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AN ALTERNATIVE PROOF OF SOME RESULTS ON THE FRAMED BORDISM CLASSES OF LOW RANK SIMPLE LIE GROUPS

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Abstract. We present a unified proof of some known results on the framed bordism classes of low rank simple Lie groups.

1. Introduction

Let G be a simple Lie group of dimension d and let $\mathcal{L}$ be its left invariant framing. Then the pair $(G, \mathcal{L})$ determines the element $[G, \mathcal{L}]$ in the the stable homotopy group π_d^S of spheres. Concerning the identification of these elements and well-known generators of the corresponding homotopy groups, we have the following results, which have been obtained in [2, 4, 5, 6, 7, 8, 9, 14, 16] and perhaps others.

(i) $[SU(2), \mathcal{L}] = \nu \in \pi_3^S$, (ii) $[SO(3), \mathcal{L}] = 2\nu$,

(iii) $[SU(3), \mathcal{L}] = \bar{\nu} \in \pi_8^S$, (iv) $[Sp(2), \mathcal{L}] = \beta_1 \in \pi_{10(3)}^S$,

(v) $[SO(5), \mathcal{L}] = 2\beta_1$, (vi) $[G_2, \mathcal{L}] = \kappa \in \pi_{14}^S$,

(vii) $[SU(4), \mathcal{L}] = \eta\kappa \in \pi_{15}^S$, (viii) $[Sp(3), \mathcal{L}] = \eta\bar{\kappa} + \sigma^3 \in \pi_{21}^S$.

These are all the nonzero $[G, \mathcal{L}]$ with the same types of G as appearing there [12]. In this note we give a unified proof of these equalities using the formula of [13] based on the computations of the unstable homotopy groups $\pi_{n+k}(S^n)$ of [15] and [10].

Let S be a circle subgroup of G and let ξ be the complex line bundle over G/S associated with the canonical principal S-bundle $G \to G/S$. Let $\beta \in \tilde{K}(S^2)$ denote the Bott element. Then [13] tells us that $[G, \mathcal{L}] \in \pi_d^S$ is given as the Kronecker product of the image of $-\beta\xi \in \tilde{K}^{-1}(S^1(G/S^+))$ by J-homomorphism $\tilde{J} : \tilde{K}^{-1}(S^1(G/S^+)) \to \pi_S^0(S^1(G/S^+))$ and the framed bordism fundamental class $[G/S] \in \pi_{d-1}^S(G/S)$; that is, $[G, \mathcal{L}] = -\langle \tilde{J}(\beta\xi), [G/S] \rangle$. Assume that in G there is a subgroup U containing S such that $U \cong SU(2)$ whose restriction to S induces an isomorphism $S \cong U(1) \subset SU(2)$. Then by [11] we know that G/S is framed null-cobordant and also by definition it follows that $\langle \tilde{J}(\beta), [G/S] \rangle$ can be written as the composite of $\tilde{J}(\beta)$ and the suspension of $[G/S]$, so we find that

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$\langle \tilde{J}(\beta), [G/S]\rangle = 0$. This allows us to replace $\tilde{J}(\beta\xi)$ by $\tilde{J}(\beta(\xi-1))$ in the above where 1 denotes the trivial complex line bundle over G/S. We therefore have

$$[G, \mathcal{L}] = -\langle \tilde{J}(\beta(\xi-1)), [G/S]\rangle. \tag{1}$$

The group G considered here, either itself or its universal covering, has a maximal closed subgroup of the form $S^1 \times H$, except for $G = SU(3)$ however which contains $S(U(1)\times U(2))$ instead. If we write $2(n-1)$ for the dimension of these homogeneous spaces of G, then we find that the classifying map $G/S \to BS^1$ of ξ factors through the complex projective space $P^{n-1}(\mathbf{C})$; hence $\beta(\xi-1)$ can be thought of as representing the composite $S^1(G/S) \to S^1(P^{n-1}(\mathbf{C})) \subset U(n)$ of its suspension and the canonical inclusion of $S^1(P^{n-1}(\mathbf{C}))$ into $U(n)$. By applying the Hopf construction this gives rise to a map

$$j(\beta(\xi-1)) : S^{2n} \wedge S^1(G/S) \to S^{2n} \tag{2}$$

where in fact we can set up this n as follows:

$$n = 2, 3, 4, 6, 4 \text{ and } 6$$

in cases (i), (iii), (iv), (vi), (vii) and (viii), respectively. Suspending appropriately we have $[E^k j(\beta(\xi-1))] = \tilde{J}(\beta(\xi-1))$ where $[f]$ denotes the homotopy class of a map f and E^k is regarded as satisfying $[E^k f] = E^k[f]$. Combining this equality with (1) we have

$$[E^k j(\beta(\xi-1)) \circ \Phi_k] = -[G, \mathcal{L}] \tag{3}$$

where Φ_k denotes the map $S^{2n+k+d} \to S^{2n+k} \wedge S^1(G/S)$ representing $[G/S]$.

