Math. J. Okayama Univ. **61** (2019), 199–204

TERWILLIGER ALGEBRAS OF SOME GROUP ASSOCIATION SCHEMES

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Abstract. The Terwilliger algebra plays an important role in the theory of association schemes. The present paper gives the explicit structures of the Terwilliger algebras of the group association schemes of the finite groups $PSL(2,7)$, A_6, and S_6.

1. Introduction

Association schemes enable us to study combinatorial problems in a unified way. We refer to [2, 6] for the foundations of association schemes. In a series of papers [10, 11, 12], Terwilliger introduced a new method, the so-called Terwilliger algebra, to investigate the commutative association schemes. Since then there have been many investigations on Terwilliger algebras (*cf.* [8, 7]). It is very important to know the explicit structure of the Terwilliger algebra. The cases of the group association schemes of S_5 and A_5 were studied in [1] along the line of the work [3]. In the present paper we determine the structures of the Terwilliger algebras of the group association schemes of the finite groups $PSL(2,7)$, A_6, and S_6.

The computations were done with Magma [5] and SageMath [9].

2. Preliminaries

We begin with the definition of a group association scheme.

Definition 1. Let G be a finite group and $C_0 = \{e\}, C_1, \ldots, C_d$ the conjugacy classes of G, where e is the identity of G. Define the relations $R_i (i = 0, 1, \ldots, d)$ on G by

$$(x, y) \in R_i \Longleftrightarrow yx^{-1} \in C_i.$$

Then $\mathfrak{X}(G) = (G, \{R_i\}_{0 \le i \le d})$ forms a commutative association scheme of class d called the *group association scheme of* G.

We associate the matrix A_i of the relation R_i as

$$(A_i)_{x,y} := \begin{cases} 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

Mathematics Subject Classification. Primary 35C07; Secondary 35K57.

Key words and phrases. Terwilliger algebra, group association scheme.

Then we have

$$A_iA_j = \sum_{k=0}^{d} p_{ij}^k A_k$$

and $A_0, \ldots, A_d$ generate the so-called *Bose-Mesner algebra* $\mathfrak{A}$. The intersection numbers p_{ij}^k of the group association scheme $\mathfrak{X}(G)$ are given by

$$|\{(x,y) \in C_i \times C_j | xy = z, z \in C_k\}|.$$

The algebra $\mathfrak{A}$ has a second basis $E_0, \ldots, E_d$ of primitive idempotents, and

$$E_i \circ E_j = \frac{1}{|G|} q_{ij}^k E_k,$$

where $\circ$ denotes Hadamard (entry-wise) multiplication. For each $i = 0, \ldots, d$, let E_i^* and A_i^* be the diagonal matrices of size $|G| \times |G|$ which are defined as follows.

$$(E_i^*)_{x,x} := \begin{cases} 1, & \text{if } x \in C_i \\ 0, & \text{if } x \notin C_i \end{cases} \qquad (x \in G)\,,$$

$$(A_i^*)_{x,x} := |G|(E_i)_{e,x} \qquad (x \in G).$$

Then $E_0^*, \ldots, E_d^*$ form a basis for the *dual Bose-Mesner algebra* $\mathfrak{A}^*$. The intersection numbers provide information for our structural results to follow. We refer to the following relations [10].

$$\begin{aligned} E_i^* A_j E_k^* = 0 \quad &\Leftrightarrow \quad p_{ij}^k = 0 \quad (0 \le i,j,k \le d), \\ E_i A_j^* E_k = 0 \quad &\Leftrightarrow \quad q_{ij}^k = 0 \quad (0 \le i,j,k \le d). \end{aligned}$$

We need to fix the ordering of the conjugacy classes. The following table gives the representatives and the orders of conjugacy classes.

(1) $PSL(2,7)$

	C_0	C_1	C_2
rep.	(1)	(357)(468)	(2354786)
$\lvert C_i\rvert$:	1	56	24

	C_3	C_4	C_5
rep.	(2465837)	(12)(34)(58)(67)	(1235)(4876)
$\lvert C_i\rvert$:	24	21	42

(2) A_6

	C_0	C_1	C_2	C_3	C_4
rep.	(1)	(12)(34)	(123)	(123)(456)	(1234)(56)
$\lvert C_i\rvert$	1	45	40	40	90

	C_5	C_6
rep.	(12345)	(12346)
$\lvert C_i\rvert$	72	72

(3) S_6

	C_0	C_1	C_2	C_3
rep.	(1)	(12)	(12)(34)	(12)(34)(56)
$\lvert C_i \rvert$	1	15	45	15
	C_4	C_5	C_6	C_7
rep.	(123)	(123)(45)	(123)(456)	(1234)
	40	120	40	90
	C_8	C_9	C_{10}	
	(1234)(56)	(12345)	(123456)	
	90	144	120	

Finally we give the definition of the Terwilliger algebra of the group association scheme. We shall denote by $\mathcal{M}_k$ the ring of $k \times k$ matrices over the complex number $\mathbf{C}$.

