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UNSTABLE HIGHER TODA BRACKETS

Dedicated to memories of the Arakis

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Abstract. We define new unstable n-fold Toda brackets $\{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)}$ and $\{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)}$ for every composable sequence $\vec{\boldsymbol{f}} = (f_n, \dots, f_1)$ of pointed maps between well-pointed spaces $X_{n+1} \xleftarrow{f_n} \cdots \xleftarrow{f_2} X_2 \xleftarrow{f_1} X_1$ with $n \geq 3$. The brackets agree with the classical Toda bracket when $n = 3$, and they are subsets of both the unstable n-fold Toda brackets of Gershenson and Cohen for every $n \geq 3$.

1. Introduction

The Toda bracket [**23**, **24**, **25**, **18**] is one of the basic tools in homotopy theory and often called a secondary composition or a 3-fold bracket. After [**24**] a number of definitions of a higher Toda bracket, that is, an n-fold bracket for $n \geq 3$, have appeared in the literature. Stable higher Toda brackets are comparatively investigated in [**3**, **27**] (cf. [**7**, **10**, **11**, **16**]). In this paper we study mainly unstable higher Toda brackets. A sequence $(p_3, p_4, p_5, \dots)$, where p_n is an unstable n-fold bracket, is called *a system of unstable higher Toda brackets* if it is defined systematically, and it is called *normal* if p_3 agrees with the classical Toda bracket up to sign. Systems of Spanier [**19**], Walker [**26**, **27**] (cf. Mori [**13**]), Blanc [**1**], Blanc–Markl [**2**], and Marcum–Oda [**12**] (cf. [**8**]) are normal; systems of Gershenson [**7**] and Cohen [**3**] are not normal. It seems difficult to nominate one of known systems as the standard system, because we have little information about their applications and relations between them. We provide two new candidates for the standard system by modifying the Gershenson's system which originated with [**24**], and study relations between new systems, the systems of Gershenson and Cohen, and the 4-fold bracket of Ôguchi [**14**, **15**]. Two new systems are normal. Our method is classical and not so abstract as [**1**, **2**].

Given a composable sequence $\vec{\boldsymbol{f}} = (f_n, \dots, f_1)$ of pointed maps between well-pointed spaces $f_i : X_i \to X_{i+1}$ with $n \geq 3$, we will define $\{\vec{\boldsymbol{f}}\}^{(\star)}$ which is a subset of the group $[\Sigma^{n-2} X_1, X_{n+1}]$, where $\star$ is one of twelve symbols defined in Definition 6.1.1(4). ($[\Sigma^k X, Y]$ is the set of homotopy classes of pointed maps from the k-fold pointed suspension of X to Y.) Hence we

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have twelve systems of unstable higher Toda brackets. Four of them, $\{\vec{\boldsymbol{f}}\}^{(\star)}$ for $\star = aq\ddot{s}_2, \ddot{s}_t, qs_2, q$, are essential; $\{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)}$ and $\{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)}$ are candidates for the standard n-fold bracket; $\{\vec{\boldsymbol{f}}\}^{(q)}$ is the largest of the twelve subsets and a revision of the n-fold C-composition product of Gershenson [**7**, Definition 2.2D]; they are the empty set for suitable $\vec{\boldsymbol{f}}$. For a pointed space X, we denote the set of homotopy classes of pointed homotopy equivalences $X \to X$ by $\mathcal{E}(X)$ which is a subset of $[X, X]$ and a group under the composition operation. The group $\mathcal{E}(\Sigma^{n-2}X_1)$ acts on $[\Sigma^{n-2}X_1, X_{n+1}]$ from the right by the composition:

$$[\Sigma^{n-2}X_1, X_{n+1}] \times \mathcal{E}(\Sigma^{n-2}X_1) \to [\Sigma^{n-2}X_1, X_{n+1}],\ (\alpha, \varepsilon) \mapsto \alpha \circ \varepsilon.$$

Our main results are (1.1) – (1.11) below.

(1.1) $\{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \cup \{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)} \subset \{\vec{\boldsymbol{f}}\}^{(qs_2)} \subset \{\vec{\boldsymbol{f}}\}^{(q)}$; $\{\vec{\boldsymbol{f}}\}^{(q)} \circ \varepsilon = \{\vec{\boldsymbol{f}}\}^{(q)}$ for every $\varepsilon \in \mathcal{E}(\Sigma^{n-2}X_1)$; $\{\vec{\boldsymbol{f}}\}^{(qs_2)} = \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \circ \mathcal{E}(\Sigma^{n-2}X_1) = \{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)} \circ \mathcal{E}(\Sigma^{n-2}X_1)$.

(1.2) If $\alpha \in \{\vec{\boldsymbol{f}}\}^{(q)}$, then there are $\theta, \theta' \in [\Sigma^{n-2}X_1, \Sigma^{n-2}X_1]$ such that $\alpha \circ \theta \in \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)}$ and $\alpha \circ \theta' \in \{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)}$.

(1.3) If $\{\vec{\boldsymbol{f}}\}^{(\star)}$ is not empty for some $\star$, then $\{\vec{\boldsymbol{f}}\}^{(\star)}$ is not empty for all $\star$.

(1.4) If $\{\vec{\boldsymbol{f}}\}^{(\star)}$ contains 0 for some $\star$, then $\{\vec{\boldsymbol{f}}\}^{(\star)}$ contains 0 for all $\star$.

(1.5) $\Sigma\{\vec{\boldsymbol{f}}\}^{(\star)} \subset (-1)^n\{\Sigma\vec{\boldsymbol{f}}\}^{(\star)}$ for all $\star$, where $\Sigma\vec{\boldsymbol{f}} = (\Sigma f_n, \dots, \Sigma f_1)$.

(1.6) $\{\vec{\boldsymbol{f}}\}^{(\star)}$ depends only on the homotopy classes of f_i $(1 \le i \le n)$ for all $\star$.

(1.7) $\{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \cup \{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)} \subset \langle\vec{\boldsymbol{f}}\rangle$, where $\langle\vec{\boldsymbol{f}}\rangle$ is the n-fold bracket of Cohen [**3**].

(1.8) When $n = 3$, we have $\{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} = \{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)} = \{\vec{\boldsymbol{f}}\}$, where $\{\vec{\boldsymbol{f}}\} = \{f_3, f_2, f_1\}$ is the classical unstable Toda bracket which does not necessarily coincide with either $\{\vec{\boldsymbol{f}}\}^{(qs_2)}$ or $\langle\vec{\boldsymbol{f}}\rangle$.

(1.9) When $n = 4$, we have $\{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)} = \bigcup\{f_4, [f_3, A_2, f_2], (f_2, A_1, f_1)\} \supset \{\vec{\boldsymbol{f}}\}^{(1)}$, where the union $\bigcup$ is taken over all triples (A_3, A_2, A_1) of null-homotopies $A_i : f_{i+1} \circ f_i \simeq *$ $(i = 1, 2, 3)$ such that $[f_{i+1}, A_i, f_i] \circ (f_i, A_{i-1}, f_{i-1}) \simeq *$ $(i = 2, 3)$, and $\{\vec{\boldsymbol{f}}\}^{(1)}$ is the 4-fold bracket of Ôguchi [**15**, (6.1)]. (See Section 2 for definitions of $[f_{i+1}, A_i, f_i]$ and (f_i, A_{i-1}, f_{i-1}).)

(1.10) For two pointed maps $Z \xleftarrow{f} Y \xleftarrow{g} X$, we denote by $\{f, g\}^{(\star)}$ the one point set consisting of the homotopy class of $f \circ g$. Then
  (1) If $\{f_{n-1}, \dots, f_1\}^{(q)} \ni 0$ and $\{f_n, f_{n-1}, \dots, f_k\}^{(aq\ddot{s}_2)} = \{0\}$ for all k with $2 \le k < n$, then $\{f_n, \dots, f_1\}^{(\star)}$ is not empty for all $\star$.

(2) If $\{f_n, \dots, f_2\}^{(q)} \ni 0$ and $\{f_k, \dots, f_2, f_1\}^{(aq\ddot{s}_2)} = \{0\}$ for all k with $2 \leq k < n$, then $\{f_n, \dots, f_1\}^{(\star)}$ is not empty for all $\star$.

(1.11) If a pointed map $j : A \to X$ is a cofibration in the category of non-pointed spaces, then for any pointed map $f : X \to Y$ the pointed map $1_Y \cup Cj : Y \cup_{f \circ j} CA \to Y \cup_f CX$ between pointed mapping cones is a cofibration in the category of non-pointed spaces.

It is not clear whether the n-fold bracket $\{\vec{\boldsymbol{f}}\}^{(\star)}$ agrees with one of the n-fold brackets in [**1**, **2**, **3**, **7**, **12**, **19**, **26**, **27**] when $n \geq 4$. An advantage of our definition is that it can be generalized easily to the stable version (see §6.9) and the subscripted version $\{\vec{\boldsymbol{f}}\}^{(\star)}_{\vec{\boldsymbol{m}}} \subset [\Sigma^{|\vec{\boldsymbol{m}}|+n-2} X_1, X_{n+1}]$ (cf. [**25**, p.9] when $n = 3$), where $\vec{\boldsymbol{m}} = (m_n, \dots, m_1)$ is a sequence of non-negative integers, $|\vec{\boldsymbol{m}}| = m_n + \cdots + m_1$, and $f_i : \Sigma^{m_i} X_i \to X_{i+1}$ $(1 \leq i \leq n)$. We omit details of the subscripted version because they are complicated but similar to the non subscripted version.

The referee pointed out that B. Gray defined unstable higher Toda brackets in his unpublished note. However we have not confirmed his definition because we could not get his note.

In Section 2, we recall usual notions of homotopy theory and state two propositions 2.1 and 2.2, where 2.1 is well-known and 2.2 is (1.11) above and a key to define $\{\vec{\boldsymbol{f}}\}^{(\star)}$. In Section 3, we study maps between mapping cones, that is, we prove a lemma which shall be used in Section 5, and recall results of Puppe [**17**]. In Section 4, we introduce the notion of homotopy cofibre. In Section 5, we revise the notion of shaft of Gershenson [**7**]. Section 6 consists of nine subsections §6.1–§6.9. In §6.1 we define $\{\vec{\boldsymbol{f}}\}^{(\star)}$. In §6.2 we prove (1.k) for k=1,2,3,4 and state an example. In §6.k we prove (1.k+2) for k=3,4,5,6,7. In §6.8 we prove a proposition which is the same as (1.10). In §6.9 we define stable higher Toda brackets. In Appendix A, we prove Proposition 2.2. In Appendix B, we recall the definition of $\langle \vec{\boldsymbol{f}} \rangle$ and prove $\Sigma \langle \vec{\boldsymbol{f}} \rangle \subset (-1)^n \langle \Sigma \vec{\boldsymbol{f}} \rangle$.

2. Preliminaries

Let TOP denote the category of topological spaces (*spaces* for short) and continuous maps (*maps* for short). Let I denote the unit interval $[0, 1]$, $I^n = I \times \cdots \times I$ (n-times), and ∂I^n the boundary of I^n. For a space X, we denote by $1_X : X \to X$ the identity map of X and by $i^X_t : X \to X \times I$ for $t \in I$ the map $i^X_t(x) = (x, t)$. For a map $f : X \to Y$, we denote by $1_f : X \times I \to Y$ the map $1_f(x, t) = f(x)$, and we call f *closed* if $f(A)$ is closed for every closed subset A of X. Given maps $f, g : X \to Y$, if there is a map $H : X \times I \to Y$ such that $H_0 = f$ and $H_1 = g$, then we write $f \simeq g$ or $H : f \simeq g$, where $H_t = H \circ i^X_t : X \to Y$ i.e. $H_t(x) = H(x, t)$. In the last case, the map H is often denoted by H_t and called a *homotopy* from f to g. The homotopy relation $\simeq$ is an equivalence relation on the set of maps $X \to Y$ and the equivalence class of f is called the *homotopy class* of f. Given a homotopy $H : X \times I \to Y$, the *inverse* homotopy $-H : X \times I \to Y$ is defined by $(-H)_t = H_{1-t}$; H is a *null homotopy* if H_1 is a constant map to a point of Y. A map $f : X \to Y$ is a *homotopy equivalence* (denoted by $f : X \simeq Y$) if there is a map $g : Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$, where g is called a *homotopy inverse of* f and denoted often by f^{-1}. We write $X \simeq Y$ if there is a homotopy equivalence $X \to Y$. A map $j : A \to X$ is a *cofibration* if, for any space Y and any maps $f : X \to Y$ and $G : A \times I \to Y$ such that $f \circ j = G \circ i_0^A$, there is a map $H : X \times I \to Y$ such that $H \circ (j \times 1_I) = G$ and $H \circ i_0^X = f$.

$$\begin{tikzcd}
 & A \times I \arrow[dr, "j \times 1_I"] \arrow[drr, bend left, "G"] & & \\
A \arrow[ur, "i_0^A"] \arrow[dr, "j"'] & & X \times I \arrow[r, dotted, "H"] & Y \\
 & X \arrow[ur, "i_0^X"] \arrow[urr, bend right, "f"'] & &
\end{tikzcd}$$

By [**20**, Theorem 1], every cofibration $j : A \to X$ is an embedding, that is, j gives a homeomorphism from A to the subspace $j(A)$ of X i.e. $j : A \approx j(A)$.

Let TOP^* denote the category of spaces with base points (*pointed spaces* for short) and maps preserving base points (*pointed maps* for short). We often call a space, a map, and a cofibration in TOP a *free* space, a *free* map, and a *free* cofibration, respectively. For any pointed space X, we denote the base point of X by x_0 or $*$. A pointed space X is a *well-pointed space* (*w-space* for short) (resp. *clw-space*) if the inclusion $\{x_0\} \to X$ is a free (resp. closed free) cofibration. Let TOP^w (resp. TOP^{clw}) denote the category of w-spaces (resp. clw-spaces) and pointed maps. Thus we have a sequence of categories: $\mathrm{TOP}^{clw} \rightarrowtail \mathrm{TOP}^w \rightarrowtail \mathrm{TOP}^* \twoheadrightarrow \mathrm{TOP}$, where $\twoheadrightarrow$ is the functor forgetting the base points, and $\mathcal{C} \rightarrowtail \mathcal{D}$ means that the category $\mathcal{C}$ is a full subcategory of the category $\mathcal{D}$ and $\mathcal{D}$ contains at least one object which is not in $\mathcal{C}$ (cf. Beispiele 1 and 2 [**5**, pp.32-33]). Homotopy, homotopy equivalence, cofibration, and some of other notions in TOP can be defined in other three categories of the above sequence exactly as in TOP, except that all maps and homotopies are required to respect the base points. As remarked in [**22**, p.438], the proof of [**20**, Theorem 1] can be modified to prove that all cofibrations in TOP^* are embeddings. When we set $\mathcal{C}_4 = \mathrm{TOP}^{clw}$, $\mathcal{C}_3 = \mathrm{TOP}^w$, $\mathcal{C}_2 = \mathrm{TOP}^*$, $\mathcal{C}_1 = \mathrm{TOP}$, if, for some $k > \ell$, a map $j : A \to X$ in $\mathcal{C}_k$ is a cofibration in $\mathcal{C}_\ell$, then j is a cofibration in $\mathcal{C}_k$. For pointed spaces X and Y, $[X,Y]$ denotes the set of pointed homotopy classes of pointed maps $X \to Y$, and the trivial map $X \to Y$, $x \mapsto y_0$, is denoted by $*$ and its homotopy class is denoted by 0; $[X,Y]$ is regarded as a pointed set with the base point 0. For homotopies $H : Y \times I \to Z$ and $G, F : X \times I \to Y$ with $F_1 = G_0$, homotopies $H \mathbin{\bar{\circ}} G : X \times I \to Z$ and $G \bullet F : X \times I \to Y$ are defined by

$$H \mathbin{\bar{\circ}} G(x,t) = H(G(x,t),t), \quad G \bullet F(x,t) = \begin{cases} F(x,2t) & 0 \leq t \leq \frac{1}{2} \\ G(x,2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Assign to the n-sphere $\mathrm{S}^n = \{(t_1,\dots,t_{n+1}) \in \mathbb{R}^{n+1} \mid \sum t_i^2 = 1\}$ $(n = 0,1,2,\dots)$ and $I = [0,1]$ the base points $(1,0,\dots,0)$ and 1, respectively. Then, as is well-known, S^n and I are *clw*-spaces.

For pointed spaces $X_1,\dots,X_n$, we denote by $X_1 \wedge \cdots \wedge X_n$ the quotient space

$$(X_1 \times \cdots \times X_n)/(\bigcup_{i=1}^{n} X_1 \times \cdots \times X_{i-1} \times \{*_i\} \times X_{i+1} \times \cdots \times X_n),$$

where $*_i$ is the base point of X_i. In $X_1 \wedge \cdots \wedge X_n$, the point represented by $(x_1,\dots,x_n)$ is denoted by $x_1 \wedge \cdots \wedge x_n$, and $*_1 \wedge \cdots \wedge *_n$ is the base point. For pointed maps $f_i : X_i \to Y_i$, we set $f_1 \wedge \cdots \wedge f_n : X_1 \wedge \cdots \wedge X_n \to Y_1 \wedge \cdots \wedge Y_n$, $x_1 \wedge \cdots \wedge x_n \mapsto f_1(x_1) \wedge \cdots \wedge f_n(x_n)$. For a pointed space X and an integer $n \geq 0$, we set $\Sigma^n X = X \wedge \mathrm{S}^n$ which is called the *n-fold pointed suspension* of X; for a pointed map $f : X \to Y$ we set $\Sigma^n f = f \wedge 1_{\mathrm{S}^n} : \Sigma^n X \to \Sigma^n Y$.

We identify S^n $(n \geq 1)$ with $I^n/\partial I^n$ and $\mathrm{S}^1 \wedge \cdots \wedge \mathrm{S}^1$ (n-times) by the following way. Take and fix a relative homeomorphism $\psi_n : (I^n, \partial I^n) \to (\mathrm{S}^n, *)$ for each $n \geq 1$ (e.g. [**25**, p.5]). Identify $I^n/\partial I^n$ with S^n by the homeomorphism induced from ψ_n, and denote $\psi_n(t_1,\dots,t_n)$ by $\overline{t_1} \wedge \cdots \wedge \overline{t_n}$. Also identify S^n with $\mathrm{S}^1 \wedge \cdots \wedge \mathrm{S}^1$ (n-times) by the homeomorphism h_n of the following commutative square with q the quotient map. (Notice that $h_n(\overline{t_1} \wedge \cdots \wedge \overline{t_n}) = \overline{t_1} \wedge \cdots \wedge \overline{t_n}$.)

$$\begin{array}{ccc} I^n & \xrightarrow{\psi_1 \times \cdots \times \psi_1} & \mathrm{S}^1 \times \cdots \times \mathrm{S}^1 \\ {\scriptstyle \psi_n}\downarrow & & \downarrow{\scriptstyle q} \\ \mathrm{S}^n & \xrightarrow[\approx]{h_n} & \mathrm{S}^1 \wedge \cdots \wedge \mathrm{S}^1 \end{array}$$

Under the above identifications, we have $\mathrm{S}^m \wedge \mathrm{S}^n = \mathrm{S}^{m+n} = \mathrm{S}^n \wedge \mathrm{S}^m$, where, if $m, n \geq 1$, then

$$(x_1 \wedge \cdots \wedge x_m) \wedge (x_{m+1} \wedge \cdots \wedge x_{m+n}) = x_1 \wedge \cdots \wedge x_{m+n}$$
$$= (x_1 \wedge \cdots \wedge x_n) \wedge (x_{n+1} \wedge \cdots \wedge x_{m+n}) \quad (x_i \in \mathrm{S}^1 \ (1 \leq i \leq m+n)).$$

Since spheres are compact and Hausdorff, it follows that, for any pointed space X, we have the identifications:

$$\begin{aligned}\Sigma^n\Sigma^m X &= (X \wedge \mathrm{S}^m) \wedge \mathrm{S}^n = X \wedge (\mathrm{S}^m \wedge \mathrm{S}^n) = X \wedge \mathrm{S}^{m+n} \\ &= X \wedge (\mathrm{S}^n \wedge \mathrm{S}^m) = (X \wedge \mathrm{S}^n) \wedge \mathrm{S}^m = \Sigma^m\Sigma^n X.\end{aligned} \tag{2.1}$$

The switching map

$$\tau(\mathrm{S}^m, \mathrm{S}^n) : \mathrm{S}^{m+n} = \mathrm{S}^m \wedge \mathrm{S}^n \to \mathrm{S}^n \wedge \mathrm{S}^m = \mathrm{S}^{m+n},\ x \wedge y \mapsto y \wedge x, \tag{2.2}$$

is a homeomorphism of the degree $(-1)^{mn}$.

Given a space A, let TOP^A denote the category of spaces under A, that is, objects are free maps $i : A \to X$ and a morphism f from $i : A \to X$ to $i' : A \to X'$ is a free map $f : X \to X'$ with $f \circ i = i'$.

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "i'"] & \\ X \arrow[rr, "f"] & & X' \end{tikzcd} \tag{2.3}$$

Let $\mathrm{TOP}^A(i, i')$ denote the set of all morphisms from $i : A \to X$ to $i' : A \to X'$. For $f, f' \in \mathrm{TOP}^A(i, i')$, if there exists a homotopy $H : X \times I \to X'$ such that $H_0 = f,\ H_1 = f',\ H_t \in \mathrm{TOP}^A(i, i')$ for all $t \in I$, then we write $f \overset{A}{\simeq} f'$ or $H : f \overset{A}{\simeq} f'$. Note that $\mathrm{TOP}^{\{*\}} = \mathrm{TOP}^*$. The following is well-known (e.g. [**6**, (3.6)], [**4**, (5.2.5)], [**5**, (2.18)], [**9**, (6.18)]).

Proposition 2.1. *Given a commutative triangle (2.3), if i and i' are cofibrations and $f : X \to X'$ is a homotopy equivalence in* TOP, *then $f : i \to i'$ is a homotopy equivalence in* TOP^A, *that is, there exists $g \in \mathrm{TOP}^A(i', i)$ with $g \circ f \overset{A}{\simeq} 1_X$ and $f \circ g \overset{A}{\simeq} 1_{X'}$.*

For spaces X and Y, we denote by $X + Y$ the topological sum of them, that is, it is the disjoint union of them as a set and $A \subset X + Y$ is open if and only if $A \cap X$ is open in X and $A \cap Y$ is open in Y.

For a pointed space X, the *cone* CX over it and the *suspension* ΣX of it are defined by $CX = X \wedge I = (X \times I)/(\{x_0\} \times I \cup X \times \{1\})$ and $\Sigma X = (X \times I)/(\{x_0\} \times I \cup X \times \{0, 1\})$. The point of ΣX represented by $(x, t) \in X \times I$ is denoted by $x \wedge \bar{t}$. The space ΣX is based by $x_0 \wedge \bar{1}$. Usually we identify $\Sigma X = \Sigma^1 X$. For a pointed map $f : X \to Y$, two maps $Cf : CX \to CY$ and $\Sigma f : \Sigma X \to \Sigma Y$ are defined by $Cf(x \wedge t) = f(x) \wedge t$ and $\Sigma f(x \wedge \bar{t}) = f(x) \wedge \bar{t}$; the (pointed) *mapping cone* of f is the space $C_f = Y \cup_f CX$ which is the quotient of $Y + CX$ by the equivalence relation generated by the relation $f(x) \sim x \wedge 0$ $(x \in X)$ and is based by the point represented by y_0; the injection $i_f : Y \to Y \cup_f CX$ is a cofibration in TOP^*

by [**17**, Hilfssatz 6] and so an embedding; let

$$q_f : Y \cup_f CX \to (Y \cup_f CX)/Y = \Sigma X,$$
$$q'_f : (Y \cup_f CX) \cup_{i_f} CY \to ((Y \cup_f CX) \cup_{i_f} CY)/CY = \Sigma X$$

denote the quotient maps, then $q_f = q'_f \circ i_{i_f}$ and q'_f is a homotopy equivalence in TOP* by [**17**, Satz 3]; for any integer $\ell \geq 1$ let $\psi^\ell_f : \Sigma^\ell Y \cup_{\Sigma^\ell f} C\Sigma^\ell X \approx \Sigma^\ell(Y \cup_f CX)$ denote the homeomorphism defined by $\psi^\ell_f(y \wedge s_\ell) = y \wedge s_\ell$ and $\psi^\ell_f(x \wedge s_\ell \wedge t) = x \wedge t \wedge s_\ell$ for $s_\ell \in \mathrm{S}^\ell$ and $t \in I$. If the first square of the following diagram in TOP* is commutative, then there exists the map $b \cup Ca$ with the diagram commutative.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "i_f"] \arrow[d, "b"'] & Y \cup_f CX \arrow[r, "q_f"] \arrow[d, dotted, "b \cup Ca"'] & \Sigma X \arrow[d, "\Sigma a"'] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "i_{f'}"] & Y' \cup_{f'} CX' \arrow[r, "q_{f'}"] & \Sigma X'
\end{tikzcd}$$

The next proposition is the same as (1.11) and shall be used to define *induced iterated mapping cones* in Definition 5.4.

Proposition 2.2. *If a pointed map $j : A \to X$ is a free (resp. closed free) cofibration, then, for any pointed map $f : X \to Y$, $1_Y \cup Cj : Y \cup_{f \circ j} CA \to Y \cup_f CX$ is a free (resp. closed free) cofibration.*

The above proposition may be folklorish, but we have not found its proof in the literature, and so we will prove it in Appendix A for completeness.

Corollary 2.3. (1) *If a pointed map $j : A \to X$ is a free (resp. closed free) cofibration, then $\Sigma j : \Sigma A \to \Sigma X$ is a free (resp. closed free) cofibration.*

(2) *If X is a w-space (resp. clw-space), then ΣX and CX are w-spaces (resp. clw-spaces), and $i_f : Y \to Y \cup_f CX$ is a free (resp. closed free) cofibration for every pointed map $f : X \to Y$.*

(3) *If $f : X \to Y$ is a pointed map between w-spaces (resp. clw-spaces), then $Y \cup_f CX$ is a w-space (resp. clw-space).*

Proof. (1) By taking $Y = \{y_0\}$ in Proposition 2.2, the assertion follows.

(2) Let X be a w-space (resp. clw-space). Set $j : A = \{x_0\} \subset X$. The assertions about ΣX and i_f follow from (1) and Proposition 2.2. Since $i_{1_X} \circ j : \{x_0\} \to CX$ is a free (resp. closed free) cofibration, CX is a w-space (resp. clw-space).

(3) Let X and Y be w-spaces (resp. clw-spaces). Then $Y \cup_f CX$ is a w-space (resp. clw-spaces), since the composite of $\{y_0\} \subset Y$ with $i_f : Y \to Y \cup_f CX$ is a free (resp. closed free) cofibration. $\square$

Given pointed maps $f : X \to Y$ and $g : Y \to Z$ with a pointed null homotopy $H : g \circ f \simeq *$, we set

$$(g, H, f) : \Sigma X \to Z \cup_g CY, \quad x \wedge \bar{t} \mapsto \begin{cases} f(x) \wedge (1-2t) & 0 \leq t \leq \frac{1}{2} \\ H(x, 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases},$$

$$[g, H, f] : Y \cup_f CX \to Z, \quad y \mapsto g(y), \quad x \wedge t \mapsto H(x, t),$$

which are called a *coextension* of f with respect to g and an *extension* of g with respect to f, respectively ([**25**, **14**]). Given pointed maps $f_i : X_i \to X_{i+1}$ for $i = 1, 2, 3$, the *Toda bracket* $\{f_3, f_2, f_1\}$ ([**23**, **24**, **25**]) which is a subset of $[\Sigma X_1, X_4]$ is the set of homotopy classes of maps of the form $[f_3, A_2, f_2] \circ (f_2, A_1, f_1)$, where $A_j : f_{j+1} \circ f_j \simeq *$ for $j = 1, 2$. If A_1 or A_2 does not exist, then $\{f_3, f_2, f_1\}$ denotes the empty set. As is well-known, $\{f_3, f_2, f_1\}$ depends only on the homotopy classes of f_i ($i = 1, 2, 3$) (e.g. Section 3 of [**15**]).

3. Maps between mapping cones

In this section we will work in TOP*.

The following shall be used to prove Lemma 5.3 which defines induced iterated mapping cones.

Lemma 3.1. *Given two maps $j : Y \to Y'$ and $g' : Y' \to Z$, the following diagram is homotopy commutative and $i_{g'\circ j} \cup Ci_j$ is a homotopy equivalence.*

$$\begin{tikzcd}
Y \arrow[r, "j"] \arrow[d, equal] & Y' \arrow[r, "i_j"] \arrow[d, "g'"'] & Y' \cup_j CY \arrow[d, "g' \cup C1_Y"'] \\
Y \arrow[r, "g' \circ j"] & Z \arrow[r, "i_{g'\circ j}"] \arrow[d, "i_{g'}"'] & Z \cup_{g'\circ j} CY \arrow[d, "i_{g'\cup C1_Y}"] \arrow[dl, "1_Z \cup Cj"'] \\
 & Z \cup_{g'} CY' \arrow[r, "i_{g'\circ j} \cup Ci_j"'] & (Z \cup_{g'\circ j} CY) \cup_{g' \cup C1_Y} C(Y' \cup_j CY)
\end{tikzcd}$$

Proof. Obviously three squares are commutative and $(1_Z \cup Cj) \circ i_{g'\circ j} = i_{g'}$. For simplicity, we set

$$g = g' \circ j, \quad h = g' \cup C1_Y, \quad k = 1_Z \cup Cj, \quad \varphi = i_{g'\circ j} \cup Ci_j.$$

We should prove that $i_h \simeq \varphi \circ k$ and φ is a homotopy equivalence. Let $z \in Z$, $y \in Y$, $y' \in Y'$ and $s, t, u \in I$. We define

$$w : I \times I \to I, \quad G : (Z \cup_g CY) \times I \to (Z \cup_g CY) \cup_h C(Y' \cup_j CY)$$

by

$$w(s,t)=\begin{cases}0 & s\le t\\ 2s-2t & \frac{s}{2}\le t\le s\\ s & t\le\frac{s}{2}\end{cases},$$

$$G(z,t)=z,\quad G(y\wedge s,t)=y\wedge w(s,t)\wedge w(s,1-t).$$

Then $G: i_h\simeq\varphi\circ k$ rel Z. In the rest of the proof we prove that φ is a homotopy equivalence. Define $\psi:(Z\cup_g CY)\cup_h C(Y'\cup_j CY)\to Z\cup_{g'} CY'$ by

$$\psi(z)=z,\quad \psi(y\wedge t)=j(y)\wedge t,\quad \psi(y'\wedge t)=y'\wedge t,$$
$$\psi(y\wedge s\wedge t)=j(y)\wedge(s+(1-s)t).$$

As is easily seen, ψ is well-defined, continuous, and $\psi\circ\varphi=1_{C_{g'}}$. We will show $1_{C_h}\simeq\varphi\circ\psi$. We have

$$\varphi\circ\psi(z)=z,\quad \varphi\circ\psi(y\wedge t)=j(y)\wedge t=y\wedge 0\wedge t,\quad \varphi\circ\psi(y'\wedge t)=y'\wedge t,$$
$$\varphi\circ\psi(y\wedge s\wedge t)=j(y)\wedge(s+(1-s)t)=y\wedge 0\wedge(s+(1-s)t).$$

Thus it suffices to construct a map

$$H:\big((Z\cup_g CY)\cup_h C(Y'\cup_j CY)\big)\times I\to(Z\cup_g CY)\cup_h C(Y'\cup_j CY)$$

such that

$$H(z,u)=z,\quad H(y'\wedge t,u)=y'\wedge t,$$
$$H(y\wedge 0\wedge t,u)=H(j(y)\wedge t,u)=j(y)\wedge t=y\wedge 0\wedge t,$$
$$H(y\wedge t,0)=y\wedge t=y\wedge t\wedge 0,\quad H(y\wedge t,1)=y\wedge 0\wedge t,$$
$$H(y\wedge s\wedge t,0)=y\wedge s\wedge t,\quad H(y\wedge s\wedge t,1)=y\wedge 0\wedge(s+(1-s)t).$$

The space $K=I\times I\times\{0\}\cup\{0\}\times I\times I\cup I\times I\times\{1\}\cup I\times\{1\}\times I\cup\{1\}\times I\times I$ is a retract of $I\times I\times I$. Indeed a retraction $r:I\times I\times I\to K$ is defined as follows: for $P\in I\times I\times I$, $r(P)$ is the intersection of K and the half line which starts from $(\frac12,-\frac12,\frac12)$ and passes through P. Define $v':K\to I\times I$ by

$$v'(s,t,0)=(s,t),\quad v'(0,t,u)=(0,t),\quad v'(s,t,1)=(0,s+(1-s)t),$$

$$v'(s,1,u)=\begin{cases}(0,1) & s\le u\\ (2s-2u,1) & \frac{s}{2}\le u\le s\\ (s,1) & u\le\frac{s}{2}\end{cases},$$

$$v'(1,t,u)=\begin{cases}(1,t) & u\le\frac{t}{2}\\ (1,2u) & \frac{t}{2}\le u\le\frac12\\ (2-2u,1) & \frac12\le u\le 1\end{cases}.$$

Then v' is well-defined and continuous. Set $v = v' \circ r : I \times I \times I \to I \times I$. Then

$$v(\{1\} \times I \times I \cup I \times \{1\} \times I) \subset \{1\} \times I \cup I \times \{1\},$$
$$v(0,t,u) = (0,t), \quad v(s,t,0) = (s,t), \quad v(s,t,1) = (0, s + (1-s)t).$$

Write $v(s,t,u) = (v_1(s,t,u), v_2(s,t,u))$ and define H by

$$H(z,u) = z, \quad H(y \wedge s, u) = y \wedge v_1(s,0,u) \wedge v_2(s,0,u),$$
$$H(y' \wedge t, u) = y' \wedge t, \quad H(y \wedge s \wedge t, u) = y \wedge v_1(s,t,u) \wedge v_2(s,t,u).$$

Then H satisfies the desired properties. Therefore φ is a homotopy equivalence. This completes the proof. □

Definition 3.2 ((9) of [**17**])**.** Given a homotopy commutative square and a homotopy

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X' & \xrightarrow{f'} & Y' \end{array}, \quad J : b \circ f \simeq f' \circ a, \tag{3.1}$$

we define $\Phi(f,f',a,b;J) : Y \cup_f CX \to Y' \cup_{f'} CX'$ by

$$\Phi(f,f',a,b;J)(y) = b(y),$$
$$\Phi(f,f',a,b;J)(x \wedge s) = \begin{cases} J(x,2s) & 0 \leq s \leq \frac{1}{2} \\ a(x) \wedge (2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}.$$

Given a homotopy $K : b \circ f \simeq f' \circ a$, if there is a free map $\varphi : X \times I \times I \to Y'$ such that $\varphi(x,s,0) = J(x,s)$, $\varphi(x,s,1) = K(x,s)$, $\varphi(*,s,t) = *$, $\varphi(x,0,t) = b \circ f(x)$, and $\varphi(x,1,t) = f' \circ a(x)$ for every $x \in X$ and $s,t \in I$, then we write $J \overset{X}{\simeq} K$ or $\varphi : J \simeq K$.