Let Q denote the map induced by projection of $S^1(G/S)$ onto the top cell S^d. Clearly it then follows that $E^{2n+k}Q \circ \Phi_k$ is homotopic to the identity map of S^{2n+k+d}, which is used freely below. We also keep the notation as above; however we write A_G instead of A when we need to specify that we are dealing with the case of G.

2. Lemmas

Lemma 1. *Let C be a subgroup of $S \subset G$ of order 2 and suppose that there is a complex representation $\rho : G \to U(2k)$ such that $\rho(z) = k(z + \bar{z})$ for $z \in S$. Then, if we put $\bar{G} = G/C$ then we have $k[\bar{G}, \mathcal{L}] = 2k[G, \mathcal{L}]$.*

Proof. Let $\bar{\xi}$ be the complex line bundle over $\bar{G}/\bar{S}$ associated with the canonical principal $\bar{S}$-bundle $\bar{G} \to \bar{G}/\bar{S}$ where $\bar{S} = S/C$. Then identifying G/S via the canonical homeomorphism with $\bar{G}/\bar{S}$ we have $\xi^{\otimes 2} \cong \bar{\xi}$. On the other hand, by virtue of the hypothesis we have $k(\xi^{\otimes 2} \oplus 1) \cong 2k\xi$. Taken together, these two isomorphisms show that $k([\bar{\xi}]-1) = 2k([\xi]-1)$ where this $[\gamma]$ in particular denotes the isomorphism class of a line bundle γ. By applying this to (1) we obtain $k[\bar{G},\mathcal{L}] = 2k[G,\mathcal{L}]$. $\square$

Let K be a subgroup of G such that $K \supset U \supset S$ and G/K is homeomorphic to a sphere S^ℓ with base point $e_+ = eK$, e being the unit element of G. Let $G/K = S^\ell$ and consider the fibration

$$K/S \to G/S \xrightarrow{p} S^\ell.$$

Here we let S^ℓ be decomposed into hemispheres $D^\ell_\pm$ respectively equipped with centers $e_\pm$; we use implicitly the fact that p is isomorphic to the bundle obtained by gluing two product bundles over $D^\ell_\pm$ by a suitable clutching function φ.

Suppose given a map $f : S^{2n+\ell} = S^{2n} \wedge S^\ell \to S^{2n}$. Then in view of the transversality theorem we consider this f as a map from $S^{2n} \wedge S^{\ell+}$ to S^{2n} satisfying the conditions: It collapses $S^{2n} = S^{2n} \wedge \{e_+\}^+$ to the base point $* \in S^{2n}$; and it maps $S^{2n} = S^{2n} \wedge \{x\}^+$ identically onto S^{2n} for all $x \in D^\ell_-$. We denote this map by the same letter f, but then $1-f$ is taken as a map of $S^{2n} \wedge S^\ell$ to S^{2n} with $e_- \in D^\ell_-$ as its base point where 1 denotes the natural projection $S^{2n} \wedge S^{\ell+} \to S^{2n}$ which maps $u \wedge x$ to u for $u \in S^{2n}$, $x \in S^\ell$. We write $\zeta = 1 - f$. We define a map $\mu : S^{2n} \wedge S^1(G/S) \to S^{2n}$ by putting

$$\mu(u \wedge (z \wedge gS)) = j(\beta(\xi - 1))(f(u \wedge p(gS)) \wedge (z \wedge gS)) \tag{4}$$

if $u \wedge (z \wedge gS) \notin S^{2n} \wedge S^1(K/S)$; otherwise $\mu(u \wedge (z \wedge gS)) = *$ where $S^1 \wedge G/S = S^1(G/S)$.

