
\cos^{-1}: (-1, 1) \to \left(m\alpha_{k} - \frac{\pi}{2}, m\alpha_{k} + \frac{\pi}{2}\right).
$$

Then $\lim_{j \to \infty} \zeta_{j} = (0, \alpha_{k})$ and $\lim_{j \to \infty} f_{m}^{I}(\zeta_{j}) = Q_{k,t}$, where $\alpha_{k}$ is as in (11).

Hence $C_{m}^{I}$ is the closure of the image of $f_{m}^{I}$, proving the first part of (i). The second part of (i) is already proven. We next prove (ii). Since $f_{m}^{I}$ is an analytic immersion on $\Omega \setminus \Sigma_{m}$, it is sufficient to show that $C_{m}^{I}$ is parametrized analytically on a neighborhood of $L_{k}$, which gives an immersion on $L_{k} \setminus \{P_{k}\}$.

For this purpose, we set $s := (\cos m\theta)/r$. Then the $x_{j}$ ($j = 0, 1, 2, 3$) have the following expressions:

$$
x_{0} \pm x_{3} = \frac{m^{2} - 1}{4m} r^{\pm 1} \left(2r s - \frac{m}{m \pm 1} r^{m}\right),
$$

$$
x_{1} + ix_{2} = \frac{e^{i \cos^{-1}(sr)/m}}{4m} \left((m - 1)^{2} e^{i \cos^{-1}(sr)}
\quad + (m + 1)^{2} e^{-i \cos^{-1}(sr)(m^{2} - 1)r^{m}}\right).
$$

Since $(\partial(x_{1} + ix_{2})/\partial r)|_{(0, s)} \neq 0$ if $s \neq 0$, one can easily check that $f_{m}^{I}(r, s)$ is an immersion at $(0, s)$ for each $s \in \mathbb{R} \setminus \{0\}$, which proves the assertion. □
Remark 3. For the parametrization \((r, s)\) as in the proof of Theorem 2, the origin \((r, s) = (0, 0)\) is a singular point for each \(k = 0, 1, \ldots, 2m - 1\), whose image is the point \(P_k\) given in (11). One can show that this parametrization gives a wave front on a neighborhood of \((0, 0)\), and the origin is a cuspidal edge (resp. swallowtail) when \(m = 2\) (resp. \(m = 3\)), see Appendix A.

Next, we consider the endpoints of \(C_m^I\). Let
\[
\begin{align*}
p_{\pm} &:= (0, 0, \pm(\sqrt{2} - 1)) \in \partial_{\pm}S_1^3, \\
n_{\pm} &:= (0, 0, \pm(\sqrt{2} + 1)) \in \partial_{-}S_1^3,
\end{align*}
\]
where \(\partial_{\pm}S_1^3\) are the ideal boundaries given in (5). We set
\[
y := (y_1, y_2, y_3) := \Pi \circ f_{m}^{I} = \frac{1}{\delta}(x_1, x_2, x_3),
\]
where \(\delta = x_0 + \sqrt{2x_0^2 + 1}\) (cf. (1)).

Theorem 4. If \(m\) is even (resp. odd), the set of endpoints of \(C_m^I\) is \(\{p_{\pm}, n_{\pm}\}\) (resp. \(\{p_{-}, n_{\pm}\}\)). More precisely, let \(\{\zeta_j = (r_j, \theta_j)\}\) be a sequence in \(\Omega\) whose image under \(\tilde{f}_m^I\) is unbounded. Then the following cases occur:

1. \(# \lim_{j \to \infty} y(\zeta_j) = n_{-} \) holds when \(# \lim_{j \to \infty} r_j = +\infty \) (that is, \(\{\zeta_j\}\) lies in \(\Omega^+\) and diverges).

2. When \(# \lim_{j \to \infty} r_j = -\infty \), that is, if \(\{\zeta_j\}\) lies in \(\Omega^-\) and diverges, then \(# \lim_{j \to \infty} y(\zeta_j) = p_{+} \) (resp. \(n_-\)) if \(m\) is even (resp. odd).