Proposition 3.3. *Suppose that (3.1) is given.*

(1) ([**17**, Hilfssatz 7])

(a) *The following diagram is homotopy commutative such that the middle square is commutative.*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{i_f} & Y \cup_f CX & \xrightarrow{q_f} & \Sigma X \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle \Phi(f,f',a,b;J)} & & \downarrow{\scriptstyle \Sigma a} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{i_{f'}} & Y' \cup_{f'} CX' & \xrightarrow{q_{f'}} & \Sigma X' \end{array}$$

(b) *In the following diagram, the first square is commutative and the second square is homotopy commutative.*

$$\begin{tikzcd} Y \cup_f CX \arrow[r, "i_{i_f}"] \arrow[d, "\Phi(f,f',a,b;J)"'] & (Y \cup_f CX) \cup_{i_f} CY \arrow[r, "q'_f"] \arrow[d, "\Phi(i_f,i_{f'},b,\Phi(f,f',a,b;J);1_{i_{f'}\circ b})"] & \Sigma X \arrow[d, "\Sigma a"] \\ Y' \cup_{f'} CX' \arrow[r, "i_{i_{f'}}"] & (Y' \cup_{f'} CX') \cup_{i_{f'}} CY' \arrow[r, "q'_{f'}"] & \Sigma X' \end{tikzcd}$$

Also $\Phi(i_f, i_{f'}, b, \Phi(f, f', a, b; J); 1_{i_{f'}\circ b}) \simeq \Phi(f, f', a, b; J) \cup Cb$.

(c) *If* a *and* b *are homotopy equivalences, then* $\Phi(f, f', a, b; J)$ *is a homotopy equivalence.*

(d) *If furthermore* $a' : X' \to X''$, $b' : Y' \to Y''$, $f'' : X'' \to Y''$ *with* $J' : b' \circ f' \simeq f'' \circ a'$ *are given, then*

$$\Phi(f', f'', a', b'; J') \circ \Phi(f, f', a, b; J) \simeq \Phi(f, f'', a' \circ a, b' \circ b; (J' \bar{\circ} 1_a) \bullet (1_{b'} \bar{\circ} J)).$$

(2) *Define* $e_a : \Sigma X \to \Sigma X'$ *by* $e_a(x \wedge \bar{t}) = \begin{cases} a(x) \wedge \bar{0} & 0 \leq t \leq \frac{1}{2} \\ a(x) \wedge \overline{2t-1} & \frac{1}{2} \leq t \leq 1 \end{cases}$.
Then $e_a \simeq \Sigma a$ *and* $q'_{f'} \circ (\Phi(f, f', a, b; J) \cup Cb) = e_a \circ q'_f \simeq \Sigma a \circ q'_f$.

(3) *If the square in (3.1) is strictly commutative, then* $\Phi(f, f', a, b; 1_{b\circ f}) \simeq b \cup Ca$.

(4) ([**17**, p.315]) *For homotopies* $a_t : X \to X'$ *and* $b_t : Y \to Y'$, *if there exists a homotopy* $J^t : b_t \circ f \simeq f' \circ a_t$ *for every* $t \in I$ *such that the function* $X \times I \times I \to Y'$, $(x, s, t) \mapsto J^t(x, s)$, *is continuous, then the function* $\Phi : (Y \cup_f CX) \times I \to Y' \cup_{f'} CX'$, $(z, t) \mapsto \Phi(f, f', a_t, b_t; J^t)(z)$, *is continuous and so*

$$\Phi(f, f', a_0, b_0; J^0) \simeq \Phi(f, f', a_1, b_1; J^1).$$

(5) *If* $K : b \circ f \simeq f' \circ a$ *satisfies* $J \overset{X}{\simeq} K$, *then* $\Phi(f, f', a, b; J) \simeq \Phi(f, f', a, b; K)$ *as elements of* $\mathrm{TOP}^Y(i_f, i_{f'} \circ b)$.

Proof. We refer a proof of (1) to [**17**].

Define $v : I \times I \to I$ and $F : \Sigma X \times I \to \Sigma X'$ by

$$v(t, u) = \begin{cases} 0 & u \leq -2t + 1 \\ t + u/2 - 1/2 & 2t - 1 \leq u \text{ and } -2t + 1 \leq u \\ 2t - 1 & u \leq 2t - 1 \end{cases},$$

$$F(x \wedge \bar{t}, u) = a(x) \wedge \overline{v(t, u)}.$$

Then $F : e_a \simeq \Sigma a$. As is easily seen, $q'_{f'} \circ (\Phi(f, f', a, b; J) \cup Cb) = e_a \circ q'_f$. Hence we obtain (2).

(3) can be easily proved.

For (4), define $\xi : (Y + X \times I) \times I \to Y' \cup_{f'} CX'$ by

$$\xi(y,t) = b_t(y), \quad \xi(x,s,t) = \begin{cases} J^t(x, 2s) & 0 \le s \le \frac{1}{2} \\ a_t(x) \wedge (2s-1) & \frac{1}{2} \le s \le 1 \end{cases}.$$

Then it is continuous and satisfies $\xi = \Phi \circ (q \times 1_I)$, where $q : Y + X \times I \to Y \cup_f CX$ is the quotient map. Hence Φ is continuous. This proves (4).

(5) is obtained by taking $a_t = a,\ b_t = b,\ J^t(x,s) = \varphi(x,s,t)$ in (4), where $\varphi : J \simeq K$. □

4. Homotopy cofibres

In this section we will work in TOP*. Hence $i_f : Y \to Y \cup_f CX$ is always a cofibration for every map $f : X \to Y$.

Definition 4.1. A map $j : Y \to Z$ is a *homotopy cofibre* of a map $f : X \to Y$ if j is a cofibration and there exists a homotopy equivalence $a : Z \to Y \cup_f CX$ with $a \circ j \simeq i_f$.

The notion "homotopy cofibre" is not new. Indeed we have the following.

Lemma 4.2. *Given maps $f : X \to Y$ and $j : Y \to Z$, j is a homotopy cofibre of f if and only if $j : Y \to Z$ is a cofibration and $X \xrightarrow{f} Y \xrightarrow{j} Z$ is a cofibre sequence, that is, there exists a homotopy commutative diagram with b, c, d homotopy equivalences:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "b"', "\simeq"] & Y \arrow[r, "j"] \arrow[d, "c"', "\simeq"] & Z \arrow[d, "d"', "\simeq"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "i_{f'}"] & Y' \cup_{f'} CX' \end{tikzcd}$$

Proof. It suffices to prove "if"-part. Let $J : c \circ f \simeq f' \circ b$. Then $\Phi = \Phi(f, f', b, c; J) : Y \cup_f CX \to Y' \cup_{f'} CX'$ is a homotopy equivalence with $\Phi \circ i_f = i_{f'} \circ c$ by Proposition 3.3(1)(c). Set $a = \Phi^{-1} \circ d : Z \to Y \cup_f CX$. Then a is a homotopy equivalence such that $a \circ j = \Phi^{-1} \circ d \circ j \simeq \Phi^{-1} \circ i_{f'} \circ c = \Phi^{-1} \circ \Phi \circ i_f \simeq i_f$. Hence j is a homotopy cofibre of f. □

Lemma 4.3. *Let $j : Y \to Z$ be a homotopy cofibre of $f : X \to Y$.*

(1) *There is a homotopy equivalence $a \in \mathrm{TOP}^Y(j, i_f)$ and its homotopy inverse $a^{-1} \in \mathrm{TOP}^Y(i_f, j)$ such that $a^{-1} \cup C1_Y : (Y \cup_f CX) \cup_{i_f} CY \to Z \cup_j CY$ is a homotopy inverse of $a \cup C1_Y$, that is,*

$$\text{(4.1)} \qquad \begin{cases} (a^{-1} \cup C1_Y) \circ (a \cup C1_Y) \simeq 1_{Z \cup_j CY}, \\ (a \cup C1_Y) \circ (a^{-1} \cup C1_Y) \simeq 1_{(Y \cup_f CX) \cup_{i_f} CY}. \end{cases}$$

(2) *If $f' : X \to Y$ satisfies $f \simeq f'$, then j is a homotopy cofibre of f'.*

(3) *If* $h : X \to X$ *is a homotopy equivalence, then* j *is a homotopy cofibre of* $f \circ h$.

(4) *If* $f = g \circ h : X \xrightarrow{h} X' \xrightarrow{g} Y$ *with* h *a homotopy equivalence, then* j *is a homotopy cofibre of* g.

(5) *If* j *is a free cofibration, then* $\Sigma^\ell j : \Sigma^\ell Y \to \Sigma^\ell Z$ *is a homotopy cofibre of* $\Sigma^\ell f : \Sigma^\ell X \to \Sigma^\ell Y$ *for any positive integer* ℓ.

(6) *If* $h : Y \to Y'$ *and* $k : Z \to Z'$ *are homeomorphisms, then* $j' = k \circ j \circ h^{-1} : Y' \to Z'$ *is a homotopy cofibre of* $h \circ f : X \to Y'$.

(7) *Given a map* $g : Y \to W$, g *can be extended to* Z *if and only if* $g \circ f \simeq *$.

Proof. (1) Suppose that $a' : Z \to Y \cup_f CX$ is a homotopy equivalence and $g : a' \circ j \simeq i_f$ is a homotopy. Since j is a cofibration by the assumption, there exists a homotopy $H : Z \times I \to Y \cup_f CX$ with $a' = H \circ i_0^Z$ and $g = H \circ (j \times 1_I)$. Then the map $a : Z \to Y \cup_f CX, z \mapsto H(z, 1)$, is a homotopy equivalence with $a \circ j = i_f$ and so a is a homotopy equivalence in $\mathrm{TOP}^Y(j, i_f)$ by Proposition 2.1. Let $a^{-1} \in \mathrm{TOP}^Y(i_f, j)$ be a homotopy inverse of a. Then $a^{-1} \circ i_f = j$ and there exist homotopies $K : a^{-1} \circ a \overset{Y}{\simeq} 1_Z$ and $L : a \circ a^{-1} \overset{Y}{\simeq} 1_{Y \cup_f CX}$. Hence $(a^{-1} \cup C1_Y) \circ (a \cup C1_Y) = K_0 \cup C1_Y \simeq K_1 \cup C1_Y = 1_{Z \cup_j CY}$ and the second equation of (4.1) is obtained similarly. This proves (1).

In the rest of the proof $a : Z \to Y \cup_f CX$ is a homotopy equivalence such that $a \circ j = i_f$.

(2) Suppose that $J : f \simeq f'$. By Proposition 3.3(1),

$$\Phi(J) := \Phi(f, f', 1_X, 1_Y; J) : Y \cup_f CX \to Y \cup_{f'} CX$$

is a homotopy equivalence and $\Phi(J) \circ i_f = i_{f'}$. Hence $\Phi(J) \circ a : Z \to Y \cup_{f'} CX$ is a homotopy equivalence and $\Phi(J) \circ a \circ j = i_{f'}$. This proves (2).

(3) Take $J : f \simeq f \circ h \circ h^{-1}$. Then $\Phi(f, f \circ h, h^{-1}, 1_Y; J)$ is a homotopy equivalence and $\Phi(f, f \circ h, h^{-1}, 1_Y; J) \circ a \circ j = i_{f \circ h}$. Hence j is a homotopy cofibre of $f \circ h$.

(4) Since $(1_Y \cup Ch) \circ a \circ j = (1_Y \cup Ch) \circ i_f = i_g$, (4) follows.

(5) Suppose that j is a free cofibration. Then $\Sigma^\ell j$ is a free cofibration by Corollary 2.3(1). We set $a' = (\psi_f^\ell)^{-1} \circ \Sigma^\ell a$, where ψ_f^ℓ is the homeomorphism $\Sigma^\ell Y \cup_{\Sigma^\ell f} C\Sigma^\ell X \approx \Sigma^\ell (Y \cup_f CX)$ defined in the section 2. Then $a' : \Sigma^\ell Z \to \Sigma^\ell Y \cup_{\Sigma^\ell f} C\Sigma^\ell X$ is a homotopy equivalence and $a' \circ \Sigma^\ell j = i_{\Sigma^\ell f}$. Hence $\Sigma^\ell j$ is a homotopy cofibre of $\Sigma^\ell f$.

(6) Since $(h \cup C1_X) \circ a \circ k^{-1} : Z' \to Y' \cup_{h \circ f} CX$ is a homotopy equivalence and $(h \cup C1_X) \circ a \circ k^{-1} \circ j' = i_{h \circ f}$, it follows that j' is a homotopy cofibre of $h \circ f$. This proves (6).

(7) Let $a : Z \to Y \cup_f CX$ be a homotopy equivalence such that $a \circ j = i_f$, and a^{-1} a homotopy inverse of a such that $a^{-1} \circ i_f = j$. If $g \circ f \simeq *$, then there exists $\widetilde{g} : Y \cup_f CX \to W$ such that $\widetilde{g} \circ i_f = g$ and $\widetilde{g} \circ a$ is an extension of g to Z. If g has an extension $g' : Z \to W$, then $g \circ f = g' \circ a^{-1} \circ i_f \circ f \simeq g' \circ a^{-1} \circ * = *$. $\square$

Remark 4.4. A map a in Lemma 4.3(1) is not necessarily unique in the sense of $\overset{Y}{\simeq}$.

Proof. Let $\nabla : \mathrm{S}^1 \vee \mathrm{S}^1 \to \mathrm{S}^1$ be the folding map. Then $\mathrm{S}^1 \cup_\nabla C(\mathrm{S}^1 \vee \mathrm{S}^1) = \mathrm{S}^2$ and $i_\nabla : \mathrm{S}^1 \to \mathrm{S}^2$ can be identified with $j : \mathrm{S}^1 \to \mathrm{S}^2$, $(x, y) \mapsto (x, y, 0)$. Obviously j is a homotopy cofibre of ∇. We set $a : \mathrm{S}^2 \to \mathrm{S}^2$, $(x, y, z) \mapsto (x, y, -z)$. Then $a, 1_{\mathrm{S}^2}$ are homotopy equivalences in $\mathrm{TOP}^{\mathrm{S}^1}(j, i_\nabla)$. Their degrees are -1 and 1, respectively. Hence $a \not\simeq 1_{\mathrm{S}^2}$ and so $a \overset{\mathrm{S}^1}{\simeq} 1_{\mathrm{S}^2}$ does not hold. $\square$

Lemma 4.5. *If $j : Y \to Z$ is a homotopy cofibre of $f : X \to Y$ and if a map $g : Y \to W$ satisfies $g \circ f \simeq *$, then, for any homotopy $A : g \circ f \simeq *$, we have*

$$[g, A, f] \cup C1_Y \simeq (g, A, f) \circ q'_f : (Y \cup_f CX) \cup_{i_f} CY \to W \cup_g CY$$

and, for any homotopy equivalence $a : Z \to Y \cup_f CX$ satisfying $a \circ j = i_f$, we have

$$(4.2)\quad ([g, A, f] \circ a \cup C1_Y) \circ \omega^{-1} = ([g, A, f] \cup C1_Y) \circ (a \cup C1_Y) \circ \omega^{-1} \simeq (g, A, f),$$

where ω^{-1} is a homotopy inverse of $\omega = q'_f \circ (a \cup C1_Y) : Z \cup_j CY \to \Sigma X$.

Proof. Consider the following diagram.

$$\begin{tikzcd}
 & Y \arrow[r, "j"] \arrow[d, equal] & Z \arrow[r, "i_j"] \arrow[d, "a"', "\simeq"] & Z \cup_j CY \arrow[d, "a \cup C1_Y"', "\simeq"] \arrow[dr, "\omega"] & \\
X \arrow[r, "f"] & Y \arrow[r, "i_f"] \arrow[dr, "g"'] & Y \cup_f CX \arrow[r, "i_{i_f}"] \arrow[d, "{[g,A,f]}"] & (Y \cup_f CX) \cup_{i_f} CY \arrow[r, "q'_f", "\simeq"'] \arrow[d, "{[g,A,f] \cup C1_Y}"'] & \Sigma X \arrow[dl, "{(g,A,f)}"] \\
 & & W \arrow[r, "i_g"'] & W \cup_g CY &
\end{tikzcd}$$

The above diagram is commutative except the right lower triangle. Define $u : I \times I \to I$ and $H : ((Y \cup_f CX) \cup_{i_f} CY) \times I \to W \cup_g CY$ by

$$u(s,t) = \begin{cases} s & s \geq t \\ 2s - t & 2s \geq t \geq s \\ -2s + t & 2s \leq t \end{cases}, \quad H(x \wedge s, t) = \begin{cases} f(x) \wedge u(s,t) & 2s \leq t \\ A(x, u(s,t)) & 2s \geq t \end{cases},$$

$$H(y,t) = y \wedge t, \quad H(y \wedge s, t) = y \wedge \max\{s,t\}.$$

Then $H : [g,A,f] \cup C1_Y \simeq (g,A,f) \circ q'_f$. Hence

$$\begin{aligned}([g,A,f] \circ a \cup C1_Y) \circ \omega^{-1} &= ([g,A,f] \cup C1_Y) \circ (a \cup C1_Y) \circ \omega^{-1} \\ \simeq (g,A,f) \circ q'_f \circ (a \cup C1_Y) \circ \omega^{-1} &= (g,A,f) \circ \omega \circ \omega^{-1} \simeq (g,A,f).\end{aligned}$$

□

5. Iterated mapping cones

In this section we will work in TOP^w. Hence for any map $f : X \to Y$ the injection $i_f : Y \to Y \cup_f CX$ is in TOP^w and a free cofibration by Corollary 2.3(2),(3).

By replacing the words "w-space" and "free cofibration" with "clw-space" and "closed free cofibration" respectively, we can develop consideration of this section similarly in TOP^{clw}.

We will revise the notion of "shaft" of Gershenson [**7**] and rename it "iterated mapping cone". Suppose that the diagram

$$\text{(5.1)} \quad \begin{tikzcd} X_1 \arrow[d, "g_1"'] & X_2 \arrow[d, "g_2"'] & X_3 \arrow[d, "g_3"'] & \cdots & X_n \arrow[d, "g_n"'] & \\ C_1 \arrow[r, "j_1"] & C_2 \arrow[r, "j_2"] & C_3 \arrow[r, "j_3"] & \cdots \arrow[r, "j_{n-1}"] & C_n \arrow[r, "j_n"] & C_{n+1} \end{tikzcd}$$

is given with $n \geq 1$, where $j_s : C_s \to C_{s+1}$ is a "free" cofibration for every s. We denote the above diagram by

$$\mathcal{S} = (X_1, \dots, X_n; C_1, \dots, C_{n+1}; g_1, \dots, g_n; j_1, \dots, j_n).$$

We often add $C_0 = \{*\}$ and the inclusion $j_0 : C_0 \to C_1$ to the above diagram.

Definition 5.1. (1) The sequence $(g_1, j_1, \dots, j_n)$ is called the *edge* of $\mathcal{S}$.

(2) $\mathcal{S}$ is a *quasi iterated mapping cone* of depth n if $C_{s+1} \cup_{j_s} CC_s \simeq \Sigma X_s$ and $[X_s, Z] \xleftarrow{g_s^*} [C_s, Z] \xleftarrow{j_s^*} [C_{s+1}, Z]$ is exact as a sequence of pointed sets for every space Z and every $s \geq 1$ (cf. [**4**, p.68]). If we choose a homotopy equivalence $\omega_s : C_{s+1} \cup_{j_s} CC_s \simeq \Sigma X_s$ for each $s \geq 1$, then the set $\Omega = \{\omega_s \,|\, 1 \leq s \leq n\}$ is called a *quasi-structure* on $\mathcal{S}$. We set $\omega_0 = 1_{C_1} : C_1 \cup_{j_0} CC_0 = C_1 \to C_1$.

(3) $\mathcal{S}$ is an *iterated mapping cone* of depth n if j_s is a homotopy cofibre of g_s for every $s \geq 1$. In this case a homotopy equivalence $a_s : C_{s+1} \longrightarrow C_s \cup_{g_s} CX_s$ and its homotopy inverse a_s^{-1} can be taken such that

$$\text{(5.2)} \quad a_s \circ j_s = i_{g_s}, \; a_s^{-1} \circ i_{g_s} = j_s, \; a_s^{-1} \circ a_s \overset{C_s}{\simeq} 1_{C_{s+1}}, \; a_s \circ a_s^{-1} \overset{C_s}{\simeq} 1_{C_s \cup_{g_s} CX_s}.$$

If we choose such a homotopy equivalence a_s for each $s \geq 1$, then we call the set $\mathcal{A} = \{a_s \,|\, 1 \leq s \leq n\}$ a *structure* on $\mathcal{S}$, and we set

$\omega_s = q'_{g_s} \circ (a_s \cup C1_{C_s})$ and $\Omega(\mathcal{A}) = \{\omega_s \,|\, 1 \le s \le n\}$ which is a quasi-structure on $\mathcal{S}$.

(4) $\mathcal{S}$ is *reduced* if $C_2 = C_1 \cup_{g_1} CX_1$ and $j_1 = i_{g_1}$. A quasi-structure Ω on a reduced quasi iterated mapping cone is *reduced* if $\omega_1 = q'_{g_1}$. A structure $\mathcal{A}$ on a reduced iterated mapping cone is *reduced* if $a_1 = 1_{C_2}$.

(5) Given a map $f : C_1 \to Y$, we denote by $\overline{f}^s : C_s \to Y$ an extension of f to C_s, that is, $f = \begin{cases} \overline{f}^1 & s = 1 \\ \overline{f}^s \circ j_{s-1} \circ \cdots \circ j_1 & s \ge 2 \end{cases}$. We set $\overline{f}^0 = * : C_0 \to Y$.

Convention 5.2. When $\mathcal{S}$ is an iterated mapping cone of depth n with a structure $\{a_s \,|\, 1 \le s \le n\}$, we denote by a_s^{-1} a homotopy inverse of a_s such that it satisfies (5.2).

Note that an iterated mapping cone is a quasi iterated mapping cone. When $\mathcal{S}$ is a reduced iterated mapping cone, a structure $\mathcal{A}$ on $\mathcal{S}$ is reduced if and only if $\Omega(\mathcal{A})$ is reduced. Notice also that a quasi iterated mapping cone is a revised version of the one called a shaft by Gershenson in [**7**, Definition 1.2D] where he did not suppose that the cofibrations j_i are free.

Let $\mathcal{S} = (X_1, \ldots, X_n; C_1, \ldots, C_{n+1}; g_1, \ldots, g_n; j_1, \ldots, j_n)$ be a quasi iterated mapping cone of depth n with a quasi-structure $\Omega = \{\omega_s \,|\, 1 \le s \le n\}$ and $f : C_1 \to Y$ a map with an extension $\overline{f}^{n+1}$ to C_{n+1}. We define maps for $0 \le s \le n$ as follows:

(5.3)
$$\begin{cases} \overline{f}^s = \overline{f}^{n+1} \circ j_n \circ \cdots \circ j_s : C_s \to Y, \\ h_{s+1} = \overline{f}^{s+1} \cup C1_{C_s} : C_{s+1} \cup_{j_s} CC_s \to Y \cup_{\overline{f}^s} CC_s, \\ k_{s+1} = 1_Y \cup Cj_s : Y \cup_{\overline{f}^s} CC_s \to Y \cup_{\overline{f}^{s+1}} CC_{s+1}, \\ \widetilde{g}_{s+1} = \begin{cases} f : C_1 \to Y & s = 0 \\ h_{s+1} \circ \omega_s^{-1} : \Sigma X_s \to Y \cup_{\overline{f}^s} CC_s & s \ge 1 \end{cases}, \\ \xi_{s+1} : (Y \cup_{\overline{f}^{s+1}} CC_{s+1}) \cup_{k_{s+1}} C(Y \cup_{\overline{f}^s} CC_s) \\ \qquad\qquad \to (Y \cup_{1_Y} CY) \cup_{\overline{f}^{s+1} \cup C\overline{f}^s} C(C_{s+1} \cup_{j_s} CC_s), \\ \quad y \mapsto y,\ c_{s+1} \wedge t \mapsto c_{s+1} \wedge t,\ y \wedge t \mapsto y \wedge t,\ c_s \wedge u \wedge t \mapsto c_s \wedge t \wedge u, \\ \widetilde{\omega_s} = \Sigma\omega_s \circ q_{\overline{f}^{s+1} \cup C\overline{f}^s} \circ \xi_{s+1} : C_{k_{s+1}} \to \begin{cases} \Sigma C_1 & s = 0 \\ \Sigma\Sigma X_s & s \ge 1 \end{cases}, \end{cases}$$

where $y \in Y,\ c_{s+1} \in C_{s+1},\ c_s \in C_s,\ t, u \in I$, and ω_s^{-1} is a homotopy inverse of ω_s. Since ω_s^{-1} is determined by ω_s up to homotopy, so is $\widetilde{g}_{s+1}$ for $s \ge 1$.

Lemma 5.3. *Under the above situation, we have* $C_{\overline{f}^0} = Y$, $\overline{f}^1 = h_1 = \widetilde{g}_1 = f$, $k_1 = i_f$, $\widetilde{\omega_0} = q'_f$, ξ_{s+1} *is a homeomorphism,* $\widetilde{\omega_s}$ *is a homotopy* *equivalence, and the following diagram (5.4) is a reduced iterated mapping cone of depth $n+1$ with a reduced quasi-structure $\widetilde{\Omega} = \{\widetilde{\omega_s} \,|\, 0 \le s \le n\}$.*

(5.4)
$$\begin{tikzcd}
C_1 \arrow[d, "\widetilde{g}_1"'] & \Sigma X_1 \arrow[d, "\widetilde{g}_2"'] & \Sigma X_2 \arrow[d, "\widetilde{g}_3"'] & \cdots & \Sigma X_n \arrow[d, "\widetilde{g}_{n+1}"'] & \\
C_{\overline{f}^0} \arrow[r, "k_1"] & C_{\overline{f}^1} \arrow[r, "k_2"] & C_{\overline{f}^2} \arrow[r, "k_3"] & \cdots \arrow[r, "k_n"] & C_{\overline{f}^n} \arrow[r, "k_{n+1}"] & C_{\overline{f}^{n+1}}
\end{tikzcd}$$

Proof. Since $k_1 = i_{\widetilde{g}_1}$, k_1 is a free cofibration and a homotopy cofibre of $\widetilde{g}_1$, and $\widetilde{\omega_0} = q'_{\widetilde{g}_1}$. Let $1 \le s \le n$. By Proposition 2.2, k_{s+1} is a free cofibration. Take $J : h_{s+1} \simeq \widetilde{g}_{s+1} \circ \omega_s$ and set $\Phi(J, s+1) = \Phi(h_{s+1}, \widetilde{g}_{s+1}, \omega_s, 1_{C_{\overline{f}^s}}; J)$. Then we have the following diagram.

$$\begin{tikzcd}
C_s \arrow[r, "j_s"] \arrow[d, equal] & C_{s+1} \arrow[r, "i_{j_s}"] \arrow[d, "\overline{f}^{s+1}"'] & C_{s+1} \cup_{j_s} CC_s \arrow[r, "\omega_s", "\simeq"'] \arrow[d, "h_{s+1}"'] & \Sigma X_s \arrow[d, "\widetilde{g}_{s+1}"'] \\
C_s \arrow[r, "\overline{f}^s"] & Y \arrow[r, "i_{\overline{f}^s}"] \arrow[d, "i_{\overline{f}^{s+1}}"'] & Y \cup_{\overline{f}^s} CC_s \arrow[r, equal] \arrow[d, "i_{h_{s+1}}"] \arrow[dl, "k_{s+1}"'] & C_{\overline{f}^s} \arrow[d, "i_{\widetilde{g}_{s+1}}"'] \\
& C_{\overline{f}^{s+1}} \arrow[r, "\simeq", "i_{\overline{f}^s} \cup C i_{j_s}"'] & C_{h_{s+1}} \arrow[r, "\simeq", "\Phi(J,s+1)"'] & C_{\widetilde{g}_{s+1}}
\end{tikzcd}$$

By Proposition 3.3(1), $\Phi(J, s+1)$ is a homotopy equivalence and $\Phi(J, s+1) \circ i_{h_{s+1}} = i_{\widetilde{g}_{s+1}}$. By Lemma 3.1, $i_{\overline{f}^s} \cup C i_{j_s}$ is a homotopy equivalence and $(i_{\overline{f}^s} \cup C i_{j_s}) \circ k_{s+1} \simeq i_{h_{s+1}}$. Hence $\Phi(J, s+1) \circ (i_{\overline{f}^s} \cup C i_{j_s}) \circ k_{s+1} \simeq i_{\widetilde{g}_{s+1}}$. Thus k_{s+1} is a homotopy cofibre of $\widetilde{g}_{s+1}$. Hence (5.4) is a reduced iterated mapping cone of depth $n+1$. As is easily seen, ξ_{s+1} is a homeomorphism, and $q_{\overline{f}^{s+1} \cup C\overline{f}^s} : C_{\overline{f}^{s+1} \cup C\overline{f}^s} \to \Sigma(C_{s+1} \cup_{j_s} CC_s)$ and $\Sigma\omega_s$ are homotopy equivalences. Hence $\widetilde{\omega_s}$ is a homotopy equivalence. Therefore $\widetilde{\Omega}$ is a reduced quasi-structure on (5.4). □

Definition 5.4. We denote the iterated mapping cone (5.4) by $\mathcal{S}(\overline{f}^{n+1}, \Omega)$, that is,

$$\mathcal{S}(\overline{f}^{n+1}, \Omega) = \big(C_1, \Sigma X_1, \ldots, \Sigma X_n; Y, C_{\overline{f}^1}, \ldots, C_{\overline{f}^{n+1}}; \\ f, \widetilde{g}_2, \ldots, \widetilde{g}_{n+1}; k_1, \ldots, k_{n+1}\big),$$

and call it the *iterated mapping cone induced from* $\mathcal{S}$ *by* $\overline{f}^{n+1}$ *and* Ω, and we call $\widetilde{\Omega} = \{\widetilde{\omega_s} \,|\, 0 \le s \le n\}$ the *typical quasi-structure* on $\mathcal{S}(\overline{f}^{n+1}, \Omega)$. When $\mathcal{S}$ is an iterated mapping cone with a structure $\mathcal{A}$, we denote the reduced iterated mapping cone $\mathcal{S}(\overline{f}^{n+1}, \Omega(\mathcal{A}))$ by $\mathcal{S}(\overline{f}^{n+1}, \mathcal{A})$ and call it the *iterated* *mapping cone induced from* $\mathcal{S}$ *by* $\overline{f}^{n+1}$ *and* $\mathcal{A}$. (Notice that we do not have typical structure on $\mathcal{S}(\overline{f}^{n+1}, \Omega)$ even if $\mathcal{S}$ is an iterated mapping cone.)

Remark 5.5. We have easily the following from definitions.

(1) The iterated mapping cone $\mathcal{S}(\overline{f}^{n+1}, \Omega)$ depends on the map $\overline{f}^{n+1}$ and spaces $X_1, \dots, X_n$ but not on maps $g_1, \dots, g_n$.
(2) The edge of $\mathcal{S}(\overline{f}^{n+1}, \Omega)$ does not depend on Ω.
(3) If two quasi iterated mapping cones $\mathcal{S}, \mathcal{S}'$ of depth n have the same edge $X_1 \overset{g_1}{\to} C_1 \overset{j_1}{\to} C_2 \overset{j_2}{\to} \cdots \overset{j_n}{\to} C_{n+1}$ and if a map $f : C_1 \to Y$ has an extension $\overline{f}^{n+1}$ to C_{n+1}, then two iterated mapping cones $\mathcal{S}(\overline{f}^{n+1}, \Omega), \mathcal{S}'(\overline{f}^{n+1}, \Omega')$ have the same edge for any quasi-structures Ω, Ω' on $\mathcal{S}, \mathcal{S}'$, respectively.

If the following problem is solved affirmatively, the number of systems which shall be defined in the next section decreases by 2 to 10.

Problem 5.6. Is every quasi iterated mapping cone of depth 1 an iterated mapping cone?

6. Unstable higher Toda brackets

In this section we will work in TOP^w. (As indicated in the previous section we can develop our consideration of this section similarly in TOP^{clw}.)

Sometimes, without particular comments, we do not distinguish in notation between a map and its homotopy class.

Throughout the section 6, we denote by $\vec{\boldsymbol{\alpha}} = (\alpha_n, \dots, \alpha_1)$ a sequence of homotopy classes

$$\alpha_i \in [X_i, X_{i+1}] \quad (i = 1, 2, \dots, n;\ n \geq 3). \tag{6.1}$$

If a map $f_i : X_i \to X_{i+1}$ represents α_i, then the sequence $\vec{\boldsymbol{f}} = (f_n, \dots, f_1)$ is called a *representative* of $\vec{\boldsymbol{\alpha}}$. We denote by $\mathrm{Rep}(\vec{\boldsymbol{\alpha}})$ the set of representatives of $\vec{\boldsymbol{\alpha}}$.

6.1. Definition of higher Toda brackets.

Given $\vec{\boldsymbol{f}} \in \mathrm{Rep}(\vec{\boldsymbol{\alpha}})$, we consider collections $\{\mathcal{S}_r, \overline{f_r}, \Omega_r \,|\, 2 \leq r \leq n\}$, $\{\mathcal{S}_2, \overline{f_2}, \Omega_2\} \cup \{\mathcal{S}_r, \overline{f_r}, \mathcal{A}_r \,|\, 3 \leq r \leq n\}$, and $\{\mathcal{S}_r, \overline{f_r}, \mathcal{A}_r \,|\, 2 \leq r \leq n\}$ (provided $\mathcal{S}_2$ is an iterated mapping cone) which satisfy the following (i), (ii), and (iii). (There is a possibility that such collections do not exist for suitable $\vec{\boldsymbol{f}}$.)

(i) $\mathcal{S}_2$ is a quasi iterated mapping cone of depth 1 as displayed in

$$\begin{tikzcd} X_1 \arrow[d, "f_1"'] & \\ X_2 \arrow[r, "j_{2,1}"] & C_{2,2} \end{tikzcd}$$

with Ω_2 a quasi-structure and $\mathcal{A}_2$ a structure provided $\mathcal{S}_2$ is an iterated mapping cone.

(ii) $\mathcal{S}_r$ is an iterated mapping cone of depth $r-1$ for $3 \leq r \leq n$ as displayed in

$$\begin{tikzcd} X_{r-1} \arrow[d, "f_{r-1}"'] & \Sigma X_{r-2} \arrow[d, "g_{r,2}"'] & \Sigma^2 X_{r-3} \arrow[d, "g_{r,3}"'] & \cdots & \Sigma^{r-2} X_1 \arrow[d, "g_{r,r-1}"'] & \\ X_r \arrow[r, "j_{r,1}"] & C_{r,2} \arrow[r, "j_{r,2}"] & C_{r,3} \arrow[r, "j_{r,3}"] & \cdots \arrow[r, "j_{r,r-2}"] & C_{r,r-1} \arrow[r, "j_{r,r-1}"] & C_{r,r} \end{tikzcd}$$

with Ω_r a quasi-structure and $\mathcal{A}_r$ a structure.

