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A NON-SYMMETRIC DIFFUSION PROCESS ON THE WIENER SPACE

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Abstract. We discuss a non-symmetric diffusion process on the Wiener space. The process we consider is generated by $A = L + b$, L being the Ornstein-Uhlenbeck operator and b being a vector field. Under suitable integrability condition for b, we show the existence of associated diffusion process. We also investigate the domain of the generator. Further we consider a similar problem in the finite dimensional Euclidean space.

1. Introduction

We construct a non-symmetric diffusion process on an abstract Wiener space (B, H, μ). Here B is a real separable Banach space, μ is a Gaussian measure with a reproducing kernel Hilbert space H. The generator of the diffusion process we consider is of the form $A = L + b$, where L is the Ornstein-Uhlenbeck operator and b is an H-valued function. We regard b as a first order differential operator and so our generator A is, using the H-derivative D, given by

$$Af = Lf + \langle b, Df\rangle. \tag{1.1}$$

We assume the following condition:

$$\exp\{|b|^2\} \in L^{2+}(\mu) := \bigcup_{p>2} L^p(\mu). \tag{A.1}$$

We construct a diffusion process in the framework of Dirichlet form (to be precise, semi-Dirichlet form). In our case, the bilinear form $\mathcal{E}$ is given by

$$\mathcal{E}(f, g) = (-Af, g) = \int_B \{(Df, Dg)_{H^*} - \langle Df, gb\rangle\} d\mu. \tag{1.2}$$

To show the existence, we appeal to the general theory of Ma-Overbeck-Röckner [4] which ensure the existence of diffusion processes associated with non-symmetric semi-Dirichlet forms. We also discuss an issue of the domain of the generator. Further we consider a similar problem in the finite dimensional Euclidean space.

The organization of this paper is as follows. We review non-symmetric diffusion processes in the section 2. We recall conditions to ensure the existence of diffusion processes. We will check all conditions for our diffusion

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process. A main point is to see the equivalence between our bilinear form and that of the Ornstein-Uhlenbeck process. In the section 3, we determine the domain of the generator. We invoke the perturbation theorem and see that the domain coincides with the domain of the Ornstein-Uhlenbeck operator. Our main tool is the logarithmic Sobolev inequality. In the finite dimensional case, we can make use of the Sobolev inequality and so we consider similar problem in the finite dimensional Euclidean space in the section 4.

2. Non-symmetric diffusion

We first recall necessary notions from Ma-Overbeck-Röckner [4] which ensure the existence of diffusion processes.

Since our space is a separable Banach space, it is a good space as a topological space, i.e, a Polish space. We need to check the following conditions:

(1) There exists a constant $\gamma \geq 0$ so that $\tilde{\mathcal{E}}_\gamma = \tilde{\mathcal{E}} + \gamma(\cdot\,,\,\cdot)$ is positive definite and closed. Here $(\ ,\ )$ is the inner product in $L^2(\mu)$ and $\tilde{\mathcal{E}}$ is the symmetric part of $\mathcal{E}$:

$$\tilde{\mathcal{E}}(f,g) = \frac{1}{2}\big\{\mathcal{E}(f,g) + \mathcal{E}(g,f)\big\}.$$

(2) There exists a constant $K \geq 1$ such that

$$|\mathcal{E}(f,g)| \leq K\tilde{\mathcal{E}}_\gamma(f,f)^{1/2}\tilde{\mathcal{E}}_\gamma(g,g)^{1/2}.$$

(3) If $f \in \mathrm{Dom}(\mathcal{E})$, then $f^+ \wedge 1 \in \mathrm{Dom}(\mathcal{E})$ and

$$\mathcal{E}(f + f^+ \wedge 1, f - f^+ \wedge 1) \geq 0.$$

(4) Each element of $\mathrm{Dom}(\mathcal{E})$ admits a quasi-continuous version.

(5) $\mathcal{E}_\gamma$-capacity is tight.

Even though the case $\gamma = 0$ was treated in [4], there is no essential difference.

