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NECESSARY AND SUFFICIENT TAUBERIAN CONDITIONS FOR THE A^r METHOD OF SUMMABILITY

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Abstract. Móricz and Rhoades determined the necessary and sufficient Tauberian conditions for certain weighted mean methods of summability in [Acta. Math. Hungar. **102(4)** (2004), 279–285]. In the present paper, we deal with the necessary and sufficient Tauberian conditions for the A^r method which was introduced by Başar in [Fırat Üniv. Fen & Müh. Bil. Dergisi **5**(1)(1993), 113–117].

1. Introduction

By a *sequence space*, we understand a linear subspace of the space $\omega = \mathbb{C}^{\mathbb{N}}$ of all complex sequences which contains ϕ, the set of all finitely non-zero sequences, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. We write c for the space of all convergent sequences.

Let λ, μ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we write $Ax = \{(Ax)_n\}$, the *A-transform* of the sequence $x = (x_k) \in \lambda$, if $(Ax)_n = \sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. If $x \in \lambda$ implies that $Ax \in \mu$ then we say that A defines a *matrix transformation* from λ into μ and denote it by $A : \lambda \to \mu$. By $(\lambda : \mu)$, we mean the class of all infinite matrices A such that $A : \lambda \to \mu$.

Definition 1. [11, pp. 222–223] *Suppose that $A = (a_{nk})$ is any infinite matrix of complex numbers and λ is any sequence space. Then, by λ_A we denote all those $x = (x_k) \in \omega$ such that the A-transform of x exists and is in λ. In the case $\lambda = c$ we have $c_A = \{x = (x_k) \in \omega : Ax \in c\}$ and c_A is called the convergence domain of the matrix A.*

Definition 2. *(cf. Boos* [7, p. 167]*) For given matrix methods A and B with $c_B \subset c_A$, by a Tauberian condition we mean the determination of a subset L of ω, such that $x \in L \cap c_A$ implies $x \in c_B$.*

Essentially, we consider the case that $B = I$; that is, we aim to conclude from $x \in L \cap c_A$ that $x \in c$.

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2. A^r Matrices

In this section, we summarize the required knowledge on the A^r matrices. Let $0 < r < 1$. Then the class $A^r = (a_{nk}^r)$ of Toeplitz matrices, introduced by Başar [6], is given by

$$a_{nk}^r = \begin{cases} \frac{1+r^k}{n+1} & , \quad 0 \le k \le n, \\ 0 & , \quad k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$. A straightforward calculation shows that the inverse matrix $B^r = (b_{nk}^r)$ of the matrix $A^r = (a_{nk}^r)$ is given by

$$b_{nk}^r = \begin{cases} \frac{(-1)^{n-k}(k+1)}{1+r^n} & , \quad n-1 \le k \le n, \\ 0 & , \quad 0 \le k \le n-2 \ \text{ or } \ k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$.

Definition 3. *(cf. Başar [6]) The A^r-transform of a sequence $(x_k)_{k\in\mathbb{N}} \in \omega$ is defined by*

$$(A^r x)_n = \sigma_n^r = \frac{1}{n+1}\sum_{k=0}^{n}(1+r^k)x_k$$

for all $n \in \mathbb{N}$. We say that (x_k) is summable A^r to l if

$$\lim_{n\to\infty} \sigma_n^r = l. \tag{2.1}$$

We assume unless stated otherwise that $0 < r < 1$. We should note here that a number of papers were published on the sequence spaces defined by the domain of the A^r matrices in some normed and paranormed sequence spaces, (see Aydın and Başar [1, 2, 3, 4, 5]).

Example 2.1. *Define the sequence $x = (x_k)$ by $x_k = (-1)^k/(1+r^k)$ for all $k \in \mathbb{N}$. Then, it is easy to see that*

$$(A^r x)_n = \sigma_n^r = \frac{1}{n+1}\sum_{k=0}^{n}(1+r^k)\frac{(-1)^k}{1+r^k} = 0$$

for all $n \in \mathbb{N}$. It is immediate that (x_k) is A^r-summable to zero while it does not converge to 0.

3. Main results

In the present section, we give the necessary and sufficient Tauberian conditions for the A^r method. Throughout this section, by λ_n we denote the integral part of the product λn, i.e. $\lambda_n := [\lambda n]$.