(iii) $\overline{f_r} : C_{r,r} \to X_{r+1}$ is an extension of f_r to $C_{r,r}$ for $2 \leq r \leq n-1$, and $\overline{f_n} : C_{n,n-1} \to X_{n+1}$ is an extension of f_n to $C_{n,n-1}$.

We use the following notations:

- $C_{r,0} = \{*\}$, $C_{r,1} = X_r$, $j_{r,0} = * : C_{r,0} \to C_{r,1}$, $f_r^0 = f_r \circ j_{r,0} : C_{r,0} \to X_{r+1}$ for $1 \leq r \leq n$, and $\overline{f_1} = f_1 : C_{1,1} \to X_2$;
- $g_{r,1} = f_{r-1}$ for $2 \leq r \leq n$;
- $\overline{f_r}^s = \begin{cases} f_r & 1 = s \leq r \leq n \\ \overline{f_r} \circ j_{r,r-1} \circ \cdots \circ j_{r,s} & 0 \leq s < r \leq n-1 \\ \overline{f_r} & 1 \leq s = r \leq n-1 \end{cases} : C_{r,s} \to X_{r+1};$
- $\overline{f_n}^s = \begin{cases} \overline{f_n} \circ j_{n,n-2} \circ \cdots \circ j_{n,s} & 0 \leq s \leq n-2 \\ \overline{f_n} & s = n-1 \end{cases} : C_{n,s} \to X_{n+1};$
- $\Omega_r = \{\omega_{r,s} \,|\, 1 \leq s < r\}$ and $\omega_{r,0} = 1_{X_r}$ for $2 \leq r \leq n$, where $\omega_{r,s} : C_{r,s+1} \cup_{j_{r,s}} CC_{r,s} \to \Sigma\Sigma^{s-1} X_{r-s}$;
- $\mathcal{A}_r = \{a_{r,s} \,|\, 1 \leq s < r\}$ and $\Omega(\mathcal{A}_r) = \{\omega_{r,s} \,|\, 1 \leq s < r\}$, where $a_{r,s} : C_{r,s+1} \to C_{r,s} \cup_{g_{r,s}} C\Sigma^{s-1} X_{r-s}$ and

$$\omega_{r,s} = q'_{g_{r,s}} \circ (a_{r,s} \cup C1_{C_{r,s}}) : C_{r,s+1} \cup_{j_{r,s}} CC_{r,s} \simeq \Sigma\Sigma^{s-1} X_{r-s},$$

and $a_{r,s}^{-1}$ is a homotopy inverse of $a_{r,s}$ such that

$$a_{r,s}^{-1} \circ i_{g_{r,s}} = j_{r,s}, \quad a_{r,s}^{-1} \circ a_{r,s} \overset{C_{r,s}}{\simeq} 1_{C_{r,s+1}}, \quad a_{r,s} \circ a_{r,s}^{-1} \overset{C_{r,s}}{\simeq} 1_{C_{g_{r,s}}}.$$

Definition 6.1.1. Various presentations of $\vec{\boldsymbol{f}}$ and related notions are defined as follows (if Problem 5.6 is affirmative, (a) (resp. (d)) equals with (a$'$) (resp. (d$'$)).

(1) A collection $\{\mathcal{S}_r,\ \overline{f_r},\ \Omega_r \,|\, 2 \leq r \leq n\}$ is
  (a) a *q-presentation* if $\mathcal{S}_{r+1} = \mathcal{S}_r(\overline{f_r}, \Omega_r)$ for $2 \leq r < n$;
  (a′) a *qs_2-presentation* if $\mathcal{S}_2$ is an iterated mapping cone and $\mathcal{S}_{r+1} = \mathcal{S}_r(\overline{f_r}, \Omega_r)$ for $2 \leq r < n$;
  (b) a *$q\dot{s}_2$-presentation* if it is a qs_2-presentation and $\mathcal{S}_2$ is reduced;
  (c) a *$q\ddot{s}_2$-presentation* if it is a $q\dot{s}_2$-presentation and Ω_2 is reduced;
  (d) an *aq-presentation* if $\mathcal{S}_{r+1} = \mathcal{S}_r(\overline{f_r}, \Omega_r)$ and $\Omega_{r+1} = \widetilde{\Omega_r}$ for $2 \leq r < n$;
  (d′) an *aqs_2-presentation* if $\mathcal{S}_2$ is an iterated mapping cone and $\mathcal{S}_{r+1} = \mathcal{S}_r(\overline{f_r}, \Omega_r)$ and $\Omega_{r+1} = \widetilde{\Omega_r}$ for $2 \leq r < n$;
  (e) an *$aq\dot{s}_2$-presentation* if it is an aqs_2-presentation and $\mathcal{S}_2$ is reduced;
  (f) an *$aq\ddot{s}_2$-presentation* if it is an $aq\dot{s}_2$-presentation and Ω_2 is reduced.

(2) A collection $\{\mathcal{S}_2, \overline{f_2}, \Omega_2\} \cup \{\mathcal{S}_r, \overline{f_r}, \mathcal{A}_r \,|\, 3 \leq r \leq n\}$ is a *q_2-presentation* if $\mathcal{S}_3 = \mathcal{S}_2(\overline{f_2}, \Omega_2)$, $\mathcal{S}_{r+1} = \mathcal{S}_r(\overline{f_r}, \mathcal{A}_r)$ $(3 \leq r < n)$, and $\mathcal{A}_r$ is reduced for $3 \leq r \leq n$.

(3) A collection $\{\mathcal{S}_r,\ \overline{f_r},\ \mathcal{A}_r \,|\, 2 \leq r \leq n\}$ is
  (g) an *s_t-presentation* if $\mathcal{S}_{r+1} = \mathcal{S}_r(\overline{f_r}, \mathcal{A}_r)$ and $\mathcal{A}_{r+1}$ is reduced for $2 \leq r < n$;
  (h) an *$\dot{s}_t$-presentation* if it is an s_t-presentation and $\mathcal{S}_2$ is reduced;
  (i) an *$\ddot{s}_t$-presentation* if it is an $\dot{s}_t$-presentation and $\mathcal{A}_2$ is reduced.

(4) Let $\star$ denote one of the following: q, aq, qs_2, $q\dot{s}_2$, $q\ddot{s}_2$, aqs_2, $aq\dot{s}_2$, $aq\ddot{s}_2$, s_t, $\dot{s}_t$, $\ddot{s}_t$, and q_2. $\vec{\boldsymbol{f}}$ is *$\star$-presentable* if it has a $\star$-presentation, and $\vec{\boldsymbol{\alpha}}$ is *$\star$-presentable* if it has a $\star$-presentable representative.

In the above definitions we used the following abbreviations: q=“quasi-structure”; s_2=“$\mathcal{S}_2$ is an iterated mapping cone”; $\dot{s}_2$=“$\mathcal{S}_2$ is a reduced iterated mapping cone”; $\ddot{s}_2$=“$\mathcal{S}_2$ is a reduced iterated mapping cone with Ω_2 reduced”; a=“asymptotic”; s_t=“structure”; $\dot{s}_t$=“s_t and $\dot{s}_2$”; $\ddot{s}_t$=“$\dot{s}_t$ and $\mathcal{A}_2$ is reduced”.

Definition 6.1.2. We denote the set of homotopy classes of $\overline{f_n} \circ g_{n,n-1}$ for all $\star$-presentations of $\vec{\boldsymbol{f}}$ by $\{\vec{\boldsymbol{f}}\}^{(\star)}$ or $\{f_n, \ldots, f_1\}^{(\star)}$ which is called the *$\star$-bracket* of $\vec{\boldsymbol{f}}$. It is a subset of $[\Sigma^{n-2}X_1, X_{n+1}]$ and there is a possibility that it is the empty set. For convenience we denote by $\{f_2, f_1\}^{(\star)}$ the one point set consisting of the homotopy class of $f_2 \circ f_1$.

Notice that $\vec{\boldsymbol{f}}$ is $\star$-presentable if and only if $\{\vec{\boldsymbol{f}}\}^{(\star)}$ is not empty. As shall be seen in §6.4, we can denote $\{\vec{\boldsymbol{f}}\}^{(\star)}$ by $\{\vec{\boldsymbol{\alpha}}\}^{(\star)}$ for any $\vec{\boldsymbol{f}} \in \mathrm{Rep}(\vec{\boldsymbol{\alpha}})$.

Remark 6.1.3. It follows from definitions that if $\vec{\boldsymbol{\alpha}}$ is q-presentable, then $\alpha_{r+1} \circ \alpha_r = 0$ for $1 \leq r \leq n-1$, and that we have the commutative diagram

(6.1.1)

$$\begin{tikzcd}
\{\vec{\boldsymbol{f}}\}^{(q\ddot{s}_2)} \arrow[r, "!!"] & \{\vec{\boldsymbol{f}}\}^{(q\dot{s}_2)} \arrow[drr, "!"] & & \{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)} \arrow[r, "!"] & \{\vec{\boldsymbol{f}}\}^{(\dot{s}_t)} \arrow[d, "!"] \\
\{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \arrow[u] \arrow[r] & \{\vec{\boldsymbol{f}}\}^{(aq\dot{s}_2)} \arrow[u] \arrow[r, "!"] & \{\vec{\boldsymbol{f}}\}^{(aqs_2)} \arrow[r] \arrow[d, "\#"] & \{\vec{\boldsymbol{f}}\}^{(qs_2)} \arrow[d, "\#"] & \{\vec{\boldsymbol{f}}\}^{(s_t)} \arrow[l] \arrow[d] \\
& & \{\vec{\boldsymbol{f}}\}^{(aq)} \arrow[r] & \{\vec{\boldsymbol{f}}\}^{(q)} & \{\vec{\boldsymbol{f}}\}^{(q_2)} \arrow[l]
\end{tikzcd}$$

where arrows are inclusions, !! is $=$ for $n \geq 4$, and four !'s are $=$ as shall be shown in Theorem 6.2.1. Notice that if Problem 5.6 is affirmative, two #'s are $=$.

The following two propositions are easy consequences of definitions.

Proposition 6.1.4. *Let* $\{\mathcal{S}_r, \overline{f_r}, \Omega_r \,|\, 2 \leq r \leq n\}$ *be a* q*-presentation of* $\vec{\boldsymbol{f}}$*. Then*

$$\begin{aligned}
C_{r,2} &= X_r \cup_{\overline{f_{r-1}}^1} CX_{r-1} \ (3 \leq r \leq n), \quad C_{3,3} = X_3 \cup_{\overline{f_2}} CC_{2,2}, \\
C_{r,s} &= X_r \cup_{\overline{f_{r-1}}^{s-1}} C(X_{r-1} \cup_{\overline{f_{r-2}}^{s-2}} C(X_{r-2} \cup \cdots \\
&\qquad \cup_{\overline{f_{r-s+2}}^2} C(X_{r-s+2} \cup_{\overline{f_{r-s+1}}^1} CX_{r-s+1}) \cdots)) \ (3 \leq s < r \leq n), \\
C_{r,r} &= X_r \cup_{\overline{f_{r-1}}^{r-1}} C(X_{r-1} \cup_{\overline{f_{r-2}}^{r-2}} C(\cdots \cup_{\overline{f_3}^3} C(X_3 \cup_{\overline{f_2}^2} CC_{2,2}) \cdots)) \\
&\qquad\qquad (4 \leq r \leq n).
\end{aligned}$$

Proposition 6.1.5. *If* $\{\vec{\boldsymbol{f}}\}^{(\star)}$ *is not empty, then* $\{f_m, f_{m-1}, \ldots, f_\ell\}^{(\star)}$ *contains* 0 *for* $1 \leq \ell < m \leq n$, $(\ell, m) \neq (1, n)$.

Definition 6.1.6. If there exist null-homotopies $A_i : f_{i+1} \circ f_i \simeq * \ (1 \leq i \leq n-1)$ such that $[f_{i+2}, A_{i+1}, f_{i+1}] \circ (f_{i+1}, A_i, f_i) \simeq * \ (1 \leq i \leq n-2)$, then we call $\vec{\boldsymbol{f}}$ and $(\vec{\boldsymbol{f}}; \vec{\boldsymbol{A}})$ *admissible*, where $\vec{\boldsymbol{A}} = (A_{n-1}, \ldots, A_1)$. We call $\vec{\boldsymbol{\alpha}}$ *admissible* if it has an admissible representative.

It follows from Proposition 2.11 of [**15**] that if $\vec{\boldsymbol{\alpha}}$ is admissible, then every representative of it is admissible. From results in forthcoming sub-sections, we can prove the following without difficulties: when $n = 3$, $\vec{\boldsymbol{\alpha}}$ is admissible if and only if $\{\vec{\boldsymbol{\alpha}}\}^{(\star)}$ contains 0 for all $\star$; when $n = 4$, $\vec{\boldsymbol{\alpha}}$ is admissible if and only if $\vec{\boldsymbol{\alpha}}$ is $\star$-presentable for all $\star$; when $n \geq 5$, if $\vec{\boldsymbol{\alpha}}$ is $\star$-presentable for some $\star$, then $\vec{\boldsymbol{\alpha}}$ is admissible.

Remark 6.1.7. The following is obvious by definitions: if $f_i : X_i \to X_{i+1}$ is a map in TOP^{clw} for every i, then the $\star$-brackets of $\vec{\boldsymbol{f}}$ in TOP^{clw} and TOP^{w} are the same for $\star = q\dot{s}_2, q\ddot{s}_2, aq\dot{s}_2, aq\ddot{s}_2, \dot{s}_t, \ddot{s}_t$.

6.2. **Relations between twelve brackets.** In this subsection we prove three results and state an example. From 6.2.1, 6.2.2, and (6.1.1), we have (1.1)–(1.4).

Theorem 6.2.1. (1) $\{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)} = \{\vec{\boldsymbol{f}}\}^{(\dot{s}_t)} = \{\vec{\boldsymbol{f}}\}^{(s_t)}$.

(2) $\{\vec{\boldsymbol{f}}\}^{(aqs_2)} = \{\vec{\boldsymbol{f}}\}^{(aq\dot{s}_2)} = \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \circ \Sigma^{n-3}\mathcal{E}(\Sigma X_1)$, $\{\vec{\boldsymbol{f}}\}^{(qs_2)} = \{\vec{\boldsymbol{f}}\}^{(q\dot{s}_2)} = \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \circ \mathcal{E}(\Sigma^{n-2}X_1)$, *and*

$$\{\vec{\boldsymbol{f}}\}^{(q\ddot{s}_2)} = \begin{cases} \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} & n = 3 \\ \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \circ \mathcal{E}(\Sigma^{n-2}X_1) & n \geq 4 \end{cases}.$$

(3) $\{\vec{\boldsymbol{f}}\}^{(q)} = \{\vec{\boldsymbol{f}}\}^{(aq)} \circ \mathcal{E}(\Sigma^{n-2}X_1)$.

(4) $\{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)} \circ \mathcal{E}(\Sigma^{n-2}X_1) = \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \circ \mathcal{E}(\Sigma^{n-2}X_1)$.

(5) *If* $\alpha \in \{\vec{\boldsymbol{f}}\}^{(q)}$, *then there are* $\theta, \theta' \in [\Sigma^{n-2}X_1, \Sigma^{n-2}X_1]$ *such that* $\alpha \circ \theta \in \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)}$ *and* $\alpha \circ \theta' \in \{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)}$.

Corollary 6.2.2. (1) $\{\vec{\boldsymbol{f}}\}^{(q)} \circ \varepsilon = \{\vec{\boldsymbol{f}}\}^{(q)}$ *and* $\{\vec{\boldsymbol{f}}\}^{(qs_2)} \circ \varepsilon = \{\vec{\boldsymbol{f}}\}^{(qs_2)}$ *for every* $\varepsilon \in \mathcal{E}(\Sigma^{n-2}X_1)$, *and* $\{\vec{\boldsymbol{f}}\}^{(aqs_2)} \circ \Sigma^{n-3}\gamma = \{\vec{\boldsymbol{f}}\}^{(aqs_2)}$ *for every* $\gamma \in \mathcal{E}(\Sigma X_1)$.

(2) $\{\vec{\boldsymbol{f}}\}^{(aq)} \circ \Sigma^{n-2}\gamma = \{\vec{\boldsymbol{f}}\}^{(aq)}$ *for every* $\gamma \in \mathcal{E}(X_1)$, *and* $-\{\vec{\boldsymbol{f}}\}^{(aq)} = \{\vec{\boldsymbol{f}}\}^{(aq)}$.

(3) *If the suspension* $\Sigma^{n-2} : \mathcal{E}(X_1) \to \mathcal{E}(\Sigma^{n-2}X_1)$ *is surjective, for example if* X_1 *is a sphere of positive dimension, then* $\{\vec{\boldsymbol{f}}\}^{(q)} = \{\vec{\boldsymbol{f}}\}^{(aq)}$.

(4) *If* $\{\vec{\boldsymbol{f}}\}^{(\star)}$ *is not empty for some* $\star$, *then* $\{\vec{\boldsymbol{f}}\}^{(\star)}$ *is not empty for all* $\star$.

(5) *If* $\{\vec{\boldsymbol{f}}\}^{(\star)}$ *contains* 0 *for some* $\star$, *then* $\{\vec{\boldsymbol{f}}\}^{(\star)}$ *contains* 0 *for all* $\star$.

(6) $-\{\vec{\boldsymbol{f}}\}^{(\star)} = \{\vec{\boldsymbol{f}}\}^{(\star)}$ *for* $\star = q$, qs_2, aq, aqs_2.

(7) *If* $n \geq 4$ *and* $\vec{\boldsymbol{f}}$ *is* $\star$*-presentable for some* $\star$, *then* $\vec{\boldsymbol{f}}$ *is admissible and* $\star$*-presentable for all* $\star$.

(8) *If* $\{\vec{\boldsymbol{f}}\}^{(\star)} = \{0\}$ *for some* $\star$, *then* $\{\vec{\boldsymbol{f}}\}^{(\star)} = \{0\}$ *for all* $\star$ *except* aq, q, q_2.

Proposition 6.2.3 (cf. p.26, p.25, and p.33 of [**27**]). *Given maps* $f_{n+1} : X_{n+1} \to X_{n+2}$ *and* $f_0 : X_0 \to X_1$, *we have*

$$f_{n+1} \circ \{f_n, \dots, f_1\}^{(\star)} \subset \{f_{n+1} \circ f_n, f_{n-1}, \dots, f_1\}^{(\star)}, \tag{6.2.1}$$

$$\{f_{n+1} \circ f_n, f_{n-1}, \dots, f_1\}^{(\star)} \subset \{f_{n+1}, f_n \circ f_{n-1}, f_{n-2}, \dots, f_1\}^{(\star)}, \tag{6.2.2}$$

$$\begin{aligned}(6.2.3)\qquad \{f_n,\dots,f_1\}^{(aq\ddot{s}_2)}\circ\Sigma^{n-2}f_0 &\subset \{f_n,\dots,f_2,f_1\circ f_0\}^{(aq\ddot{s}_2)}\\ &\subset \{f_n,\dots,f_3,f_2\circ f_1,f_0\}^{(aq\ddot{s}_2)}.\end{aligned}$$

Remark 6.2.4. We can prove the following analogues relations of (6.2.3).

$$\begin{aligned}\{f_n,\dots,f_1\}^{(\ddot{s}_t)}\circ\Sigma^{n-2}f_0 &\subset \{f_n,\dots,f_2,f_1\circ f_0\}^{(\ddot{s}_t)}\\ &\subset \{f_n,\dots,f_3,f_2\circ f_1,f_0\}^{(\ddot{s}_t)}.\end{aligned}$$

Details shall appear elsewhere.

Example 6.2.5 (cf. Lemma 4.10 of [**24**], Lemma 5.1 of [**7**], Example 6.6.2(2) below). Let p be an odd prime and $\alpha_1(3):\mathrm{S}^{2p}\to\mathrm{S}^3$ a map of which the homotopy class is of order p. For every integer $n\geq 3$, we set $\alpha_1(n)=\Sigma^{n-3}\alpha_1(3):\mathrm{S}^{n+2p-3}\to\mathrm{S}^n$ and $\Xi=\{\alpha_1(n),\alpha_1(n+2p-3),\alpha_1(n+2(2p-3)),\dots,\alpha_1(n+(p-1)(2p-3))\}^{(\star)}\subset\pi_{n+2p(p-1)-2}(\mathrm{S}^n)$. Take n such that $n\geq 2p(p-1)$. Then the p-primary component of $\pi_{n+2p(p-1)-2}(\mathrm{S}^n)$ is $\mathbb{Z}_p$, and the following can be proved: Ξ contains an element of order p and the order of any element of Ξ is a multiple of p so that Ξ does not contain 0. We need an argument for the proof, but we omit details.

Proof of Theorem 6.2.1(1). It suffices to show that $\{\vec{\boldsymbol{f}}\}^{(s_t)}\subset\{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)}$. Let $\alpha\in\{\vec{\boldsymbol{f}}\}^{(s_t)}$ and $\{\mathcal{S}_r,\overline{f_r},\mathcal{A}_r\,|\,2\leq r\leq n\}$ an s_t-presentation of $\vec{\boldsymbol{f}}$ with $\alpha=\overline{f_n}\circ g_{n,n-1}$, where

$$\begin{aligned}\mathcal{S}_r=(X_{r-1},\Sigma X_{r-2},\dots,\Sigma^{r-2}X_1;C_{r,1},\dots,C_{r,r};&\\ g_{r,1},\dots,g_{r,r-1};j_{r,1},\dots,j_{r,r-1});&\end{aligned}$$

$C_{r,1}=X_r,\ g_{r,1}=f_{r-1},\ \mathcal{A}_r=\{a_{r,s}\,|\,1\leq s<r\},\ \Omega(\mathcal{A}_r)=\{\omega_{r,s}\,|\,1\leq s<r\}$; if $3\leq r\leq n$, then $C_{r,2}=X_r\cup_{f_{r-1}}CX_{r-1},\ j_{r,1}=i_{f_{r-1}}$, and $a_{r,1}=1_{C_{r,2}}$.

We are going to construct an $\ddot{s}_t$-presentation $\{\mathcal{S}'_r,\overline{f_r}',\mathcal{A}'_r\,|\,2\leq r\leq n\}$ of $\vec{\boldsymbol{f}}$ such that $\overline{f_n}'\circ g'_{n,n-1}=\alpha$.

First we set $\mathcal{S}'_2=(X_1;X_2,X_2\cup_{f_1}CX_1;f_1;i_{f_1})$, $a'_{2,1}=1_{C'_{2,2}}$, $\mathcal{A}'_2=\{a'_{2,1}\}$, $\Omega'_2=\Omega(\mathcal{A}'_2)=\{q'_{f_1}\}$, $e_2=a_{2,1}^{-1}:C'_{2,2}\to C_{2,2}$, and $\overline{f_2}'=\overline{f_2}\circ e_2$. Then $C'_{2,1}=C_{2,1}$, $e_2\circ j'_{2,1}=j_{2,1}$ and

$$\begin{aligned}\omega_{2,1}\circ(e_2\cup C1_{X_2})&=q'_{f_1}\circ(a_{2,1}\cup C1_{X_2})\circ(e_2\cup C1_{X_2})\\ &\simeq q'_{f_1}=\omega'_{2,1}\quad\text{(by (4.1))}.\end{aligned}$$

Secondly we set $\mathcal{S}'_3=\mathcal{S}'_2(\overline{f_2}',\mathcal{A}'_2)$ and

$$e_3=1_{X_3}\cup Ce_2:C'_{3,3}=X_3\cup_{\overline{f_2}'}C(X_2\cup_{f_1}CX_1)\to C_{3,3}=X_3\cup_{\overline{f_2}}CC_{2,2}.$$

Then $C'_{3,s} = C_{3,s}$ $(s=1,2)$, $j'_{3,1} = j_{3,1}$, $j'_{3,2} = 1_{X_3} \cup Cj'_{2,1}$, $e_3 \circ j'_{3,2} = j_{3,2}$, and

$$\begin{aligned} g'_{3,2} &= (\overline{f_2}' \cup C1_{X_2}) \circ \omega'^{-1}_{2,1} = (\overline{f_2} \cup C1_{X_2}) \circ (e_2 \cup C1_{X_2}) \circ \omega'^{-1}_{2,1} \\ &\simeq (\overline{f_2} \cup C1_{X_2}) \circ \omega^{-1}_{2,1} = g_{3,2}. \end{aligned}$$

Take a homotopy $K^3 : g_{3,2} \simeq g'_{3,2}$ and set

$$\begin{aligned} &\Phi(K^3) = \Phi(g_{3,2}, g'_{3,2}, 1_{\Sigma X_1}, 1_{X_3 \cup_{f_2} CX_2}; K^3) \\ &\qquad\qquad : (X_3 \cup_{f_2} CX_2) \cup_{g_{3,2}} C\Sigma X_1 \to (X_3 \cup_{f_2} CX_2) \cup_{g'_{3,2}} C\Sigma X_1, \\ &a'_{3,2} = \Phi(K^3) \circ a_{3,2} \circ e_3 : C'_{3,3} \to (X_3 \cup_{f_2} CX_2) \cup_{g'_{3,2}} C\Sigma X_1, \\ &a'_{3,1} = 1_{C'_{3,2}},\ \mathcal{A}'_3 = \{a'_{3,1}, a'_{3,2}\},\ \overline{f_3}' = \begin{cases} \overline{f_3} : C'_{3,2} = C_{3,2} \to X_4 & n = 3 \\ \overline{f_3} \circ e_3 : C'_{3,3} \to X_4 & n \geq 4 \end{cases}. \end{aligned}$$

Then $\mathcal{A}'_3$ is a reduced structure on $\mathcal{S}'_3$. When $n=3$, $\{\mathcal{S}'_r, \overline{f_r}', \mathcal{A}'_r \,|\, r = 2,3\}$ is an $\ddot{s}_t$-presentation of $\vec{\boldsymbol{f}}$ such that $\overline{f_3}' \circ g'_{3,2} = \alpha$. When $n \geq 4$, by repeating the above process, we have $\mathcal{S}'_r, \overline{f_r}', \mathcal{A}'_r$ and $e_r : C'_{r,r} \simeq C_{r,r}$ for $4 \leq r \leq n$ such that

$$\begin{cases} \mathcal{S}'_r = \mathcal{S}'_{r-1}(\overline{f_{r-1}}', \mathcal{A}'_{r-1}),\ e_r = 1_{X_r} \cup Ce_{r-1}; \\ C'_{r,s} = C_{r,s}\ (1 \leq s \leq r-1),\ C'_{r,r} = X_r \cup_{\overline{f_{r-1}}'} CC'_{r-1,r-1}; \\ j'_{r,s} = j_{r,s},\ a'_{r,s} = a_{r,s},\ g'_{r,s} = g_{r,s}\ (1 \leq s \leq r-2); \\ \omega'_{r,r-1} \simeq \omega_{r,r-1} \circ (e_r \cup C1_{C_{r,r-1}}),\ g'_{r,r-1} \simeq g_{r,r-1}; \\ \overline{f_r}' = \begin{cases} \overline{f_n} : C'_{n,n-1} = C_{n,n-1} \to X_{n+1} & r = n \\ \overline{f_r} \circ e_r : C'_{r,r} \to X_{r+1} & r < n \end{cases}; \\ a'_{r,r-1} = \Phi(K^r) \circ a_{r,r-1} \circ e_r : C'_{r,r} \to C'_{r,r-1} \cup_{g'_{r,r-1}} C\Sigma^{r-2} X_1, \end{cases}$$

where $K^r : g_{r,r-1} \simeq g'_{r,r-1}$ and

$$\begin{aligned} &\Phi(K^r) = \Phi(g_{r,r-1}, g'_{r,r-1}, 1_{\Sigma^{r-2} X_1}, 1_{C_{r,r-1}}; K^r) \\ &\qquad\qquad : C_{r,r-1} \cup_{g_{r,r-1}} C\Sigma^{r-2} X_1 \to C'_{r,r-1} \cup_{g'_{r,r-1}} C\Sigma^{r-2} X_1. \end{aligned}$$

Then $\mathcal{A}'_r$ is a reduced structure on $\mathcal{S}'_r$. Therefore $\{\mathcal{S}'_r, \overline{f_r}', \mathcal{A}'_r \,|\, 2 \leq r \leq n\}$ is an $\ddot{s}_t$-presentation of $\vec{\boldsymbol{f}}$ such that $\overline{f_n}' \circ g'_{n,n-1} \simeq \overline{f_n} \circ g_{n,n-1}$ and hence $\alpha \in \{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)}$. This proves Theorem 6.2.1(1). □

Proof of Theorem 6.2.1(2). First we prove $\{\vec{\boldsymbol{f}}\}^{(aqs_2)} \subset \{\vec{\boldsymbol{f}}\}^{(aq\dot{s}_2)}$ which is equivalent to the first equality. Let $\alpha \in \{\vec{\boldsymbol{f}}\}^{(aqs_2)}$ and $\{\mathcal{S}_r, \overline{f_r}, \Omega_r \,|\, 2 \leq r \leq n\}$ an aqs_2-presentation of $\vec{\boldsymbol{f}}$ with $\alpha = \overline{f_n} \circ g_{n,n-1}$. It suffices to construct an $aq\dot{s}_2$-presentation $\{\mathcal{S}'_r, \overline{f_r}', \Omega'_r \,|\, 2 \leq r \leq n\}$ with $\alpha = \overline{f_n}' \circ g'_{n,n-1}$. Set $\mathcal{S}'_2 = (X_1; X_2, X_2 \cup_{f_1} CX_1; f_1; i_{f_1})$. Since $\mathcal{S}_2$ is an iterated mapping cone, we can take $e_2 : C'_{2,2} = X_2 \cup_{f_1} CX_1 \simeq C_{2,2}$ such that $e_2 \circ j'_{2,1} = j_{2,1}$. Set $\overline{f_2}' = \overline{f_2} \circ e_2$, $\omega'_{2,1} = \omega_{2,1} \circ (e_2 \cup C1_{X_2}) : C'_{2,2} \cup_{j'_{2,1}} CC'_{2,1} \to \Sigma X_1$, and $\Omega'_2 = \{\omega'_{2,1}\}$. Set $\mathcal{S}'_3 = \mathcal{S}'_2(\overline{f_2}', \Omega'_2)$, $\Omega'_3 = \widetilde{\Omega'_2}$, $e_3 = 1_{X_3} \cup Ce_2 : C'_{3,3} \to C_{3,3}$, and $\overline{f_3}' = \begin{cases} \overline{f_3} : C'_{3,2} = C_{3,2} \to X_4 & n = 3 \\ \overline{f_3} \circ e_3 : C'_{3,3} \to X_4 & n \geq 4 \end{cases}$. Then $\omega'_{3,2} = \omega_{3,2} \circ (e_3 \cup C1_{C_{3,2}})$ and $g'_{3,2} = (\overline{f_2}' \cup C1_{X_2}) \circ \omega'^{-1}_{2,1} \simeq (\overline{f_2} \cup C1_{X_2}) \circ \omega^{-1}_{2,1} = g_{3,2}$. By continuing the construction inductively, we obtain an $aq\dot{s}_2$-presentation $\{\mathcal{S}'_r, \overline{f_r}', \Omega'_r \,|\, 2 \leq r \leq n\}$ and $e_r : C'_{r,r} \simeq C_{r,r}$ such that $C'_{r,s} = C_{r,s}$ $(1 \leq s < r < n)$, $\omega'_{r,r-1} = \omega_{r,r-1} \circ (e_r \cup 1_{C_{r,r-1}}) : C'_{r,r} \cup CC'_{r,r-1} \to C_{r,r} \cup CC_{r,r-1}$ for $r < n$, and $\overline{f_r}' = \begin{cases} \overline{f_n} : C'_{n,n-1} = C_{n,n-1} \to X_{n+1} & r = n \\ \overline{f_r} \circ e_r : C'_{r,r} \to X_{r+1} & r < n \end{cases}$ so that $g'_{n,n-1} = (\overline{f_{n-1}}' \cup C1_{n-1,n-2}) \circ \omega'^{-1}_{n-1,n-2} \simeq g_{n,n-1}$. Hence $\overline{f_n}' \circ g'_{n,n-1} \simeq \overline{f_n} \circ g_{n,n-1}$. This proves the first equality in (2).

Secondly we prove the second equality in (2). Let $\alpha \in \{\vec{\boldsymbol{f}}\}^{(aq\dot{s}_2)}$ and $\{\mathcal{S}_r, \overline{f_r}, \Omega_r \,|\, 2 \leq r \leq n\}$ an $aq\dot{s}_2$-presentation of $\vec{\boldsymbol{f}}$ with $\alpha = \overline{f_n} \circ g_{n,n-1}$. Set $\mathcal{S}'_2 = \mathcal{S}_2$, $\omega'_{2,1} = q'_{f_1}$, $\Omega'_2 = \{\omega'_{2,1}\}$, and $\theta = \omega_{2,1} \circ \omega'^{-1}_{2,1} \in \mathcal{E}(\Sigma X_1)$. By Remark 5.5(3), we define inductively $\mathcal{S}'_3 = \mathcal{S}'_2(\overline{f_2}, \Omega'_2)$, $\Omega'_3 = \widetilde{\Omega'_2}$; $\ldots$; $\mathcal{S}'_n = \mathcal{S}'_{n-1}(\overline{f_{n-1}}, \Omega'_{n-1})$, $\Omega'_n = \widetilde{\Omega'_{n-1}}$. Then $\{\mathcal{S}'_r, \overline{f_r}, \Omega'_r \,|\, 2 \leq r \leq n\}$ is an $aq\ddot{s}_2$-presentation of $\vec{\boldsymbol{f}}$ such that $\omega'_{r,s} = \omega_{r,s}$ for $1 \leq s \leq r-2$, and $\Sigma^{r-2}\theta \circ \omega'_{r,r-1} = \omega_{r,r-1}$. Hence $\alpha \circ \Sigma^{n-3}\theta = \overline{f_n} \circ g'_{n,n-1} \in \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)}$ and so $\alpha \in \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \circ \Sigma^{n-3}\theta^{-1} \subset \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \circ \Sigma^{n-3}\mathcal{E}(\Sigma X_1)$. Thus $\{\vec{\boldsymbol{f}}\}^{(aq\dot{s}_2)} \subset \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \circ \Sigma^{n-3}\mathcal{E}(\Sigma X_1)$.