Let R denote the restriction of $j(\beta(\xi - 1))$ to $S^{2n} \wedge S^1(K/S)$; it can be explicitly written as $R = E^{2(n-n_K)} j(\beta(\xi_K - 1))$. Then we have

Lemma 2. *If R is written as the composition of $E^{2n}Q_K$ with a map $r : S^{2n} \wedge S^{d-\ell} \to S^{2n}$, that is, $R \simeq r \circ E^{2n}Q_K$, then there exists a map $h : S^{2n} \wedge S^\ell \wedge S^{d-\ell} \to S^{2n}$ such that*

$$Q^*([h]) = [j(\beta(\xi - 1))] - [\mu].$$

Proof. Consider the homotopy sum $j(\beta(\xi-1))+(-\mu)$. It becomes homotopic to the constant map on $S^{2n} \wedge (D^\ell_-)^+ \wedge S^1(K/S^+)$ at the base point. Since the homotopy between them is compatible with φ, its restriction to $S^{2n} \wedge (D^\ell_+)^+ \wedge S^1(K/S^+)$ defines a map of $S^{2n} \wedge S^\ell \wedge S^1(K/S^+)$ to S^{2n} where S^ℓ is regarded as $D^\ell_+/\partial D^\ell_+$. This can be deformed continuously into the map H from $S^{2n} \wedge S^\ell \wedge S^1(K/S)$ to S^{2n}; it can be explicitly given by the formula $H(u \wedge x \wedge y) = R(\zeta(u \wedge x) \wedge y)$, hence from the assumption we see that the desired map h can be defined by replacing R by r, which clearly satisfies the equation above. $\square$

We consider how to explicitly specify h above. Regard h as a map from $S^{2n} \wedge S^{\ell+} \wedge S^{d-\ell}$ to S^{2n} satisfying the conditions: $h(u \wedge e_- \wedge y) = *$ and $h(u \wedge e_+ \wedge y_0) = u$ for some $y_0 \in S^{d-\ell}$. Then we can view the restriction of it to $S^{2n} \wedge S^{d-\ell} = S^{2n} \wedge \{e_+\}^+ \wedge S^{d-\ell}$ as r and ζ as a map given by $\zeta(u \wedge x) = h(u \wedge x \wedge y_0)$; so from the argument above we may consider that $h(u \wedge x \wedge y) = r(\zeta(u \wedge x) \wedge y)$. This tells us that if $[r]$ and $[\zeta]$ represent generators of their respective corresponding groups, then $[h]$ can be a generator of $\pi^{2n}(S^{2n} \wedge S^d)$, written $\alpha = [h]$; and that in a similar way its converse holds. Below, suppose that α is a generator, then we have by Lemma 2

$$\operatorname{ord}(\alpha) = \operatorname{ord}([r]), \quad \operatorname{ord}(\alpha) \mid \operatorname{ord}([\zeta]) \tag{5}$$

where $\operatorname{ord}(x)$ denotes the order of an element x. In the next section we use Lemma 2 combined with (5), together with (6) below, in order to prove the assertions here.

The assumption above about R implies further that for large k the composition $E^k\mu \circ \Phi_k : S^{2n+k+d} \to S^{2n+k} \wedge S^1(G/S) \to S^{2n+k}$ can be continuously deformed into the composite $\gamma \circ F : S^{2n+k+d} \to S^{2n+k+d-\ell} \to S^{2n+k}$ where $\gamma = E^k r$ and $F = E^{k+(d-\ell)} f$. Hence by applying (3) we obtain from Lemma 2

$$[G, \mathcal{L}] = -E^k\alpha + [\gamma][F]. \tag{6}$$

3. Proof

In this section we prove the assertions (i) - (viii) using the previous sections.

(i) Let $S = U(1) \subset SU(2)$ and consider the fibration $S \to SU(2) \xrightarrow{p} S^2 = SU(2)/S$. Since $\xi - 1$ can be identified with $-\beta$ through the natural homeomorphism $SU(2)/S \approx S^2$, by [2] we have $e'_R \tilde{J}(\beta(\xi - 1)) = -1/24$ so by (1), we get $[SU(2), \mathcal{L}] = \nu \in \pi_3^S = \mathbf{Z}_{24}$. This means that $j(\beta(\xi - 1)) : S^4 \wedge S^3 \to S^4$ in (2) gives the generator -1 of the first summand of $\pi^4(S^4 \wedge S^3) = \mathbf{Z} \oplus \mathbf{Z}_{12}$, so we have $[E^1 j(\beta(\xi - 1))] = -\nu$ in $\pi^5(S^5 \wedge S^3) = \pi_3^S = \mathbf{Z}_{24}$.

The proof of the following five cases (iii), (iv), (vi) - (viii) proceeds in three steps. The observation just presented above gives the first step to start this process. In doing this we rely essentially on the corresponding results on $\pi_{n+k}(S^n)$ in [15] and [10]; but we use freely them without references.