We now turn to our case $A = L + b$. The associated bilinear form $\mathcal{E}$ is given by (1.2). We use a perturbation method. Our basic generator is the Ornstein-Uhlenbeck operator and so we need several facts for the Ornstein-Uhlenbeck operator L. We denote the bilinear form associated with the Ornstein-Uhlenbeck operator by $\mathcal{E}^{\text{O-U}}$, i.e.,

$$\mathcal{E}^{\text{O-U}}(f,g) = \int_B (Df, Dg)_{H^*}\,d\mu. \tag{2.1}$$

Our main tool is the following logarithmic Sobolev inequality (due to Gross [2]): for $f \in \mathrm{Dom}(\mathcal{E}^{\text{O-U}})$,

$$\int_B |f|^2 \log\{|f|/\|f\|_2\}\,d\mu \leq \int_B |Df|^2\,d\mu. \tag{2.2}$$

Here $\|\cdot\|_2$ denotes the L^2 norm. When $p \geq 2$, by substituting f by $|f|^{p/2}\operatorname{sgn}(f)$, we easily have

$$\int_B |f|^p \log\{|f|/\|f\|_p\} d\mu \leq \frac{p}{2}\int_B |Df|^2|f|^{p-2} d\mu. \tag{2.3}$$

We use this inequality later to show that A is dissipative in L^p. Now let us check the first condition (1). First of all, we have to give the domain of $\mathcal{E}$. By the following proposition, $\mathcal{E}$ is well-defined on $\mathrm{Dom}(\mathcal{E}^{\text{O-U}})$.

Proposition 2.1. *There exist $\gamma > 0$ and $C > 0$ such that*

$$\mathcal{E}(f,f) \leq C\{\mathcal{E}^{\text{O-U}}(f,f) + (f,f)\}, \tag{2.4}$$

$$C^{-1}\mathcal{E}^{\text{O-U}}(f,f) \leq \mathcal{E}(f,f) + \gamma(f,f). \tag{2.5}$$

Proof. First we note

$$|(fb, Df)_{H^*}| \leq \frac{1}{2}\{|Df|^2 + |f|^2|b|^2\}.$$

To estimate the second term in the right hand side, we recall the Hausdorff-Young inequality $st \leq s\log s - s + e^t$. Using this, we have

$$|f|^2|b|^2 \leq \frac{|f|^2}{2(1+\varepsilon)}\log\frac{|f|^2}{2(1+\varepsilon)} - \frac{|f|^2}{2(1+\varepsilon)} + e^{2(1+\varepsilon)|b|^2}.$$

When $\|f\|_2 = 1$, we have

$$\begin{aligned}
\int_B |f|^2|b|^2 d\mu &\leq \frac{1}{1+\varepsilon}\int_B |f|^2\log|f| d\mu - \frac{1}{2(1+\varepsilon)}\int_B |f|^2 \log 2(1+\varepsilon) d\mu \\
&\quad - \int_B \frac{|f|^2}{2(1+\varepsilon)} d\mu + \|e^{2(1+\varepsilon)|b|^2}\|_1 \\
&\leq \frac{1}{1+\varepsilon}\int_B (Df, Df)_{H^*} d\mu \\
&\quad - \frac{1}{2(1+\varepsilon)}\{1 + \log 2(1+\varepsilon)\} + \|e^{2(1+\varepsilon)|b|^2}\|_1.
\end{aligned}$$

Hence, for general f,

$$\begin{aligned}
\int_B |f|^2|b|^2 d\mu &\leq \frac{1}{1+\varepsilon}\int_B (Df, Df)_{H^*} d\mu \\
&\qquad + \Big\{\|e^{2(1+\varepsilon)|b|^2}\|_1 - \frac{1}{2(1+\varepsilon)}\log 2e(1+\varepsilon)\Big\}\|f\|_2^2 \\
&\leq \frac{1}{1+\varepsilon}\int_B (Df, Df)_{H^*} d\mu + 2\gamma\|f\|_2^2
\end{aligned}$$

where we set

$$2\gamma = \|e^{2(1+\varepsilon)|b|^2}\|_1 - \frac{1}{2(1+\varepsilon)}\log 2e(1+\varepsilon).$$

Thus we have

$$\begin{aligned}\mathcal{E}(f,f) &\geq \int_B |Df|^2 d\mu - \frac{1}{2}\int_B \big(|Df|^2 + |f|^2|b|^2\big) d\mu \\ &\geq \int_B |Df|^2 d\mu - \frac{1}{2}\int_B |Df|^2 d\mu - \frac{1}{2(1+\varepsilon)}\int_B |Df|^2 d\mu - \gamma\|f\|_2^2 \\ &\geq \frac{\varepsilon}{2(1+\varepsilon)}\int_B |Df|^2 d\mu - \gamma\|f\|_2^2.\end{aligned}$$