Conversely let $\alpha \in \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)}$ and $\theta \in \mathcal{E}(\Sigma X_1)$. Let $\{\mathcal{S}'_r, \overline{f_r}', \Omega'_r \,|\, 2 \leq r \leq n\}$ be an $aq\ddot{s}_2$-presentation of $\vec{\boldsymbol{f}}$ with $\alpha = \overline{f_n}' \circ g'_{n,n-1}$. Let $\mathcal{S}_r$ be the iterated mapping cone which is obtained from $\mathcal{S}'_r$ by replacing $g'_{r,r-1}$ with $g'_{r,r-1} \circ \Sigma^{r-3}\theta$, and Ω_r the quasi-structure on $\mathcal{S}_r$ which is obtained from Ω'_r by replacing $\omega'_{r,r-1}$ with $\Sigma^{r-2}\theta^{-1} \circ \omega'_{r,r-1}$. Then $\{\mathcal{S}_r, \overline{f_r}', \Omega_r \,|\, 2 \leq r \leq n\}$ is an $aq\dot{s}_2$-presentation of $\vec{\boldsymbol{f}}$ such that $g_{r,s} = g'_{r,s}$ for $1 \leq s \leq r-2$ and $g_{r,r-1} = g'_{r,r-1} \circ \Sigma^{r-3}\theta$. Hence $\alpha \circ \Sigma^{n-3}\theta = \overline{f_n}' \circ g'_{n,n-1} \circ \Sigma^{n-3}\theta = \overline{f_n}' \circ g_{n,n-1} \in \{\vec{\boldsymbol{f}}\}^{(aq\dot{s}_2)}$. Thus $\{\vec{\boldsymbol{f}}\}^{(aq\dot{s}_2)} \supset \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \circ \Sigma^{n-3}\mathcal{E}(\Sigma X_1)$. This proves the second equality in (2).

Thirdly we prove the third and fourth equalities in (2). We prove

$$\{\vec{\boldsymbol{f}}\}^{(qs_2)} \subset \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \circ \mathcal{E}(\Sigma^{n-2}X_1), \tag{6.2.4}$$

$$\text{(6.2.5)} \qquad \{\vec{\boldsymbol f}\}^{(aq\ddot{s}_2)} \circ \mathcal{E}(\Sigma^{n-2}X_1) \subset \begin{cases} \{\vec{\boldsymbol f}\}^{(q\dot{s}_2)} & n=3 \\ \{\vec{\boldsymbol f}\}^{(q\ddot{s}_2)} & n \geq 4 \end{cases}.$$

If these are proved, then

$$\text{(6.2.6)} \qquad \begin{aligned} \{\vec{\boldsymbol f}\}^{(qs_2)} &\subset \{\vec{\boldsymbol f}\}^{(aq\ddot{s}_2)} \circ \mathcal{E}(\Sigma^{n-2}X_1) \\ &\subset \begin{cases} \{\vec{\boldsymbol f}\}^{(q\dot{s}_2)} \subset \{\vec{\boldsymbol f}\}^{(qs_2)} & n=3 \\ \{\vec{\boldsymbol f}\}^{(q\ddot{s}_2)} \subset \{\vec{\boldsymbol f}\}^{(q\dot{s}_2)} \subset \{\vec{\boldsymbol f}\}^{(qs_2)} & n \geq 4 \end{cases} \end{aligned}$$

so that the third and fourth equalities in (2) follow. To prove (6.2.4), let $\alpha \in \{\vec{\boldsymbol f}\}^{(qs_2)}$ and $\{\mathcal{S}_r, \overline{f_r}, \Omega_r \,|\, 2 \leq r \leq n\}$ a qs_2-presentation of $\vec{\boldsymbol f}$ with $\alpha = \overline{f_n} \circ g_{n,n-1}$. Set

$$\mathcal{S}'_2 = (X_1; X_2, X_2 \cup_{f_1} CX_1; f_1; i_{f_1}), \quad j'_{2,1} = i_{f_1}, \quad \omega'_{2,1} = q'_{f_1}, \quad \Omega'_2 = \{\omega'_{2,1}\}.$$

Since $j_{2,1}$ is a homotopy cofibre of f_1 by the hypothesis, there exists a homotopy equivalence $e_2 : C'_{2,2} = X_2 \cup_{f_1} CX_1 \to C_{2,2}$ such that $e_2 \circ j'_{2,1} = j_{2,1}$. Set $\overline{f_2}' = \overline{f_2} \circ e_2$. Then $\overline{f_2}'$ is an extension of f_2 to $C'_{2,2}$. Set

$$\mathcal{S}'_3 = \mathcal{S}'_2(\overline{f_2}', \Omega'_2),\ \Omega'_3 = \widetilde{\Omega'_2},\ e_3 = 1_{X_3} \cup Ce_2 : C'_{3,3} \to C_{3,3},$$

$$\overline{f_3}' = \begin{cases} \overline{f_3} : C'_{3,2} = C_{3,2} \to X_4 & n=3 \\ \overline{f_3} \circ e_3 : C'_{3,3} \to X_4 & n \geq 4 \end{cases}.$$

Proceeding with the construction, we have an $aq\ddot{s}_2$-presentation $\{\mathcal{S}'_r, \overline{f_r}', \Omega'_r \,|\, 2 \leq r \leq n\}$ of $\vec{\boldsymbol f}$ such that

$$C'_{r,s} = C_{r,s}\ (1 \leq s \leq r-1),\ j'_{r,s} = j_{r,s}\ (1 \leq s \leq r-2),$$
$$e_r = 1_{X_r} \cup Ce_{r-1} : C'_{r,r} \to C_{r,r}\ (3 \leq r \leq n),$$
$$\overline{f_r}' = \begin{cases} \overline{f_r} \circ e_r : C'_{r,r} \to X_{r+1} & r<n \\ \overline{f_n} : C'_{n,n-1} = C_{n,n-1} \to X_{n+1} & r=n \end{cases}.$$

Set $\theta = \omega_{n-1,n-2} \circ (e_{n-1} \cup C1_{C_{n-1,n-2}}) \circ \omega'^{-1}_{n-1,n-2} : \Sigma^{n-2}X_1 \to \Sigma^{n-2}X_1$. Then θ is a homotopy equivalence and

$$\begin{aligned} \alpha &= \overline{f_n} \circ g_{n,n-1} = \overline{f_n} \circ (\overline{f_{n-1}} \cup C1_{C_{n-1,n-2}}) \circ \omega^{-1}_{n-1,n-2} \\ &= \overline{f_n} \circ (\overline{f_{n-1}} \cup C1_{C_{n-1,n-2}}) \circ (e_{n-1} \cup C1_{C_{n-1,n-2}}) \circ \omega'^{-1}_{n-1,n-2} \circ \theta^{-1} \\ &= \overline{f_n}' \circ (\overline{f_{n-1}}' \cup C1_{n-1,n-2}) \circ \omega'^{-1}_{n-1,n-2} \circ \theta^{-1} \\ &\in \{\vec{\boldsymbol f}\}^{(aq\ddot{s}_2)} \circ \theta^{-1} \subset \{\vec{\boldsymbol f}\}^{(aq\ddot{s}_2)} \circ \mathcal{E}(\Sigma^{n-2}X_1). \end{aligned}$$

This proves (6.2.4).

To prove (6.2.5), let $\alpha \in \{\vec{\boldsymbol f}\}^{(aq\ddot{s}_2)}$, $\varepsilon \in \mathcal{E}(\Sigma^{n-2}X_1)$, and $\{\mathcal{S}_r, \overline{f_r}, \Omega_r | 2 \leq r \leq n\}$ an $aq\ddot{s}_2$-presentation of $\vec{\boldsymbol f}$ such that $\alpha = \overline{f_n} \circ g_{n,n-1}$. Let $\{\mathcal{S}'_r, \overline{f_r}', \Omega'_r \,|$

$2 \leq r \leq n\}$ be obtained from $\{\mathcal{S}_r, \overline{f_r}, \Omega_r | 2 \leq r \leq n\}$ by replacing $\omega_{n-1,n-2}$ and $\mathcal{S}_n$ with $\varepsilon^{-1} \circ \omega_{n-1,n-2}$ and $\mathcal{S}'_n = \mathcal{S}_{n-1}(\overline{f_{n-1}}, \Omega'_{n-1})$, respectively. Then $\{\mathcal{S}'_r, \overline{f_r}', \Omega'_r \,|\, 2 \leq r \leq n\}$ is a $q\dot{s}_2$-presentation of $\vec{\boldsymbol{f}}$ if $n = 3$ and a $q\ddot{s}_2$-presentation of $\vec{\boldsymbol{f}}$ if $n \geq 4$, and $g'_{n,n-1} = g_{n,n-1} \circ \varepsilon$ for $n \geq 3$. Hence $\alpha \circ \varepsilon = \overline{f_n} \circ g_{n,n-1} \circ \varepsilon = \overline{f_n}' \circ g'_{n,n-1} \in \begin{cases} \{\vec{\boldsymbol{f}}\}^{(q\dot{s}_2)} & n = 3 \\ \{\vec{\boldsymbol{f}}\}^{(q\ddot{s}_2)} & n \geq 4 \end{cases}$. This proves (6.2.5).

Fourthly it follows from (6.2.6) that

$$\{\vec{\boldsymbol{f}}\}^{(q\dot{s}_2)} = \begin{cases} \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \circ \mathcal{E}(\Sigma X_1) & n = 3 \\ \{\vec{\boldsymbol{f}}\}^{(q\ddot{s}_2)} & n \geq 4 \end{cases}.$$

By definition, we have $\{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} = \{\vec{\boldsymbol{f}}\}^{(q\ddot{s}_2)}$ for $n = 3$. Hence we obtain the fifth equality in (2). □

Proof of Theorem 6.2.1(3). First we prove $\{\vec{\boldsymbol{f}}\}^{(q)} \subset \{\vec{\boldsymbol{f}}\}^{(aq)} \circ \mathcal{E}(\Sigma^{n-2} X_1)$. Let $\alpha \in \{\vec{\boldsymbol{f}}\}^{(q)}$ and $\{\mathcal{S}_r, \overline{f_r}, \Omega_r \,|\, 2 \leq r \leq n\}$ a q-presentation of $\vec{\boldsymbol{f}}$ with $\alpha = \overline{f_n} \circ g_{n,n-1}$. We define inductively

$$\mathcal{S}'_2 = \mathcal{S}_2,\ \Omega'_2 = \Omega_2;\ \mathcal{S}'_{k+1} = \mathcal{S}'_k(\overline{f_k}, \Omega'_k),\ \Omega'_{k+1} = \widetilde{\Omega'_k}\ (2 \leq k < n).$$

By Remark 5.5(3), this definition is possible and $\mathcal{S}'_r, \mathcal{S}_r$ have the same edge. Then $\{\mathcal{S}'_r, \overline{f_r}, \Omega'_r \,|\, 2 \leq r \leq n\}$ is an aq-presentation of $\vec{\boldsymbol{f}}$ and

$$\begin{aligned} \{\vec{\boldsymbol{f}}\}^{(aq)} \ni \overline{f_n} \circ g'_{n,n-1} &= \overline{f_n} \circ (\overline{f_{n-1}} \cup C1_{C_{n-1,n-2}}) \circ \omega'^{-1}_{n-1,n-2} \\ &= \overline{f_n} \circ (\overline{f_{n-1}} \cup C1_{C_{n-1,n-2}}) \circ \omega^{-1}_{n-1,n-2} \circ \varepsilon_0 = \alpha \circ \varepsilon_0 \end{aligned}$$

where $\varepsilon_0 = \omega_{n-1,n-2} \circ \omega'^{-1}_{n-1,n-2} \in \mathcal{E}(\Sigma^{n-2} X_1)$. Hence

$$\alpha \in \{\vec{\boldsymbol{f}}\}^{(aq)} \circ \varepsilon_0^{-1} \subset \{\vec{\boldsymbol{f}}\}^{(aq)} \circ \mathcal{E}(\Sigma^{n-2} X_1)$$

and so $\{\vec{\boldsymbol{f}}\}^{(q)} \subset \{\vec{\boldsymbol{f}}\}^{(aq)} \circ \mathcal{E}(\Sigma^{n-2} X_1)$.

Secondly we prove $\{\vec{\boldsymbol{f}}\}^{(q)} \supset \{\vec{\boldsymbol{f}}\}^{(aq)} \circ \mathcal{E}(\Sigma^{n-2} X_1)$. Let $\alpha \in \{\vec{\boldsymbol{f}}\}^{(aq)}$, $\varepsilon \in \mathcal{E}(\Sigma^{n-2} X_1)$, and $\{\mathcal{S}_r, \overline{f_r}, \Omega_r \,|\, 2 \leq r \leq n\}$ an aq-presentation of $\vec{\boldsymbol{f}}$ with $\alpha = \overline{f_n} \circ g_{n,n-1}$. We set

$$\Omega'_{n-1} = \begin{cases} \{\varepsilon^{-1} \circ \omega_{2,1}\} & n = 3 \\ \{\omega_{n-1,s},\ \varepsilon^{-1} \circ \omega_{n-1,n-2} \,|\, 1 \leq s \leq n-3\} & n \geq 4 \end{cases}$$

which is a quasi-structure on $\mathcal{S}_{n-1}$. Set $\mathcal{S}'_n = \mathcal{S}_{n-1}(\overline{f_{n-1}}, \Omega'_{n-1})$. Since $\mathcal{S}'_n$ is obtained from $\mathcal{S}_n$ by replacing $g_{n,n-1}$ with $g_{n,n-1} \circ \varepsilon$, it follows that

$$\{\mathcal{S}_r, \overline{f_r}, \Omega_r \,|\, 2 \leq r \leq n-2\} \cup \{\mathcal{S}_{n-1}, \overline{f_{n-1}}, \Omega'_{n-1}, \mathcal{S}'_n, \overline{f_n}, \Omega_n\}$$

is a q-presentation of $\vec{\boldsymbol{f}}$ and it represents $\alpha\circ\varepsilon$. Hence $\alpha\circ\varepsilon\in\{\vec{\boldsymbol{f}}\}^{(q)}$. Thus $\{\vec{\boldsymbol{f}}\}^{(q)}\supset\{\vec{\boldsymbol{f}}\}^{(aq)}\circ\mathcal{E}(\Sigma^{n-2}X_1)$.

Therefore $\{\vec{\boldsymbol{f}}\}^{(q)}=\{\vec{\boldsymbol{f}}\}^{(aq)}\circ\mathcal{E}(\Sigma^{n-2}X_1)$. □

Proof of Theorem 6.2.1(4). We prove

$$\{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)}\subset\{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)}\circ\mathcal{E}(\Sigma^{n-2}X_1), \tag{6.2.7}$$

$$\{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)}\circ\mathcal{E}(\Sigma^{n-2}X_1)\supset\{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)}. \tag{6.2.8}$$

If these are done, then, by applying $\mathcal{E}(\Sigma^{n-2}X_1)$ to them from the right, we have the equality. To prove (6.2.7), let $\alpha\in\{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)}$ and $\{\mathcal{S}_r,\overline{f_r},\mathcal{A}_r\,|\,2\le r\le n\}$ an $\ddot{s}_t$-presentation of $\vec{\boldsymbol{f}}$ with $\alpha=\overline{f_n}\circ g_{n,n-1}$. We define inductively

$$\mathcal{S}'_2=\mathcal{S}_2,\Omega'_2=\Omega(\mathcal{A}_2);\ \mathcal{S}'_{k+1}=\mathcal{S}'_k(\overline{f_k},\Omega'_k),\ \Omega'_{k+1}=\widetilde{\Omega'_k}\ (2\le k<n).$$

By Remark 5.5(3), this definition is possible, and $\mathcal{S}'_r,\mathcal{S}_r$ have the same edge. Then $\{\mathcal{S}'_r,\overline{f_r},\Omega'_r\,|\,2\le r\le n\}$ is an $aq\ddot{s}_2$-presentation of $\vec{\boldsymbol{f}}$ and

$$\begin{aligned}\{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)}\ni\overline{f_n}\circ g'_{n,n-1}&=\overline{f_n}\circ(\overline{f_{n-1}}\cup C1_{C_{n-1,n-2}})\circ\omega'^{-1}_{n-1,n-2}\\&=\overline{f_n}\circ(\overline{f_{n-1}}\cup C1_{C_{n-1,n-2}})\circ\omega^{-1}_{n-1,n-2}\circ\varepsilon_0=\alpha\circ\varepsilon_0,\end{aligned}$$

where $\varepsilon_0=\omega_{n-1,n-2}\circ\omega'^{-1}_{n-1,n-2}\in\mathcal{E}(\Sigma^{n-2}X_1)$. Hence

$$\alpha\in\{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)}\circ\varepsilon_0^{-1}\subset\{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)}\circ\mathcal{E}(\Sigma^{n-2}X_1).$$

This proves (6.2.7).

To prove (6.2.8), let $\alpha\in\{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)}$ and $\{\mathcal{S}_r,\overline{f_r},\Omega_r\,|\,2\le r\le n\}$ an $aq\ddot{s}_2$-presentation of $\vec{\boldsymbol{f}}$ with $\alpha=\overline{f_n}\circ g_{n,n-1}$. Set $\mathcal{S}'_2=\mathcal{S}_2$ and $\mathcal{A}'_2=\{1_{C_{2,2}}\}$. We define inductively $\mathcal{S}'_{r+1}=\mathcal{S}'_r(\overline{f_r},\mathcal{A}'_r)$ and $\mathcal{A}'_{r+1}$ is a reduced structure on $\mathcal{S}'_{r+1}$ for $r\ge 2$. By Remark 5.5(3), this definition is possible, and $\mathcal{S}'_r,\mathcal{S}_r$ have the same edge. Then $\{\mathcal{S}'_r,\overline{f_r},\mathcal{A}'_r\,|\,2\le r\le n\}$ is an $\ddot{s}_t$-presentation of $\vec{\boldsymbol{f}}$ and

$$\begin{aligned}\{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)}\ni\overline{f_n}\circ g'_{n,n-1}&=\overline{f_n}\circ(\overline{f_{n-1}}\cup C1_{C_{n-1,n-2}})\circ\omega'^{-1}_{n-1,n-2}\\&=\overline{f_n}\circ(\overline{f_{n-1}}\cup C1_{C_{n-1,n-2}})\circ\omega^{-1}_{n-1,n-2}\circ\varepsilon_0=\alpha\circ\varepsilon_0,\end{aligned}$$

where $\Omega(\mathcal{A}'_{n-1})=\{\omega'_{n-1,s}\,|\,1\le s<n-1\}$ and $\varepsilon_0=\omega_{n-1,n-2}\circ\omega'^{-1}_{n-1,n-2}\in\mathcal{E}(\Sigma^{n-2}X_1)$. Hence $\alpha\in\{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)}\circ\varepsilon_0^{-1}\subset\{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)}\circ\mathcal{E}(\Sigma^{n-2}X_1)$. This proves (6.2.8) and completes the proof of Theorem 6.2.1(4). □

Proof of Theorem 6.2.1(5). Let $\alpha\in\{\vec{\boldsymbol{f}}\}^{(q)}$ and $\{\mathcal{S}_r,\overline{f_r},\Omega_r\,|\,2\le r\le n\}$ a q-presentation of $\vec{\boldsymbol{f}}$ with $\alpha=\overline{f_n}\circ g_{n,n-1}$. We are going to define an $aq\ddot{s}_2$-presentation $\{\mathcal{S}'_r,\overline{f_r}',\Omega'_r\,|\,2\le r\le n\}$ of $\vec{\boldsymbol{f}}$ such that $\overline{f_n}'\circ g'_{n,n-1}=\alpha\circ\theta$

for some map $\theta : \Sigma^{n-2}X_1 \to \Sigma^{n-2}X_1$ (notice that θ is not necessarily a homotopy equivalence). Now set

$$\mathcal{S}'_2 = (X_1; X_2, X_2 \cup_{f_1} CX_1; f_1; i_{f_1}), \quad \Omega'_2 = \{q'_{f_1}\}.$$

Since $\mathcal{S}_2$ is a quasi iterated mapping cone and $j'_{2,1} = i_{f_1}$ is a cofibration, there exists a map (not necessarily a homotopy equivalence) $e_2 : C'_{2,2} \to C_{2,2}$ such that $e_2 \circ j'_{2,1} = j_{2,1}$. Set $\overline{f_2}' = \overline{f_2} \circ e_2$. Then $\overline{f_2}' \circ j'_{2,1} = f_2$ and so $\overline{f_2}'$ is an extension of f_2 to $C'_{2,2}$. Set

$$\mathcal{S}'_3 = \mathcal{S}'_2(\overline{f_2}', \Omega'_2),\ \Omega'_3 = \widetilde{\Omega'_2},\ e_3 = 1_{X_3} \cup Ce_2 : C'_{3,3} \to C_{3,3},$$
$$\overline{f_3}' = \begin{cases} \overline{f_3} : C'_{3,2} = C_{3,2} \to X_4 & n = 3 \\ \overline{f_3} \circ e_3 : C'_{3,3} \to X_4 & n \geq 4 \end{cases}.$$

Proceeding with the construction, we have an $aq\ddot{s}_2$-presentation $\{\mathcal{S}'_r, \overline{f_r}', \Omega'_r \mid 2 \leq r \leq n\}$ of $\vec{\boldsymbol{f}}$ and maps $e_r : C'_{r,r} \to C_{r,r}$ $(2 \leq r \leq n-1)$ such that

$$C'_{r,s} = C_{r,s}\ (1 \leq s \leq r-1),\ j'_{r,s} = j_{r,s}\ (1 \leq s \leq r-2),$$
$$e_r \circ j'_{r,r-1} = j_{r,r-1}\ (2 \leq r \leq n-1),$$
$$\overline{f_r}' = \begin{cases} \overline{f_n} : C'_{n,n-1} = C_{n,n-1} \to X_{n+1} & r = n \\ \overline{f_r} \circ e_r : C'_{r,r} \to X_{r+1} & r < n \end{cases}.$$

Set $\theta = \omega_{n-1,n-2} \circ (e_{n-1} \cup C1_{C'_{n-1,n-2}}) \circ \omega'^{-1}_{n-1,n-2} : \Sigma^{n-2}X_1 \to \Sigma^{n-2}X_1$. Then $\omega^{-1}_{n-1,n-2} \circ \theta = (e_{n-1} \cup C1_{C'_{n-1,n-2}}) \circ \omega'^{-1}_{n-1,n-2}$ and

$$\begin{aligned}
\{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \ni \overline{f_n}' \circ g'_{n,n-1} &= \overline{f_n}' \circ (\overline{f_{n-1}}' \cup C1_{C_{n-1,n-2}}) \circ \omega'^{-1}_{n-1,n-2} \\
&= \overline{f_n} \circ (\overline{f_{n-1}} \cup C1_{C_{n-1,n-2}}) \circ (e_{n-1} \cup C1_{C_{n-1,n-2}}) \circ \omega'^{-1}_{n-1,n-2} \\
&= \overline{f_n} \circ (\overline{f_{n-1}} \cup C1_{C_{n-1,n-2}}) \circ \omega^{-1}_{n-1,n-2} \circ \theta \\
&= \alpha \circ \theta.
\end{aligned}$$

Since $\{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \subset \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \circ \mathcal{E}(\Sigma^{n-2}X_1) = \{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)} \circ \mathcal{E}(\Sigma^{n-2}X_1)$ by (4), we have $\alpha \circ \theta = \beta \circ \gamma$ for some $\beta \in \{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)}$ and $\gamma \in \mathcal{E}(\Sigma^{n-2}X_1)$. Set $\theta' = \theta \circ \gamma^{-1}$. Then $\alpha \circ \theta' = \beta \in \{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)}$. □

Proof of Corollary 6.2.2. To prove (1), let $\varepsilon \in \mathcal{E}(\Sigma^{n-2}X_1)$. By composing ε from the right to equalities $\{\vec{\boldsymbol{f}}\}^{(q)} = \{\vec{\boldsymbol{f}}\}^{(aq)} \circ \mathcal{E}(\Sigma^{n-2}X_1)$ in Theorem 6.2.1(3) and $\{\vec{\boldsymbol{f}}\}^{(qs_2)} = \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \circ \mathcal{E}(\Sigma^{n-2}X_1)$ in Theorem 6.2.1(2), we have $\{\vec{\boldsymbol{f}}\}^{(q)} \circ \varepsilon = \{\vec{\boldsymbol{f}}\}^{(q)}$ and $\{\vec{\boldsymbol{f}}\}^{(qs_2)} \circ \varepsilon = \{\vec{\boldsymbol{f}}\}^{(qs_2)}$. Let $\gamma \in \mathcal{E}(\Sigma X_1)$. By composing $\Sigma^{n-3}\gamma$ from the right to equality $\{\vec{\boldsymbol{f}}\}^{(aqs_2)} = \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \circ \Sigma^{n-3}\mathcal{E}(\Sigma X_1)$ in Theorem 6.2.1(2), we have $\{\vec{\boldsymbol{f}}\}^{(aqs_2)} \circ \Sigma^{n-3}\gamma = \{\vec{\boldsymbol{f}}\}^{(aqs_2)}$.

To prove (2), let $\alpha \in \{\vec{\boldsymbol f}\}^{(aq)}$ and $\{\mathcal{S}_r, \overline{f_r}, \Omega_r \,|\, 2 \le r \le n\}$ an aq-presentation of $\vec{\boldsymbol f}$ with $\alpha = \overline{f_n} \circ g_{n,n-1}$. For the first equality in (2), let $\gamma \in \mathcal{E}(X_1)$, and set

$$\omega'_{r,s} = \begin{cases} \omega_{r,s} & 1 \le s \le r-2 \\ \Sigma^{r-1}\gamma \circ \omega_{r,r-1} & 1 \le s = r-1 \end{cases}.$$

Then $\Omega'_r = \{\omega'_{r,s} \,|\, 1 \le s < r\}$ is a quasi-structure on $\mathcal{S}_r$ and $\Omega'_{r+1} = \widetilde{\Omega'_r}$. For $r \ge 2$ and $1 \le s < r < n$, we set $g'_{r+1,s+1} = \begin{cases} g_{r+1,s+1} & s \le r-2 \\ g_{r+1,r} \circ \Sigma^{r-1}\gamma^{-1} & s = r-1 \end{cases}$, that is,

$$g'_{r+1,s+1} = \begin{cases} (\overline{f_r}^{s+1} \cup C1_{C_{r,s}}) \circ \omega_{r,s}^{-1} & s \le r-2 \\ (\overline{f_r} \cup C1_{C_{r,r-1}}) \circ \omega_{r,r-1}^{-1} \circ \Sigma^{r-1}\gamma^{-1} & s = r-1 \end{cases}.$$

Set $\mathcal{S}'_2 = \mathcal{S}_2$ and let $\mathcal{S}'_{r+1}$ be the iterated mapping cone obtained from $\mathcal{S}_{r+1}$ by replacing $g_{r+1,r}$ with $g'_{r+1,r}$ for $2 \le r \le n-1$. Then $\mathcal{S}'_r$ and $\mathcal{S}_r$ have the same edge, and $\{\mathcal{S}'_r, \overline{f_r}, \Omega'_r \,|\, 2 \le r \le n\}$ is an aq-presentation of $\vec{\boldsymbol f}$ such that

$$\overline{f_n} \circ g'_{n,n-1} = \overline{f_n} \circ g_{n,n-1} \circ \Sigma^{n-2}\gamma^{-1} = \alpha \circ \Sigma^{n-2}\gamma^{-1}.$$

Hence $\{\vec{\boldsymbol f}\}^{(aq)} \circ \Sigma^{n-2}\gamma^{-1} \subset \{\vec{\boldsymbol f}\}^{(aq)}$ and so $\{\vec{\boldsymbol f}\}^{(aq)} \subset \{\vec{\boldsymbol f}\}^{(aq)} \circ \Sigma^{n-2}\gamma$. By taking γ^{-1} instead of γ, we have $\{\vec{\boldsymbol f}\}^{(aq)} \circ \Sigma^{n-2}\gamma \subset \{\vec{\boldsymbol f}\}^{(aq)}$. Therefore we obtain the first equality in (2). For the second equality in (2), set $\omega^*_{2,1} = (-1_{\Sigma X_1}) \circ \omega_{2,1}$, $\omega^*_{3,2} = \Sigma(-1_{\Sigma X_1}) \circ \omega_{3,2}, \dots, \omega^*_{n,n-1} = \Sigma^{n-2}(-1_{\Sigma X_1}) \circ \omega_{n,n-1}$, $\Omega^*_2 = \{\omega^*_{2,1}\}$, and $\Omega^*_r = \{\omega_{r,1}, \dots, \omega_{r,r-2}, \omega^*_{r,r-1}\}$ for $3 \le r \le n$. Set $\mathcal{S}^*_2 = \mathcal{S}_2$ and, for $r \ge 3$, let $\mathcal{S}^*_r$ be the iterated mapping cone obtained from $\mathcal{S}_r$ by replacing $g_{r,r-1}$ with $g^*_{r,r-1} = g_{r,r-1} \circ \Sigma^{r-3}(-1_{\Sigma X_1})$. Then $\{\mathcal{S}^*_r, \overline{f_r}, \Omega^*_r \,|\, 2 \le r \le n\}$ is an aq-presentation of $\vec{\boldsymbol f}$ by Lemma 4.3(3) such that $-\alpha = \overline{f_n} \circ g^*_{n,n-1} \in \{\vec{\boldsymbol f}\}^{(aq)}$. Hence $-\{\vec{\boldsymbol f}\}^{(aq)} \subset \{\vec{\boldsymbol f}\}^{(aq)}$. By composing $-1_{\Sigma^{n-2}X_1}$ from the right to the last relation, we have $\{\vec{\boldsymbol f}\}^{(aq)} \subset -\{\vec{\boldsymbol f}\}^{(aq)}$. Therefore $-\{\vec{\boldsymbol f}\}^{(aq)} = \{\vec{\boldsymbol f}\}^{(aq)}$.

The assertion (3) follows from (2) and Theorem 6.2.1(3).

If $\{\vec{\boldsymbol f}\}^{(\star)}$ is not empty for some $\star$, then $\{\vec{\boldsymbol f}\}^{(q)}$ is not empty by (6.1.1) so that $\{\vec{\boldsymbol f}\}^{(aq\ddot{s}_2)}$ and $\{\vec{\boldsymbol f}\}^{(\ddot{s}_t)}$ are not empty by Theorem 6.2.1(5), and so $\{\vec{\boldsymbol f}\}^{(\star)}$ is not empty for every $\star$ by (6.1.1) and Theorem 6.2.1. This proves (4).

If $\{\vec{\boldsymbol f}\}^{(\star)}$ contains 0 for some $\star$, then $\{\vec{\boldsymbol f}\}^{(q)}$ contains 0 by (6.1.1), and so $\{\vec{\boldsymbol f}\}^{(\star)}$ contains 0 for every $\star$ by Theorem 6.2.1(5) and (6.1.1). This proves (5).

By setting $\varepsilon = -1_{\Sigma^{n-2}X_1}$ and $\gamma = -1_{\Sigma X_1}$ in (1), we have (6) for $\star = q, qs_2, aqs_2$. The assertion (6) for $\star = aq$ was proved in (2).

Suppose that $n \geq 4$ and $\vec{\boldsymbol{f}}$ is $\star$-presentable for some $\star$. Then $\vec{\boldsymbol{f}}$ is $\star$-presentable for all $\star$ by (4) so that in particular it is $aq\ddot{s}_2$-presentable and so it is admissible by definitions and (4.2). This proves (7).

We have (8) from (6.1.1) and Theorem 6.2.1(1),(2),(4). $\square$

Proof of Proposition 6.2.3. (6.2.1) is easily obtained from definitions.

About (6.2.2), it suffices to prove it for $\star = aq\ddot{s}_2, \ddot{s}_t, aq, q_2$ by Theorem 6.2.1(1)-(3). We prove (6.2.2) for $\star = aq\ddot{s}_2$, because other cases can be proved similarly. Let $\alpha \in \{f_{n+1}\circ f_n, f_{n-1}, \dots, f_1\}^{(aq\ddot{s}_2)}$ and $\{\mathcal{S}_r, \overline{f_r}, \Omega_r \,|\, 2 \leq r < n\} \cup \{\mathcal{S}_n, \overline{f_{n+1}\circ f_n}, \Omega_n\}$ an $aq\ddot{s}_2$-presentation of $(f_{n+1}\circ f_n, f_{n-1}, \dots, f_1)$ such that $\alpha = \overline{f_{n+1}\circ f_n}\circ g_{n,n-1}$. Then $C_{n,r} = X_n \cup_{\overline{f_{n-1}}^{r-1}} CC_{n-1,r-1}$ and $\overline{f_{n+1}\circ f_n}$ is an extension of $f_{n+1}\circ f_n$ to $C_{n,n-1}$. Set $\mathcal{S}'_n = \mathcal{S}_{n-1}(f_n\circ \overline{f_{n-1}}, \Omega_{n-1})$ and Ω'_n the typical quasi-structure on $\mathcal{S}'_n$. Then $C'_{n,r} = X_{n+1} \cup_{f_n\circ\overline{f_{n-1}}^{r-1}} CC_{n-1,r-1}$. Define $\overline{f_n}' : C'_{n,n-1} = X_{n+1}\cup_{f_n\circ\overline{f_{n-1}}^{n-2}} CC_{n-1,n-2} \to X_{n+2}$ by $\overline{f_n}'|_{X_{n+1}} = f_{n+1}$ and $\overline{f_n}'|_{CC_{n-1,n-2}} = \overline{f_{n+1}\circ f_n}|_{CC_{n-1,n-2}}$. The map $\overline{f_n}'$ is a well-defined extension of f_{n+1} to $C'_{n,n-1}$ and $\overline{f_n}'\circ(f_n\cup C1_{C_{n-1,n-2}}) = \overline{f_{n+1}\circ f_n}$. Hence $\{\mathcal{S}_r, \overline{f_r}, \Omega_r | r \leq n-2\}\cup\{\mathcal{S}_{n-1}, f_n\circ\overline{f_{n-1}}, \Omega_{n-1}\}\cup\{\mathcal{S}'_n, \overline{f_n}', \Omega'_n\}$ is an $aq\ddot{s}_2$-presentation of $(f_{n+1}, f_n\circ f_{n-1}, \dots, f_1)$ and it represents $\overline{f_n}'\circ(f_n\cup C1_{C_{n-1,n-2}})\circ g_{n,n-1} = \alpha$ so that $\alpha \in \{f_{n+1}, f_n\circ f_{n-1}, \dots, f_1\}^{(aq\ddot{s}_2)}$. This proves (6.2.2) for $\star = aq\ddot{s}_2$.