(ii) Consider the fibrations $S = Spin(2) \to Spin(3) \to S^2 = Spin(3)/S$. The spin representation Δ of $Spin(3)$ satisfies $\Delta(z) = z + \bar{z}, z \in S$, so applying Lemma 1 from the result of (i) we have $[SO(3), \mathcal{L}] = 2[Spin(3), \mathcal{L}] = 2\nu$.

(iii) Consider the fibration $SU(2)/S \to SU(3)/S \xrightarrow{p} S^5 = SU(3)/SU(2)$ where $S = U(1) \subset SU(2)$ and the map $j(\beta(\xi - 1)) : S^6 \wedge S^1(SU(3)/S) \to S^6$

given in (2). Then $R = E^2 j(\beta(\xi_{SU(2)} - 1))$, so when viewed $SU(2)/S = S^2$, we have $r = R$ and by virtue of (i) it follows that $[r] = -\nu$ in $\pi^6(S^6 \wedge S^3) = \mathbf{Z}_{24}$. We recall the decompositions

$$\pi^6(S^6 \wedge S^8) = \mathbf{Z}_{24} \cdot a \oplus \mathbf{Z}_2 \cdot b, \quad \pi^{10}(S^{10} \wedge S^8) = \pi_8^S = \mathbf{Z}_2 \cdot \bar{\nu} \oplus \mathbf{Z}_2 \cdot \epsilon$$

where a, b are chosen to satisfy $E^4 a = \bar{\nu}$, $E^4 b = \epsilon$ respectively. Then since $\mathrm{ord}(a) = \mathrm{ord}([r])$ we can take $\alpha = a$ and also, since $\pi^6(S^6 \wedge S^5) = \mathbf{Z}$, the choice of ζ is unique up to sign; hence properly assigning to ζ a map representing its generator we have by Lemma 2

$$(E^6 Q)^*(a) = [j(\beta(\xi - 1))] - [\mu]$$

because of (5). Besides we have $\pi^7(S^7 \wedge S^5) = 0$. This implies that $E^1 f$ is null homotopic, whence $E^4[\mu] = 0$. Hence the relation $E^4 a = \bar{\nu}$ mentioned above yields

$$E^4[j(\beta(\xi - 1))] = (E^{10} Q)^*(\bar{\nu}). \tag{7}$$

Therefore applying (6) we obtain $[SU(3), \mathcal{L}] = \bar{\nu}$.

(iv) The argument for this case is basically analogous to the case (iii). Consider the fibration $Sp(1)/S \to Sp(2)/S \xrightarrow{p} S^7 = Sp(2)/Sp(1)$ where $S = U(1) \subset SU(2) = Sp(1)$. Let $j(\beta(\xi - 1)) : S^8 \wedge S^1(Sp(2)/S) \to S^8$ be due to (2); similarly as in the above case we have $R = E^4 j(\beta(\xi_{Sp(2)} - 1))$, so we may view $r = R$ and hence $[r] = -\nu$ in $\pi^8(S^8 \wedge S^3) = \mathbf{Z}_{24}$, so that $\mathrm{ord}([r]) = 24$. Recalling the decompositions

$$\pi^8(S^8 \wedge S^{10}) = \mathbf{Z}_{24} \cdot a \oplus \mathbf{Z}_{24} \cdot b \oplus \mathbf{Z}_2, \quad \pi^9(S^9 \wedge S^{10}) = \mathbf{Z}_{24} \cdot c \oplus \mathbf{Z}_2$$

where a, b and c are chosen to satisfy $E^1 a = E^1 b = c$, the argument as in (iii) suggests that in this case we can take $\alpha = a + b$. In fact, then the relations between a, b and c stated above tells us to select as ζ a map representing a generator, particularly, of the second summand of $\pi^8(S^8 \wedge S^7) = \mathbf{Z} \oplus \mathbf{Z}_{120}$ for the reason that E^1 maps each generator of the first and second summands of this group to once and twice a generator of $\pi^9(S^9 \wedge S^7) = \mathbf{Z}_{240}$, respectively. Similarly as in (iii) we can view that with these choices (5) holds and so by Lemma 2 we have

$$(E^8 Q)^*(a + b) = [j(\beta(\xi - 1))] - [\mu].$$

Besides operating E^1 we obtain

$$E^1[j(\beta(\xi - 1))] - E^1[\mu] = (E^9 Q)^*(2c). \tag{8}$$

This also implies that the operation of E^1 on $j(\beta(\xi - 1))$ works in the same way as on f and hence this choice is only one unique choice up to sign.