3. When \(# r_j \to 0 \) and \(\{\zeta_j\}\) is contained in \(A_m^+\) (resp. \(A_m^-, B_m^+, B_m^-\)), the limit of \(y(\zeta_j)\) is obtained as in the following table:

<table>
<thead>
<tr>
<th>The domain containing ({\zeta_j})</th>
<th>(A_m^+)</th>
<th>(A_m^-)</th>
<th>(B_m^+)</th>
<th>(B_m^-)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lim_{j \to \infty} y(\zeta_j)) for even (m)</td>
<td>(p_-)</td>
<td>(n_+)</td>
<td>(n_+)</td>
<td>(p_-)</td>
</tr>
<tr>
<td>(\lim_{j \to \infty} y(\zeta_j)) for odd (m)</td>
<td>(p_-)</td>
<td>(n_-)</td>
<td>(n_+)</td>
<td>(n_+)</td>
</tr>
</tbody>
</table>

Proof. We rewrite (16) as
\[
y_l = \frac{x_l/x_0}{1 + \text{sgn}(x_0)\sqrt{2 + 1/(x_0)^2}} \quad (l = 1, 2, 3),
\]
where \(\text{sgn}(x_0)\) denotes the sign of \(x_0\). By (6) and (7),
\[
\lim_{r \to \pm \infty} \frac{x_3}{x_0} = 1, \quad \lim_{r \to \pm \infty} \frac{x_l}{x_0} = 0 \quad (l = 1, 2),
\]
\[
\lim_{r \to \mp \infty} x_0 = -\infty, \quad \lim_{r \to -\infty} (-1)^m x_0 = \infty,
\]
proving (1) and (2).
We prove (3) for the case that \( \{ \zeta_j \} \subset B^-_m \). Noticing that \( r_j < 0 \), (13) implies that

\[
(-1)^m \left( r_j^{m-1} + \frac{\cos m\theta}{r_j} \right) > 0
\]

holds for each \( j \). Since \( \{ x_0(\zeta_j) \} \) is unbounded, so is \( (\cos m\theta_j)/r_j \). Then the sign of \( (\cos m\theta)/r_j \) is equal to \( (-1)^m \) for sufficiently large \( j \) because \( r_j \) tends to 0. Then by (6), \( \text{sgn}(x_0(\zeta_j)) = (-1)^m \). On the other hand, (6) implies that \( \lim_{j \to \infty} x_3(\zeta_j)/x_0(\zeta_j) = -1 \). Thus, we have

\[
\lim_{j \to \infty} y_3(\zeta_j) = \frac{-1}{1 + (-1)^m \sqrt{2}} = 1 - (-1)^m \sqrt{2}.
\]

Since \( x_1 \) and \( x_2 \) are bounded near \( r = 0 \), \( y_l(\zeta_j) \) tends to 0 for \( l = 1, 2 \). Thus we have the conclusion. The other cases can be proved similarly. \( \square \)

3. Exceptional catenoids of type II.

Here we show that the image of the exceptional catenoid \( f^\Pi_m \) in \( S^3 \) has an analytic extension. For each integer \( m \geq 2 \), we set

\[
f^\Pi_m(r, \theta) = (x_0(r, \theta), x_1(r, \theta), x_2(r, \theta), x_3(r, \theta)),
\]

with \( z = re^{i\theta} \) \( r > 0, \theta \in [0, 2\pi) \). By (4), \( f^\Pi_m \)'s components are

\[
x_0 = \frac{1 - m^2}{4m} \left( r + \frac{1}{r} \right) \cos m\theta,
\]

\[
x_3 = \frac{1 - m^2}{4m} \left( r - \frac{1}{r} \right) \cos m\theta,
\]

\[
x_1 = -\frac{(m^2 + 1) \cos m\theta \cos \theta + 2m \sin m\theta \sin \theta}{2m},
\]

\[
x_2 = -\frac{(m^2 + 1) \cos m\theta \sin \theta - 2m \sin m\theta \cos \theta}{2m},
\]

where \( z = re^{i\theta} \) \( r > 0, \theta \in [0, 2\pi) \). The secondary Gauss map \( g_m \) of \( f^\Pi_m \) is a meromorphic function on \( C \cup \{ \infty \} \) given by (cf. \( 1, (39) \))

\[
g_m = (z^m - 1)/(z^m + 1).
\]