To prove the first containment of (6.2.3), let $\alpha \in \{f_n, \dots, f_1\}^{(aq\ddot{s}_2)}$ and $\{\mathcal{S}_r, \overline{f_r}, \Omega_r \,|\, 2 \leq r \leq n\}$ an $aq\ddot{s}_2$-presentation of $(f_n, \dots, f_1)$ with $\overline{f_n}\circ g_{n,n-1} = \alpha$. We are going to construct an $aq\ddot{s}_2$-presentation $\{\mathcal{S}'_r, \overline{f_r}', \Omega'_r \,|\, 2 \leq r \leq n\}$ of $(f_n, \dots, f_2, f_1\circ f_0)$ with $\overline{f_n}'\circ g'_{n,n-1} = \alpha\circ\Sigma^{n-2}f_0$. We set

$$X'_1 = X_0,\ X'_k = X_k\ (2 \leq k \leq n+1),\ f'_1 = f_1\circ f_0,\ f'_k = f_k\ (2 \leq k \leq n),$$
$$e_{1,0} = 1_{\{*\}} : C'_{1,0} \to C_{1,0},\ e_{1,1} = f_0 : C'_{1,1} = X_0 \to C_{1,1} = X_1,$$
$$\mathcal{S}'_2 = (X'_1; X'_2, X'_2\cup_{f'_1} CX'_1; f'_1; i_{f'_1}),\ \Omega'_2 = \{q'_{f'_1}\},$$
$$e_{2,0} = 1_{\{*\}} : C'_{2,0} \to C_{2,0},\ e_{2,s} = 1_{X_2}\cup Ce_{1,s-1} : C'_{2,s} \to C_{2,s}\ (s = 1, 2),$$
$$\overline{f_2}' = \overline{f_2}\circ e_{2,2} : C'_{2,2} \to X_3,\ \mathcal{S}'_3 = \mathcal{S}'_2(\overline{f_2}', \Omega'_2),\ \Omega'_3 = \widetilde{\Omega'_2}.$$

Then $C'_{3,s} = C_{3,s}$ for $s = 0, 1, 2$. We set

$$e_{3,0} = 1_{\{*\}} : C'_{3,0} \to C_{3,0},\ e_{3,s} = 1_{X_3}\cup Ce_{2,s-1} : C'_{3,s} \to C_{3,s}\ (1 \leq s \leq 3),$$
$$\overline{f_3}' = \begin{cases} \overline{f_3}\circ e_{3,2} : C'_{3,2} \to X_4 & n = 3 \\ \overline{f_3}\circ e_{3,3} : C'_{3,3} \to X_4 & n \geq 4 \end{cases}.$$

Then $e_{3,s} = 1_{C_{3,s}}$ for $0 \leq s \leq 2$ and $g_{3,2}\circ\Sigma f_0 \simeq g'_{3,2}$. By repeating the process, we have an $aq\ddot{s}_2$-presentation $\{\mathcal{S}'_r, \overline{f_r}', \Omega_r \,|\, 2 \leq r \leq n\}$

of $(f_n, \dots, f_2, f_1 \circ f_0)$ such that $C'_{r,s} = C_{r,s}$ $(1 \leq s < r)$, $\overline{f_n}' = \overline{f_n}$ and $g_{n,n-1} \circ \Sigma^{n-2} f_0 \simeq g'_{n,n-1}$ so that $\overline{f_n} \circ g_{n,n-1} \circ \Sigma^{n-2} f_0 \simeq \overline{f_n}' \circ g'_{n,n-1}$. Hence $\alpha \circ \Sigma^{n-2} f_0 \in \{f_n, \dots, f_2, f_1 \circ f_0\}^{(aq\ddot{s}_2)}$. This proves the first containment of (6.2.3).

In the rest of the proof we prove the second containment of (6.2.3). Let $\alpha \in \{f_n, \dots, f_2, f_1 \circ f_0\}^{(aq\ddot{s}_2)}$ and $\{\mathcal{S}_r, \overline{f_r}, \Omega_r \,|\, 2 \leq r \leq n\}$ an $aq\ddot{s}_2$-presentation of $(f_n, \dots, f_2, f_1 \circ f_0)$ with $\overline{f_n} \circ g_{n,n-1} = \alpha$. Set

$$\begin{gathered}
X_1^* = X_0, \quad X_2^* = X_1, \quad f_1^* = f_0, \quad X_k^* = X_k \ (3 \leq k \leq n), \\
f_2^* = f_2 \circ f_1, \quad f_k^* = f_k \ (3 \leq k \leq n), \\
\mathcal{S}_2^* = (X_1^*; X_2^*, X_2^* \cup_{f_1^*} CX_1^*; f_1^*; i_{f_1^*}), \quad \Omega_2^* = \{q'_{f_1^*}\}, \\
e_{2,2} = f_1 \cup C1_{X_0} : C_{2,2}^* = X_1 \cup_{f_0} CX_0 \to C_{2,2} = X_2 \cup_{f_1 \circ f_0} CX_0, \\
e_{2,1} = f_1 : C_{2,1}^* \to C_{2,1}, \quad \overline{f_2}^* = \overline{f_2} \circ e_{2,2} : C_{2,2}^* \to X_3, \\
\mathcal{S}_3^* = \mathcal{S}_2^*(\overline{f_2}^*, \Omega_2^*), \quad \Omega_3^* = \widetilde{\Omega_2^*}, \quad e_{3,1} = 1_{X_3}, \\
e_{3,2} = 1_{X_3} \cup Cf_1 : C_{3,2}^* = X_3 \cup_{f_2 \circ f_1} CX_1 \to X_3 \cup_{f_2} CX_2 = C_{3,2}, \\
e_{3,3} = 1_{X_3} \cup Ce_{2,2} : C_{3,3}^* = X_3 \cup_{\overline{f_2}^*} CC_{2,2}^* \to X_3 \cup_{\overline{f_2}} CC_{2,2} = C_{3,3}, \\
\overline{f_3}^* = \begin{cases} \overline{f_3} \circ e_{3,2} : C_{3,2}^* \to X_4 & n = 3 \\ \overline{f_3} \circ e_{3,3} : C_{3,3}^* \to X_4 & n \geq 4 \end{cases}.
\end{gathered}$$

Then $e_{3,s+1} \circ j_{3,s}^* = j_{3,s} \circ e_{3,s}$ $(s = 1, 2)$ and $e_{3,2} \circ g_{3,2}^* \simeq g_{3,2}$. When $n = 3$, $\{\mathcal{S}_r^*, \overline{f_r}^*, \Omega_r^* \,|\, r = 2, 3\}$ is an $aq\ddot{s}_2$-presentation of $(f_3, f_2 \circ f_1, f_0)$ and $\overline{f_3} \circ g_{3,2} \simeq \overline{f_3}^* \circ g_{3,2}^*$ so that $\alpha \in \{f_3, f_2 \circ f_1, f_0\}^{(aq\ddot{s}_2)}$. Suppose $n \geq 4$. Set $\mathcal{S}_4^* = \mathcal{S}_3^*(\overline{f_3}^*, \Omega_3^*)$ and $\Omega_4^* = \widetilde{\Omega_3^*}$. Then $C_{4,s}^* = C_{4,s}$ $(s = 1, 2)$. Set

$$\begin{aligned}
&e_{4,s} = 1_{C_{4,s}} : C_{4,s}^* \to C_{4,s} \ (s = 1, 2), \\
&e_{4,s+1} = 1_{X_4} \cup Ce_{3,s} \\
&\qquad : C_{4,s+1}^* = X_4 \cup_{\overline{f_3}^{*s}} CC_{3,s}^* \to C_{4,s+1} = X_4 \cup_{\overline{f_3}^s} CC_{3,s} \ (s = 2, 3), \\
&\overline{f_4}^* = \begin{cases} \overline{f_4} \circ e_{4,3} : C_{4,3}^* \to X_5 & n = 4 \\ \overline{f_4} \circ e_{4,4} : C_{4,4}^* \to X_5 & n \geq 5 \end{cases}.
\end{aligned}$$

Then $e_{4,s+1} \circ j_{4,s}^* = j_{4,s} \circ e_{4,s}$ $(1 \leq s \leq 3)$ and $e_{4,3} \circ g_{4,3}^* \simeq g_{4,3}$. By repeating the process, we obtain $\{\mathcal{S}_r^*, \overline{f_r}^*, \Omega_r^* \,|\, 2 \leq r \leq n\}$ which is an $aq\ddot{s}_2$-presentation of $(f_n, \dots, f_2 \circ f_1, f_0)$ such that $e_{n,n-1} \circ g_{n,n-1}^* \simeq g_{n,n-1}$ and $\overline{f_n}^* = \overline{f_n} \circ e_{n,n-1}$ so that $\overline{f_n}^* \circ g_{n,n-1}^* \simeq \overline{f_n} \circ g_{n,n-1}$. This shows $\alpha \in \{f_n, \dots, f_2 \circ f_1, f_0\}^{(aq\ddot{s}_2)}$ and completes the proof of Proposition 6.2.3. □

6.3. **Suspension of higher Toda brackets.** Let $\vec{\boldsymbol{f}} = (f_n, \dots, f_1)$. Given $\ell \geq 1$, $\{\Sigma^\ell \vec{\boldsymbol{f}}\}^{(\star)}$ can be considered, since $\Sigma^\ell X_i$ is well-pointed for every i by Corollary 2.3(2). We prove the following which implies (1.5) (cf. [**7**, Lemma 2.3D], [**27**, Lemma 2.3]).

Theorem 6.3.1. (1) $\Sigma^\ell\{\vec{\boldsymbol{f}}\}^{(\star)} \subset (-1)^{\ell n}\{\Sigma^\ell \vec{\boldsymbol{f}}\}^{(\star)}$ *for all* $\ell \geq 1$ *and* $\star$.
(2) $\Sigma^\ell\{\vec{\boldsymbol{f}}\}^{(\star)} \subset \{\Sigma^\ell \vec{\boldsymbol{f}}\}^{(\star)}$ *for all* $\ell \geq 1$ *and* $\star = q$, qs_2, aq, aqs_2.

Proof. (2) follows from (1) and Corollary 6.2.2(6). We are going to prove (1). Note that

$$\Sigma^\ell\{\vec{\boldsymbol{f}}\}^{(\star)} \subset [\Sigma^\ell\Sigma^{n-2}X_1, \Sigma^\ell X_{n+1}],\ \{\Sigma^\ell\vec{\boldsymbol{f}}\}^{(\star)} \subset [\Sigma^{n-2}\Sigma^\ell X_1, \Sigma^\ell X_{n+1}],$$

where $\Sigma^\ell\Sigma^{n-2}X_1 = \Sigma^{n-2}\Sigma^\ell X_1$ by the identification (2.1). Hence (1) is equivalent to

$$\Sigma^\ell\{\vec{\boldsymbol{f}}\}^{(\star)} \subset \{\Sigma^\ell\vec{\boldsymbol{f}}\}^{(\star)} \circ (1_{X_1} \wedge \tau(\mathrm{S}^{n-2}, \mathrm{S}^\ell)), \tag{6.3.1}$$

where $\tau(\mathrm{S}^{n-2}, \mathrm{S}^\ell) : \mathrm{S}^{n-2+\ell} = \mathrm{S}^{n-2} \wedge \mathrm{S}^\ell \to \mathrm{S}^\ell \wedge \mathrm{S}^{n-2} = \mathrm{S}^{n-2+\ell}$ is the switching homeomorphism defined in (2.2). We prove (6.3.1) when $\ell = 1$, because (6.3.1) for $\ell \geq 2$ is obtained by an induction. By Theorem 6.2.1(1)-(4), it suffices to prove (6.3.1) for $\star = \ddot{s}_t$, $aq\ddot{s}_2$, aq, $q\ddot{s}_2$, q_2. We prove (6.3.1) for only the case of $\star = \ddot{s}_t$, because the cases of $\star = aq\ddot{s}_2$, aq, $q\ddot{s}_2$, q_2 can be treated similarly or more easily. Let $\alpha \in \{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)}$ and $\{\mathcal{S}_r, \overline{f_r}, \mathcal{A}_r \,|\, 2 \leq r \leq n\}$ an $\ddot{s}_t$-presentation of $\vec{\boldsymbol{f}}$ with $\alpha = \overline{f_n} \circ g_{n,n-1}$. Set $C^*_{r,1} = \Sigma X_r$ $(2 \leq r \leq n)$ and $f^*_r = \Sigma f_r : \Sigma X_r \to \Sigma X_{r+1}$ $(1 \leq r \leq n)$. We are going to construct an $\ddot{s}_t$-presentation $\{\mathcal{S}^*_r, \overline{f^*_r}, \mathcal{A}^*_r \,|\, 2 \leq r \leq n\}$ of $\Sigma\vec{\boldsymbol{f}}$ such that $\Sigma(\overline{f_n} \circ g_{n,n-1}) \simeq (\overline{f^*_n} \circ g^*_{n,n-1}) \circ (1_{X_1} \wedge \tau(\mathrm{S}^{n-2}, \mathrm{S}^1))$, where

$$\begin{aligned}\mathcal{S}^*_r = (\Sigma X_{r-1}, \Sigma^2 X_{r-2}, \dots, \Sigma^{r-1}X_1; C^*_{r,1}, \dots, C^*_{r,r};\\ g^*_{r,1}, \dots, g^*_{r,r-1}; j^*_{r,1}, \dots, j^*_{r,r-1}),\end{aligned}$$
$$g^*_{r,1} = f^*_{r-1}, \quad C^*_{r,2} = \Sigma X_r \cup_{f^*_{r-1}} C\Sigma X_{r-1}, \quad j^*_{r,1} = i_{g^*_{r,1}}.$$

Set $\mathcal{S}^*_2 = (\Sigma X_1; \Sigma X_2, \Sigma X_2 \cup_{f^*_1} C\Sigma X_1; f^*_1; i_{f^*_1})$. Then $\mathcal{S}^*_2$ is an iterated mapping cone with a reduced structure $\mathcal{A}^*_2 = \{a^*_{2,1}\}$ and a reduced quasi-structure $\Omega(\mathcal{A}^*_2) = \{\omega^*_{2,1}\}$, where $a^*_{2,1} = 1_{C^*_{2,2}}$ and $\omega^*_{2,1} = q'_{f^*_1}$. Set $e_{2,s} = \begin{cases} 1_{\{*\}} & s = 0 \\ \psi_{f_1^{s-1}} & s = 1, 2 \end{cases} : C^*_{2,s} \approx \Sigma C_{2,s}$ and $\overline{f^*_2} = \Sigma\overline{f_2} \circ e_{2,2} : C^*_{2,2} \to \Sigma X_3$. Then $e_{2,1} = 1_{\Sigma X_2}$, $\overline{f^*_2}$ is an extension of f^*_2 to $C^*_{2,2}$, and

$$a^*_{2,1} = (e^{-1}_{2,1} \cup C(1_{X_1} \wedge \tau(\mathrm{S}^0, \mathrm{S}^1))) \circ (\psi^1_{g_{2,1}})^{-1} \circ \Sigma a_{2,1} \circ e_{2,2},$$
$$e_{2,s+1} \circ j^*_{2,s} = \Sigma j_{2,s} \circ e_{2,s} \ (s = 0, 1),$$
$$\overline{f^*_2}^s = \Sigma\overline{f_2}^s \circ e_{2,s} : C^*_{2,s} \to \Sigma X_3 \ (s = 0, 1, 2),$$

$$\omega^*_{2,1} = (1_{X_1} \wedge \tau(\mathrm{S}^0 \wedge \mathrm{S}^1, \mathrm{S}^1)) \circ \Sigma\omega_{2,1} \circ \psi^1_{j_{2,1}} \circ (e_{2,2} \cup Ce_{2,1})$$
$$: C^*_{2,2} \cup_{j^*_{2,1}} CC^*_{2,1} \to \Sigma\Sigma X_1.$$

Set $\mathcal{S}^*_3 = \mathcal{S}^*_2(\overline{f^*_2}, \mathcal{A}^*_2)$ and

$$e_{3,s} = \begin{cases} 1_{\{*\}} & s = 0 \\ \psi^1_{\overline{f_2}^{s-1}} \circ (1_{\Sigma X_3} \cup Ce_{2,s-1}) & s = 1,2,3 \end{cases} : C^*_{3,s} \approx \Sigma C_{3,s},$$
$$\overline{f^*_3} = \begin{cases} \Sigma\overline{f_3} \circ e_{3,2} : C^*_{3,2} \to \Sigma X_4 & n = 3 \\ \Sigma\overline{f_3} \circ e_{3,3} : C^*_{3,3} \to \Sigma X_4 & n \geq 4 \end{cases}.$$

Then $\mathcal{S}^*_3$ is an iterated mapping cone, $\overline{f^*_3}$ is an extension of f^*_3 to $C^*_{3,2}$ or $C^*_{3,3}$ according as $n = 3$ or $n \geq 4$, and

$$e_{3,1} = 1_{\Sigma X_3}, \quad e_{3,s+1} \circ j^*_{3,s} = \Sigma j_{3,s} \circ e_{3,s} \ (s = 0,1,2),$$
$$\overline{f^*_3}^s = \Sigma\overline{f_3}^s \circ e_{3,s} \ (s = 1,2),$$
$$g^*_{3,1} = f^*_2, \quad g^*_{3,2} = (\overline{f^*_2} \cup C1_{C^*_{2,1}}) \circ {\omega^*_{2,1}}^{-1} : \Sigma\Sigma X_1 \to C^*_{3,2}.$$

The next relation holds when $r = 3$.

(6.3.2)
$$g^*_{r,s} \simeq e^{-1}_{r,s} \circ \Sigma g_{r,s} \circ (1_{X_{r-s}} \wedge \tau(\mathrm{S}^{s-1}, \mathrm{S}^1))^{-1} : \Sigma^{s-1}\Sigma X_{r-s} \to C^*_{r,s} \ (1 \leq s < r).$$

Indeed

$$\begin{aligned}
g^*_{r,s} &= (\overline{f^*_{r-1}}^s \cup C1_{C^*_{r-1,s-1}}) \circ {\omega^*_{r-1,s-1}}^{-1} \\
&= (\Sigma\overline{f_{r-1}}^s \circ e_{r-1,s} \cup C1_{C^*_{r-1,s-1}}) \circ {\omega^*_{r-1,s-1}}^{-1} \\
&= (1_{\Sigma X_r} \cup Ce_{r-1,s-1})^{-1} \circ (\Sigma\overline{f_{r-1}}^s \cup C1_{\Sigma C_{r-1,s-1}}) \circ (e_{r-1,s} \cup Ce_{r-1,s-1}) \\
&\qquad \circ {\omega^*_{r-1,s-1}}^{-1} \\
&\simeq (1_{\Sigma X_r} \cup Ce_{r-1,s-1})^{-1} \circ (\Sigma\overline{f_{r-1}}^s \cup C1_{\Sigma C_{r-1,s-1}}) \circ (e_{r-1,s} \cup Ce_{r-1,s-1}) \\
&\qquad \circ (e_{r-1,s} \cup Ce_{r-1,s-1})^{-1} \circ (\psi^1_{j_{r-1,s-1}})^{-1} \\
&\qquad \circ (\Sigma\omega_{r-1,s-1})^{-1} \circ (1_{X_{r-s}} \wedge \tau(\mathrm{S}^{s-1}, \mathrm{S}^1))^{-1} \\
&= (1_{\Sigma X_r} \cup Ce_{r-1,s-1})^{-1} \circ (\Sigma\overline{f_{r-1}}^s \cup C1_{\Sigma C_{r-1,s-1}}) \circ (\psi^1_{j_{r-1,s-1}})^{-1} \\
&\qquad \circ (\Sigma\omega_{r-1,s-1})^{-1} \circ (1_{X_{r-s}} \wedge \tau(\mathrm{S}^{s-1}, \mathrm{S}^1))^{-1} \\
&= (1_{\Sigma X_r} \cup Ce_{r-1,s-1})^{-1} \circ (\psi^1_{\overline{f_{r-1}}^{s-1}})^{-1} \circ \Sigma(\overline{f_{r-1}}^s \cup C1_{C_{r-1,s-1}}) \\
&\qquad \circ (\Sigma\omega_{r-1,s-1})^{-1} \circ (1_{X_{r-s}} \wedge \tau(\mathrm{S}^{s-1}, \mathrm{S}^1))^{-1} \\
&= e^{-1}_{r,s} \circ \Sigma(\overline{f_{r-1}}^s \cup C1_{C_{r-1,s-1}}) \circ (\Sigma\omega_{r-1,s-1})^{-1} \circ (1_{X_{r-s}} \wedge \tau(\mathrm{S}^{s-1}, \mathrm{S}^1))^{-1} \\
&\simeq e^{-1}_{r,s} \circ \Sigma((\overline{f_{r-1}}^s \cup C1_{C_{r-1,s-1}}) \circ \omega^{-1}_{r-1,s-1}) \circ (1_{X_{r-s}} \wedge \tau(\mathrm{S}^{s-1}, \mathrm{S}^1))^{-1}
\end{aligned}$$

$= e_{r,s}^{-1} \circ \Sigma g_{r,s} \circ (1_{X_{r-s}} \wedge \tau(\mathrm{S}^{s-1}, \mathrm{S}^1))^{-1}.$

Set

$$\begin{aligned} a_{3,s}^* = (e_{3,s}^{-1} \cup C(1_{X_{3-s}} \wedge \tau(\mathrm{S}^{s-1}, \mathrm{S}^1))) \circ (\psi^1_{g_{3,s}})^{-1} \circ \Sigma a_{3,s} \circ e_{3,s+1} \\ : C_{3,s+1}^* \simeq C_{3,s}^* \cup_{g_{3,s}^*} C\Sigma^s X_{3-s} \quad (s = 1, 2). \end{aligned}$$

Then $\mathcal{A}_3^* = \{a_{3,1}^*, a_{3,2}^*\}$ is a reduced structure on $\mathcal{S}_3^*$. Indeed, $a_{3,1}^* = 1_{C_{3,2}^*}$ is obvious and, from the commutative diagram below for $r = s = 3$, we have $a_{3,2}^* \circ j_{3,2}^* = i_{g_{3,2}^*}$.

(6.3.3)

$$\begin{tikzcd}
C^*_{r,s-1} \arrow[r, "e_{r,s-1}"] \arrow[d, "j^*_{r,s-1}"] & \Sigma C_{r,s-1} \arrow[d, "\Sigma j_{r,s-1}"] \\
C^*_{r,s} \arrow[r, "e_{r,s}"] \arrow[d, "a^*_{r,s-1}"] & \Sigma C_{r,s} \arrow[d, "\Sigma a_{r,s-1}"] \\
C^*_{r,s-1} \cup_{g^*_{r,s-1}} C\Sigma^{s-2}\Sigma X_{r-s+1} & \Sigma(C_{r,s-1} \cup_{g_{r,s-1}} C\Sigma^{s-2} X_{r-s+1}) \arrow[d, "(\psi^1_{g_{r,s-1}})^{-1}"] \\
& \Sigma C_{r,s-1} \cup C\Sigma\Sigma^{s-2} X_{r-s+1} \arrow[ul, "e^{-1}_{r,s-1} \cup C(1_{X_{r-s+1}} \wedge \tau(\mathrm{S}^{s-2}, \mathrm{S}^1))"]
\end{tikzcd}$$

Hence $\mathcal{S}_3^*$ is an iterated mapping cone with a reduced structure $\mathcal{A}_3^*$ and a reduced quasi-structure $\Omega(\mathcal{A}_3^*) = \{\omega_{3,s}^* \mid s = 1, 2\}$, where $\omega_{3,s}^* = q'_{g_{3,s}^*} \circ (a_{3,s}^* \cup C1_{C_{3,s}^*})$. We are going to prove

$$\begin{aligned} \omega_{3,s}^* = (1_{X_{3-s}} \wedge \tau(\mathrm{S}^{s-1} \wedge \mathrm{S}^1, \mathrm{S}^1)) \circ \Sigma\omega_{3,s} \circ \psi^1_{j_{3,s}} \circ (e_{3,s+1} \cup Ce_{3,s}) \\ : C_{3,s+1}^* \cup CC_{3,s}^* \to \Sigma\Sigma^{s-1}\Sigma X_{3-s} \quad (s = 1, 2), \end{aligned}$$

that is, we are going to prove that the following diagram is commutative when $r = 3$.

(6.3.4)

$$\begin{tikzcd}
C^*_{r,s+1} \cup CC^*_{r,s} \arrow[r, "e_{r,s+1} \cup Ce_{r,s}"] \arrow[d, "a^*_{r,s} \cup C1_{C^*_{r,s}}"'] & \Sigma C_{r,s+1} \cup C\Sigma C_{r,s} \arrow[d, "\psi^1_{j_{r,s}}"'] \\
(C^*_{r,s} \cup_{g^*_{r,s}} C\Sigma^{s-1}\Sigma X_{r-s}) \cup CC^*_{r,s} \arrow[d, "q'_{g^*_{r,s}}"'] & \Sigma(C_{r,s+1} \cup CC_{r,s}) \arrow[d, "\Sigma\omega_{r,s}"'] \\
(\Sigma\Sigma^{s-1})\Sigma X_{r-s} & \Sigma(\Sigma\Sigma^{s-1})X_{r-s} \arrow[l, "1_{X_{r-s}} \wedge \tau(\mathrm{S}^{s-1} \wedge \mathrm{S}^1, \mathrm{S}^1)"]
\end{tikzcd}$$

Recall that

$$\omega_{r,s} = q'_{g_{r,s}} \circ (a_{r,s} \cup C1_{C_{r,s}})$$
$$: C_{r,s+1} \cup_{j_{r,s}} CC_{r,s} \to (C_{r,s} \cup_{g_{r,s}} C\Sigma^{s-1}X_{r-s}) \cup_{i_{g_{r,s}}} CC_{r,s} \to \Sigma\Sigma^{s-1}X_{r-s}.$$

Since $CC^*_{r,s}$ is mapped finally to $*$ by the maps in the last diagram, it suffices to show that the two maps from $C^*_{r,s+1}$ to $(\Sigma\Sigma^{s-1})\Sigma X_{r-s}$ are the same, that is, $q_{g^*_{r,s}} \circ a^*_{r,s} = (1_{X_{r-s}} \wedge \tau(\mathrm{S}^{s-1} \wedge \mathrm{S}^1, \mathrm{S}^1)) \circ \Sigma q_{g_{r,s}} \circ \Sigma a_{r,s} \circ e_{r,s+1}$. Since

$$a^*_{r,s} = (e^{-1}_{r,s} \cup (1_{X_{r-s}} \wedge \tau(\mathrm{S}^{s-1}, \mathrm{S}^1)) \circ (\psi^1_{g_{r,s}})^{-1} \circ \Sigma a_{r,s} \circ e_{r,s+1}$$

by the definition, it suffices to show

$$\begin{aligned} &q_{g^*_{r,s}} \circ (e^{-1}_{r,s} \cup C(1_{X_{r-s}} \wedge \tau(\mathrm{S}^{s-1}, \mathrm{S}^1)) \circ (\psi^1_{g_{r,s}})^{-1} \\ &= (1_{X_{r-s}} \wedge \tau(\mathrm{S}^{s-1} \wedge \mathrm{S}^1, \mathrm{S}^1)) \circ \Sigma q_{g_{r,s}} \\ &: \Sigma(C_{r,s} \cup_{g_{r,s}} C\Sigma^{s-1}X_{r-s}) \to \Sigma\Sigma^{s-1}\Sigma X_{r-s}. \end{aligned}$$

The last equality is proved easily. Hence (6.3.4) is commutative when $r = 3$.

When $n = 3$, $\{\mathcal{S}^*_r, \overline{f^*_r}, \mathcal{A}^*_r \mid r = 2, 3\}$ is an $\ddot{s}_t$-presentation of $\Sigma\vec{\boldsymbol{f}}$ and

$$\begin{aligned} \overline{f^*_3} \circ g^*_{3,2} &= \Sigma\overline{f_3} \circ e_{3,2} \circ g^*_{3,2} \\ &\simeq \Sigma\overline{f_3} \circ e_{3,2} \circ e^{-1}_{3,2} \circ \Sigma g_{3,2} \circ (1_{X_1} \wedge \tau(\mathrm{S}^1, \mathrm{S}^1))^{-1} \text{ (by (6.3.2))} \\ &= \Sigma(\overline{f_3} \circ g_{3,2}) \circ (1_{X_1} \wedge \tau(\mathrm{S}^1, \mathrm{S}^1))^{-1}. \end{aligned}$$

Hence $\Sigma\{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)} \subset \{\Sigma\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)} \circ (1_{X_1} \wedge \tau(\mathrm{S}^1, \mathrm{S}^1))$.

When $n \geq 4$, we set $\mathcal{S}^*_4 = \mathcal{S}^*_3(\overline{f^*_3}, \mathcal{A}^*_3)$ and

$$e_{4,s} = \begin{cases} 1_{\{*\}} & s = 0 \\ \psi^1_{\overline{f_3}^{s-1}} \circ (1_{\Sigma X_4} \cup Ce_{3,s-1}) & s = 1, 2, 3, 4 \end{cases} : C^*_{4,s} \approx \Sigma C_{4,s},$$

$$\overline{f^*_4} = \begin{cases} \Sigma\overline{f_4} \circ e_{4,3} : C^*_{4,3} \to \Sigma X_5 & n = 4 \\ \Sigma\overline{f_4} \circ e_{4,4} : C^*_{4,4} \to \Sigma X_5 & n \geq 5 \end{cases}.$$

Then $\mathcal{S}^*_4$ is an iterated mapping cone, $\overline{f^*_4}$ is an extension of f^*_4 to $C^*_{4,3}$ or $C^*_{4,4}$ according as $n = 4$ or $n \geq 5$, and

$$e_{4,1} = 1_{\Sigma X_4}, \quad e_{4,s+1} \circ j^*_{4,s} = \Sigma j_{4,s} \circ e_{4,s} \ (s = 0, 1, 2, 3),$$
$$\overline{f^*_4}^s = \Sigma\overline{f_4}^s \circ e_{4,s} \ (s = 1, 2, 3),$$

$$g^*_{4,s} = \begin{cases} f^*_3 & s = 1 \\ (\overline{f^*_3}^s \cup C1_{C^*_{3,s-1}}) \circ {\omega^*_{3,s-1}}^{-1} & s = 2, 3 \end{cases} : \Sigma\Sigma^{s-2}\Sigma X_{4-s} \to C^*_{4,s}$$

We can prove

$$g^*_{4,s} \simeq e^{-1}_{4,s} \circ \Sigma g_{4,s} \circ (1_{X_{4-s}} \wedge \tau(\mathrm{S}^{s-1}, \mathrm{S}^1))^{-1} \ (s = 1, 2, 3)$$

by the method which was used to prove (6.3.2) for $r = 3$. Set

$$a^*_{4,s} = (e^{-1}_{4,s} \cup C(1_{X_{4-s}} \wedge \tau(\mathrm{S}^{s-1}, \mathrm{S}^1)) \circ (\psi^1_{g_{4,s}})^{-1} \circ \Sigma a_{4,s} \circ e_{4,s+1}$$
$$: C^*_{4,s+1} \to C^*_{4,s} \cup_{g^*_{4,s}} C\Sigma^s X_{4-s} \ (s = 1, 2, 3).$$

Then the diagrams (6.3.3) and (6.3.4) are commutative for $r = 4$, that is, $\mathcal{A}^*_4 = \{a^*_{4,s} \,|\, 1 \le s \le 3\}$ is a reduced structure on $\mathcal{S}^*_4$ and, if we set $\Omega(\mathcal{A}^*_4) = \{\omega^*_{4,s} \,|\, 1 \le s \le 3\}$, then

$$\omega^*_{4,s} = q'_{g^*_{4,s}} \circ (a^*_{4,s} \cup C1_{C^*_{4,s}})$$
$$= (1_{X_{4-s}} \wedge \tau(\mathrm{S}^{s-1} \wedge \mathrm{S}^1, \mathrm{S}^1)) \circ \Sigma\omega_{4,s} \circ \psi^1_{j_{4,s}} \circ (e_{4,s+1} \cup Ce_{4,s}).$$

When $n = 4$, $\{\mathcal{S}^*_r, \overline{f^*_r}, \mathcal{A}^*_r \,|\, 2 \le r \le 4\}$ is an $\ddot{s}_t$-presentation of $\Sigma\vec{\boldsymbol{f}}$ and

$$\overline{f^*_4} \circ g^*_{4,3} \simeq \Sigma\overline{f_4} \circ e_{4,3} \circ e^{-1}_{4,3} \circ \Sigma g_{4,3} \circ (1_{X_1} \wedge \tau(\mathrm{S}^2, \mathrm{S}^1))^{-1}$$
$$= \Sigma(\overline{f_4} \circ g_{4,3}) \circ (1_{X_1} \wedge \tau(\mathrm{S}^2, s^1))^{-1}.$$

Hence $\Sigma\{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)} \subset \{\Sigma\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)} \circ (1_{X_1} \wedge \tau(\mathrm{S}^2, \mathrm{S}^1))$.

By repeating the above process, we have an $\ddot{s}_t$-presentation $\{\mathcal{S}^*_r, \overline{f^*_r}, \mathcal{A}^*_r | 2 \le r \le n\}$ of $\Sigma\vec{\boldsymbol{f}}$ such that $\overline{f^*_n} \circ g^*_{n,n-1} \simeq \Sigma(\overline{f_n} \circ g_{n,n-1}) \circ (1_{X_1} \wedge \tau(\mathrm{S}^{n-2}, \mathrm{S}^1))^{-1}$ so that $\Sigma\{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)} \subset \{\Sigma\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)} \circ (1_{X_1} \wedge \tau(\mathrm{S}^{n-2}, \mathrm{S}^1))$. This completes the proof of Theorem 6.3.1 for $\star = \ddot{s}_t$. □

6.4. **Homotopy invariance of higher Toda brackets.** We prove the following which is (1.6) (cf. [**7**, Theorem 3.4] for $\{\vec{\boldsymbol{f}}\}^{(q)}$) and allows us to use the notation $\{\vec{\boldsymbol{\alpha}}\}^{(\star)}$ instead of $\{\vec{\boldsymbol{f}}\}^{(\star)}$ for every $\star$.

Theorem 6.4.1. *If $\vec{\boldsymbol{f}}, \vec{\boldsymbol{f}}' \in \mathrm{Rep}(\vec{\boldsymbol{\alpha}})$, then $\{\vec{\boldsymbol{f}}\}^{(\star)} = \{\vec{\boldsymbol{f}}'\}^{(\star)}$ for all $\star$.*

For $\vec{\boldsymbol{f}} = (f_n, \ldots, f_1) \in \mathrm{Rep}(\vec{\boldsymbol{\alpha}})$ and $i \in \{1, 2, \ldots, n\}$, let $\vec{\boldsymbol{f}}_i \in \mathrm{Rep}(\vec{\boldsymbol{\alpha}})$ denote a sequence obtained from $\vec{\boldsymbol{f}}$ by replacing f_i with f'_i such that $f'_i \simeq f_i$, for example $\vec{\boldsymbol{f}}_2 = (f_n, \ldots, f_3, f'_2, f_1)$ with $f'_2 \simeq f_2$.