We need the following lemmas in proving our theorems.

Lemma 3.1. *Let us define $\langle\lambda\rangle$ for every $\lambda > 0$ by $\langle\lambda\rangle = \lambda - [\lambda]$. Then, the following statements hold:*

(i) *If $\lambda > 1$, then $\lambda_n > n$ for each $n \in \mathbb{N} \setminus \{0\}$ with $n \geq \langle\lambda\rangle^{-1}$.*
(ii) *If $0 < \lambda < 1$, then $\lambda_n < n$ for each $n \in \mathbb{N} \setminus \{0\}$.*

Proof. Obviously, $0 \leq \langle\lambda\rangle < 1$. For every $\lambda > 0$ and each $n \in \mathbb{N}$, we have

$$\lambda_n = [\lambda n] = n[\lambda] + [n\langle\lambda\rangle]. \tag{3.1}$$

Since $\lambda = [\lambda] + \langle\lambda\rangle$ and $\lambda n = n[\lambda] + n\langle\lambda\rangle$.

Let us suppose that $\lambda > 1$. In the case $\lambda \geq 2$, the relation (3.1) leads to the inequalities, $\lambda_n \geq n[\lambda] \geq 2n \geq n$. In the case $1 < \lambda < 2$, we have $[\lambda] = 1$ and $0 < \langle\lambda\rangle < 1$. So, we can assume $n \geq \langle\lambda\rangle^{-1}$. Thus, it follows from (3.1) that $\lambda n = n + [n\langle\lambda\rangle] \geq n + 1 \geq n$.

Let $0 < \lambda < 1$. Then we have $\lambda = \langle\lambda\rangle$. This implies that $\lambda n = \langle\lambda\rangle n < n$. Meanwhile, we have

$$\lambda n = [\lambda n] + \langle\lambda n\rangle \geq [\lambda n] = \lambda_n.$$

As a result, we reach the desired inequality: $\lambda_n < n$ for each $n \in \mathbb{N}\setminus\{0\}$. □

Lemma 3.2. *We have the following statements:*

(i) *Let $\lambda > 1$. For each $n \in \mathbb{N} \setminus \{0\}$ with $n \geq (3\lambda - 1)/\lambda(\lambda - 1)$, we have*

$$\frac{\lambda}{\lambda - 1} < \frac{\lambda_n + 1}{\lambda_n - n} < \frac{2\lambda}{\lambda - 1}. \tag{3.2}$$

(ii) *If $0 < \lambda < 1$, for each $n \in \mathbb{N} \setminus \{0\}$ with $n > \lambda^{-1}$ we have*

$$0 < \frac{\lambda_n + 1}{n - \lambda_n} < \frac{2\lambda}{1 - \lambda}. \tag{3.3}$$

Proof. (i) Let $\lambda > 1$ and for each $n \in \mathbb{N} \setminus \{0\}$

$$n \geq \frac{3\lambda - 1}{\lambda(\lambda - 1)}. \tag{3.4}$$

This implies

$$n \geq \frac{\lambda + (2\lambda - 1)}{\lambda(\lambda - 1)} \geq \frac{1}{\lambda - 1}. \tag{3.5}$$

So

$$\frac{2\lambda}{\lambda - 1} - \frac{\lambda n + 1}{\lambda n - n - 1} = \frac{\lambda}{\lambda n - n - 1}\left(n - \frac{3\lambda - 1}{\lambda(\lambda - 1)}\right) \geq 0.$$

Since for $n \geq \langle\lambda\rangle^{-1}$, $n \leq [\lambda n] \leq \lambda n$ and $0 \leq \langle\lambda n\rangle < 1$, we have the following inequality:

$$\frac{\lambda}{\lambda - 1} = \frac{\lambda n}{\lambda n - n} \leq \frac{\lambda n}{[\lambda n] - n} = \frac{[\lambda n] + \langle\lambda n\rangle}{[\lambda n] - n} < \frac{\lambda_n + 1}{\lambda_n - n}. \tag{3.6}$$

on the other hand we note that $\langle\lambda n\rangle - 1 < 0$ and by the inequalities (3.5) we have

$$[\lambda n] - n + (\langle\lambda n\rangle - 1) = \lambda n - n - 1 = (\lambda - 1)n - 1 > 0.$$