From this

$$\mathcal{E}(f,f) + \gamma\|f\|_2^2 \geq \frac{\varepsilon}{2(1+\varepsilon)}\int_B |Df|^2 d\mu. \tag{2.6}$$

Similarly we have

$$\mathcal{E}(f,f) \leq \frac{4+3\varepsilon}{2(1+\varepsilon)}\int_B |Df|^2 d\mu + \gamma\|f\|_2^2$$

which shows (2.4). □

By the above proposition, $\tilde{\mathcal{E}}$ is equivalent with the following form

$$\mathcal{E}_1^{\text{O-U}}(f,g) = \int_B (Df, Dg)_{H^*} d\mu + (f,g)_{L^2},$$

but it is well known that this form is closed. Now we have shown that $\tilde{\mathcal{E}}_\gamma$ with the domain $\text{Dom}(\mathcal{E}^{\text{O-U}})$ is positive definite and closed. This shows (1).

(2) is shown in the similar manner.

Instead of (3), we will show the following:

$$\mathcal{E}(f^+ \wedge 1, f - f^+ \wedge 1) \geq 0.$$

(Equivalence can be seen in [5, Theorem I.4.4].) Moreover it is enough to show the following (see, e.g., [5, Proposition I.4.7]): for any $\varepsilon > 0$ there exists $\varphi_\varepsilon \colon \mathbb{R} \to [-\varepsilon, 1+\varepsilon]$ such that $\varphi_\varepsilon(t) = t$ for $t \in [0,1]$, $0 \leq \varphi_\varepsilon(t_2) - \varphi_\varepsilon(t_1) \leq t_2 - t_1$ if $t_1 \leq t_2$, $\varphi_\varepsilon(f) \in \text{Dom}(\mathcal{E})$ and

$$\liminf_{\varepsilon\to 0} \mathcal{E}(\varphi_\varepsilon(f), f - \varphi_\varepsilon(f)) \geq 0.$$

We take any smooth function φ_ε such that

$$\varphi_\varepsilon(t) = \begin{cases} -\varepsilon, & t \leq -2\varepsilon, \\ 1+\varepsilon, & t \geq 1+2\varepsilon. \end{cases}$$

Now

$$\begin{aligned}\mathcal{E}(\varphi_\varepsilon(f), f - \varphi_\varepsilon(f)) = &\int_B (D\varphi_\varepsilon(f), D(f - \varphi_\varepsilon(f)))_{H^*} d\mu \\ &- \int_B (D\varphi_\varepsilon(f), b(f - \varphi_\varepsilon(f)))_{H^*} d\mu\end{aligned}$$

$$= \int_B \varphi_\varepsilon'(f)(1-\varphi_\varepsilon'(f))|Df|^2_{H^*}d\mu$$
$$- \int_B \varphi_\varepsilon'(f)(Df,b)_{H^*}(f-\varphi_\varepsilon(f))d\mu.$$

On the other hand,

$$\int_B |\varphi_\varepsilon'(f)(Df,b)_{H^*}(f-\varphi_\varepsilon(f))|d\mu \le \int_{\{-2\varepsilon\le f\le 0\}\cup\{1\le f\le 1+2\varepsilon\}} |Df||b||f|d\mu.$$

Hence

$$\limsup_{\varepsilon\to 0}\int_B |\varphi_\varepsilon'(f)(Df,b)_{H^*}(f-\varphi_\varepsilon(f))|d\mu \le \int_{\{f=0\}\cup\{f=1\}} |Df||b||f|d\mu = 0.$$

Here we used that for any constant c,

$$\int_{\{f=c\}} |Df|d\mu = 0.$$

Thus we have

$$\liminf_{\varepsilon\to 0}\mathcal{E}(\varphi_\varepsilon(f), f-\varphi_\varepsilon(f)) \ge 0$$

as desired.

(4) can be shown by a standard method because $\mathrm{Dom}(\mathcal{E})\cap C(B)$ is dense in $\mathrm{Dom}(\mathcal{E})$.