Now since E^3c represents a generator of $\pi^{12}(S^{12} \wedge S^{10}) = \pi_{10}^S = \mathbf{Z}_6$, putting $E^3c = \beta_1$ we have from (8)

$$E^4[j(\beta(\xi - 1))] - E^4[\mu] = (E^{12}Q)^*(2\beta_1). \tag{9}$$

Since $E^1[f]$ is twice an element, we see that the 2-component of $[\gamma][F]$ is 0; hence from the fact that β_1 can not be represented as a product of two elements follows that $[\gamma][F] = 0$. This is because if β_1 can be written as at least a product of elements of π_3^S and π_7^S, then c must be done so, which means that the same thing occurs for both of a and b. But if so, then the difference between their property of the two generators of $\pi^8(S^8 \wedge S^7)$ noticed above leads to a contradiction. Hence applying (6) to (9) we obtain $[Sp(2), \mathcal{L}] = \beta_1$.

(v) The proof of this case is quite analogous to that of case (ii). Since $Sp(2) \cong Spin(5)$ [1], we take $S = Spin(2) \subset Spin(5)$ and consider the fibrations $S \to Spin(5) \to Spin(5)/S$. Let Δ be the spin representation of $Spin(5)$. Then it holds that $\Delta(z) = 2z + 2\bar{z}$ for $z \in S$. Hence using Lemma 1 we have $4[SO(5), \mathcal{L}] = 2[Spin(5), \mathcal{L}]$. But according to [12] the 2-component of $[SO(5), \mathcal{L}]$ is zero, so it follows that $[SO(5), \mathcal{L}] = 2\beta_1$.

(vi) The remaining three cases are proceeded in a similar way to the above two cases (iii) and (iv). The first two and the last cases are reduced to the these two cases using the equalities (7) and (9), respectively.

For the present case consider the inclusion $SU(3) \subset G_2$ with $G_2/SU(3) = S^6$ ([17], [1]), which induces the fibration $SU(3)/S \to G_2/S \xrightarrow{p} S^6$ where $S = U(1) \subset SU(2) \subset SU(3)$. Let $j(\beta(\xi - 1)) : S^{14} \wedge S^{14} \to S^{14}$ denote the two-fold suspension of the map given in (2). Then $R = E^8 j(\beta(\xi_{SU(3)} - 1))$. This together with (7) shows that we can write $R \simeq r \circ E^{14}Q_{SU(3)}$ where $r : S^{14} \wedge S^8 \to S^{14}$ denotes a map representing $\bar{\nu}$, i.e., $\mathrm{ord}([r]) = 2$. Hence if we choose as α the generator b given in the decompositions

$$\pi^{14}(S^{14} \wedge S^{14}) = \mathbf{Z}_8 \cdot a \oplus \mathbf{Z}_2 \cdot b, \quad \pi^{16}(S^{16} \wedge S^{14}) = \pi_{14}^S = \mathbf{Z}_2 \cdot \sigma^2 \oplus \mathbf{Z}_2 \cdot \kappa$$

where a, b are chosen to satisfy $E^2a = \sigma^2$, $E^2b = \kappa$, respectively, then we find that r above coincides up to homotopy with that derived from this α for the same reason as before. Moreover, since $\pi^{14}(S^{14} \wedge S^6) = \mathbf{Z}_2$ the choice of ζ is unique; in fact, it is taken to be its generator. These choices allow us to apply Lemma 2 together with (5) and thereby we get

$$[j(\beta(\xi - 1))] - [\mu] = (E^{14}Q)^*(b),$$

hence it follows that $E^2[H(\varrho - 1)] - E^2[\mu] = (E^{16}Q)^*(\kappa)$ due to the relation $E^2b = \kappa$.

Now since $[\gamma] \in \pi_8^S = (\mathbf{Z}_2)^2$, $[\gamma]\nu$ is at least twice an element of $(\pi_{11}^S)_{(2)} = \mathbf{Z}_8$, which implies that $[\gamma][F] = 0$ due to $\nu^2 = [F]$. Therefore by applying (6) we obtain $[G_2, \mathcal{L}] = \kappa$.