Thus, we conclude the inequality

$$\frac{\lambda_n + 1}{\lambda_n - 1} = \frac{[\lambda n] + 1}{[\lambda n] - n} < \frac{\lambda_n + 1}{\lambda_n - n + (\langle\lambda n\rangle - 1)} = \frac{\lambda n + 1}{\lambda n - n - 1}.$$

(ii) Let $0 < \lambda < 1$ and $n \in \mathbb{N} \setminus \{0\}$ with $n > \lambda^{-1}$. The straightforward computation leads to

$$\frac{2\lambda}{1-\lambda} - \frac{\lambda n + 1}{n - \lambda n} = \frac{\lambda n - 1}{n(1-\lambda)} > 0,$$

which implies that

$$\frac{\lambda n + 1}{n - \lambda n} < \frac{2\lambda}{1-\lambda}.$$

Additionally, since $0 \leq \langle\lambda n\rangle < 1$, we have

$$\frac{\lambda_n + 1}{n - \lambda_n} = \frac{[\lambda n] + 1}{n - [\lambda n]} = \frac{\lambda n - \langle\lambda n\rangle + 1}{n - \lambda n + \langle\lambda n\rangle} \leq \frac{\lambda n + 1}{n - \lambda n + \langle\lambda n\rangle} \leq \frac{\lambda n + 1}{n - \lambda n}.$$

Therefore, we eventually obtain the inequality (3.3). □

Lemma 3.3. *If a sequence (x_k) is summable A^r to a finite number l, then for each $\lambda > 1$*

$$\lim_{n\to\infty} \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} (1 + r^k)x_k = l \tag{3.7}$$

and for each $0 < \lambda < 1$

$$\lim_{n\to\infty} \frac{1}{n - \lambda_n} \sum_{k=\lambda_n+1}^{n} (1 + r^k)x_k = l. \tag{3.8}$$

Proof. (i) Let $\lambda > 1$. For each $n \in \mathbb{N} \setminus \{0\}$ with $n \geq \langle\lambda\rangle^{-1}$ we have following equality:

$$\frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} (1 + r^k)x_k = \sigma_n^r + \frac{\lambda_n + 1}{\lambda_n - n} \left(\sigma_{\lambda_n}^r - \sigma_n^r\right). \tag{3.9}$$

Now, (3.7) follows from (2.1) and (3.2).

(ii) Let $0 < \lambda < 1$. In this situation, for each $n \in \mathbb{N} \setminus \{0\}$, we make use of the following equality:

$$\frac{1}{n - \lambda_n} \sum_{k=\lambda_n+1}^{n} (1 + r^k)x_k = \sigma_n^r + \frac{\lambda_n + 1}{n - \lambda_n} \left(\sigma_n^r - \sigma_{\lambda_n}^r\right) \tag{3.10}$$

Now, (3.8) follows from (2.1) and (3.3). □

Theorem 3.4. *Let (x_k) be a sequence of real numbers which is summable A^r to a finite limit l. Then*

$$\lim_{n\to\infty} x_n = l \tag{3.11}$$

if and only if the following two conditions are satisfied:

$$\sup_{\lambda>1} \liminf_{n\to\infty} \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} \left[(1+r^k)x_k - x_n\right] \geq 0, \tag{3.12}$$

$$\sup_{0<\lambda<1} \liminf_{n\to\infty} \frac{1}{n-\lambda_n} \sum_{k=\lambda_n+1}^{n} \left[x_n - (1+r^k)x_k\right] \geq 0. \tag{3.13}$$

Proof. Necessity. Assume that both (3.11) and (2.1) are satisfied. Then, an application Lemma 3.3 yields (3.12) for all $\lambda > 1$ and (3.13) for all $0 < \lambda < 1$.

Sufficiency. Assume that (2.1), (3.12) and (3.13) are satisfied.