Lemma 6.4.2. *If $\vec{\boldsymbol{f}} \in \mathrm{Rep}(\vec{\boldsymbol{\alpha}})$, then $\{\vec{\boldsymbol{f}}\}^{(\star)} = \{\vec{\boldsymbol{f}}_i\}^{(\star)}$ for all $\star$ and i.*

From the lemma, the theorem is proved as follows:

$$\{\vec{\boldsymbol{f}}\}^{(\star)} = \{f_n, \ldots, f_2, f'_1\}^{(\star)} = \{f_n, \ldots, f_3, f'_2, f'_1\}^{(\star)} = \ldots$$
$$= \{f'_n, \ldots, f'_1\}^{(\star)} = \{\vec{\boldsymbol{f}}'\}^{(\star)}.$$

Proof of Lemma 6.4.2. By Theorem 6.2.1, it suffices to prove the lemma for the cases $\star = \ddot{s}_t, aq\ddot{s}_2, aq, q_2$. We consider only the case of $\star = \ddot{s}_t$, because other cases can be treated similarly or more easily. For simplicity we abbreviate $\{\ \}^{(\ddot{s}_t)}$ as $\{\ \}$. Let $\alpha \in \{\vec{\boldsymbol{f}}\}$, $\{\mathcal{S}_r, \overline{f_r}, \mathcal{A}_r \,|\, 2 \le r \le n\}$ an $\ddot{s}_t$-presentation of $\vec{\boldsymbol{f}} = (f_n, \ldots, f_1)$ such that $\alpha = \overline{f_n} \circ g_{n,n-1}$, $\mathcal{A}_r = \{a_{r,s} \,|\, 1 \le$

$s < r\}$ with $a_{r,1} = 1_{C_{r,2}}$, and $\Omega(\mathcal{A}_r) = \{\omega_{r,s} \,|\, 1 \le s < r\}$. We are going to construct an $\ddot{s}_t$-presentation $\{\mathcal{S}'_r, \overline{f_r}', \mathcal{A}'_r \,|\, 2 \le r \le n\}$ of $\vec{\boldsymbol{f}}_i$ with $\overline{f_n}' \circ g'_{n,n-1} = \alpha$. If this is done, then $\{\vec{\boldsymbol{f}}\} \subset \{\vec{\boldsymbol{f}}_i\}$, and by interchanging $\vec{\boldsymbol{f}}$ with $\vec{\boldsymbol{f}}_i$ each other we have $\{\vec{\boldsymbol{f}}_i\} \subset \{\vec{\boldsymbol{f}}\}$ so that $\{\vec{\boldsymbol{f}}\} = \{\vec{\boldsymbol{f}}_i\}$.

We divide the proof into three cases: $i = n$; $i = 1$; $2 \le i \le n-1$.

First we consider the case: $i = n$. Let $\vec{\boldsymbol{f}}_n = (f'_n, f_{n-1}, \dots, f_1)$ with $J^n : f_n \simeq f'_n$ and set $j = j_{n,n-2} \circ \cdots \circ j_{n,2} \circ j_{n,1}$. Since j is a free cofibration, there exists a map $H : C_{n,n-1} \times I \to X_{n+1}$ such that $H \circ i_0^{C_{n,n-1}} = \overline{f_n}$ and $H \circ (j \times 1_I) = J^n$. Let $\{\mathcal{S}'_r, \overline{f_r}', \mathcal{A}'_r \,|\, 2 \le r \le n\}$ be the collection obtained from $\{\mathcal{S}_r, \overline{f_r}, \mathcal{A}_r \,|\, 2 \le r \le n\}$ by replacing $\overline{f_n}$ with H_1. Then the new collection is an $\ddot{s}_t$-presentation of $\vec{\boldsymbol{f}}_n$ such that it represents $\alpha = H_1 \circ g_{n,n-1} \in \{\vec{\boldsymbol{f}}_n\}$. Hence $\{\vec{\boldsymbol{f}}\} = \{\vec{\boldsymbol{f}}_n\}$.

Secondly we consider the case: $i = 1$. Let $\vec{\boldsymbol{f}}_1 = (f_n, \dots, f_2, f'_1)$ with $J^1 : f_1 \simeq f'_1$. Set

$$\mathcal{S}'_2 = (X_1; X_2, X_2 \cup_{f'_1} CX_1; f'_1; i_{f'_1}),$$
$$e_{2,2} = \Phi(f'_1, f_1, 1_{X_1}, 1_{X_2}; -J^1) : C'_{2,2} = X_2 \cup_{f'_1} CX_1 \to C_{2,2} = X_2 \cup_{f_1} CX_1,$$
$$e_{2,1} = 1_{X_2}, \quad \overline{f_2}' = \overline{f_2} \circ e_{2,2} : C'_{2,2} \to X_3, \quad a'_{2,1} = 1_{C'_{2,2}}.$$

Then $e_{2,2} \circ j'_{2,1} = j_{2,1}$, and $\omega'_{2,1} \simeq e_{1_{X_1}} \circ \omega'_{2,1} = \omega_{2,1} \circ (e_{2,2} \cup Ce_{2,1})$ by Proposition 3.3(2). Set $\mathcal{S}'_3 = \mathcal{S}'_2(\overline{f_2}', \mathcal{A}'_2)$. Then $C'_{3,s} = C_{3,s}$ for $s = 1, 2$ and $j'_{3,1} = j_{3,1}$. Set

$$e_{3,3} = 1_{X_3} \cup Ce_{2,2} : C'_{3,3} = X_3 \cup_{\overline{f_2}'} CC'_{2,2} \to C_{3,3} = X_3 \cup_{\overline{f_2}} CC_{2,2},$$
$$e_{3,s} = 1_{C_{3,s}} \ (s = 1, 2), \quad \overline{f_3}' = \begin{cases} \overline{f_3} \circ e_{3,2} & n = 3 \\ \overline{f_3} \circ e_{3,3} & n \ge 4 \end{cases}.$$

We have $e_{3,3} \circ j'_{3,2} = j_{3,2} \circ e_{3,2}$, $g'_{3,1} = g_{3,1}$, and

$$\begin{aligned} g'_{3,2} &= (\overline{f_2}' \cup C1_{X_2}) \circ \omega'^{-1}_{2,1} = (\overline{f_2} \cup C1_{X_2}) \circ (e_{2,2} \cup C1_{X_2}) \circ \omega'^{-1}_{2,1} \\ &\simeq (\overline{f_2} \cup C1_{X_2}) \circ \omega^{-1}_{2,1} = g_{3,2}. \end{aligned}$$

Take $K^3 : g_{3,2} \simeq g'_{3,2}$ and set

$$\Phi(K^3) = \Phi(g_{3,2}, g'_{3,2}, 1_{\Sigma X_1}, 1_{C_{3,2}}; K^3) : C_{3,2} \cup_{g_{3,2}} C\Sigma X_1 \to C'_{3,2} \cup_{g'_{3,2}} C\Sigma X_1,$$
$$a'_{3,2} = \Phi(K^3) \circ a_{3,2} \circ e_{3,3} : C'_{3,3} \to C'_{3,2} \cup_{g'_{3,2}} C\Sigma X_1.$$

Then $\mathcal{A}'_3 = \{a_{3,1}, a'_{3,2}\}$ is a reduced structure on $\mathcal{S}'_3$. We have $\omega'_{3,1} = \omega_{3,1} = \omega_{3,1} \circ (e_{3,2} \cup Ce_{3,1})$ and

$$\omega'_{3,2} = q'_{g'_{3,2}} \circ (a'_{3,2} \cup C1_{C'_{3,2}})$$

$$
\begin{aligned}
&= q'_{g'_{3,2}} \circ (\Phi(K^3) \cup C1_{C_{3,2}}) \circ (a_{3,2} \cup C1_{C_{3,2}}) \circ (e_{3,3} \cup Ce_{3,2}) \\
&\simeq q'_{g_{3,2}} \circ (a_{3,2} \cup C1_{C_{3,2}}) \circ (e_{3,3} \cup Ce_{3,2}) \quad \text{(by Proposition 3.3(2))} \\
&= \omega_{3,2} \circ (e_{3,3} \cup Ce_{3,2}).
\end{aligned}
$$

When $n = 3$, $\{\mathcal{S}'_r, \overline{f_r}', \mathcal{A}'_r \,|\, r = 2, 3\}$ is an $\ddot{s}_t$-presentation of (f_3, f_2, f'_1) such that $\overline{f_3}' \circ g'_{3,2} \simeq \overline{f_3} \circ g_{3,2}$ and so $\{\vec{\boldsymbol{f}}\} = \{\vec{\boldsymbol{f}}_1\}$. When $n \geq 4$, set $\mathcal{S}'_4 = \mathcal{S}'_3(\overline{f_3}', \mathcal{A}'_3)$. By repeating the above process we obtain $\{\mathcal{S}'_r, \overline{f_n}', \mathcal{A}'_r \,|\, 2 \leq r \leq n\}$, an $\ddot{s}_t$-presentation of $\vec{\boldsymbol{f}}_1$, and $e_{r,s} : C'_{r,s} \simeq C_{r,s}$ such that

$$
\begin{gathered}
C'_{n,s} = C_{n,s},\ e_{n,s} = 1_{C_{n,s}}\ (1 \leq s \leq n-1); \\
a'_{n,s} = a_{n,s},\ \omega'_{n,s} = \omega_{n,s}\ (1 \leq s \leq n-2); \\
\omega'_{n,n-1} \simeq \omega_{n,n-1} \circ (e_{n,n} \cup Ce_{n,n-1}); \\
\overline{f_n}' = \overline{f_n} : C'_{n,n-1} = C_{n,n-1} \to X_{n+1}; \quad g'_{n,n-1} \simeq g_{n,n-1}
\end{gathered}
$$

so that $\overline{f_n}' \circ g'_{n,n-1} \simeq \overline{f_n} \circ g_{n,n-1}$ and hence $\{\vec{\boldsymbol{f}}\} = \{\vec{\boldsymbol{f}}_1\}$.

Thirdly we consider the case: $2 \leq i \leq n-1$. We prove only the case $i = 2$, because other cases can be proved similarly. Let $\vec{\boldsymbol{f}}_2 = (f_n, \ldots, f_3, f'_2, f_1)$ with $J^2 : f_2 \simeq f'_2$.

Step 1: Set $\mathcal{S}'_2 = \mathcal{S}_2$ and $\mathcal{A}'_2 = \mathcal{A}_2$. Then $C'_{2,s} = C_{2,s}$, $j'_{2,1} = j_{2,1} = i_{f_1}$, and $\omega'_{2,1} = \omega_{2,1} = q'_{f_1}$. Set $e_{2,s} = 1_{C_{2,s}}$ for $s = 1, 2$. Since $j_{2,1} = i_{f_1} : X_2 \to C_{2,2} = X_2 \cup_{f_1} CX_1$ is a free cofibration, there exists $H^2 : C_{2,2} \times I \to X_3$ such that $H^2 \circ i_0^{C_{2,2}} = \overline{f_2}$ and $H^2 \circ (i_{f_1} \times 1_I) = J^2$. Set $\overline{f_2}' = H^2_1 : C'_{2,2} = C_{2,2} \to X_3$.

Step 2: Set

$$
\begin{aligned}
&\mathcal{S}'_3 = \mathcal{S}'_2(\overline{f_2}', \mathcal{A}'_2), \quad a'_{3,1} = 1_{C'_{3,2}}, \\
&e_{3,3} = \Phi(\overline{f_2}', \overline{f_2}, 1_{C_{2,2}}, 1_{X_3}; -H^2) : C'_{3,3} = X_3 \cup_{\overline{f_2}'} CC'_{2,2} \to \\
&\qquad\qquad C_{3,3} = X_3 \cup_{\overline{f_2}} CC_{2,2}, \\
&e_{3,2} = \Phi(f'_2, f_2, 1_{X_2}, 1_{X_3}; -J^2) : C'_{3,2} = X_3 \cup_{f'_2} CX_2 \to C_{3,2} = X_3 \cup_{f_2} CX_2, \\
&e_{3,1} = 1_{X_3} : C'_{3,1} \to C_{3,1}, \quad \overline{f_3}' = \begin{cases} \overline{f_3} \circ e_{3,2} : C'_{3,2} \to X_4 & n = 3 \\ \overline{f_3} \circ e_{3,3} : C'_{3,3} \to X_4 & n \geq 4 \end{cases}.
\end{aligned}
$$

Then $e_{3,3} \circ j'_{3,2} = j_{3,2} \circ e_{3,2}$, $e_{3,2} \circ j'_{3,1} = j_{3,1} \circ e_{3,1}$, and $e_{3,1} \circ g'_{3,1} = f'_2 \simeq f_2 = g_{3,1}$. We will prove

$$
e_{3,2} \circ g'_{3,2} \simeq g_{3,2} \quad i.e. \quad e_{3,2} \circ (\overline{f_2}' \cup C1_{X_2}) \circ \omega'^{-1}_{2,1} \simeq (\overline{f_2} \cup C1_{X_2}) \circ \omega^{-1}_{2,1}.
$$

It suffices to prove $e_{3,2} \circ (\overline{f_2}' \cup C1_{X_2}) \simeq \overline{f_2} \cup C1_{X_2}$, since $\omega'_{2,1} = \omega_{2,1}$. We have

$$
e_{3,2} \circ (\overline{f_2}' \cup C1_{X_2})
$$

$\simeq \Phi(f_2', f_2, 1_{X_2}, 1_{X_3}; -J^2) \circ \Phi(i_{f_1}, f_2', 1_{X_2}, \overline{f_2}'; 1_{f_2'})$ (by Proposition 3.3(3))

$\simeq \Phi(i_{f_1}, f_2, 1_{X_2}, \overline{f_2}'; ((-J^2)\bar{\circ}1_{1_{X_2}}) \bullet (1_{1_{X_3}}\bar{\circ}1_{f_2'}))$ (by Proposition 3.3(1)(d))

$\simeq \Phi(i_{f_1}, f_2, 1_{X_2}, \overline{f_2}'; -J^2)$

(by Proposition 3.3(5) and $((-J^2)\bar{\circ}1_{1_{X_2}}) \bullet (1_{1_{X_3}}\bar{\circ}1_{f_2'}) \overset{X_2}{\simeq} -J^2$).

We define $\widetilde{J}^t : X_2 \times I \to X_3$ for $t \in I$ by $\widetilde{J}^t(x_2, u) = (-J^2)(x_2, t + u - tu)$. Then $\widetilde{J}^t : (-H^2)_t \circ i_{f_1} \simeq f_2 \circ 1_{X_2}$ and so

$$\begin{aligned}\Phi(i_{f_1}, f_2, 1_{X_2}, \overline{f_2}'; -J^2) &\simeq \Phi(i_{f_1}, f_2, 1_{X_2}, \overline{f_2}; 1_{f_2}) \text{ (by Proposition 3.3(4))}\\ &\simeq \overline{f_2} \cup C1_{X_2} \text{ (by Proposition 3.3(3))}.\end{aligned}$$

Therefore $e_{3,2} \circ (\overline{f_2}' \cup C1_{X_2}) \simeq \overline{f_2} \cup C1_{X_2}$ as desired.

Let $e_{3,2}^{-1} : C_{3,2} \to C_{3,2}'$ be a homotopy inverse of $e_{3,2}$. Take $N : e_{3,2}^{-1} \circ e_{3,2} \simeq 1_{C_{3,2}'}$ and set $L = 1_{i_{g_{3,2}'}} \bar{\circ} N : i_{g_{3,2}'} \circ e_{3,2}^{-1} \circ e_{3,2} \simeq i_{g_{3,2}'}$. Take $K^{3,2} : e_{3,2}^{-1} \circ g_{3,2} \simeq g_{3,2}'$ and set $\Phi(K^{3,2}) = \Phi(g_{3,2}, g_{3,2}', 1_{\Sigma X_1}, e_{3,2}^{-1}; K^{3,2})$ and

$$a_{3,2}'' = \Phi(K^{3,2}) \circ a_{3,2} \circ e_{3,3} : C_{3,3}' \to C_{3,2}' \cup_{g_{3,2}'} C\Sigma X_1.$$

Then $L : a_{3,2}'' \circ j_{3,2}' = i_{g_{3,2}'} \circ e_{3,2}^{-1} \circ e_{3,2} \simeq i_{g_{3,2}'}$. Since $j_{3,2}'$ is a free cofibration, there exists a homotopy $M : C_{3,3}' \times I \to C_{3,2}' \cup_{g_{3,2}'} C\Sigma X_1$ such that $M_0 = a_{3,2}''$ and $M \circ (j_{3,2}' \times 1_I) = L$. Set $a_{3,2}' = M_1$, $\mathcal{A}_3' = \{1_{C_{3,2}'}, a_{3,2}'\}$, and $\Omega_3' = \Omega(\mathcal{A}_3')$. Then $\mathcal{A}_3'$ is a reduced structure on $\mathcal{S}_3'$. We will prove

$$\omega_{3,s}' \simeq \omega_{3,s} \circ (e_{3,s+1} \cup Ce_{3,s}) \ (s = 1, 2).$$

Since $\omega_{3,1} \circ (e_{3,2} \cup Ce_{3,1}) = e_{1_{X_1}} \circ \omega_{3,1}' \simeq \omega_{3,1}'$ by Proposition 3.3(2), the case of $s = 1$ holds. By the definition, we have $M_t \circ j_{3,2}' = L_t = i_{g_{3,2}'} \circ N_t$ for every $t \in I$. Set $H^t = 1_{M_t \circ j_{3,2}'} : M_t \circ j_{3,2}' = i_{g_{3,2}'} \circ N_t$. Since the function $C_{3,2}' \times I \times I \to C_{3,2}' \cup_{g_{3,2}'} C\Sigma X_1$, $(z, s, t) \mapsto H^t(z, s) = M_t(j_{3,2}'(z))$, is continuous, it follows from Proposition 3.3(4) that $\Phi(j_{3,2}', i_{g_{3,2}'}, N_0, M_0; H^0) \simeq \Phi(j_{3,2}', i_{g_{3,2}'}, N_1, M_1; H^1)$. By Proposition 3.3(3), we have

$$\begin{aligned}\Phi(j_{3,2}', i_{g_{3,2}'}, N_0, M_0; H^0) &\simeq M_0 \cup CN_0 = a_{3,2}'' \cup C(e_{3,2}^{-1} \circ e_{3,2}),\\ \Phi(j_{3,2}', i_{g_{3,2}'}, N_1, M_1; H^1) &\simeq M_1 \cup CN_1 = a_{3,2}' \cup C1_{C_{3,2}'}\end{aligned}$$

and so

$$\begin{aligned}\omega_{3,2}' = q_{g_{3,2}'}' \circ (a_{3,2}' \cup C1_{C_{3,2}'}) &\simeq q_{g_{3,2}'}' \circ (a_{3,2}'' \cup C(e_{3,2}^{-1} \circ e_{3,2}))\\ &= q_{g_{3,2}'}' \circ (\Phi(K^{3,2}) \cup Ce_{3,2}^{-1}) \circ (a_{3,2} \cup C1_{C_{3,2}}) \circ (e_{3,3} \cup Ce_{3,2})\\ &\simeq q_{g_{3,2}}' \circ (a_{3,2} \circ C1_{C_{3,2}}) \circ (e_{3,3} \cup Ce_{3,2}) \quad \text{(by Proposition 3.3(2))}\end{aligned}$$

$= \omega_{3,2} \circ (e_{3,3} \cup Ce_{3,2})$.

Step 3: When $n = 3$, $\{\mathcal{S}'_r, \overline{f_r}', \mathcal{A}'_r \,|\, r = 2,3\}$ is an $\ddot{s}_t$-presentation of (f_3, f'_2, f_1) such that $\overline{f_3}' \circ g'_{3,2} = \overline{f}_3 \circ e_{3,2} \circ g'_{3,2} \simeq \overline{f_3} \circ g_{3,2}$ so that $\{\vec{\boldsymbol{f}}\} = \{\vec{\boldsymbol{f}}_2\}$. When $n \geq 4$, by repeating the above process, we have $\mathcal{S}'_r$, $\overline{f_r}'$, $\mathcal{A}'_r$, and $e_{r,s} : C'_{r,s} \simeq C_{r,s}$ for $4 \leq r \leq n$ such that $\mathcal{A}'_r$ is a reduced structure on $\mathcal{S}'_r$ and

$$\begin{cases} \mathcal{S}'_r = \mathcal{S}'_{r-1}(\overline{f_{r-1}}', \mathcal{A}'_{r-1}), \\ e_{r,1} = 1_{X_r},\ e_{r,s} = 1_{X_r} \cup Ce_{r-1,s-1} : C'_{r,s} \simeq C_{r,s}\ (2 \leq s \leq r), \\ e_{r,s+1} \circ j'_{r,s} = j_{r,s} \circ e_{r,s},\ \omega'_{r,s} \simeq \omega_{r,s} \circ (e_{r,s+1} \cup Ce_{r,s}), \\ \qquad\qquad e_{r,s} \circ g'_{r,s} \simeq g_{r,s}\ (1 \leq s < r), \\ \overline{f_r}' = \begin{cases} \overline{f_n} \circ e_{n,n-1} : C'_{n,n-1} \to X_{n+1} & r = n \\ \overline{f_r} \circ e_{r,r} : C'_{r,r} \to X_{r+1} & r < n \end{cases}. \end{cases}$$

Then $\{\mathcal{S}'_r, \overline{f_r}', \mathcal{A}'_r \,|\, 2 \leq r \leq n\}$ is an $\ddot{s}_t$-presentation of $\vec{\boldsymbol{f}}_2$ such that $\overline{f_n}' \circ g'_{n,n-1} = \overline{f_n} \circ e_{n,n-1} \circ g'_{n,n-1} \simeq \overline{f_n} \circ g_{n,n-1}$. Therefore $\{\vec{\boldsymbol{f}}\} = \{\vec{\boldsymbol{f}}_2\}$. $\square$

6.5. **Relations with the J. Cohen's higher Toda bracket.** Let $\langle \vec{\boldsymbol{f}} \rangle$ be the Cohen's n-fold bracket of $\vec{\boldsymbol{f}}$ which shall be recalled and denoted by $\langle \vec{\boldsymbol{f}} \rangle^w$ in Appendix B. The purpose of this subsection is to prove the following which is (1.7).

Theorem 6.5.1. $\{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \cup \{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)} \subset \langle \vec{\boldsymbol{f}} \rangle$.

Let $\alpha \in \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)} \cup \{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)}$. Let $\{\mathcal{S}_r, \overline{f_r}, \Omega_r | 2 \leq r \leq n\}$ and $\{\mathcal{S}_r, \overline{f_r}, \mathcal{A}_r \,|\, 2 \leq r \leq n\}$ be an $aq\ddot{s}_2$-presentation of $\vec{\boldsymbol{f}}$ and an $\ddot{s}_t$-presentation of $\vec{\boldsymbol{f}}$ with $\alpha = \overline{f_n} \circ g_{n,n-1}$ according as $\alpha \in \{\vec{\boldsymbol{f}}\}^{(aq\ddot{s}_2)}$ or $\alpha \in \{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)}$. We prepare two lemmas.

Lemma 6.5.2. *$C_{n,n-1}$ is a finitely filtered space of type $(f_{n-1}, \ldots, f_2)$ (see Appendix B for the definition).*

Proof. We can assume that the free cofibration $j_{n,s} : C_{n,s} \to C_{n,s+1}$ is an inclusion map. Then by setting $F_k = C_{n,k}\,(0 \leq k \leq n-1)$, where $C_{n,0} = \{*\}$, we obtain the filtration of $C_{n,n-1}$: $F_0 = \{*\} \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} = C_{n,n-1}$. By Proposition 6.1.4 and Lemma B.5, we have the canonical homeomorphism

$$F_{k+1}/F_k = C_{n,k+1}/C_{n,k} \approx \Sigma^k X_{n-k} \quad (0 \leq k \leq n-2).$$

By this homeomorphism, we identify F_{k+1}/F_k with $\Sigma^k X_{n-k}$. We set

$$g_k = (-1)^k 1_{\Sigma^k X_{n-k}} : \Sigma^k X_{n-k} \to C_{n,k+1}/C_{n,k} = \Sigma^k X_{n-k}.$$

The assertion we must prove is that the skew pentagon of the following diagram is homotopy commutative for $1 \leq k \leq n-2$.

$$\begin{tikzcd}
\Sigma^k X_{n-k+1} \arrow[dd, bend right, "\Sigma g_{k-1}"'] & \Sigma^k X_{n-k} \arrow[l, "\Sigma^k f_{n-k}"'] \arrow[d, "g_k"'] & & \\
\Sigma F_k \arrow[d, "\Sigma(/F_{k-1})"] & F_{k+1}/F_k \arrow[l, "\delta"'] \arrow[d, equal] & F_{k+1} \arrow[l, "q"'] \arrow[d, "/F_{k-1}"'] & F_k \arrow[l, "j_{n,k}"'] \arrow[d, "/F_{k-1}"'] \\
\Sigma^k X_{n-k+1} & \Sigma^k X_{n-k} \arrow[l, "-\Sigma^k f_{n-k}"] & \Sigma^{k-1} C_{f_{n-k}} \arrow[l, "\Sigma^{k-1} q_{f_{n-k}}"] & \Sigma^{k-1} X_{n-k+1} \arrow[l, "\Sigma^{k-1} i_{f_{n-k}}"]
\end{tikzcd}$$

Since lower two rows are cofibre sequences and three squares are homotopy commutative, we have $\Sigma(/F_{k-1})\circ\delta\circ g_k \simeq (-\Sigma^k f_{n-k})\circ g_k = (-1)^{k+1}\Sigma^k f_{n-k} = (-1)^{k-1}\Sigma^k f_{n-k} = \Sigma g_{k-1} \circ \Sigma^k f_{n-k}$. Hence the skew pentagon of the above diagram is homotopy commutative. This proves Lemma 6.5.2. $\square$

Lemma 6.5.3. *The next square is homotopy commutative for* $2 \leq r \leq n$*, where* q_r *is the quotient.*

$$\begin{tikzcd}
\Sigma^{r-2} X_1 \arrow[r, "g_{r,r-1}"] \arrow[d, "\Sigma^{r-2} f_1"'] & C_{r,r-1} \arrow[d, "q_r = /C_{r,r-2}"] \\
\Sigma^{r-2} X_2 \arrow[r, "(-1)^r"] & \Sigma^{r-2} X_2 = C_{r,r-1}/C_{r,r-2}
\end{tikzcd}$$

Proof. For $r = 2$, the square is commutative. We use an induction on $r \geq 3$. The next diagram is homotopy commutative and $g_{3,2} \simeq (\overline{f_2} \cup C1_{X_2}) \circ \omega_{2,1}^{-1}$.

$$\begin{tikzcd}
 & & \Sigma X_1 \arrow[r, "-\Sigma f_1"] & \Sigma X_2 \arrow[ddd, bend left, "="] \\
X_2 \arrow[r, "i_{f_1}"] \arrow[d, equal] & C_{f_1} \arrow[ur, "q_{f_1}"] \arrow[r, "i_{i_{f_1}}"'] \arrow[d, "\overline{f_2}"'] & C_{f_1} \cup CX_2 \arrow[u, "\omega_{2,1}"', "\simeq"] \arrow[r, "i_{i_{i_{f_1}}}"] \arrow[d, "\overline{f_2}\cup C1_{X_2}"'] & (C_{f_1} \cup CX_2) \cup CC_{f_1} \arrow[u, "q'_{i_{f_1}}"', "\simeq"] \arrow[d, "(\overline{f_2}\cup C1_{X_2})\cup C\overline{f_2}"'] \\
X_2 \arrow[r, "f_2"] & X_3 \arrow[r, "i_{f_2}"] & X_3 \cup_{f_2} CX_2 \arrow[r, "i_{i_{f_2}}"] \arrow[dr, "q_{f_2}"'] & (X_3 \cup CX_2) \cup CX_3 \arrow[d, "q'_{f_2}"', "\simeq"] \\
 & & & \Sigma X_2
\end{tikzcd}$$

Then $q_{f_2} \circ g_{3,2} \circ \omega_{2,1} \simeq q_{f_2} \circ (\overline{f_2} \cup C1_{X_2}) \simeq (-\Sigma f_1) \circ \omega_{2,1}$ and so $q_{f_2} \circ g_{3,2} \simeq -\Sigma f_1$. This proves the assertion for $r = 3$. Suppose that the assertion is true for some r with $3 \leq r < n$.

First we consider the case of $\ddot{s}_t$-presentation. Set $Y = C_{r,r-1} \cup_{g_{r,r-1}} C\Sigma^{r-2}X_1$ and consider the following diagram.

$$\begin{tikzcd}
\Sigma^{r-1}X_1 \arrow[r, "-\Sigma g_{r,r-1}"] & \Sigma C_{r,r-1} \arrow[rdd, bend left, "="] & \\
Y \cup_{i_{g_{r,r-1}}} CC_{r,r-1} \arrow[u, "q'_{g_{r,r-1}}", "\simeq"'] \arrow[r, "i_{i_{g_{r,r-1}}}"] & (Y \cup CC_{r,r-1}) \cup CY \arrow[u, "q'_{i_{g_{r,r-1}}}", "\simeq"'] & \\
C_{r,r} \cup_{j_{r,r-1}} CC_{r,r-1} \arrow[u, "a_{r,r-1} \cup C1_{C_{r,r-1}}", "\simeq"'] \arrow[r, "i_{i_{j_{r,r-1}}}"] \arrow[d, "\overline{f_r} \cup C1_{C_{r,r-1}}"] & (C_{r,r} \cup CC_{r,r-1}) \cup CC_{r,r} \arrow[u, "(a_{r,r-1} \cup C1_{C_{r,r-1}}) \cup Ca_{r,r-1}", "\simeq"'] \arrow[r, "q'_{j_{r,r-1}}", "\simeq"'] \arrow[d, "(\overline{f_r} \cup C1_{C_{r,r-1}}) \cup C\overline{f_r}"] & \Sigma C_{r,r-1} \arrow[d, equal] \\
X_{r+1} \cup_{\overline{f_r}^{r-1}} CC_{r,r-1} \arrow[r, "i_{i_{\overline{f_r}^{r-1}}}"] \arrow[d, equal] & (X_{r+1} \cup CC_{r,r-1}) \cup CX_{r+1} \arrow[r, "q'_{\overline{f_r}^{r-1}}", "\simeq"'] & \Sigma C_{r,r-1} \arrow[d, "\Sigma q_r"] \\
C_{r+1,r} \arrow[rr, "q_{r+1} = /C_{r+1,r-1}"] & & \Sigma^{r-1}X_2
\end{tikzcd}$$

The diagram is commutative except the first square which is homotopy commutative. It follows from definitions that $\omega_{r,r-1} = q'_{g_{r,r-1}} \circ (a_{r,r-1} \cup C1_{C_{r,r-1}})$ and $g_{r+1,r} \circ \omega_{r,r-1} \simeq \overline{f_r} \cup C1_{C_{r,r-1}}$ so that

$$\begin{aligned}
(\ /C_{r+1,r-1}) \circ g_{r+1,r} \circ \omega_{r,r-1} &\simeq (\ /C_{r+1,r-1}) \circ (\overline{f_r} \cup C1_{C_{r,r-1}}) = \Sigma q_r \circ q_{j_{r,r-1}} \\
&\simeq \Sigma q_r \circ (-\Sigma g_{r,r-1}) \circ \omega_{r,r-1} \\
&\simeq (-1)^{r+1}\Sigma^{r-1} f_1 \circ \omega_{r,r-1} \text{ (by the inductive assumption).}
\end{aligned}$$

Hence $(\ /C_{r+1,r-1}) \circ g_{r+1,r} \simeq (-1)^{r+1}\Sigma^{r-1} f_1$. This completes the induction and proves Lemma 6.5.3 for the case of $\ddot{s}_t$-presentation.

Secondly we consider the case of $aq\ddot{s}_2$-presentation. It suffices to prove

$$q_{r+1} \circ g_{r+1,r} \circ \omega_{r,r-1} \simeq (-1)^{r+1} 1_{\Sigma^{r-1}X_2} \circ \Sigma^{r-1} f_1 \circ \omega_{r,r-1}. \tag{6.5.1}$$

We have

$$q_{r+1} \circ g_{r+1,r} \circ \omega_{r,r-1} \simeq q_{r+1} \circ (\overline{f_r} \cup C1_{C_{r,r-1}}) = \Sigma q_r \circ q'_{j_{r,r-1}} \circ i_{i_{j_{r,r-1}}} = \Sigma q_r \circ q_{j_{r,r-1}}$$

and

$$\begin{aligned}
&(-1)^{r+1} 1_{\Sigma^{r-1}X_2} \circ \Sigma^{r-1} f_1 \circ \omega_{r,r-1} \\
&\simeq \Sigma q_r \circ (-\Sigma g_{r,r-1}) \circ \omega_{r,r-1} \quad \text{(by the inductive assumption)} \\
&\simeq \Sigma q_r \circ (-\Sigma(\overline{f_{r-1}} \cup C1_{C_{r-1,r-2}}) \circ \Sigma\omega_{r-1,r-2}^{-1}) \circ \omega_{r,r-1} \\
&= \Sigma q_r \circ (-\Sigma(\overline{f_{r-1}} \cup C1_{C_{r-1,r-2}}) \circ \Sigma\omega_{r-1,r-2}^{-1}) \circ \widetilde{\omega_{r-1,r-2}} \\
&= \Sigma q_r \circ (-\Sigma(\overline{f_{r-1}} \cup C1_{C_{r-1,r-2}}) \circ \Sigma\omega_{r-1,r-2}^{-1}) \circ \Sigma\omega_{r-1,r-2}
\end{aligned}$$

$$\circ\, q_{\overline{f_{r-1}}\cup C\overline{f_{r-1}}^{r-2}} \circ \xi$$
$$\simeq \Sigma q_r \circ (-\Sigma(\overline{f_{r-1}} \cup C1_{C_{r-1,r-2}})) \circ q_{\overline{f_{r-1}}\cup C\overline{f_{r-1}}^{r-2}} \circ \xi,$$

where

$$\begin{aligned}\xi : C_{r,r} &\cup_{j_{r,r-1}} CC_{r,r-1}\\ &= (X_r \cup_{\overline{f_{r-1}}} CC_{r-1,r-1}) \cup_{1_{X_r}\cup Cj_{r-1,r-2}} C(X_r \cup_{\overline{f_{r-1}}^{r-2}} CC_{r-1,r-2})\\ &\to (X_r \cup_{1_{X_r}} CX_r) \cup_{\overline{f_{r-1}}\cup C\overline{f_{r-1}}^{r-2}} C(C_{r-1,r-1} \cup_{j_{r-1,r-2}} CC_{r-1,r-2})\end{aligned}$$

is the homeomorphism defined in (5.3). Set

$$\begin{aligned} M &= \Sigma q_r \circ q_{j_{r,r-1}},\\ N &= \Sigma q_r \circ (-\Sigma(\overline{f_{r-1}} \cup C1_{C_{r-1,r-2}})) \circ q_{\overline{f_{r-1}}\cup C\overline{f_{r-1}}^{r-2}} \circ \xi.\end{aligned}$$

Then (6.5.1) is equivalent to $M \simeq N$. Let π denote the composite of

$$C_{r,r} \cup_{j_{r,r-1}} CC_{r,r-1} \xrightarrow{q_{j_{r,r-1}}} \Sigma C_{r,r-1} = \Sigma(X_r \cup_{\overline{f_{r-1}}^{r-2}} CC_{r-1,r-2})$$
$$\xrightarrow{\Sigma q_{\overline{f_{r-1}}^{r-2}}} \Sigma\Sigma C_{r-1,r-2}.$$

Define $M', N' : \Sigma\Sigma C_{r-1,r-2} \to \Sigma^{r-1}X_2$ by

$$M'(y \wedge \overline{t} \wedge \overline{u}) = q_{r-1}(y) \wedge \overline{t} \wedge \overline{u}, \quad N'(y \wedge \overline{t} \wedge \overline{u}) = q_{r-1}(y) \wedge \overline{u} \wedge \overline{1-t}.$$

Since $q_r = \Sigma q_{r-1} \circ q_{\overline{f_{r-1}}^{r-2}}$, we have $M = M' \circ \pi$ and $N = N' \circ \pi$. As is easily seen, $M' \simeq N'$ so that we have $M \simeq N$ as desired. This completes the induction and the proof of Lemma 6.5.3. □

Proof of Theorem 6.5.1. Under the notations of [**3**, **15**] and Lemma 6.5.2, we have

$$j_{C_{n,n-1}} : X_n = F_1 \subset F_{n-1} = C_{n,n-1},$$
$$\sigma_{C_{n,n-1}} : C_{n,n-1} = F_{n-1} \xrightarrow{/F_{n-2}} F_{n-1}/F_{n-2} = \Sigma^{n-2}X_2 \xrightarrow{(-1)^n} \Sigma^{n-2}X_2.$$

By Lemma 6.5.3, we have the following homotopy commutative diagram.

$$\begin{tikzcd}
 & \Sigma^{n-2}X_1 \arrow[dl, "\Sigma^{n-2}f_1"'] \arrow[d, "g_{n,n-1}"] & \\
\Sigma^{n-2}X_2 & C_{n,n-1} \arrow[l, "\sigma_{C_{n,n-1}}"] \arrow[d, "\overline{f_n}"'] & X_n \arrow[l, "j_{C_{n,n-1}}"'] \arrow[dl, "f_n"] \\
 & X_{n+1} &
\end{tikzcd}$$

Hence $\alpha = \overline{f_n} \circ g_{n,n-1} \in \langle \vec{\boldsymbol{f}}\,\rangle$ and so $\{\vec{\boldsymbol{f}}\,\}^{(aq\ddot{s}_2)} \cup \{\vec{\boldsymbol{f}}\,\}^{(\ddot{s}_t)} \subset \langle \vec{\boldsymbol{f}}\,\rangle$. This proves Theorem 6.5.1. □

6.6. **3-fold brackets.** We denote the classical unstable Toda bracket of $\vec{\boldsymbol{f}} = (f_3, f_2, f_1)$ by $\{\vec{\boldsymbol{f}}\,\}$ or $\{f_3, f_2, f_1\}$ (see the end of Section 2). We have (1.8) from Theorem 6.6.1 and Example 6.6.2 below.