Definition 2. Let G be a finite group. The *Terwilliger algebra* $T(G)$ of the group association scheme $\mathfrak{X}(G)$ is a sub-algebra of $\mathcal{M}_{|G|}$ generated by $\mathfrak{A}$ and $\mathfrak{A}^*$.

Since $T(G)$ is closed under the conjugate-transpose, $T(G)$ is semi-simple. In the next section, we investigate the Terwilliger algebras of the group association schemes of $PSL(2,7)$, A_6 and S_6.

3. Results

In [1], Balmaceda and Oura gave the structures of the Terwilliger algebra of the group association schemes of S_5 and A_5. Following their method, we determine the Terwilliger algebras for the cases $PSL(2,7)$, A_6, and S_6.

Theorem 3.1. *The dimensions of* $T(PSL(2,7))$, A_6 *and* $T(S6)$ *are given as follows.*

$$\begin{aligned} \dim T(PSL(2,7)) &= 165, \\ \dim T(A_6) &= 336, \\ \dim T(S_6) &= 758. \end{aligned}$$

Proof. We compute a set of linearly independent elements among $E_i^* A_j E_k^*$ and $E_i^* A_j E_k^* \cdot E_k^* A_l E_m^* = E_i^* A_j E_k^* A_l E_m^*$. By direct calculation we can see that any form $E_i^* A_{i_1} E_j^* \cdot E_j^* A_{i_2} E_k^* \cdot E_k^* A_{i_3} E_l^*$ linearly depends on the $E_i^* A_{i_4} E_l^*$'s and the $E_i^* A_{i_5} E_{k_1}^* \cdot E_{k_1}^* A_{i_6} E_l^*$'s. Therefore the products of more than two elements of the form $E_i^* A_j E_k^*$ give no new elements of a basis. [1] $\square$

[1] This answers a question raised by Prof.Terwilliger. Indeed our original argument had a gap. He informed us the reference [4]. Our result $\dim T(S_6) = 758$ is violated to Conjecture 3.5.

We provide the matrices below to show how many elements of a basis occur. As these matrices are symmetric, we omit the entries below diagonal. These matrices are indexed by the conjugacy classes in the order assumed earlier. The entries of matrices indicate the dimension of each position. For example, the entry 6 in the (C_2, C_2)-position for the group $PSL(2,7)$ comes from the dimension of subspace that is the product of entry $E_2^*A_iE_j^*$ and $E_k^*A_lE_2^*$. The dimension coming from $E_2^*A_iE_2^*$ is 5 and the product of $E_2^*A_iE_j^*$ and $E_k^*A_lE_2^*$ has dimension 6.

$$PSL(2,7): \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ & 13 & 7 & 7 & 5 & 10 \\ & & 6 & 6 & 3 & 6 \\ & & & 6 & 3 & 6 \\ & & & & 4 & 5 \\ & & & & & 9 \end{pmatrix}$$

$$A_6: \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 9 & 5 & 5 & 9 & 7 & 7 \\ & & 8 & 8 & 9 & 8 & 8 \\ & & & 8 & 9 & 8 & 8 \\ & & & & 16 & 13 & 13 \\ & & & & & 12 & 12 \\ & & & & & & 12 \end{pmatrix}$$

$$S_6: \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 3 & 3 & 4 & 4 & 5 & 3 & 2 & 4 & 2 & 3 \\ & & 6 & 4 & 6 & 9 & 8 & 2 & 6 & 4 & 6 \\ & & & 8 & 8 & 8 & 7 & 4 & 8 & 4 & 8 \\ & & & & 12 & 13 & 13 & 4 & 12 & 6 & 13 \\ & & & & & 19 & 16 & 3 & 13 & 6 & 12 \\ & & & & & & 23 & 3 & 13 & 8 & 16 \\ & & & & & & & 3 & 4 & 3 & 5 \\ & & & & & & & & 12 & 6 & 13 \\ & & & & & & & & & 6 & 9 \\ & & & & & & & & & & 19 \end{pmatrix}$$

We denote by $Z(T(G))$ the center of the Terwilliger algebra $T(G)$ of a finite group G.