First, we consider the case $\lambda > 1$. Given any $\varepsilon > 0$, by (3.12) there exists $\lambda > 1$ such that

$$\liminf_{n\to\infty} \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} \left[(1+r^k)x_k - x_n\right] \geq -\varepsilon. \tag{3.14}$$

For each $n \in \mathbb{N} \setminus \{0\}$ with $n \geq \langle\lambda\rangle^{-1}$, it follows from (3.9) that

$$\begin{aligned} x_n - \sigma_n^r &= \frac{\lambda_n + 1}{\lambda_n - n}\left(\sigma_{\lambda_n}^r - \sigma_n^r\right) \\ &\quad -\frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} \left[(1+r^k)x_k - x_n\right]. \end{aligned} \tag{3.15}$$

On the other hand, by (3.2), we have

$$\left|\frac{\lambda_n + 1}{\lambda_n - n}\left(\sigma_{\lambda_n}^r - \sigma_n^r\right)\right| \leq \frac{2\lambda}{\lambda - 1}\left|\sigma_{\lambda_n}^r - \sigma_n^r\right|$$

and so

$$\lim_{n\to\infty} \frac{\lambda_n + 1}{\lambda_n - n}\left(\sigma_{\lambda_n}^r - \sigma_n^r\right) = 0. \tag{3.16}$$

Combining (3.15)-(3.16) gives that

$$\begin{aligned} \limsup_{n\to\infty}\left(x_n - \sigma_n^r\right) &\leq \limsup_{n\to\infty} \frac{\lambda_n + 1}{\lambda_n - n}\left(\sigma_{\lambda_n}^r - \sigma_n^r\right) \\ &\quad + \limsup_{n\to\infty}\left\{-\frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} \left[\left(1+r^k\right)x_k - x_n\right]\right\} \end{aligned}$$

$$
\begin{aligned}
&\leq \quad -\liminf_{n\to\infty}\left\{\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}\left[(1+r^k)x_k - x_n\right]\right\}\\
&\leq \quad \varepsilon.
\end{aligned}
$$

Consequently, for each $\varepsilon > 0$

$$
\limsup_{n\to\infty} x_n \leq l + \varepsilon. \tag{3.17}
$$

Second, we consider the case $0 < \lambda < 1$. For each $n \in \mathbb{N} \setminus \{0\}$, it follows from (3.10) that

$$
\begin{aligned}
x_n - \sigma_n^r \quad &= \quad \frac{\lambda_n + 1}{n - \lambda_n}\left(\sigma_n^r - \sigma_{\lambda_n}^r\right)\\
&\quad + \frac{1}{n-\lambda_n}\sum_{k=\lambda_n+1}^{n}\left[x_n - (1+r^k)x_k\right].
\end{aligned} \tag{3.18}
$$

Using a similar argument as above, by virtue of (3.13) and (3.3), for any $\varepsilon > 0$ we conclude that

$$
\begin{aligned}
\liminf_{n\to\infty}\left(x_n - \sigma_n^r\right) \quad &\geq \quad \liminf_{n\to\infty}\frac{\lambda_n + 1}{n - \lambda_n}\left(\sigma_n^r - \sigma_{\lambda_n}^r\right)\\
&\quad + \liminf_{n\to\infty}\left\{\frac{1}{n-\lambda_n}\sum_{k=\lambda_n+1}^{n}\left[x_n - (1+r^k)x_k\right]\right\}\\
&\geq \quad -\varepsilon.
\end{aligned}
$$

Consequently, for each $\varepsilon > 0$

$$
\liminf_{n\to\infty} x_n \geq l - \varepsilon. \tag{3.19}
$$

We conclude (3.11) by combining (3.17) and (3.19). □

Remark. From the proof of Theorem 3.4 it turns out that even more is true: If the conditions (2.1) and (3.11) or equivalently, the conditions (2.1), (3.12) and (3.13) are satisfied, then we necessarily have

$$
\lim_{n\to\infty}\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}\left[(1+r^k)x_k - x_n\right] = 0 \tag{3.20}
$$

for all $\lambda > 1$, and

$$
\lim_{n\to\infty}\frac{1}{n-\lambda_n}\sum_{k=\lambda_n+1}^{n}\left[x_n - (1+r^k)x_k\right] = 0 \tag{3.21}
$$

for all $0 < \lambda < 1$.