Since the singular set \( \Sigma_m \) of the map \( f^\Pi_m \) is

\[
\Sigma_m = \{ z \in C \setminus \{ 0 \} : |g_m(z)| = 1 \} = \{ re^{i\theta} \in C \setminus \{ 0 \} : \cos m\theta = 0 \},
\]

we have \( \Sigma_m = \sigma_0 \cup \sigma_1 \cup \cdots \cup \sigma_{2m-1} \), where

\[
\sigma_k := \{ z = re^{i\alpha_k} : r > 0 \} \quad (\alpha_k := \frac{(2k+1)\pi}{2m})
\]
for \(k = 0, \ldots, 2m - 1\). In particular, if we set
\[
\Omega_k := \left\{ re^{i\theta} : \frac{(2k-1)\pi}{2m} < \theta < \frac{(2k+1)\pi}{2m}, \quad r > 0 \right\},
\]
then the union of the \(\Omega_k (k = 0, \ldots, 2m - 1)\) is the regular set of \(f_m^\Pi\), that is, the regular set consists of a disjoint union of \(2m\) sectors.

**Proposition 5.** The map \(f_m^\Pi\) satisfies:

(i) For each \(m \geq 2\), the image \(f_m^\Pi(\sigma_k)\) consists of a point. More precisely,
\[
f_m^\Pi(\sigma_k) = (-1)^k (0, -\sin \alpha_k, \cos \alpha_k, 0),
\]
where \(\alpha_k\) is as in (19) \((k = 0, \ldots, 2m - 1)\).

(ii) The endpoints of the image of \(f_m^\Pi\) are the four points \(p_\pm\) and \(n_\pm\) as in (15).

**Proof.** Substituting \(\theta = \alpha_k\) into (18) and using that \(\cos m\theta = 0\) and \(\sin m\theta = (-1)^k\) on \(\sigma_k\), we get the first assertion.

To prove the second assertion, we remark that
\[
\text{sgn}(x_0) = (-1)^{k+1} \quad \text{(on } \Omega_k)\,
\]
for each \(k\), since \(\text{sgn}(\cos m\theta) = (-1)^k\). Take a sequence \(\{z_j\}\) on \(C \setminus \{0\}\) such that \(\Pi \circ f_m^\Pi(z_j)\) converges to one of the points in the ideal boundary. By (i), we may assume that each \(z_j \not\in \Sigma_m\). With finitely many sectors, we may also assume \(\{z_j\} \subset \Omega_k\) for some \(k\). Then \(x_0(z_j)\) diverges to \(\infty\) or \(-\infty\) as \(j \to \infty\), that is, \(\{r_j + r_j^{-1}\}_{j=1,2,\ldots}\) is unbounded, where \(r_j := |z_j|\). Taking a subsequence, we may assume
\[
\lim_{j \to \infty} r_j = 0 \quad \text{or} \quad \lim_{j \to \infty} r_j = \infty.
\]

We set \(y := \Pi \circ f_m^\Pi\). Since \(x_1\) and \(x_2\) are bounded (cf. (18)), \(y(l) \to 0\) for \(l = 1, 2\), where \(y = (y_1, y_2, y_3)\). On the other hand, by (18) and (22), we have \(\lim_{j \to \infty} (x_3(z_j)/x_0(z_j)) = \pm 1\). Thus, we have \(\lim_{j \to \infty} y_3(z_j) = \pm \sqrt{2} \pm 1\), which proves (ii). \(\square\)

It should be remarked that \(x_1, x_2\) depend only on the variable \(\theta\), and
\[
(x_1(\theta), x_2(\theta)) = -2\gamma_m(\theta)
\]
holds. Here, \(\gamma_m\) is exactly the same hypo-trochoid as given in Proposition 1. For fixed \(\theta\), the image of the curve defined by \(r \mapsto (x_0(r, \theta), x_3(r, \theta))\) coincides with
\[
\left\{(t, z) \in \mathbb{R}^2_1 : t^2 - z^2 = \frac{(m^2 - 1)^2}{(2m)^2} \cos^2 m\theta, \quad \text{sgn}(\cos m\theta)t < 0 \right\}.
\]
In particular, it is half of a hyperbola when \( \cos m \theta \neq 0 \). If \( \cos m \theta = 0 \), the image reduces to a point. So we can conclude that the real analytic extension of the image of \( f_m^\| \) coincides with the set

\[
C_m^\| := \left\{ (t, x, y, z) \in \mathbb{R}^4_1; (x, y) = -2\gamma_m(\theta), \quad t^2 - z^2 = \frac{(m^2 - 1)^2}{(2m)^2} \cos^2 m \theta, \quad \theta \in [0, 2\pi) \right\}.
\]