Theorem 6.6.1. *When $n = 3$, we have*

$$\begin{aligned}\{\vec{\boldsymbol{f}}\,\}^{(\ddot{s}_t)} &= \{\vec{\boldsymbol{f}}\,\}^{(aq\ddot{s}_2)} = \{\vec{\boldsymbol{f}}\,\}^{(q\ddot{s}_2)} = \{\vec{\boldsymbol{f}}\,\}\\ &\subset \{\vec{\boldsymbol{f}}\,\}^{(aqs_2)} = \{\vec{\boldsymbol{f}}\,\}^{(qs_2)} = \{\vec{\boldsymbol{f}}\,\}^{(q\dot{s}_2)} = \{\vec{\boldsymbol{f}}\,\}^{(aq\ddot{s}_2)} \circ \mathcal{E}(\Sigma X_1)\\ &\subset \{\vec{\boldsymbol{f}}\,\}^{(aq)} = \{\vec{\boldsymbol{f}}\,\}^{(q_2)} = \{\vec{\boldsymbol{f}}\,\}^{(q)},\end{aligned}$$

so that systems $\{\ \}^{(\star)}$ of unstable higher Toda brackets for $\star = \ddot{s}_t, aq\ddot{s}_2, q\ddot{s}_2$ are normal, and there exist $\vec{\boldsymbol{f}}$ and $\vec{\boldsymbol{f}}'$ such that $\{\vec{\boldsymbol{f}}\,\}^{(q)}$ is empty, $\langle \vec{\boldsymbol{f}}\,\rangle$ is not empty, and $\{\vec{\boldsymbol{f}}'\}^{(q)}$ is a non-empty proper subset of $\langle \vec{\boldsymbol{f}}'\,\rangle$ so that the Cohen's system of unstable higher Toda brackets is not normal.

Proof. The relations $\{\vec{\boldsymbol{f}}\,\}^{(\ddot{s}_t)} = \{\vec{\boldsymbol{f}}\,\}^{(aq\ddot{s}_2)} = \{\vec{\boldsymbol{f}}\,\}^{(q\ddot{s}_2)} \subset \{\vec{\boldsymbol{f}}\,\}^{(aq)} = \{\vec{\boldsymbol{f}}\,\}^{(q_2)} = \{\vec{\boldsymbol{f}}\,\}^{(q)} \supset \{\vec{\boldsymbol{f}}\,\}^{(qs_2)} \supset \{\vec{\boldsymbol{f}}\,\}^{(aqs_2)} \supset \{\vec{\boldsymbol{f}}\,\}^{(aq\ddot{s}_2)}$ hold immediately from the definitions. We have $\{\vec{\boldsymbol{f}}\,\}^{(aqs_2)} = \{\vec{\boldsymbol{f}}\,\}^{(aq\ddot{s}_2)} \circ \mathcal{E}(\Sigma X_1) = \{\vec{\boldsymbol{f}}\,\}^{(qs_2)} = \{\vec{\boldsymbol{f}}\,\}^{(q\dot{s}_2)}$ by Theorem 6.2.1(2). The equality $\{\vec{\boldsymbol{f}}\,\}^{(q\ddot{s}_2)} = \{\vec{\boldsymbol{f}}\,\}$ follows easily from (4.2).

Let $p^1 : \mathrm{S}^7 \to \mathrm{S}^4 = \mathbb{H}P^1$ be the projection, where $\mathbb{H}P^m$ is the quaternionic projective m-space, and $*^1_3$ the trivial map $\mathrm{S}^4 \to \mathrm{S}^3$. Set $\vec{\boldsymbol{f}} = (*^1_3, p^1, 2\Sigma^3 p^1)$. Then it follows from [**15**, Remark B.5] that $\langle \vec{\boldsymbol{f}}\,\rangle$ contains 0 and $\{\vec{\boldsymbol{f}}\,\}$ is empty so that $\{\vec{\boldsymbol{f}}\,\}^{(q)}$ is empty by Corollary 6.2.2(4).

Set $\vec{\boldsymbol{f}}' = (2\iota_5, \nu_5\eta_8, 2\iota_9)$ (see [**25**] for notations). Then, since $\pi_{10}(\mathrm{S}^5) = \mathbb{Z}_2\{\nu_5\eta_8^2\}$ and $\{\vec{\boldsymbol{f}}'\,\} = \{\nu_5\eta_8^2\}$ by [**25**, Proposition 5.9, Theorem 13.4, Corollary 3.7], we have $\{\vec{\boldsymbol{f}}'\,\}^{(q)} = \{\nu_5\eta_8^2\}$ by Theorem 6.2.1(5). Also we have $\langle \vec{\boldsymbol{f}}'\,\rangle = \pi_{10}(\mathrm{S}^5)$ by Remark B.5(1) of [**15**]. Hence $\{\vec{\boldsymbol{f}}'\,\}^{(q)}$ is a non-empty proper subset of $\langle \vec{\boldsymbol{f}}'\,\rangle$ so that the Cohen's system of unstable higher Toda brackets is not normal. This completes the proof of the theorem. □

Example 6.6.2. We use freely notations and results in [**25**].

(1) If $\ell \geq 8$, then $\{\Sigma^\ell \nu_5, 8\iota_{\ell+8}, \Sigma^{\ell+1}\sigma'\} = (-1)^\ell \zeta_{\ell+5}$,

$$\{\Sigma^\ell \nu_5, 8\iota_{\ell+8}, \Sigma^{\ell+1}\sigma'\}^{(qs_2)} = \{\zeta_{\ell+5}, -\zeta_{\ell+5}\} \subset \pi_{\ell+16}(\mathrm{S}^{\ell+5}) = \mathbb{Z}_8\{\zeta_{\ell+5}\} \oplus \mathbb{Z}_{63},$$

and the order of any element of $\{\Sigma^\ell \nu_5, 8\iota_{\ell+8}, \Sigma^{\ell+1}\sigma'\}^{(q)}$ is a multiple of 8.

(2) If $\ell \geq 7$, then $\{\Sigma^\ell \alpha_1(5), \Sigma^\ell \alpha_1(9), \Sigma^\ell \alpha_1(12)\} = (-1)^\ell \beta_1(\ell+5)$,

$$\{\Sigma^\ell \alpha_1(5), \Sigma^\ell \alpha_1(9), \Sigma^\ell \alpha_1(12)\}^{(qs_2)} = \{\beta_1(\ell+5), -\beta_1(\ell+5)\}$$

$$\subset \pi_{\ell+15}(\mathrm{S}^{\ell+5}) = \mathbb{Z}_3\{\beta_1(\ell+5)\} \oplus \mathbb{Z}_2,$$

and the order of any element of $\{\Sigma^\ell\alpha_1(5), \Sigma^\ell\alpha_1(9), \Sigma^\ell\alpha_1(12)\}^{(q)}$ is a multiple of 3.

6.7. **A 4-fold bracket.** When $\vec{\boldsymbol{\alpha}} = (\alpha_4, \alpha_3, \alpha_2, \alpha_1)$ is admissible, we have considered the next four non-empty subsets of $[\Sigma^2 X_1, X_5]$ which are called tertiary compositions [**15**]: $\{\vec{\boldsymbol{\alpha}}\}^{(0)} \subset \{\vec{\boldsymbol{\alpha}}\}^{(1)} \subset \{\vec{\boldsymbol{\alpha}}\}^{(2)} \subset \{\vec{\boldsymbol{\alpha}}\}^{(3)}$. Here $\{\vec{\boldsymbol{\alpha}}\}^{(1)} = \bigcup_{\vec{\boldsymbol{f}}\in \mathrm{Rep}(\vec{\boldsymbol{\alpha}})}\{\vec{\boldsymbol{f}}\}^{(1)}$ is the tertiary composition of Ôguchi [**14**, **15**], and

$$\{\vec{\boldsymbol{\alpha}}\}^{(2)} = \bigcup_{\vec{\boldsymbol{A}}} \big(\{f_4, [f_3, A_2, f_2], (f_2, A_1, f_1)\} \cap \{[f_4, A_3, f_3], (f_3, A_2, f_2), -\Sigma f_1\}\big), \tag{6.7.1}$$

$$\{\vec{\boldsymbol{\alpha}}\}^{(3)} = \bigcup_{\vec{\boldsymbol{A}}}\{[f_4, A_3, f_3], i_{f_3} \circ [f_3, A_2, f_2], (f_2, A_1, f_1)\}, \tag{6.7.2}$$

where $\vec{\boldsymbol{f}} = (f_4, f_3, f_2, f_1)$ is a representative of $\vec{\boldsymbol{\alpha}}$ and unions $\bigcup_{\vec{\boldsymbol{A}}}$ are taken over all $\vec{\boldsymbol{A}} = (A_3, A_2, A_1)$ such that $(\vec{\boldsymbol{f}}; \vec{\boldsymbol{A}})$ is admissible. As remarked in [**15**, p.56], the terms on right hand sides of (6.7.1) and (6.7.2) do not depend on the choice of a representative $\vec{\boldsymbol{f}}$. The following theorem implies (1.9).

Theorem 6.7.1. *The sequence $\vec{\boldsymbol{\alpha}} = (\alpha_4, \alpha_3, \alpha_2, \alpha_1)$ is admissible if and only if it is $\star$-presentable for some and hence all $\star$. If $\vec{\boldsymbol{\alpha}}$ is admissible and $\vec{\boldsymbol{f}} = (f_4, f_3, f_2, f_1) \in \mathrm{Rep}(\vec{\boldsymbol{\alpha}})$, then*

$$\{\vec{\boldsymbol{\alpha}}\}^{(2)} \subset \{\vec{\boldsymbol{\alpha}}\}^{(\ddot{s}_t)} = \{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)} = \bigcup \{f_4, [f_3, A_2, f_2], (f_2, A_1, f_1)\} \subset \{\vec{\boldsymbol{\alpha}}\}^{(3)}, \tag{6.7.3}$$

where $\bigcup$ is taken over all $\vec{\boldsymbol{A}} = (A_3, A_2, A_1)$ such that $(\vec{\boldsymbol{f}}; \vec{\boldsymbol{A}})$ is admissible.

Corollary 6.7.2 (cf. Theorem 2.7 of [**27**]). *Given a map $f_0 : X_0 \to X_1$, if $\{f_2, f_1, f_0\} = \{0\}$, then $\{f_4, f_3, f_2, f_1\}^{(\ddot{s}_t)} \circ \Sigma^2 f_0 \subset f_4 \circ \{f_3, f_2, f_1, f_0\}^{(\ddot{s}_t)}$.*

Proof of Theorem 6.7.1. First we suppose that $\vec{\boldsymbol{\alpha}}$ is admissible. Let $\vec{\boldsymbol{f}} = (f_4, f_3, f_2, f_1)$ be a representative of $\vec{\boldsymbol{\alpha}}$. We are going to show

$$\bigcup \{f_4, [f_3, A_2, f_2], (f_2, A_1, f_1)\} \subset \{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)} \tag{6.7.4}$$

where $\bigcup$ is taken over all $\vec{\boldsymbol{A}} = (A_3, A_2, A_1)$ such that $(\vec{\boldsymbol{f}}; \vec{\boldsymbol{A}})$ is admissible.

Let $(\vec{\boldsymbol{f}}; \vec{\boldsymbol{A}})$ be an admissible representative of $\vec{\boldsymbol{\alpha}}$. Then $[f_4, A_3, f_3] \circ (f_3, A_2, f_2) \simeq *$ and $[f_3, A_2, f_2] \circ (f_2, A_1, f_1) \simeq *$, and it follows from [**14**, Proposition (5.11)] (or [**15**, Lemma 3.6]) that

$$\begin{aligned} f_4 \circ [f_3, A_2, f_2] &= [f_4, A_3, f_3] \circ i_{f_3} \circ [f_3, A_2, f_2] \\ &\simeq [f_4, A_3, f_3] \circ (f_3, A_2, f_2) \circ q_{f_2} \simeq *. \end{aligned}$$

Hence $\{f_4, [f_3, A_2, f_2], (f_2, A_1, f_1)\}$ is non-empty. Take α from it. Then there exist $F : f_4 \circ [f_3, A_2, f_2] \simeq *$ and $B_2 : [f_3, A_2, f_2] \circ (f_2, A_1, f_1) \simeq *$ with

$$\alpha = [f_4, F, [f_3, A_2, f_2]] \circ ([f_3, A_2, f_2], B_2, (f_2, A_1, f_1)).$$

We are going to construct an $\ddot{s}_t$-presentation $\{\mathcal{S}_r, \overline{f_r}, \mathcal{A}_r \,|\, r = 2, 3, 4\}$ of $\vec{\boldsymbol{f}}$ with $\overline{f_4} \circ g_{4,3} = \alpha$. If this is done, then $\vec{\boldsymbol{f}}$ is $\ddot{s}_t$-presentable and hence $\star$-presentable for every $\star$ by Corollary 6.2.2(4), and (6.7.4) is proved.

Set

$$\mathcal{S}_2 = \big(X_1; X_2, X_2 \cup_{f_1} CX_1; f_1; i_{f_1}\big),\ \overline{f_2} = [f_2, A_1, f_1],\ \mathcal{A}_2 = \{1_{X_2 \cup_{f_1} CX_1}\},$$
$$\mathcal{S}_3 = \mathcal{S}_2(\overline{f_2}, \mathcal{A}_2).$$

Then $g_{3,2} = (\overline{f_2} \cup C1_{X_2}) \circ q'^{-1}_{f_1} \simeq (f_2, A_1, f_1)$ by (4.2). Since $j_{3,2}$ is a homotopy cofibre of $g_{3,2}$, there exists $a_{3,2} : C_{3,3} \simeq C_{3,2} \cup_{g_{3,2}} C\Sigma X_1$ such that $a_{3,2} \circ j_{3,2} = i_{g_{3,2}}$. Set $\mathcal{A}_3 = \{1_{C_{3,2}}, a_{3,2}\}$. Take $H : g_{3,2} \simeq (f_2, A_1, f_1)$ and set

$$\begin{aligned}
\Phi(H) &= \Phi(g_{3,2}, (f_2, A_1, f_1), 1_{\Sigma X_1}, 1_{C_{3,2}}; H)\\
&\quad : C_{3,2} \cup_{g_{3,2}} C\Sigma X_1 \to C_{3,2} \cup_{(f_2, A_1, f_1)} C\Sigma X_1,\\
\overline{f_3} &= [[f_3, A_2, f_2], B_2, (f_2, A_1, f_1)] \circ \Phi(H) \circ a_{3,2} : C_{3,3} \to X_4,\\
\mathcal{S}_4 &= \mathcal{S}_3(\overline{f_3}, \mathcal{A}_3).
\end{aligned}$$

Since $\overline{f_3}^2 = \overline{f_3} \circ j_{3,2} = [f_3, A_2, f_2]$, we can set

$$\overline{f_4} = [f_4, F, [f_3, A_2, f_2]] : C_{4,3} = X_4 \cup_{\overline{f_3}^2} CC_{3,2} \to X_5.$$

Let $\mathcal{A}_4$ be any reduced structure on $\mathcal{S}_4$. Then $\{\mathcal{S}_r, \overline{f_r}, \mathcal{A}_r \,|\, r = 2, 3, 4\}$ is an $\ddot{s}_t$-presentation of $\vec{\boldsymbol{f}}$ and

$$\begin{aligned}
g_{4,3} &= (\overline{f_3} \cup C1_{C_{3,2}}) \circ \omega_{3,2}^{-1}\\
&\simeq ([[f_3, A_2, f_2], B_2, (f_2, A_1, f_1)] \cup C1_{C_{3,2}}) \circ (\Phi(H) \cup C1_{C_{3,2}})\\
&\qquad\qquad \circ (a_{3,2} \cup C1_{C_{3,2}}) \circ \omega_{3,2}^{-1}\\
&\simeq ([[f_3, A_2, f_2], B_2, (f_2, A_1, f_1)] \cup C1_{C_{3,2}}) \circ (\Phi(H) \cup C1_{C_{3,2}}) \circ q'^{-1}_{g_{3,2}}\\
&\simeq ([[f_3, A_2, f_2], B_2, (f_2, A_1, f_1)] \cup C1_{C_{3,2}}) \circ q'^{-1}_{(f_2, A_1, f_1)}\\
&\qquad\qquad \text{(by Proposition 3.3(2))}\\
&\simeq ([f_3, A_2, f_2], B_2, (f_2, A_1, f_1)) \quad \text{(by (4.2))}.
\end{aligned}$$

Hence $\overline{f_4} \circ g_{4,3} \simeq [f_4, F, [f_3, A_2, f_2]] \circ ([f_3, A_2, f_2], B_2, (f_2, A_1, f_1)$. Thus (6.7.4) is proved.

Secondly suppose that $\vec{\boldsymbol{\alpha}}$ is $\ddot{s}_t$-presentable. Then it has an $\ddot{s}_t$-presentable representative $\vec{\boldsymbol{f}} = (f_4, f_3, f_2, f_1)$. We are going to show that $\vec{\boldsymbol{f}}$ and $\vec{\boldsymbol{\alpha}}$ are admissible, and

$$\bigcup\{f_4,[f_3,A_2,f_2],(f_2,A_1,f_1)\}\supset\{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)}, \tag{6.7.5}$$

where $\bigcup$ is taken over all $\vec{\boldsymbol{A}}=(A_3,A_2,A_1)$ such that $(\vec{\boldsymbol{f}};\vec{\boldsymbol{A}})$ is admissible. If this is done, then

$$\{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)}=\bigcup\{f_4,[f_3,A_2,f_2],(f_2,A_1,f_1)\} \tag{6.7.6}$$

by (6.7.4) and (6.7.5).

Let $\alpha\in\{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)}$ and $\{\mathcal{S}_r,\,\overline{f_r},\,\mathcal{A}_r\,|\,r=2,3,4\}$ an $\ddot{s}_t$-presentation of $\vec{\boldsymbol{f}}$ with $\alpha=\overline{f_4}\circ g_{4,3}$. Take $A_1:f_2\circ f_1\simeq *$ and $A_2:f_3\circ f_2\simeq *$ such that $\overline{f_2}=[f_2,A_1,f_1]$ and $\overline{f_3}^2=[f_3,A_2,f_2]$. We have $g_{3,2}=(\overline{f_2}\cup C1_{X_2})\circ q'^{-1}_{f_1}\simeq(f_2,A_1,f_1)$ and $g_{4,2}=(\overline{f_3}^2\cup C1_{X_3})\circ q'^{-1}_{f_2}\simeq(f_3,A_2,f_2)$ by (4.2). Since $j_{3,2}$ is a homotopy cofibre of $g_{3,2}$ and $\overline{f_3}$ is an extension of $\overline{f_3}^2$ on $C_{3,3}$, we have $\overline{f_3}^2\circ g_{3,2}\simeq *$ by Lemma 4.3(7). Hence

$$[f_3,A_2,f_2]\circ(f_2,A_1,f_1)\simeq *.$$

Since $\overline{f_4}^2:X_4\cup_{f_3}CX_3\to X_5$ is an extension of f_4, there exists $A_3:f_4\circ f_3\simeq *$ such that $\overline{f_4}^2=[f_4,A_3,f_3]$. Since $j_{4,2}$ is a homotopy cofibre of $g_{4,2}$ and $\overline{f_4}$ is an extension of $\overline{f_4}^2$ on $C_{4,3}$, we have $\overline{f_4}^2\circ g_{4,2}\simeq *$ by Lemma 4.3(7), that is,

$$[f_4,A_3,f_3]\circ(f_3,A_2,f_2)\simeq *.$$

Thus $(\vec{\boldsymbol{f}};\vec{\boldsymbol{A}})$ is admissible. Since $\overline{f_4}$ is an extension of f_4 on $X_4\cup_{\overline{f_3}^2}CC_{3,2}$, there exists a homotopy $D:f_4\circ\overline{f_3}^2\simeq *$ such that $\overline{f_4}=[f_4,D,\overline{f_3}^2]$. Let $B:\overline{f_3}^2\circ g_{3,2}\simeq *$ be a homotopy such that $\overline{f_3}\circ a_{3,2}^{-1}=[\overline{f_3}^2,B,g_{3,2}]$. Then

$$\begin{aligned}
g_{4,3}&=(\overline{f_3}\cup C1_{C_{3,2}})\circ\omega_{3,2}^{-1}\\
&\simeq(\overline{f_3}\cup C1_{C_{3,2}})\circ(a_{3,2}^{-1}\cup C1_{C_{3,2}})\circ(a_{3,2}\cup C1_{C_{3,2}})\circ\omega_{3,2}^{-1}\\
&=(\overline{f_3}\circ a_{3,2}^{-1}\circ a_{3,2}\cup C1_{C_{3,2}})\circ\omega_{3,2}^{-1}=([\overline{f_3}^2,B,g_{3,2}]\circ a_{3,2}\cup C1_{C_{3,2}})\circ\omega_{3,2}^{-1}\\
&\simeq(\overline{f_3}^2,B,g_{3,2})\quad(\text{by }(4.2)).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\alpha&=\overline{f_4}\circ g_{4,3}=[f_4,D,\overline{f_3}^2]\circ(\overline{f_3}^2,B,g_{3,2})\\
&\in\{f_4,\overline{f_3}^2,g_{3,2}\}=\{f_4,[f_3,A_2,f_2],(f_2,A_1,f_1)\}.
\end{aligned}$$

This proves (6.7.5) and (6.7.6) holds.

Therefore $\vec{\boldsymbol{\alpha}}$ is admissible if and only if it is $\ddot{s}_t$-presentable and hence $\star$-presentable for every $\star$. Also if $\vec{\boldsymbol{\alpha}}$ is admissible, then (6.7.6) holds for every representative $\vec{\boldsymbol{f}}$ of $\vec{\boldsymbol{\alpha}}$.

In the rest of the proof we suppose that $\vec{\boldsymbol{\alpha}}$ is admissible and $\vec{\boldsymbol{f}} \in \mathrm{Rep}(\vec{\boldsymbol{\alpha}})$. Then $\{\vec{\boldsymbol{\alpha}}\}^{(2)} \subset \{\vec{\boldsymbol{\alpha}}\}^{(\ddot{s}_t)} = \{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)} = \bigcup_{\vec{\boldsymbol{A}}}\{f_4, [f_3, A_2, f_2], (f_2, A_1, f_1)\}$ by (6.7.1), (6.7.6), and Theorem 6.5.1. Since $f_4 = [f_4, A_3, f_3] \circ i_{f_3}$, it follows from [**25**, Proposition 1.2 III)] that

$$\{f_4, [f_3, A_2, f_2], (f_2, A_1, f_1)\} \subset \{[f_4, A_3, f_3], i_{f_3} \circ [f_3, A_2, f_2], (f_2, A_1, f_1)\}$$

so that $\{\vec{\boldsymbol{f}}\}^{(\ddot{s}_t)} \subset \{\vec{\boldsymbol{\alpha}}\}^{(3)}$ by (6.7.2). This completes the proof of Theorem 6.7.1. □

Proof of Corollary 6.7.2. By Theorem 6.7.1, we have

$$\{f_4, f_3, f_2, f_1\}^{(\ddot{s}_t)} \circ \Sigma^2 f_0 = \bigcup \left(\{f_4, [f_3, A_2, f_2], (f_2, A_1, f_1)\} \circ \Sigma^2 f_0\right),$$

where $\bigcup$ is taken over all (A_3, A_2, A_1) such that $(f_4, f_3, f_2, f_1; A_3, A_2, A_1)$ is admissible. Take any such (A_3, A_2, A_1). By the assumption, $f_1 \circ f_0 \simeq *$ and for any $A_0 : f_1 \circ f_0 \simeq *$, we have $[f_2, A_2, f_1] \circ (f_1, A_0, f_0) \simeq *$ so that $(f_3, f_2, f_1, f_0; A_2, A_1, A_0)$ is admissible. Hence $\{f_3, [f_2, A_1, f_1], (f_1, A_0, f_0)\} \subset \{f_3, f_2, f_1, f_0\}^{(\ddot{s}_t)}$ by Theorem 6.7.1. Now we have

$$\begin{aligned}
&\{f_4, [f_3, A_2, f_2], (f_2, A_1, f_1)\} \circ \Sigma^2 f_0 \\
&= -\{f_4, [f_3, A_2, f_2], (f_2, A_1, f_1)\} \circ \Sigma q_{f_1} \circ \Sigma(f_1, A_0, f_0) \\
&\qquad\qquad\qquad\qquad (\text{since } q_{f_1} \circ (f_1, A_0, f_0) \simeq -\Sigma f_0) \\
&\subset -\{f_4, [f_3, A_2, f_2], (f_2, A_1, f_1) \circ q_{f_1}\} \circ \Sigma(f_1, A_0, f_0) \\
&\qquad\qquad\qquad\qquad (\text{by } [\mathbf{25}, \text{ Proposition 1.2(i)}]) \\
&= -\{f_4, [f_3, A_2, f_2], i_{f_2} \circ [f_2, A_1, f_1]\} \circ \Sigma(f_1, A_0, f_0) \ (\text{by } [\mathbf{14}, \ (5.11)]) \\
&\subset -\{f_4, [f_3, A_2, f_2] \circ i_{f_2}, [f_2, A_1, f_1]\} \circ \Sigma(f_1, A_0, f_0) \\
&\qquad\qquad\qquad\qquad (\text{by } [\mathbf{25}, \text{ Proposition 1.2(ii)}]) \\
&= -\{f_4, f_3, [f_2, A_1, f_1]\} \circ \Sigma(f_1, A_0, f_0) \\
&= f_4 \circ \{f_3, [f_2, A_1, f_1], (f_1, A_0, f_0)\} \ (\text{by } [\mathbf{25}, \text{ Proposition 1.4}]).
\end{aligned}$$

Hence we have the assertion. □

Remark 6.7.3. (1) It follows from Theorem 6.7.1, Theorem 6.5.1 and [**15**, Proposition B.6] that if $\vec{\boldsymbol{f}} = (f_4, f_3, f_2, f_1)$ is admissible, then

$$\langle \vec{\boldsymbol{f}} \rangle \supset \bigcup \Big[\{f_4, [f_3, A_2, f_2], (f_2, A_1, f_1)\} \cup \{[f_4, A_3, f_3], (f_3, A_2, f_2), -\Sigma f_1\}\Big],$$

where $\bigcup$ is taken over all $\vec{\boldsymbol{A}} = (A_3, A_2, A_1)$ such that $(\vec{\boldsymbol{f}}; \vec{\boldsymbol{A}})$ is admissible.

(2) When we work in TOP^{clw}, it can be shown that the Walker's 4-fold product of $\vec{\boldsymbol{\alpha}} = (\alpha_4, \alpha_3, \alpha_2, \alpha_1)$ is not empty if and only if $\vec{\boldsymbol{\alpha}}$ is admissible.

6.8. **Proof of (1.10).** We prove the following which is the same as (1.10).

Proposition 6.8.1. (1) *If* $\{f_{n-1},\dots,f_1\}^{(q)} \ni 0$ *and* $\{f_n,\dots,f_k\}^{(aq\ddot{s}_2)} = \{0\}$ *for all* k *with* $2 \le k < n$, *then* $\{f_n,\dots,f_1\}^{(\star)}$ *is not empty for all* $\star$.

(2) (*cf.* [**27**, Lemma 2.2]) *If* $\{f_n,\dots,f_2\}^{(q)} \ni 0$ *and* $\{f_k,\dots,f_1\}^{(aq\ddot{s}_2)} = \{0\}$ *for all* k *with* $2 \le k < n$, *then* $\{f_n,\dots,f_1\}^{(\star)}$ *is not empty for all* $\star$.

Proof. When $n = 3$, the assertions hold by Theorem 6.6.1 and Corollary 6.2.2(4). Hence we suppose $n \ge 4$.

(1) By Corollary 6.2.2(5) and the assumptions, there exists an $aq\ddot{s}_2$-presentation $\{\mathcal{S}_r, \overline{f_r}, \Omega_r \,|\, 2 \le r < n\}$ of $(f_{n-1},\dots,f_1)$ such that $\overline{f_{n-1}} \circ g_{n-1,n-2} \simeq *$. Since $j_{n-1,n-2}$ is a homotopy cofibre of $g_{n-1,n-2}$, there is a homotopy equivalence $e : C_{n-1,n-1} \to C_{n-1,n-2} \cup_{g_{n-1,n-2}} C\Sigma^{n-3}X_1$ such that $e \circ j_{n-1,n-2} = i_{g_{n-1,n-2}}$. Let $\overline{f_{n-1}}' : C_{n-1,n-2} \cup_{g_{n-1,n-2}} C\Sigma^{n-3}X_1 \to X_n$ be an extension of $\overline{f_{n-1}}$. Set $\overline{f_{n-1}}^* = \overline{f_{n-1}}' \circ e : C_{n-1,n-1} \to X_n$, $\mathcal{S}_n = \mathcal{S}_{n-1}(\overline{f_{n-1}}^*, \Omega_{n-1})$, and $\Omega_n = \widetilde{\Omega_{n-1}}$. Since $f_n \circ f_{n-1} \simeq *$ by the assumption $\{f_n, f_{n-1}\}^{(aq\ddot{s}_2)} = \{0\}$, f_n has an extension $\overline{f_n}^2 : C_{n,2} = X_n \cup_{f_{n-1}} CX_{n-1} \to X_{n+1}$. Since $\overline{f_n}^2 \circ g_{n,2}$ represents an element of $\{f_n, f_{n-1}, f_{n-2}\}^{(aq\ddot{s}_2)} = \{0\}$, and since $j_{n,2}$ is a homotopy cofibre of $g_{n,2}$, $\overline{f_n}^2$ has an extension $\overline{f_n}^3 : C_{n,3} = X_n \cup_{\overline{f_{n-1}}^{*2}} CC_{n-1,2} \to X_{n+1}$. We inductively have a map $\overline{f_n} : C_{n,n-1} = X_n \cup_{\overline{f_{n-1}}^{*n-2}} CC_{n-1,n-2} \to X_{n+1}$ which is an extension of f_n. Then the collection obtained from $\{\mathcal{S}_r, \overline{f_r}, \Omega_r \,|\, 2 \le r \le n\}$ by replacing $\overline{f_{n-1}}$ with $\overline{f_{n-1}}^*$ is an $aq\ddot{s}_2$-presentation of $(f_n,\dots,f_1)$. Thus $\{f_n,\dots,f_1\}^{(aq\ddot{s}_2)}$ is not empty and so $\{f_n,\dots,f_1\}^{(\star)}$ is not empty for all $\star$ by Corollary 6.2.2(4).