Lastly we show (5), the tightness of the capacity. Before proving this let us review the capacity. In the following, we take $\alpha > \gamma$ so that $\tilde{\mathcal{E}}_\alpha$ is equivalent with $\mathcal{E}^{\text{O-U}}$. Hence $\tilde{\mathcal{E}}_\alpha$ is positive definite and the resolvent G_α exists for $\alpha > \gamma$. For any function h on B and an open set U, set

$$\mathcal{L}_{h,U} = \{w \in \mathrm{Dom}(\mathcal{E})\,|w \ge h \text{ a.e. on } U\}. \tag{2.7}$$

Then it holds that (see Ma-Röckner [5, Chapter III.1.])

i) there exists a unique $h_U \in \mathcal{L}_{h,U}$ such that

$$\mathcal{E}_\alpha(h_U, w) \ge \mathcal{E}_\alpha(h_U, h_U) \quad {}^\forall w \in \mathcal{L}_{h,U}. \tag{2.8}$$

ii) h_U is α-excessive and if $w \in \mathrm{Dom}(\mathcal{E})$ and $w \ge 0$ a.e. on U, then $\mathcal{E}_\alpha(h_U, w) \ge 0$.

iii) If, in addition, h is α-excessive, then

$$0 \le h_U \le h, \quad h_U = h \text{ on } U.$$

Using this notation, the capacity is defined as follows. We take any α-excessive functions h, and co-α-excessive $\hat{g}$ and fix them. For any open set U, set

$$\mathrm{Cap}_{h,g}(U) := \mathcal{E}_\alpha(h_U, \hat{g}_U). \tag{2.9}$$

In particular, take any $\phi \in L^2$ such that $\phi > 0$ a.e., and set $h = G_\alpha\phi$, $\hat{g} = \hat{G}_\alpha\phi$. Then,

$$\mathrm{Cap}_{h,g}(U) = (h_U, \phi) = (\phi, \hat{g}_U) = \mathcal{E}_\alpha(h_U, \hat{g}). \tag{2.10}$$

Here, we used that $\hat{g}_U = \hat{g}$ a.e. on U.

In the following we take $\phi = 1$ and we write Cap in place of $\mathrm{Cap}_{h,g}$. Since $A1 = 0$, we have $\alpha G_\alpha 1 = 1$, i.e., $h = 1/\alpha$. Since the Ornstein-Uhlenbeck operator is self-adjoint, the associated capacity is characterized as follows:

$$\mathrm{Cap}^{\text{O-U}}(U) = \inf\{\mathcal{E}_1^{\text{O-U}}(w, w)\,; w \geq 1 \text{ a.e. on } U\} \tag{2.11}$$

This capacity can be extended for arbitrary sets as an outer capacity.

Proposition 2.2. *There exists $C > 0$ such that for any set E,*

$$\mathrm{Cap}(E)^2 \leq C\,\mathrm{Cap}^{\text{O-U}}(E). \tag{2.12}$$

Proof. It is enough to show this for an open set U. First we note that

$$\begin{aligned}\mathrm{Cap}(U) &= \mathcal{E}_\alpha(h_U, \hat{g}) \\ &\leq K_\alpha \mathcal{E}_\alpha(h_U, h_U)^{1/2}\mathcal{E}_\alpha(\hat{g}, \hat{g})^{1/2}.\end{aligned}$$

Here, for the inequality in the second line, see e.g., [7, (1.1.3)]. Further, for any $w \in \mathcal{L}_{h,U}$, i.e., $w \geq 1/\alpha$ a.e. on U,

$$\begin{aligned}\mathcal{E}_\alpha(h_U, h_U) &\leq \mathcal{E}_\alpha(h_U, w) \\ &\leq K_\alpha \mathcal{E}_\alpha(h_U, h_U)^{1/2}\mathcal{E}_\alpha(w, w)^{1/2}\end{aligned}$$

which implies

$$\begin{aligned}\mathcal{E}_\alpha(h_U, h_U) &\leq K_\alpha^2 \mathcal{E}_\alpha(w, w) \\ &\leq K_1 \int_B (|Dw|^2 + |w|^2)d\mu.\end{aligned}$$

Thus we have

$$\mathrm{Cap}(U)^2 \leq K_\alpha^2 K_1 \mathcal{E}_\alpha(\hat{g}, \hat{g}) \inf\left\{\int_B (|Dw|^2 + |w|^2)d\mu\,; w \geq 1/\alpha \text{ a.e. on } U\right\}. \tag{2.13}$$

This means that Cap is dominated by the capacity associated with the Ornstein-Uhlenbeck process up to constant. □

It is well-known that the capacity associated with the Ornstein-Uhlenbeck process is tight. Now the tightness of our capacity follows.

The above proposition is sufficient to show the tightness. But one may ask whether the reversed estimate holds and we have

Proposition 2.3. Cap *and* $\mathrm{Cap}^{\text{O-U}}$ *have the common capacity zero sets, i.e.,* $\mathrm{Cap}(V) = 0$ *if and only if* $\mathrm{Cap}^{\text{O-U}}(V) = 0$.