(vii) The proof for this case is exactly the same as that in the case (vi). We use the fibration $SU(3)/S \to SU(4)/S \xrightarrow{p} S^7 = SU(4)/SU(3)$ where $S = U(1) \subset SU(2) \subset SU(3)$. Let $j(\beta(\xi - 1)) : S^{12} \wedge S^1(SU(4)/S) \to S^{12}$ denote the four-fold suspension of the map given in (2), then as seen below μ can be defined using a map representing the generator σ of $\pi^{12}(S^{12} \wedge S^7) = \pi_7^S = \mathbf{Z}_{240}$. Besides as in the previous case to α we can assign b in the decompositions

$$\pi^{12}(S^{12} \wedge S^{15}) = \mathbf{Z}_{240} \oplus \mathbf{Z}_2 \cdot b. \quad \pi^{17}(S^{17} \wedge S^{15}) = \pi_{15}^S = \mathbf{Z}_{480} \oplus \mathbf{Z}_2 \cdot \eta\kappa$$

where b satisfies $E^5 b = \eta\kappa$. Then owing to the same reasoning as in (vi) we have

$$[j(\beta(\xi - 1))] - [\mu] = (E^{12}Q)^*(b).$$

We check that the relations $\sigma\epsilon = \sigma\bar{\nu} = 0$ hold: Since σ is the E^4-image of a generator of the free summand of $\pi^8(S^8 \wedge S^7) = \mathbf{Z} \oplus \mathbf{Z}_{120}$ it follows that $\eta\sigma^2 = 0$, so we have $\sigma\epsilon = \sigma\bar{\nu}$ using the relation $\eta\sigma = \epsilon + \bar{\nu}$. Hence taken into consideration $\epsilon = \langle \nu^2, 2, \eta \rangle$ and $\sigma\nu^2 = 0 \in \pi_{13}^S = \mathbf{Z}_3$, we get $\sigma\epsilon = 0$, so $\sigma\bar{\nu} = 0$. This implies that applying (6) to the above equality we obtain $[SU(4), \mathcal{L}] = \eta\kappa$.

(viii) Consider the fibration $Sp(2)/S \to Sp(3)/S \xrightarrow{p} S^{11} = Sp(3)/Sp(2)$ where $S = U(1) \subset SU(2) = Sp(1) \subset Sp(2)$. Let $j(\beta(\xi - 1)) : S^{12} \wedge S^1(Sp(3)/S) \to S^{12}$ be the map given in (2). Then $R = E^4 j(\beta(\xi_{Sp(2)} - 1))$. Hence from (9) it follows that $R \simeq r \circ E^{12}Q_{Sp(2)}$ where r denotes a map of $S^{12} \wedge S^{10}$ into S^{12} such that $[r] = 2E^3c$, in the notation of (8), E^3c being a generator of $\pi^{12}(S^{12} \wedge S^{10}) = \mathbf{Z}_6$. If we take $\alpha = a_1 + a_2$ in the decompositions

$$\pi^{12}(S^{12} \wedge S^{21}) = \mathbf{Z}_6 \oplus (\mathbf{Z}_2)^2 \oplus \mathbf{Z}_2 \cdot a_1 \oplus \mathbf{Z}_2 \cdot a_2,$$
$$\pi^{23}(S^{23} \wedge S^{21}) = \pi_{21}^S = \mathbf{Z}_2 \cdot \sigma^3 \oplus \mathbf{Z}_2 \cdot \eta\bar{\kappa}$$

where a_1, a_2 are chosen to satisfy $E^{11}a_1 = \sigma^3$, $E^{11}a_2 = \eta\bar{\kappa}$ respectively, then r above coincides (up to homotopy) with that derived from α. Recall the decompositions

$$\pi^{12}(S^{12} \wedge S^{11}) = \mathbf{Z} \cdot c_1 \oplus \mathbf{Z}_{504} \cdot c_2, \qquad \pi^{13}(S^{13} \wedge S^{11}) = \mathbf{Z}_{504} \cdot c$$

where c_1 and c_2 are chosen to satisfy $E^1c_1 = E^1c_2 = c$. Then owing to the choice of α we find that a map representing $c_1 + c_2$ must be chosen to be ζ

and so invoking (5) from Lemma 2 we have

$$(E^{12}Q)^*(\alpha) = [j(\beta(\xi - 1))] - [\mu].$$

Further, since $2\pi_{21}^S = 0$, from the fact that both $E^1 c_1$ and $E^1 c_2$ equal c it follows that $\langle E^k[\mu], [\Phi_k]\rangle = 0$ for large k. Hence applying the equality above to (6) we get $[Sp(3), \mathcal{L}] = \sigma^3 + \eta\bar{\kappa}$. This completes the proof of the assertions here.

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