For each \( k = 0, \ldots, 2m - 1 \), the analytic extension \( C_m^\| \) contains a union of two light-like lines

\[
L_k^\pm := \left\{ (t, -\sin \alpha_k, \cos \alpha_k, \pm t) ; t \in \mathbb{R} \right\},
\]

where \( \alpha_k \) is as in (19). Moreover, \( C_m^\| \) is symmetric with respect to the isometric involution

\[
S_3^1 \ni (t, x, y, z) \mapsto (t, \cos(2\alpha_k)x + \sin(2\alpha_k)y, \sin(2\alpha_k)x - \cos(2\alpha_k)y, z) \in S_3^1.
\]

This involution fixes the two lines \( L_k^+ \) and \( L_k^- \). Suppose that \( m \) is an odd integer. By (a) of Proposition 1, \( \gamma_m \) is \( \pi \)-periodic. In this case, one half of the hyperbola at \( \theta + \pi \) is just the other half of the hyperbola (23) at \( \theta \), and \( C_m^\| \) coincides with the closure of the image of \( f_m^\| \).

In the case \( m \) is even, \( C_m^\| \) does not coincide with the closure of the image of \( f_m^\| \). Moreover, \( C_m^\| \) contains the image of the map \( \iota \circ f_m^\| \), which is congruent to \( f_m^\| \), where \( \iota \) is the involution as in (10), and \( C_m^\| \) is just the closure of the union of the images of \( f_m^\| \) and \( f_m^\| \). Figure 4 shows \( C_m^\| \) and the image of \( f_m^\| \) for \( m = 2 \).

Summarizing the above, we get the following:

**Theorem 6.** For each \( m = 2, 3, \ldots \), the set \( C_m^\| \) gives the real analytic extension of the exceptional catenoid \( f_m^\| \), and has the following properties:

(i) The projection of \( C_m^\| \) into the xy-plane in \( \mathbb{R}^4_1 \) is the hypo-trochoid \(-2\gamma_m \). Furthermore, the section of \( C_m^\| \) by a plane containing a point of the hypo-trochoid and perpendicular to the xy-plane is a hyperbola unless the plane passes through the cone-like singularity of \( C_m^\| \).

(ii) \( C_m^\| \) is almost immersed and has four endpoints. Two of them lie in \( \partial_+ S_3^1 \) and the others lie in \( \partial_- S_3^1 \). Moreover, \( C_m^\| \) is almost embedded if \( m = 2, 3 \).

(iii) If \( m \) is odd, then \( C_m^\| \) is the closure of the image of \( f_m^\| \). On the other hand, if \( m \) is even, then the closure of the image of \( f_m^\| \) is just half of \( C_m^\| \). The other half can be obtained by the isometric involution of \( S_3^1 \) given in (10).
When \( m \geq 4 \), \( C_m \) has self-intersections. It should be remarked that similar phenomena occur for parabolic or hyperbolic catenoids in the class of space-like maximal surfaces in \( \mathbb{R}^3 \) (see [3]).