(2) We set $X'_r = X_{r+1}$ $(1 \le r \le n)$ and $f'_r = f_{r+1}$ $(1 \le r < n)$. By the assumptions and Corollary 6.2.2(5), $\{f'_{n-1},\dots,f'_1\}^{(aq\ddot{s}_2)}$ contains 0. Let $\{\mathcal{S}'_r, \overline{f'_r}, \Omega'_r \,|\, 2 \le r < n\}$ be an $aq\ddot{s}_2$-presentation of $(f'_{n-1},\dots,f'_1)$ with $\overline{f'_{n-1}} \circ g'_{n-1,n-2} \simeq *$. Set $\mathcal{S}_2 = (X_1; X_2, X_2 \cup_{f_1} CX_1; f_1; i_{f_1})$ and $\Omega_2 = \{q'_{f_1}\}$. Let $\overline{f_2} : C_{2,2} = X_2 \cup_{f_1} CX_1 \to X_3$ be an extension of f_2. Set $\mathcal{S}_3 = \mathcal{S}_2(\overline{f_2}, \Omega_2)$ and $\Omega_3 = \widetilde{\Omega_2}$. Then $C_{3,s} = C'_{2,s}$ $(1 \le s \le 2)$, $j_{3,1} = j'_{2,1}$, $g_{3,1} = g'_{2,1}$, and Ω_3 contains Ω'_2. Set $\overline{f_3}^2 = \overline{f'_2} : C_{3,2} \to X_4$. By the assumptions, $\overline{f_3}^2 \circ g_{3,2} \simeq *$ so that $\overline{f_3}^2$ can be extended to a map $\overline{f_3} : C_{3,3} \to X_4$ which is an extension of f_3. Set $\mathcal{S}_4 = \mathcal{S}_3(\overline{f_3}, \Omega_3)$ and $\Omega_4 = \widetilde{\Omega_3}$. Then $C_{4,s} = C'_{3,s}$ $(1 \le s \le 3)$, $j_{4,s} = j'_{3,s}$ $(1 \le s \le 2)$, $g_{4,s} = g'_{3,s}$ $(1 \le s \le 2)$, and Ω_4 contains Ω'_3. When $n = 4$, $\overline{f_4} := \overline{f'_3} : C_{4,3} \to X_5$ is an extension of f_4 and we obtain $aq\ddot{s}_2$-presentation $\{\mathcal{S}_r, \overline{f_r}, \Omega_r \,|\, 2 \le r \le 4\}$ of $(f_4,\dots,f_1)$ so that $\{f_4,\dots,f_1\}^{(\star)}$ is

not empty for all $\star$. When $n \geq 5$, by repeating the above process, we have an $aq\ddot{s}_2$-presentation of $(f_n, \dots, f_1)$. Hence $\{f_n, \dots, f_1\}^{(\star)}$ is not empty for all $\star$. This completes the proof of Proposition 6.8.1. $\square$

6.9. **Stable higher Toda brackets.** A stable n-fold bracket for $n \geq 3$ was defined in [**3**, **27**] (cf. [**10**, **16**]). We will give another definition which is a generalization of [**25**, p.32]. We set $\{X, Y\} = \varinjlim_k [\Sigma^k X, \Sigma^k Y]$. Given $\beta_i \in \{X_i, X_{i+1}\}$ $(1 \leq i \leq n)$, we will define $\{\beta_n, \dots, \beta_1\}^{(\star)} \subset \{\Sigma^{n-2}X_1, X_{n+1}\}$. Take a non-negative integer m such that β_i is represented by $f_i^m : \Sigma^m X_i \to \Sigma^m X_{i+1}$ for all i and set $\vec{\boldsymbol{f}^m} = (f_n^m, \dots, f_1^m)$. The following square is commutative for every integer $M \geq 0$.

$$\begin{array}{ccc}
[\Sigma^m\Sigma^{n-2}X_1, \Sigma^m X_{n+1}] & \xrightarrow{\Sigma^M} & [\Sigma^M\Sigma^m\Sigma^{n-2}X_1, \Sigma^M\Sigma^m X_{n+1}] \\
{\scriptstyle (1_{X_1}\wedge\tau(\mathrm{S}^{n-2},\mathrm{S}^m))^*}\uparrow & & \uparrow{\scriptstyle (1_{X_1}\wedge\tau(\mathrm{S}^{n-2},\mathrm{S}^m)\wedge 1_{\mathrm{S}^M})^*} \\
[\Sigma^{n-2}\Sigma^m X_1, \Sigma^m X_{n+1}] & \xrightarrow{\Sigma^M} & [\Sigma^M\Sigma^{n-2}\Sigma^m X_1, \Sigma^M\Sigma^m X_{n+1}]
\end{array}$$

We have

$$\begin{aligned}
&\Sigma^M(\{\vec{\boldsymbol{f}^m}\}^{(\star)} \circ (1_{X_1} \wedge \tau(\mathrm{S}^{n-2}, \mathrm{S}^m))) \\
&= \Sigma^M\{\vec{\boldsymbol{f}^m}\}^{(\star)} \circ (1_{X_1} \wedge \tau(\mathrm{S}^{n-2}, \mathrm{S}^m) \wedge 1_{\mathrm{S}^M}) \\
&\subset \{\Sigma^M\vec{\boldsymbol{f}^m}\}^{(\star)} \circ (1_{\Sigma^m X_1} \wedge \tau(\mathrm{S}^{n-2}, \mathrm{S}^M)) \circ (1_{X_1} \wedge \tau(\mathrm{S}^{n-2}, \mathrm{S}^m) \wedge 1_{\mathrm{S}^M}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (6.3.1))} \\
&= \{\Sigma^M\vec{\boldsymbol{f}^m}\}^{(\star)} \circ (1_{X_1} \wedge \tau(\mathrm{S}^{n-2}, \mathrm{S}^m \wedge \mathrm{S}^M)).
\end{aligned}$$

Hence the sequence $\{\{\Sigma^M\vec{\boldsymbol{f}^m}\}^{(\star)} \circ (1_{X_1} \wedge \tau(\mathrm{S}^{n-2}, \mathrm{S}^m \wedge \mathrm{S}^M))\}_{M\geq 0}$ defines a subset of $\{\Sigma^{n-2}X_1, X_{n+1}\}$. We denote it by $\{\beta_n, \dots, \beta_1\}^{(\star)}$. It does not depend on the choice of $\vec{\boldsymbol{f}^m}$. For another $\vec{\boldsymbol{f}^k}$, there exist M, K such that $\Sigma^M f_i^m \simeq \Sigma^K f_i^k$ for all i. In this case $m + M = k + K$ and $\{\Sigma^M\vec{\boldsymbol{f}^m}\}^{(\star)} = \{\Sigma^K\vec{\boldsymbol{f}^k}\}^{(\star)}$ by Theorem 6.4.1. Hence $\{\Sigma^M\vec{\boldsymbol{f}^m}\}^{(\star)} \circ (1_{X_1} \wedge \tau(\mathrm{S}^{n-2}, \mathrm{S}^m \wedge \mathrm{S}^M)) = \{\Sigma^K\vec{\boldsymbol{f}^k}\}^{(\star)} \circ (1_{X_1} \wedge \tau(\mathrm{S}^{n-2}, \mathrm{S}^k \wedge \mathrm{S}^K))$. Thus $\{\beta_n, \dots, \beta_1\}^{(\star)}$ is well-defined.

Appendix A. Proof of Proposition 2.2

Given a free space X, we set $\Gamma X = (X \times I)/(X \times \{1\})$ which is called the *unpointed cone* on X and whose point represented by $(x, t) \in X \times I$ is denoted by $x \wedge t$. We regard X as a subspace of ΓX by the identification $x = x \wedge 0$ $(x \in X)$. We set $SX = \Gamma X/X$ which is called the *unpointed suspension* of X and whose point represented by $(x, t) \in X \times I$ is denoted by $x \wedge \bar{t}$. For a map $f : X \to Y$, we define $\Gamma f : \Gamma X \to \Gamma Y$ and $Sf : SX \to SY$ by $\Gamma f(x \wedge t) = f(x) \wedge t$ and $Sf(x \wedge \bar{t}) = f(x) \wedge \bar{t}$, and we denote by $Y \cup_f \Gamma X$

the quotient space of $Y+\Gamma X$ by the equivalence relation generated by the relation $f(x)\sim x\wedge 0$ $(x\in X)$.

Given two free maps $X \xleftarrow{u} A \xrightarrow{v} Y$, let $X \overset{uAv}{+} Y$ denote the quotient space of $X+Y$ by the equivalence relation generated by the relation $u(a)\sim v(a)$ $(a\in A)$. Let $X \xrightarrow{i_X} X+Y \xleftarrow{i_Y} Y$ be the inclusion maps and $q: X+Y\to X \overset{uAv}{+} Y$ the quotient map. Then the following is a push-out diagram in TOP.

(A.1)
$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "u"'] & Y \arrow[d, "q\circ i_Y"] \\ X \arrow[r, "q\circ i_X"] & X \overset{uAv}{+} Y \end{tikzcd}$$

The space $X \overset{uAi}{+} \Gamma A$ which is induced from $X \xleftarrow{u} A \overset{i}{\subset} \Gamma A$ is denoted by $X\cup_u \Gamma A$ and called the *unpointed mapping cone* of u.

Lemma A.1. *Given the push-out diagram (A.1), if u is a free (resp. closed free) cofibration, then $q\circ i_Y$ is a free (resp. closed free) cofibration.*

Proof. Suppose that u is a free cofibration. Then, as is well-known (for example [**4**, (5.1.8)]), $q\circ i_Y$ is a free cofibration. Since u is injective, the equality $q^{-1}(q\circ i_Y(B)) = u(v^{-1}(B))+B$ holds for every subset B of Y. Hence if u is closed then $q\circ i_Y$ is closed. □

Note that, given a pointed map $f: X\to Y$, we have the following commutative diagram in which all maps are quotient maps.

$$\begin{tikzcd} Y+X\times I \arrow[r, "1_Y+q"] \arrow[dr, "1_Y+q"'] & Y+\Gamma X \arrow[r, "q"] \arrow[d, "1_Y+q"] & Y\cup_f \Gamma X \arrow[d, "1_Y\cup q"] \\ & Y+CX \arrow[r, "q"] & Y\cup_f CX \end{tikzcd}$$

Proof of Proposition 2.2. We can suppose that $j: A\subset X$ by [**20**, Theorem 1]. Consider the following commutative diagram.

$$\begin{tikzcd} Y+A\times I \arrow[r, "\pi_1"] \arrow[d, "i"'] & Y\cup_{f\circ j}\Gamma A \arrow[r, "q_1"] \arrow[dd, "1_Y\cup\Gamma j"] & Y\cup_{f\circ j} CA \arrow[dd, "1_Y\cup Cj"] \\ Y+X\times\partial I\cup A\times I \arrow[d, "i'"'] \arrow[ur, dotted, "\varphi_f"'] & & \\ Y+X\times I \arrow[r, "\pi_2"] & Y\cup_f \Gamma X \arrow[r, "q_2"] & Y\cup_f CX \end{tikzcd}$$

where π_1, π_2, q_1, q_2 are quotient maps, i, i' are inclusions, and φ_f is defined by

$$\varphi_f|_Y = \pi_1|_Y, \quad \varphi_f(x,0) = \pi_1 f(x),$$
$$\varphi_f(x,1) = \pi_1(a,1)\ (a \in A), \quad \varphi_f|_{A\times I} = \pi_1|_{A\times I}.$$

First we show that φ_f is continuous. By [**21**, Theorem 2], $X \times \{0\} \cup A \times I$ and $X \times \{1\} \cup A \times I$ are retracts of $X \times I$. Therefore, since φ_f is continuous on the subspaces $X \times \{0\}$, $X \times \{1\}$, $A \times I$ of the space $X \times I$, it follows from [**21**, Lemma 3] that φ_f is continuous on the subspaces $X \times \{0\} \cup A \times I$, $X \times \{1\} \cup A \times I$ of the space $X \times I$. Hence φ_f is continuous on the open subspaces $X \times \{0\} \cup A \times [0,1)$, $X \times \{1\} \cup A \times (0,1]$ of the space $X \times \partial I \cup A \times I$. Therefore φ_f is continuous on $X \times \{0\} \cup A \times [0,1) \cup X \times \{1\} \cup A \times (0,1] = X \times \partial I \cup A \times I$ so that φ_f is continuous.

Secondly we show that $(\pi_2, 1_Y \cup \Gamma j)$ is a push-out of (i', φ_f) in TOP. For any space Z and any maps $Y + X \times I \overset{g}{\to} Z \overset{h}{\leftarrow} Y \cup_{f\circ j} \Gamma A$ such that $g \circ i' = h \circ \varphi_f$, there exists only one map $k : Y \cup_f \Gamma X \to Z$ with $k \circ \pi_2 = g$. It is obvious that $k \circ (1_Y \cup \Gamma j) = h$. Hence $(\pi_2, 1_Y \cup \Gamma j)$ is a push-out of (i', φ_f) in TOP.

Thirdly we show that $1_Y \cup \Gamma j$ is a free (resp. closed free) cofibration. By the last assertion and Lemma A.1, it suffices to show that i' is a free (resp. closed free) cofibration. Since the inclusion $\partial I \subset I$ is a closed free cofibration, it follows from [**21**, Theorem 6] that the inclusion $X \times \partial I \cup A \times I \subset X \times I$ is a free (resp. closed free) cofibration so that i' is a free (resp. closed free) cofibration.

Fourthly we show that $1_Y \cup Cj : Y \cup_{f\circ j} CA \to Y \cup_f CX$ is a free (resp. closed free) cofibration. By Lemma A.1 it suffices to show that the last square of the above diagram is a push-out in TOP. Let Z be any space and $Y \cup_f \Gamma X \overset{g}{\to} Z \overset{h}{\leftarrow} Y \cup_{f\circ j} CA$ any maps such that $g \circ (1_Y \cup \Gamma j) = h \circ q_1$. Then there is only one map $k : Y \cup_f CX \to Z$ with $k \circ q_2 = g$. It is obvious that $k \circ (1_Y \cup Cj) = h$. Hence the last square of the above diagram is a push-out in TOP. □

The following corollary overlaps with [**9**, (6.13)].

Corollary A.2. *If $j : A \to X$ is a free (resp. closed free) cofibration, then $\Gamma j : \Gamma A \to \Gamma X$ and $Sj : SA \to SX$ are free (resp. closed free) cofibrations. If in addition j is pointed, then $Cj : CA \to CX$ is a free (resp. closed free) cofibration.*

Proof. We can suppose that $j : A \subset X$. Consider the commutative diagram

$$\begin{tikzcd}
A + A \times I \arrow[r, "q_1"] \arrow[d, "i_1"'] & A \cup_{1_A} \Gamma A = \Gamma A \arrow[d, "j\cup\Gamma 1_A"] & \\
X + A \times I \arrow[r, "q_2"] \arrow[d, "i_2"'] & X \cup_j \Gamma A \arrow[r, "/X"] \arrow[dd, "1_X\cup\Gamma j"] & SA \arrow[dd, "Sj"] \\
X + X \times \partial I \cup A \times I \arrow[ur, dotted, "\varphi_{1_X}"'] \arrow[d, "i_3"'] & & \\
X + X \times I \arrow[r, "q_3"] & X \cup_{1_X} \Gamma X = \Gamma X \arrow[r, "/X"] & SX
\end{tikzcd}$$

where i_1, i_2, i_3 are inclusions, q_1, q_2, q_3 are quotients, and φ_{1_X} is defined as in the proof of Proposition 2.2. When we take off i_2, the remaining three squares of the diagram are push-outs in TOP. Since i_1, i_3 are free (resp. closed free) cofibrations, it follows that $j \cup \Gamma 1_A, 1_X \cup \Gamma j$ are free (resp. closed free) cofibrations so that $\Gamma j = (1_X \cup \Gamma j) \circ (j \cup \Gamma 1_A)$ and Sj are free (resp. closed free) cofibrations.

Suppose that j is pointed. Consider the commutative diagram

$$\begin{tikzcd}
A \cup_{1_A} \Gamma A = \Gamma A \arrow[r, "q_4"] \arrow[d, "j\cup\Gamma 1_A"'] & A \cup_{1_A} CA = CA \arrow[d, "j\cup C1_A"] \\
X \cup_j \Gamma A \arrow[r, "q_5"] \arrow[d, "1_X\cup\Gamma j"'] & X \cup_j CA \arrow[d, "1_X\cup Cj"] \\
X \cup_{1_X} \Gamma X \arrow[r, "q_6"] & X \cup_{1_X} CX = CX
\end{tikzcd}$$

where q_4, q_5, q_6 are quotients. Since the two squares of the diagram are push-outs in TOP, $Cj = (1_X \cup Cj) \circ (j \cup C1_A)$ is a free (resp. closed free) cofibration by Lemma A.1. This completes the proof of Corollary A.2. $\square$

Appendix B. J. Cohen's higher Toda brackets

First we recall from [**21**, Theorem 2] that the inclusion $j : X \subset Y$ is a free cofibration if and only if there exists a retraction $r : Y \times I \to Y \times \{0\} \cup X \times I$. When j is pointed, the pointed map $\delta : Y/X \to \Sigma X$ which makes the following diagram to be commutative was called "canonical" in [**3**].

$$\begin{tikzcd}
Y \arrow[r, "i_1^Y"] \arrow[d, "q"'] & Y \times I \arrow[r, "r"] & Y \times \{0\} \cup X \times I \arrow[d, "p"] \\
Y/X \arrow[rr, "\delta"'] & & \Sigma X
\end{tikzcd}$$

Here q is the quotient, $p(Y \times \{0\}) = *$, and $p(x, t) = x \wedge \bar{t}$. An important property of δ is the following.

Lemma B.1. *The canonical map δ is a connecting map in the cofibre sequence*

$$X \xrightarrow{j} Y \xrightarrow{q} Y/X \xrightarrow{\delta} \Sigma X \xrightarrow{-\Sigma j} \Sigma Y \xrightarrow{-\Sigma q} \cdots .$$

Proof. Consider the following diagram, where π is the usual homotopy equivalence [**17**, Satz 2].

(B.1)

$$\begin{tikzcd}
X \arrow[r, "j"] & Y \arrow[r, "i_j"] \arrow[d, equal] & Y \cup_j CX \arrow[r, "i_{i_j}"] \arrow[d, "\simeq"', "\pi"] & (Y \cup_j CX) \cup_{i_j} CY \arrow[r, "i_{i_{i_j}}"] \arrow[d, "\simeq"', "q'_j"] & \cdots \\
& Y \arrow[r, "q"] & Y/X \arrow[r, "\delta"] & \Sigma X \arrow[r, "-\Sigma j"] & \cdots
\end{tikzcd}$$

We define $u : I \times I \to I$ and $H : (Y \cup_j CX) \times I \to \Sigma X$ by

$$u(s,t) = \begin{cases} s+t & s+t \leq 1 \\ 1 & s+t \geq 1 \end{cases}, \quad H(y,t) = p \circ r(y,t), \quad H(x \wedge s, t) = x \wedge \overline{u(s,t)}.$$

Then $H : q'_j \circ i_{i_j} \simeq \delta \circ \pi$. Hence the second square of (B.1) is homotopy commutative. Since the first square of (B.1) is commutative, this completes the proof. □

J. Cohen [**3**] defined an n-fold bracket $\langle \vec{\boldsymbol{f}} \rangle$ in the category TOP^*, where $\vec{\boldsymbol{f}} = (f_n, \ldots, f_1)$ and $f_i : X_i \to X_{i+1}$ is a map in TOP^* ($1 \leq i \leq n$; $n \geq 3$). We are going to modify $\langle \vec{\boldsymbol{f}} \rangle$ to $\langle \vec{\boldsymbol{f}} \rangle^w$ (resp. $\langle \vec{\boldsymbol{f}} \rangle^{clw}$) by restricting TOP^* to its full-subcategory TOP^w (resp. TOP^{clw}).

Let $\circledast$ denote $*$, w or clw.

By $\vec{\boldsymbol{f}} = (f_n, \ldots, f_1) \in \mathrm{TOP}^{\circledast}$, we mean that $f_i : X_i \to X_{i+1}$ is in $\mathrm{TOP}^{\circledast}$ for every i. To avoid confusions, we paraphrase Cohen's expression "$X \in \{f_{n-1}, \ldots, f_2\}$" in "$X$ is a finitely filtered space of type $(f_{n-1}, \ldots, f_2)$" [**15**]. Given $(f_{n-1}, \ldots, f_2) \in \mathrm{TOP}^{\circledast}$, where $f_i : X_i \to X_{i+1}$, a pointed space X is a *finitely filtered space of type* $(f_{n-1}, \ldots, f_2)$ *in* $\mathrm{TOP}^{\circledast}$ if the following (1) and (2) are satisfied.

(1) The pointed space X has a filtration $F_0X = \{*\} \subset F_1X \subset \cdots \subset F_{n-1}X = X$ such that the inclusion $F_kX \subset F_{k+1}X$ is a free, free or closed free cofibration for every k according as $\circledast$ is $*$, w or clw. (Hence $X, F_kX \in \mathrm{TOP}^w$.)

(2) There exists $g_k : \Sigma^k X_{n-k} \simeq F_{k+1}X/F_kX$ for $0 \leq k \leq n-2$ such that the next diagram is homotopy commutative for $1 \leq k \leq n-2$.

(B.2)
$$\begin{tikzcd}
\Sigma\Sigma^{k-1}X_{n+1-k} \arrow[d, "\Sigma g_{k-1}"'] & & \Sigma^k X_{n-k} \arrow[ll, "\Sigma^k f_{n-k}"'] \arrow[d, "g_k"] \\
\Sigma(F_kX/F_{k-1}X) & \Sigma F_kX \arrow[l, "\Sigma q"] & F_{k+1}X/F_kX \arrow[l, "\delta"]
\end{tikzcd}$$

Under the above situation, we set

(B.3)
$$\begin{cases} j_X : X_n = \Sigma^0 X_n \xrightarrow{g_0} F_1X \subset X, \\ \sigma_X : X = F_{n-1}X \xrightarrow{q} F_{n-1}X/F_{n-2}X \xrightarrow{g_{n-2}^{-1}} \Sigma^{n-2}X_2. \end{cases}$$

We define $\langle \vec{\boldsymbol{f}} \rangle^{\circledast}$ for $\vec{\boldsymbol{f}} \in \mathrm{TOP}^{\circledast}$ to be the set of all $\alpha \in [\Sigma^{n-2}X_1, X_{n+1}]$ such that there is a finitely filtered space X of type $(f_{n-1}, \ldots, f_2)$ in $\mathrm{TOP}^{\circledast}$ and a couple of maps g, h which make the following diagram homotopy commutative and α is the homotopy class of $h \circ g$.

(B.4)
$$\begin{tikzcd}
 & \Sigma^{n-2}X_1 \arrow[dl, "\Sigma^{n-2}f_1"'] \arrow[d, "g"] & \\
\Sigma^{n-2}X_2 & X \arrow[l, "\sigma_X"] \arrow[d, "h"'] & X_n \arrow[l, "j_X"'] \arrow[dl, "f_n"] \\
 & X_{n+1} &
\end{tikzcd}$$

Note that $\langle \vec{\boldsymbol{f}} \rangle^{*}$ is the bracket $\langle \vec{\boldsymbol{f}} \rangle$ defined by Cohen, $\langle \vec{\boldsymbol{f}} \rangle^{w} = \langle \vec{\boldsymbol{f}} \rangle^{*}$ if $\vec{\boldsymbol{f}} \in \mathrm{TOP}^w$, and $\langle \vec{\boldsymbol{f}} \rangle^{clw} \subset \langle \vec{\boldsymbol{f}} \rangle^{w} = \langle \vec{\boldsymbol{f}} \rangle^{*}$ if $\vec{\boldsymbol{f}} \in \mathrm{TOP}^{clw}$. If $\vec{\boldsymbol{f}} = (f_n, \ldots, f_1)$ and $\vec{\boldsymbol{f}}' = (f_n', \ldots, f_1')$ are in $\mathrm{TOP}^{\circledast}$ and satisfy $f_i \simeq f_i'$ for all i, and if X is a finitely filtered space of type $(f_{n-1}, \ldots, f_2)$ in $\mathrm{TOP}^{\circledast}$, then X is a finitely filtered space of type $(f_{n-1}', \ldots, f_2')$ in $\mathrm{TOP}^{\circledast}$ so that $\langle \vec{\boldsymbol{f}} \rangle^{\circledast} = \langle \vec{\boldsymbol{f}}' \rangle^{\circledast}$.

The following holds obviously from definitions.

Proposition B.2. *Given a map $f_0 : X_0 \to X_1$ in $\mathrm{TOP}^{\circledast}$, we have*

$$\langle f_n, \ldots, f_1 \rangle^{\circledast} \circ \Sigma^{n-2} f_0 \subset \langle f_n, \ldots, f_2, f_1 \circ f_0 \rangle^{\circledast}.$$

Lemma B.3. *Given $\vec{\boldsymbol{f}} \in \mathrm{TOP}^{\circledast}$, if X is a finitely filtered space of type $(f_{n-1}, \dots, f_2)$ in $\mathrm{TOP}^{\circledast}$, then ΣX is a finitely filtered space of type $(\Sigma f_{n-1}, \dots, \Sigma f_2)$ in $\mathrm{TOP}^{\circledast}$ such that $j_{\Sigma X} = \Sigma j_X : \Sigma X_n \to \Sigma X$ and $\sigma_{\Sigma X} \simeq (-1)^n \Sigma \sigma_X : \Sigma X \to \Sigma^{n-1} X_2$.*

Proof. From the definitions, there is a filtration $F_0 X = \{*\} \subset F_1 X \subset \cdots \subset F_{n-1} X = X$ with $F_{k-1} X \subset F_k X$ a free or closed free cofibration for $k \geq 1$ and maps j_X, σ_X of (B.3). Define $F_k \Sigma X = \Sigma F_k X$ for $0 \leq k \leq n-1$. Set $g_k^* = \Sigma g_k \circ (1_{X_{n-k}} \wedge \tau(\mathrm{S}^1, \mathrm{S}^k))$ for $0 \leq k \leq n-2$. Then g_k^* is a homotopy equivalence and $g_k^* \simeq (-1)^k \Sigma g_k$ under the identification (2.1): $\Sigma^k \Sigma X_{n-k} = \Sigma \Sigma^k X_{n-k}$. By suspending (B.2), the diagram

$$\begin{tikzcd}
 & \Sigma^k \Sigma X_{n+1-k} \arrow[d, "1_{X_{n+1-k}} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{k-1} \wedge \mathrm{S}^1)"'] & \Sigma^k \Sigma X_{n-k} \arrow[l, "\Sigma^k \Sigma f_{n-k}"'] \arrow[d, "1_{X_{n-k}} \wedge \tau(\mathrm{S}^1, \mathrm{S}^k)"'] \arrow[dd, bend left, "g_k^*"] \\
\Sigma \Sigma^{k-1} \Sigma X_{n+1-k} \arrow[d, "\Sigma g_{k-1}^*"'] & \Sigma \Sigma^k X_{n+1-k} \arrow[l, "1_{X_{n+1-k}} \wedge \tau(\mathrm{S}^{k-1}, \mathrm{S}^1) \wedge 1_{\mathrm{S}^1}"] \arrow[d, "\Sigma^2 g_{k-1}"] & \Sigma \Sigma^k X_{n-k} \arrow[l, "\Sigma \Sigma^k f_{n-k}"] \arrow[d, "\Sigma g_k"'] \\
\Sigma^2 (F_k X / F_{k-1} X) \arrow[r, equal] & \Sigma^2 (F_k X / F_{k-1} X) & \Sigma (F_{k+1} X / F_k X) \arrow[l, "\Sigma^2 q_{k-1} \circ \Sigma \delta_k"']
\end{tikzcd}$$

is homotopy commutative for $1 \leq k \leq n-2$, where δ_k is a connecting map of the cofibre sequence $F_k X \overset{j_k}{\subset} F_{k+1} X \overset{q_k}{\to} F_{k+1} X / F_k X$ and $q_{k-1} : F_k X \to F_k X / F_{k-1} X$ is the quotient map. Set

$$\delta_k^* = -\Sigma \delta_k : F_{k+1} \Sigma X / F_k \Sigma X = \Sigma (F_{k+1} X / F_k X) \to \Sigma F_k \Sigma X = \Sigma^2 F_k X$$

which is a connecting map of the cofibre sequence $F_k \Sigma X \xrightarrow{-\Sigma j_k} F_{k+1} \Sigma X \xrightarrow{-\Sigma q_k} F_{k+1} \Sigma X / F_k \Sigma X$. We have

$$\begin{aligned}
&\Sigma^2 q_{k-1} \circ \delta_k^* \circ g_k^* = \Sigma^2 q_{k-1} \circ (-\Sigma \delta_k) \circ g_k^* \\
&\simeq (-1_{\Sigma (F_k \Sigma X / F_{k-1} \Sigma X)}) \circ \Sigma^2 q_{k-1} \circ \Sigma \delta_k \circ g_k^* \\
&\simeq (-1_{\Sigma (F_k \Sigma X / F_{k-1} \Sigma X)}) \circ \Sigma^2 g_{k-1} \circ (1_{X_{n+1-k}} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{k-1} \wedge \mathrm{S}^1)) \circ \Sigma^k \Sigma f_{n-k} \\
&= (-1_{\Sigma (F_k \Sigma X / F_{k-1} \Sigma X)}) \circ \Sigma g_{k-1}^* \circ (1_{X_{n+1-k}} \wedge \tau(\mathrm{S}^{k-1}, \mathrm{S}^1) \wedge 1_{\mathrm{S}^1}) \\
&\qquad\qquad \circ (1_{X_{n+1-k}} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{k-1} \wedge \mathrm{S}^1)) \circ \Sigma^k \Sigma f_{n-k} \\
&\simeq \Sigma g_{k-1}^* \circ (-1_{\Sigma \Sigma^{k-1} \Sigma X}) \circ (1_{X_{n+1-k}} \wedge \tau(\mathrm{S}^{k-1}, \mathrm{S}^1) \wedge 1_{\mathrm{S}^1}) \\
&\qquad\qquad \circ (1_{X_{n+1-k}} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{k-1} \wedge \mathrm{S}^1)) \circ \Sigma^k \Sigma f_{n-k} \\
&\simeq \Sigma g_{k-1}^* \circ \Sigma^k \Sigma f_{n-k}.
\end{aligned}$$

Hence ΣX is a finitely filtered space of type $(\Sigma f_{n-1}, \dots, \Sigma f_2)$ such that $j_{\Sigma X} = \Sigma j_X$ and $\sigma_{\Sigma X} = g_{n-2}^{*-1} \circ \Sigma q_{n-2} \simeq (-1)^{n-2} \Sigma g_{n-2}^{-1} \circ \Sigma q_{n-2} = (-1)^n \Sigma \sigma_X$. □

Theorem B.4. $\Sigma\langle \vec{\boldsymbol{f}} \rangle^{\circledast} \subset (-1)^n \langle \Sigma \vec{\boldsymbol{f}} \rangle^{\circledast}$.

Proof. Take $\alpha \in \langle \vec{\boldsymbol{f}} \rangle^{\circledast}$. Then there is a finitely filtered space X of type $(f_{n-1}, \dots, f_2)$ in $\mathrm{TOP}^{\circledast}$ and the homotopy commutative diagram (B.4) with $\alpha = h \circ g$. Suspending (B.4), we have the next homotopy commutative diagram, where $\tau = 1_{X_2} \wedge \tau(\mathrm{S}^{n-2}, \mathrm{S}^1)$.

$$\begin{tikzcd}
 & & \Sigma^{n-2}\Sigma X_1 \arrow[dll, "\Sigma^{n-2}\Sigma f_1"'] \arrow[d, "1_{X_1}\wedge\tau(\mathrm{S}^1,\mathrm{S}^{n-2})"] & \\
 & & \Sigma\Sigma^{n-2}X_1 \arrow[dl, "\Sigma\Sigma^{n-2}f_1"'] \arrow[d, "\Sigma g"] & \\
\Sigma^{n-2}\Sigma X_2 & \Sigma\Sigma^{n-2}X_2 \arrow[l, "\tau"] & \Sigma X \arrow[l, "\Sigma\sigma_X"] \arrow[d, "\Sigma h"'] & \Sigma X_n \arrow[l, "\Sigma j_X"'] \arrow[dl, "\Sigma f_n"] \\
 & & \Sigma X_{n+1} &
\end{tikzcd}$$

Since ΣX is a finitely filtered space of type $(\Sigma f_{n-1}, \dots, \Sigma f_2)$ in $\mathrm{TOP}^{\circledast}$, we have $\tau \circ \Sigma\sigma_X \simeq \sigma_{\Sigma X}$ and $\Sigma j_X = j_{\Sigma X}$ by Lemma B.3. It follows from the homotopy commutativity of the above diagram that

$$\sigma_{\Sigma X} \circ \Sigma g \circ (1_{X_1} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n-2})) \simeq \Sigma^{n-2}\Sigma f_1,$$
$$(-1)^n \Sigma\alpha = \Sigma h \circ \Sigma g \circ (1_{X_1} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n-2})) \in \langle \Sigma \vec{\boldsymbol{f}} \rangle^{\circledast},$$

and so $\Sigma\alpha \in (-1)^n \langle \Sigma \vec{\boldsymbol{f}} \rangle^{\circledast}$. This completes the proof. □

The following lemma was used in the proof of Lemma 6.4.2. It can be proved easily, so we omit details.

Lemma B.5. *Let $j : A \subset X$ be a pointed inclusion map which is a free cofibration, and $f : X \to Y$ a pointed map. Then there are natural homeomorphisms*

$$(Y \cup_f CX)/(Y \cup_{f\circ j} CA) \approx \Sigma X/\Sigma A \approx \Sigma(X/A).$$

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References

[**1**] D. Blanc, *Higher homotopy operations and the realizability of homotopy groups*, Proc. London Math. Soc., **70** (1995), 214–240.

[**2**] D. Blanc and M. Markl, *Higher homotopy operations*, Math. Z. **345** (2003), 1–29.

[**3**] J. M. Cohen, *The decomposition of stable homotopy*, Ann. of Math. **87** (1968), 305–320.

[**4**] T. tom Dieck, Algebraic Topology, EMS Textbooks in Math., European Math. Soc., 2008.

[**5**] T. tom Dieck, K. H. Kamps and D. Puppe, Homotopietheorie, Lecture Notes in Math. **157**, Springer-Verlag, 1970.

[**6**] A. Dold, Halbexakte Homotopiefunktoren, Lecture Notes in Math. **12**, Springer-Verlag, 1966.

[**7**] H. H. Gershenson, *Higher composition products*, J. Math. Kyoto Univ. **5** (1965), 1–37.

[**8**] K. A. Hardie, H. J. Marcum and N. Oda, *Bracket operations in the homotopy theory of a 2-category*, Rend. Ist. Mat. Univ. Trieste **33** (2001), 19–70.

[**9**] I. M. James, General topology and homotopy theory, Springer-Verlag, New York, 1984.

[**10**] S. O. Kochman, *Uniqueness of Massey products on the stable homotopy of spheres*, Can. J. Math. **32** (1980), 576–589.

[**11**] A. F. Lawrence, Matric Massey products and matric Toda brackets in the Adams spectral sequence, Ph.D Theses, Univ. Chicago, 1969.

[**12**] H. J. Marcum and N. Oda, *Long box bracket operations in homotopy theory*, Appl. Categor. Strut. **19** (2011), 137–173.

[**13**] M. Mori, *On higher Toda brackets*, Bull. Colledge of Sci. Univ. of the Ryukyus **35** (1983), 1–4.

[**14**] K, Ôguchi, *A generalization of secondary composition and its applications*, J. Fac. Sci. Univ. Tokyo **10** (1963), 29–79.

[**15**] H. Ōshima and K. Ōshima, *Quasi tertiary compositions and a Toda bracket in homotopy groups of* SU(3), Math. J. Okayama Univ. **57** (2015), 13–78.

[**16**] G. J. Porter, *Higher products*, Trans. Amer. Math. Soc. **148** (1970), 315–345.

[**17**] D. Puppe, *Homotopiemengen und ihre induzierten Abbildungen I*, Math. Z. **69** (1958), 299–344.

[**18**] E. H. Spanier, *Secondary operations on mappings and cohomology*, Ann. of Math. **75** (1962), 260–282.

[**19**] E. H. Spanier, *Higher order operations*, Trans. Amer. Math. Soc. **109** (1963), 509–539.

[**20**] A. Strøm, *Note on cofibrations*, Math. Scand. **19** (1966), 11–14.

[**21**] A. Strøm, *Note on cofibrations II*, Math. Scand. **22** (1969), 130–142.

[**22**] A. Strøm, *The homotopy category is a homotopy category*, Arch. Math. **23** (1972), 435–441.

[**23**] H. Toda, *Generalized Whitehead products and homotopy groups of spheres*, J. Inst. Poly. Osaka City Univ. **3** (1952), 43–82.

[**24**] H. Toda, *p-primary components of homotopy groups IV. Compositions and toric constructions*, Mem. Coll. Sci. Univ. Kyoto, Ser. A **32** (1959), 297–332.

[**25**] H. Toda, Composition methods in homotopy groups of spheres, Ann. of Math. Studies **49**, Princeton, 1962.

[**26**] G. Walker, *Long Toda brackets*, Proc. Adv. Studies Inst. Alg. Top., Aarhus Univ. Mat. Inst. Various Publ. Ser. **13** (1970), 612–631.

[27] G. Walker, Toda brackets and the odd primary homotopy of complex Stiefel manifolds, Manchester Centre for Pure Mathematics, Preprint No 1994/02.

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