Proof. We use a characterization of capacity zero sets by means of Dirichlet form. An increasing sequence of closed set $\{F_n\}$ is called a nest (precisely to say $\mathcal{E}_{\gamma+1}$-nest) if $\bigcup_n \mathrm{Dom}(\mathcal{E})_{F_n}$ is dense in $\mathrm{Dom}(\mathcal{E})$. Here, $\mathrm{Dom}(\mathcal{E})_{F_n}$ is a set of all elements in $\mathrm{Dom}\,\mathcal{E}$ with support in F_n. Then, a set N is of zero capacity if and only if there exists a nest $\{F_n\}$ such that $N \subseteq \bigcap_n F_n^c$.

Now the result follows easily since $\mathcal{E}$ and $\mathcal{E}^{\text{O-U}}$ have the same domain and the same topology. □

If we impose an additional condition, we can give a direct estimate between two capacities. To show this, we recall that the reduced function is defined for an arbitrary set by using the quasi-continuous function. For any set V and a function h, put

$$\mathcal{L}_{h,V} = \{w \in \mathrm{Dom}(\mathcal{E}) \,|\, \tilde{w} \geq h \text{ Cap-q.e. on } V\}. \tag{2.14}$$

Here, $\tilde{w}$ denotes the quasi-continuous modification of w with respect to the capacity Cap. We can show that there exists h_V satisfying (2.8) as well (see [5, Capter III Exercise 3.10 (ii)]). Moreover similar properties hold for h_V if we replace a.e. with q.e. We denote the reduced function with respect to $\mathcal{E}_1^{\text{O-U}}$ by e_V. Further we assume that $\hat{g}$ is $\mathrm{Cap}^{\text{O-U}}$-quasi-continuous and hence Cap-quasi continuous by Proposition 2.2.

Proposition 2.4. *There exists a constant $C > 0$ such that for any $\lambda > 0$ and any set U,*

$$\mathrm{Cap}^{\text{O-U}}(U \cap \{\hat{g} > \lambda\}) \leq C\lambda^{-1}\,\mathrm{Cap}(U \cap \{\hat{g} > \lambda\}) \tag{2.15}$$

Proof. We recall that for any set V,

$$\begin{aligned}\mathrm{Cap}(V) &= \mathcal{E}_\alpha(h_V, \hat{g}),\\ \mathrm{Cap}^{\text{O-U}}(V) &= \mathcal{E}_1^{\text{O-U}}(e_V, e_V).\end{aligned}$$

Setting $V_\lambda = U \cap \{\hat{g} > \lambda\}$, we have

$$\begin{aligned}\mathrm{Cap}^{\text{O-U}}(V_\lambda) &= \mathcal{E}_1^{\text{O-U}}(e_{V_\lambda}, e_{V_\lambda})\\ &\leq \mathcal{E}_1^{\text{O-U}}(h_{V_\lambda}, h_{V_\lambda}) \quad (\because h_{V_\lambda} \geq 1 \text{ q.e. on } V_\lambda \text{ and}\\ &\qquad\qquad\qquad e_{V_\lambda} \text{ is the minimizing element})\\ &\leq C_1\mathcal{E}_\alpha(h_{V_\lambda}, h_{V_\lambda}) \quad (\because \text{Proposition 2.1})\\ &\leq C_1\lambda^{-1}\mathcal{E}_\alpha(h_{V_\lambda}, \hat{g}) \quad (\because \lambda^{-1}\hat{g} \geq 1 \text{ on } V_\lambda)\\ &= C_1\lambda^{-1}\,\mathrm{Cap}(V_\lambda)\end{aligned}$$

which completes the proof. □

3. Domain of the generator

In this section, we determine the domain of the generator.

Let $\{T_t\}$ be the semigroup associated with the semi-Dirichlet form $\mathcal{E}$. There exists $\gamma \geq 0$ such that $\{e^{-\gamma t}T_t\}$ is a contraction semigroup in L^2. By the interpolation, $\{T_t\}$ is also a strongly continuous contraction semigroup in L^p. To specify the space, we denote the semigroup by $\{T_t^{(p)}\}$ and its generator by $A_{(p)}$.