**Remark 7.** As shown in [2], \( C_m \) is analytically complete, that is, \( C_m \) admits no analytic extension.

To end this paper, we remark that the replacement

\[
s \mapsto r = e^s
\]

of the parameter of \( f_m \) induces constant mean curvature surfaces in anti-de Sitter space. This induces a family of surfaces

\[
\tilde{f}_m := (x_0(s, \theta), x_1(\theta), x_2(\theta), x_3(s, \theta))
\]

given by \( (x_0, x_3) = \frac{1}{m^2} \cos m\theta(\cos s, \sin s) \), and \( x_1, x_2 \) as in (18), where \( m = 2, 3, 4, \ldots \). For each \( m \), the corresponding surface lies in the space form

\[
H_1^3(-1) := \{(t, x, y, z) : t^2 - x^2 - y^2 + z^2 = -1\}
\]

doing constant curvature \(-1\) realized in \((\mathbb{R}^4_2, + + - +)\). The image of \( \tilde{f}_m \) gives a compact almost immersed time-like surface of constant mean curvature one having a finite number of cone-like singularities. Moreover, if \( m \) equals 2 or 3, the surface is almost embedded in the sense given in the introduction. To draw the surfaces, we use the ‘solid torus model’ of \( H_1^3(-1) \), that is, we define the following projection

\[
\tilde{H} : H_1^3(-1) \ni (t, x, y, z) \mapsto \frac{1}{\rho} \left( \left( 1 + \frac{t}{\rho} \right) x, \left( 1 + \frac{t}{\rho} \right) y, z \right) \in \mathbb{R}^3,
\]

where \( \rho := \sqrt{x^2 + y^2} \). The image of \( \tilde{H} \) is the interior of the solid torus obtained by rotating the unit disk with center \((1, 0, 0)\) about the third axis in \( \mathbb{R}^3 \). The images of \( \tilde{H} \circ \tilde{f}_m \) for \( m = 2, 3 \) are given in Figure 8.
Appendix A. Singularities of Exceptional Catenoids of Type I

In this appendix, we discuss properties of singularities of the exceptional catenoids of type I. By the criteria in [5, Theorem 3.4], we have the following:

**Proposition A.1.** The singular set of \( f^I_m \) is
\[
\Sigma_m := \{ z = re^{i\theta} \in \mathbb{C} \setminus \{0\} : r^m + 2\cos m\theta = 0 \}.
\]
The \( m \) points
\[
z = re^{i\theta}, \quad (r, \theta) = \left( 2^{1/m}, \frac{1}{m}(2j+1)\pi \right), \quad (j = 0, \ldots, m-1)
\]
are swallowtails, and the \( 2m \) points
\[
z = re^{i\theta}, \quad (r, \theta) = \begin{cases} 
(2^{1/(2m)}, \frac{1}{m} (\frac{3}{4} + 2j) \pi), \\
(2^{1/(2m)}, \frac{1}{m} (\frac{5}{4} + 2j) \pi),
\end{cases} \quad (j = 0, \ldots, m-1)
\]
are cuspidal cross caps. Other points in \( \Sigma_m \) are cuspidal edges.

Next, we discuss singularities of the parametrization of \( C^I_m \) near the line \( L_k \) \((k = 0, \ldots, 2m - 1)\), as in the proof of Theorem 2. Without loss of generality, we may assume \( k = 0 \). Then the parametrization is expressed as
\[
\tilde{f}^I_m(r, s) := (x_0(r, s), x_1(r, s), x_2(r, s), x_3(r, s)),
\]
where
\[
x_0 + x_3 := \frac{m^2 - 1}{4m} \left( 2r^2 s - \frac{m-1}{m+1} r^{m+1} \right),
\]
(A.24) \[
x_0 - x_3 := \frac{m^2 - 1}{4m} \left( 2s - \frac{m+1}{m-1} r^{m-1} \right),
\]
\[
x_1 + i x_2 := \frac{(m-1)^2}{4m} e^{i(m+1)\theta} + \frac{(m+1)^2}{4m} e^{-i(m-1)\theta} - \frac{m^2 - 1}{4m} r^m e^{i\theta}
\]
and
\[
\theta := \theta(r, s) = \frac{1}{m} \cos^{-1}(rs),
\]
(A.25)
where we consider \( \cos^{-1}(rs) \in [0, \pi] \). As shown in Theorem 2, the map \( \tilde{f}^I_m \) is an immersion at \((0, s)\) if \( s \neq 0 \). We show the following:

**Proposition A.2.** The map \( \tilde{f}^I_m \) is a wave front near the origin, and the origin \((0, 0)\) is a cuspidal edge (resp. swallowtail) when \( m = 2 \) (resp. \( m = 3 \)).