Lemma 3.2. *The dimensions of $Z(T(G))$ for $G = PSL(2,7),\ A_6,\ S_6$ are given as follows.*

$$\dim Z(T(PSL(2,7))) = 7,$$
$$\dim Z(T(A_6)) = 10,$$

$$\dim Z(T(S_6)) = 14.$$

Proof. The result is obtained by determining a basis for the center. We solve a linear equation system $\{x_i y = y x_i\}$ ranging over all elements x_i in the basis of $T(G)$ and $y = \sum c_j b_j$, where b_j are the basis elements of $T(G)$ and c_j is any scalar. □

Let $\{e_i : 1 \le i \le s\}$ be a basis of $Z(T(G))$. Then we have $e_i e_j = \sum t_{ij}^k e_k$ and put $B_i := (t_{ij}^k)$ for $1 \le i \le s$. Since these matrices mutually commute, they are simultaneously diagonalizable. We shall denote by $v_1(i), \ldots, v_s(i)$ the diagonal entries of the diagonalized matrix of B_i and define the matrix M by $M_{ij} := v_i(j)$. Then we get the primitive central idempotents $\varepsilon_1, \ldots, \varepsilon_s$ by

$$(\varepsilon_1, \ldots, \varepsilon_s) = (e_1, \ldots, e_s)M^{-1}.$$

Theorem 3.3. *The degrees of the irreducible complex representations afforded by every idempotent are given below.*

$T(PSL(2,7))$	ε_i	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7	
	deg ε_i	1	2	3	3	5	6	9	
$T(A_6)$	ε_i	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7	ε_8
	deg ε_i	1	3	3	4	4	6	6	7
	ε_i	ε_9	ε_{10}						
	deg	8	10						
$T(S_6)$	ε_i	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7	ε_8
	deg ε_i	1	1	1	3	3	4	6	7
		ε_9	ε_{10}	ε_{11}	ε_{12}	ε_{13}	ε_{14}		
		8	8	9	9	11	15		

Proof. This is because that $T(G)\varepsilon_i \cong \mathcal{M}_{d_i}$ and that $d_i^2 = \dim T(G)\varepsilon_i$ equals the number of linearly independent elements in the set $\{x_j \varepsilon_i\}$, where x_j are the basis elements of T. □

Theorems 3.1 and 3.3 are combined as

$$165 = 1^2 + 2^2 + 3^2 + 3^2 + 5^2 + 6^2 + 9^2,$$

$$336 = 1^2 + 3^2 + 3^2 + 4^2 + 4^2 + 6^2 + 6^2 + 7^2 + 8^2 + 10^2,$$

$$758 = 1^2 + 1^2 + 1^2 + 3^2 + 3^2 + 4^2 + 6^2 + 7^2 + 8^2 + 8^2 + 9^2 + 9^2 + 11^2 + 15^2.$$

The degrees of irreducible complex representations afforded by every primitive central idempotents enable us to get the following structure theorem.

Corollary 3.4. *We have that*

$$T(PSL(2,7)) \cong \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3 \oplus \mathcal{M}_3 \oplus \mathcal{M}_5 \oplus \mathcal{M}_6 \oplus \mathcal{M}_9,$$

$$
\begin{aligned}
T(A_6) \cong & \mathcal{M}_1 \oplus \mathcal{M}_3 \oplus \mathcal{M}_3 \oplus \mathcal{M}_4 \oplus \mathcal{M}_4 \oplus \mathcal{M}_6 \oplus \mathcal{M}_6 \oplus \mathcal{M}_7 \oplus \mathcal{M}_8 \\
& \oplus \mathcal{M}_{10}, \\
T(S_6) \cong & \mathcal{M}_1 \oplus \mathcal{M}_1 \oplus \mathcal{M}_1 \oplus \mathcal{M}_3 \oplus \mathcal{M}_3 \oplus \mathcal{M}_4 \oplus \mathcal{M}_6 \oplus \mathcal{M}_7 \oplus \mathcal{M}_8 \\
& \oplus \mathcal{M}_8 \oplus \mathcal{M}_9 \oplus \mathcal{M}_9 \oplus \mathcal{M}_{11} \oplus \mathcal{M}_{15}.
\end{aligned}
$$

Acknowledgment. The second named author was supported by JSPS KAKENHI Grant Number JP25400014. The authors would like to thank Prof. Terwilliger for helpful discussions.

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(*Received October 27, 2017*)
(*Accepted November 8, 2018*)