As before let $\mathcal{E}^{\text{O-U}}$ be the Dirichlet form associated with the Ornstein-Uhlenbeck operator defined by (2.1). $\{T_t^{\text{O-U}}\}$ and L denote the semigroup and the generator. $L_{(p)}$ denotes the generator in L^p and we regard $A_{(p)}$ as a perturbation of $L_{(p)}$.

Theorem 3.1. *Under the condition* (A.1)*, we have* $\mathrm{Dom}(A_{(2)}) = \mathrm{Dom}(L_{(2)})$ *and further if* $\mathcal{D} \subseteq \mathrm{Dom}(L_{(2)})$ *is dense (with respect to the graph norm of* $L_{(2)}$*), the* $\mathcal{D}$ *is also dense in* $\mathrm{Dom}(A_{(2)})$*. When* $p > 2$*, there exists a constant* $r = r_p$ *such that if* $e^{|b|^2} \in L^{r+}$*, then the above result holds for* $A_{(p)}$*.*

To prove this theorem, we recall a general theory. Let S be a closed operator on a Banach space B. We assume that S is *hyper-dissipative* (sometimes called m-dissipative) and hence generates a contraction semigroup. Here let us explain what it means that S is hyper-dissipative. An operator S on a Banach space B is called *dissipative* if for any $u \in B$ and $\varphi \in F(u)$ it holds that

$$\langle Su, \varphi \rangle \leq 0. \tag{3.1}$$

Here F is the duality mapping, i.e., $F(u)$ is a set of all φ such that

$$\langle u, \varphi \rangle = \|u\|_B^2 = \|\varphi\|_{B^*}^2.$$

When $B = L^p$ with $p \in [0, +\infty)$, F is given by $F(u) = |u|^{p-1}\,\mathrm{sgn}(u)\|u\|_p^{2-p}$. Further, if $\mathrm{Ran}(1 - S) = B$, S is called *hyper-dissipative.*

Suppose we are given an operator T with $\mathrm{Dom}(T) \supseteq \mathrm{Dom}(S)$. Then the following fact is well-known (e.g. Pazy [8, Theorem 3.3.2.]). Suppose that T satisfies the followings:

(P.1) For any $s \in [0, 1]$, $(S + sT, \mathrm{Dom}(S))$ is dissipative.

(P.2) There exists positive constants a and b such that $0 < a < 1$ and

$$\|Tx\|_E \leq a\|Sx\|_E + b\|x\|_E, \quad {}^\forall x \in \mathrm{Dom}(S). \tag{3.2}$$

Then $(S+T, \mathrm{Dom}(S))$ is hyper-dissipative. Further if $\mathcal{D} \subseteq \mathrm{Dom}(S)$ is dense, then $\mathcal{D}$ is dense in $\mathrm{Dom}(S + T)$ with respect to the graph norm.

To apply this fact, we first see that $A_{(p)} + \gamma = L_{(p)} + b + \gamma$ is dissipative for some $\gamma > 0$. This fact was already shown when $p = 2$. In fact we can take $\varepsilon = 0$ in (2.6). When $p > 2$, let q be the conjugate exponent of p, i.e., $1/p + 1/q = 1$. By the logarithmic Sobolev inequality and the Hausdorff-Young inequality $st \le s\log s - s + e^t$, we have

$$\begin{aligned}
\int_B |f|^p|b|^2 d\mu &\le \int_B \Big\{ \frac{2|f|^p}{q^2}\log\frac{2|f|^p}{q^2} - \frac{2|f|^p}{q^2} + e^{q^2|b|^2/2}\Big\} d\mu \\
&\le \frac{2p}{q^2}\int_B |f|^p \log|f|\,d\mu \\
&\quad + \int_B \Big\{ \frac{2}{q^2}\log(2/q^2)|f|^p - \frac{2}{q^2}|f|^p + e^{q^2|b|^2/2}\Big\} d\mu \\
&\le \frac{2p}{q^2}\int_B |f|^p \log(|f|/\|f\|_p) d\mu + \frac{2p}{q^2}\|f\|_p^p \log\|f\|_p \\
&\quad + (\frac{2}{q^2}\log(2/q^2) - \frac{2}{q^2})\|f\|_p^p + \|e^{q^2|b|^2/2}\|_1 \\
&\le \frac{p^2}{q^2}\int_B |Df|^2|f|^{p-2} d\mu + \frac{2p}{q^2}\|f\|_p^p \log\|f\|_p \\
&\quad + (\frac{2}{q^2}\log(2/q^2) - \frac{2}{q^2})\|f\|_p^p + \|e^{q^2|b|^2/2}\|_1. \quad (\because (2.3))
\end{aligned}$$