**Proof.** Since
\[
x_1(0, 0) + i x_2(0, 0) = -i e^{i\pi/(2m)} = \sin \frac{\pi}{2m} - i \cos \frac{\pi}{2m},
\]
Then 
\[ \pi: S^3_1 \ni (x_0, x_1, x_2, x_3) \mapsto (x_0 + x_3, x_0 - x_3, x_1) \in \mathbb{R}^3 \]
gives a local coordinate system of \( S^3_1 \) around \( \bar{f}^m_1(0, 0) \). So it is sufficient show the conclusion for the map
\[ F(r, s) := \pi \circ f^m_1(r, s) = (X(r, s), Y(r, s), Z(r, s)), \]
where
\[ X(r, s) := 2(m + 1)r^2s - (m - 1)r^{m+1}, \]
\[ Y(r, s) := 2(m - 1)s - (m + 1)r^{m-1}, \]
\[ Z(r, s) := \frac{m - 1}{m + 1} \cos(m + 1)\theta + \frac{m + 1}{m - 1} \cos(m - 1)\theta - r^m \cos \theta. \]

By (A.25),
\[ \theta_r = -\delta s, \quad \theta_s = -\delta r \quad \left( \delta(r, s) := \frac{1}{m \sin m\theta(r, s)} \right) \]
hold. Then we have
\[ F_r = ((m + 1)r(4s - (m - 1)r^{m-1}), -(m^2 - 1)r^{m-2}, \]
\[ (2s - mr^{m-1}) \cos \theta - sr \delta \lambda \sin \theta, \]
\[ F_s = (2(m + 1)r^2, 2(m - 1), 2r \cos \theta - r^2 \delta \lambda \sin \theta), \]
where
\[ \lambda := 2s + r^{m-1}. \]

By a direct computation, we have \( F_r \times F_s = \lambda \nu \), where
\[ \nu := (\nu_1, \nu_2, \nu_3) \]
\[ \nu_1 := -(m - 1)(2 \cos \theta - r \delta(2s + (m + 1)r^{m-1}) \sin \theta), \]
\[ \nu_2 := -(m + 1)r^2(2 \cos \theta - r \delta(2s - (m - 1)r^{m-1}) \sin \theta), \]
\[ \nu_3 := 4(m^2 - 1)r. \]

Since
\[ \nu(0, 0) = (-2(m - 1) \cos(\pi/(2m)), 0, 0) \neq 0, \]
\( \nu \) is the normal vector field of \( F \), and \( \lambda \) in (A.27) is an identifier of singularities, that is, \( \{(r, s) : \lambda(r, s) = 0\} \) is the singular set. Since \( d\lambda \neq 0 \), the singular points of \( F \) are non-degenerate. Thus, the singular direction is
\[ \xi := -2 \frac{\partial}{\partial r} + (m - 1)r^{m-2} \frac{\partial}{\partial s}. \]
On other hand, since $2F_r + (m + 1)r^{m-2}F_s = 0$ when $\lambda(r, s) = 0$,
\[ \eta := 2 \frac{\partial}{\partial r} + (m + 1)r^{m-2} \frac{\partial}{\partial s} \]
is the null direction. Moreover,
\[ \text{div} (\eta)(0, 0) = (0, 0, 4(m^2 - 1)), \]
which is not proportional to $\nu(0, 0)$. Thus, the map $F$ gives a wave front near the origin.

When $m = 2$, the singular direction and the null direction are linearly independent at the origin. Hence, by the criterion in [6, Proposition 1.3], the origin is a cuspidal edge.

Finally, when $m = 3$, the singular direction and the null direction are
\[ \xi = -2 \frac{\partial}{\partial r} + 2r \frac{\partial}{\partial s}; \quad \eta = 2 \frac{\partial}{\partial r} + 4r \frac{\partial}{\partial s}, \]
which are proportional at the origin. Moreover, $\text{det}(\xi, \eta) = -12r$, where $\text{det}$ is the determinant function on the $(r, s)$-plane. Hence by the criterion in [6, Proposition 1.3], the origin is a swallowtail. \qed

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THE ANALYTIC EXTENSION OF EXCEPTIONAL CATENOIDS

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