Remark. The proof of Theorem 3.4 can be modified so that its conclusion remains valid if the conditions (3.12) and (3.13) are replaced by the following ones:

$$\inf_{\lambda>1} \limsup_{n\to\infty} \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} \Big[(1+r^k)x_k - x_n\Big] \le 0 \tag{3.22}$$

and

$$\inf_{0<\lambda<1} \limsup_{n\to\infty} \frac{1}{n-\lambda_n} \sum_{k=\lambda_n+1}^{n} \Big[x_n - (1+r^k)x_k\Big] \le 0. \tag{3.23}$$

Definition 4. A sequence (x_k) of real numbers is said to be slowly decreasing if

$$\lim_{\lambda\to 1^+} \liminf_{n\to\infty} \min_{n<k\le\lambda_n} [x_k - x_n] \ge 0 \tag{3.24}$$

or equivalently

$$\lim_{\lambda\to 1^-} \liminf_{n\to\infty} \min_{\lambda_n<k\le n} [x_n - x_k] \ge 0. \tag{3.25}$$

The right-hand limit in (3.24) exists and can be equivalently replaced by $\sup_{\lambda>1}$. Historically, the notion of slow decrease (with respect to summability C_1) goes back to Schmidt [15].

Corollary 3.5. *Let a sequence (x_k) of real numbers be slowly decreasing (or slowly incresing). Then,*

$$\lim_{n\to\infty} \sigma_n^r = l \quad \textit{implies} \quad \lim_{n\to\infty} x_n = l. \tag{3.26}$$

Proof. For $\lambda > 1$, for each $n \in \mathbb{N}\setminus\{0\}$ with $n \ge \langle\lambda\rangle^{-1}$ we have the following inequality:

$$\begin{aligned} \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} \Big[(1+r^k)x_k - x_n\Big] &\ge \min_{n<k\le\lambda_n} \Big[(1+r^k)x_k - x_n\Big] \\ &= \min_{n<k\le\lambda_n} \Big(r^k x_k\Big) + \min_{n<k\le\lambda_n} (x_k - x_n)\,. \end{aligned}$$

We have

$$x_k = \frac{(k+1)\sigma_k^r - k\sigma_{k-1}^r}{1+r^k}, \qquad \frac{x_k}{k} = \frac{\sigma_k^r - \sigma_{k-1}^r}{1+r^k} + \frac{\sigma_k^r}{k(1+r^k)}.$$

On the other hand if (x_k) is summable A^r, then we have $x_k/k \to 0$, as $k \to \infty$. Therefore $r^k x_k \to 0$, as $k \to \infty$. So, the condition (3.24) clearly implies the condition (3.12). Similarly, (3.25) implies (3.13). By Theorem 3.4, we have the implication (3.26). □

Theorem 3.6. *Let (x_k) be a sequence of complex numbers which is summable A^r. Then, (x_k) converges to the same limit if and only if one of the following two conditions is satisfied:*

$$\inf_{\lambda>1} \limsup_{n\to\infty} \left| \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} \Big[(1+r^k)x_k - x_n \Big] \right| = 0 \tag{3.27}$$

or

$$\inf_{0<\lambda<1} \limsup_{n\to\infty} \left| \frac{1}{n-\lambda_n} \sum_{k=\lambda_n+1}^{n} \Big[x_n - (1+r^k)x_k \Big] \right| = 0. \tag{3.28}$$

Proof. Necessity. The proof is similar to the proof of necessity part of Theorem 3.4.

Sufficiency. Assume that (2.1) and one of the conditions (3.27) and (3.28) are satisfied. Let any $\varepsilon > 0$ be given. By (3.27), there exists $\lambda > 1$ such that

$$\limsup_{n\to\infty} \left| \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} \Big[(1+r^k)x_k - x_n \Big] \right| < \varepsilon. \tag{3.29}$$

By (3.15), for each $n \in \mathbb{N} \setminus \{0\}$ with $n \geq \langle\lambda\rangle^{-1}$, we have

$$\begin{aligned} \limsup_{n\to\infty} |x_n - \sigma_n^r| \quad &\leq \quad \limsup_{n\to\infty} \frac{\lambda_n + 1}{\lambda_n - n} \left| \sigma_{\lambda_n}^r - \sigma_n^r \right| \\ &\quad + \limsup_{n\to\infty} \left| \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} \Big[(1+r^k)x_k - x_n \Big] \right|. \end{aligned} \tag{3.30}$$

In case $0 < \lambda < 1$, on the other hand, by (3.28) there exists $0 < \lambda < 1$ such that

$$\limsup_{n\to\infty} \left| \frac{1}{n-\lambda_n} \sum_{k=\lambda_n+1}^{n} \Big[x_n - (1+r^k)x_k \Big] \right| < \varepsilon. \tag{3.31}$$