Hence, if $\|f\|_p = 1$, then, setting $g = F(f) = |f|^{p-1}\operatorname{sgn}(f)$

$$\begin{aligned}
-(Af, g) &= \int_B (Df, Dg)_{H^*}\,d\mu - \int_B \langle Df, gb\rangle\,d\mu \\
&= (p-1)\int_B |Df|^2|f|^{p-2}d\mu - \langle Df, b|f|^{p-1}\operatorname{sgn}(f)\rangle \\
&\ge (p-1)\int_B |Df|^2|f|^{p-2}d\mu - \frac{p}{2q}\int_B |Df|^2|f|^{p-2}d\mu \\
&\quad - \int_B \frac{q}{2p}|f|^p|b|^2 d\mu \\
&= (p-1-\frac{p}{2q})\int_B |Df|^2|f|^{p-2}d\mu - \frac{q}{2p}\Big\{\frac{p^2}{q^2}\int_B |Df|^2|f|^{p-2}d\mu \\
&\quad + \frac{2p}{q^2}\|f\|_p^p\log\|f\|_p + \Big(\frac{2}{q^2}\log(2/q^2) - \frac{2}{q^2}\Big)\|f\|_p^p + \|e^{q^2|b|^2/2}\|_1\Big\} \\
&= (p-1-\frac{p}{2q}-\frac{p}{2q})\int_B |Df|^2|f|^{p-2}d\mu \\
&\quad + \Big\{\Big(\frac{2}{q^2}\log(2/q^2) - \frac{2}{q^2}\Big) + \|e^{q^2|b|^2/2}\|_1\Big\} \\
&\ge \frac{2}{q^2}\log(2/q^2) - \frac{2}{q^2} + \|e^{q^2|b|^2/2}\|_1.
\end{aligned}$$

Therefore, for a general $f \in L^p$ and $g = F(f)$, it holds that

$$(3.3) \qquad (Af, g) + \Big\{ \Big(\frac{2}{q^2} \log(2/q^2) - \frac{2}{q^2} \Big) + \|e^{q^2|b|^2/2}\|_1 \Big\} (f, g) \leq 0.$$

Now, it easily follows that $A + \gamma$ is dissipative where we set

$$(3.4) \qquad \gamma = \frac{2}{q^2} \log(2/q^2) - \frac{2}{q^2} + \|e^{q^2|b|^2/2}\|_1.$$

Next we see the condition (P.2). By the Hausdorff-Young inequality,

$$|\langle b, Df \rangle|^2 \leq |Df|^2 |b|^2 \leq \frac{|Df|^2}{2(1+\varepsilon)} \log \frac{|Df|^2}{2(1+\varepsilon)} - \frac{|Df|^2}{2(1+\varepsilon)} + e^{2(1+\varepsilon)|b|^2}.$$

When $\|Df\|_2 = 1$, we have

$$\begin{aligned}
&\int_B |\langle b, Df \rangle|^2 d\mu \\
&\quad \leq \frac{1}{2(1+\varepsilon)} \int_B |Df|^2 \log |Df|^2 d\mu - \frac{1}{2(1+\varepsilon)} \int_B |Df|^2 \log 2(1+\varepsilon) d\mu \\
&\qquad - \int_B \frac{|Df|^2}{2(1+\varepsilon)} d\mu + \|e^{2(1+\varepsilon)|b|^2}\|_1 \\
&\quad \leq \frac{1}{1+\varepsilon} \int_B (D^2 f, D^2 f) d\mu - \frac{1}{2(1+\varepsilon)} \{1 + \log 2(1+\varepsilon)\} + \|e^{2(1+\varepsilon)|b|^2}\|_1 \\
&\quad \leq \frac{1}{1+\varepsilon} \int_B |Lf|^2 d\mu - \frac{1}{2(1+\varepsilon)} \{1 + \log 2(1+\varepsilon)\} + \|e^{2(1+\varepsilon)|b|^2}\|_1.
\end{aligned}$$

Here we used the Logarithmic Sobolev inequality (2.2) for H^*-valued function Df and the following inequality (see, e.g., [9, Proposition 4.5]):

$$(3.5) \qquad \int_E |D^2 f|^2 d\mu \leq \int_E |Lf|^2 d\mu.$$

We set

$$(3.6) \qquad 2\gamma = \|e^{2(1+\varepsilon)|b|^2}\|_1 - \frac{1}{2(1+\varepsilon)} \log 2e(1+\varepsilon).$$