By (3.18), for each $n \in \mathbb{N} \setminus \{0\}$ with $n > \lambda^{-1}$, we have

$$\begin{aligned} \limsup_{n\to\infty} |x_n - \sigma_n^r| \quad &\leq \quad \limsup_{n\to\infty} \frac{\lambda_n + 1}{n - \lambda_n} \left| \sigma_n^r - \sigma_{\lambda_n}^r \right| \\ &\quad + \limsup_{n\to\infty} \left| \frac{1}{n-\lambda_n} \sum_{k=\lambda_n+1}^{n} \Big[x_n - (1+r^k)x_k \Big] \right|. \end{aligned} \tag{3.32}$$

By (3.30) or (3.32), in either case we obtain

$$\limsup_{n\to\infty} |x_n - \sigma_n^r| = 0 \tag{3.33}$$

whence it follows that

$$\lim_{n\to\infty} |x_n - \sigma_n^r| = 0. \tag{3.34}$$

Now, we conclude (3.11) from (2.1) and (3.34). $\square$

We recall that a sequence (x_k) of *complex numbers* is said to be slowly oscillating

$$\lim_{\lambda\to 1^+} \limsup_{n\to\infty} \max_{n<k\le\lambda_n} |x_k - x_n| = 0 \tag{3.35}$$

or equivalently

$$\lim_{\lambda\to 1^-} \limsup_{n\to\infty} \max_{\lambda_n<k\le n} |x_k - x_n| = 0. \tag{3.36}$$

The right-hand limit in (3.35) can be equivalently replaced by $\inf_{\lambda>1}$.

Corollary 3.7. *Let a sequence (x_k) of complex numbers be slowly oscillating. Then, the implication (3.26) holds.*

Proof. For $\lambda > 1$, for each $n \in \mathbb{N}\setminus\{0\}$ with $n \geq \langle\lambda\rangle^{-1}$; we have the following inequality:

$$\begin{aligned} \left|\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}\Big[(1+r^k)x_k - x_n\Big]\right| &\le \max_{n<k\le\lambda_n}\left|(1+r^k)x_k - x_n\right| \\ &\le \max_{n<k\le\lambda_n}\left|r^k x_k\right| + \max_{n<k\le\lambda_n}|x_k - x_n|. \end{aligned}$$

On the other hand, if (x_k) is summable A^r; then $\lim_{k\to\infty}(r^k x_k) = 0$. So, the condition (3.35) clearly implies the condition (3.27). Similarly, (3.36) implies (3.28). By Theorem 3.6, we have the implication (3.26). $\square$

4. Conclusion

In 1995, Móricz and Rhoades [12] obtained the necessary and sufficient Tauberian conditions for weighted mean. In [12], Theorem 1 gives a one-sided Tauberian result and Theorem 2 is an extension of Theorem 1 to complex sequences. Later, Móricz and Rhoades derived the weaker Tauberian conditions under which convergence of the sequence (s_n) follows from its weighted mean (N, p). They firstly considered real sequences and gave a one-sided Taberian theorem. Secondly, they considered complex sequences and gave a two-sided Tauberian theorem. These are more general than Theorem 1 and Theorem 2 of [13], respectively. In [10], Dik et al. introduce some classical and neoclassical Tauberian-like conditions to retrieve subsequential convergence of a real sequence (u_n) and some other sequences related to it out of the boundedness of the sequence (u_n). In [8], Çanak and Totur generalize a result of Č.V. Stanojević and V.B. Stanojević given in [16] for the general control modulo the oscillatory behavior of order m, where m is any positive integer.

Following Móricz and Rhoades [12, 13], we have derived the necessary and sufficient Tauberian conditions for the method A^r of summability in the present work. Although Rhoades [14, Corollary 2.2] proved the equivalence of the matrix A^r to the Cesàro matrix C_1 of order one, the main results are new since they are independently derived from the existing results. We should note that as a natural continuation of this paper, it is meaningful to obtain the necessary and sufficient Tauberian conditions for the Euler means E^r, the generalized difference matrix $B(r,s)$ and factorable matrix $G(u,v)$.

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