Then, for general f, we have

$$\int_B |\langle b, Df \rangle|^2 d\mu \leq \frac{1}{1+\varepsilon} \int_B |Lf|^2 d\mu + 2\gamma \|Df\|_2^2.$$

Now we note that there exists a constant $K > 0$ such that

$$2\gamma \|Df\|_2^2 \leq \frac{\varepsilon/2}{1+\varepsilon} \|Lf\|_2^2 + K\|f\|_2^2.$$

Therefore, we have

$$\int_B |\langle b, Df \rangle|^2 d\mu \leq \frac{1+\varepsilon/2}{1+\varepsilon} \|Lf\|_2^2 + K\|f\|_2^2.$$

Thus we have

$$\left\{\int_B |\langle b, Df\rangle|^2 d\mu\right\}^{1/2} \leq \sqrt{\frac{1+\varepsilon/2}{1+\varepsilon}}\|Lf\|_2 + \sqrt{K}\|f\|_2.$$

This completes the proof when $p = 2$.

When $p > 2$, we have to modify the above proof. We need the following proposition:

Proposition 3.2. *There exist positive constants β, K_1 and K_2 such that*

$$\int_B |Df|^p \log_+^{p/2} |Df| d\mu \leq \beta\|(\lambda - L)f\|_p^p + K_1\|Df\|_p^p \log_+^{p/2} \|(\lambda - L)f\|_p + K_2\|Df\|_p^p \tag{3.7}$$

where $\log_+ x = \max\{\log x, 0\}$.

To prove this proposition, we recall the Orlicz space. Let Φ be a strictly increasing convex function with $\Phi(0) = 0$. We assume that Φ satisfies Δ_2 condition (i.e., there exists a constant K such that $\Phi(2x) \leq K\Phi(x)$ for all $x \geq 0$). The Orlicz space L_Φ is the set of all functions f satisfying

$$\int \Phi(|f|) d\mu < \infty$$

with the norm

$$\|f\|_\Phi := \inf\{\lambda > 0\,; \int \Phi(|f|/\lambda) d\mu \leq 1\}. \tag{3.8}$$

We take Φ to be equal to $t^p \log^{p/2} t$ for large $t > 0$. In this case, the space L_Φ is denoted by $L^p \log^{p/2} L$. We also assume that $\Phi(t) \geq t^p \log_+^{p/2} t$ for $t > 0$. On the Wiener space, it is known (see, Bakry-Meyer [1]) that $\sqrt{\lambda - L}^{-1}$ is a bounded operator from $L^p(\mu)$ into $L^p \log^{p/2} L(\mu)$. This fact also holds for a vector valued function. Therefore we have that there exists a constant c such that

$$\|Df\|_\Phi \leq c\|\sqrt{\lambda - L} Df\|_p.$$

By combining this with Meyer's inequality, we get

$$\|Df\|_\Phi \leq \kappa\|(\lambda - L)f\|_p \tag{3.9}$$

for some constant κ. Now we are ready to prove Proposition 3.2.

Proof of Proposition 3.2. From (3.9), if $\kappa\|(\lambda - L)f\|_p = 1$,

$$\int_B |Df|^p \log_+^{p/2} |Df| d\mu \leq \int_B \Phi(|Df|) d\mu \leq 1.$$

For general f, taking $f/\kappa\|(\lambda-L)f\|_p$ in the above argument, we have

$$\int_B \frac{|Df|^p}{\kappa^p\|(\lambda-L)f\|_p^p}\log_+^{p/2}\frac{|Df|}{\kappa\|(\lambda-L)f\|_p}d\mu \le 1.$$

Now we use the inequality $\log_+ ab \le \log_+ a + \log_+ b$. By the Hölder inequality, for any $\varepsilon > 0$, there exists a constant K such that

$$(a+b)^{p/2} \le (1+\varepsilon)a^{p/2} + Kb^{p/2}. \tag{3.10}$$

Together with this inequality, we have that for any $\varepsilon > 0$, there exists K such that

$$\log_+^{p/2} ab \le (1+\varepsilon)\log_+^{p/2} a + K\log_+^{p/2} b.$$

Using this inequality, we get

$$\begin{aligned}\log_+^{p/2}|Df| &= \log_+^{p/2}\Big\{\frac{|Df|}{\kappa\|(\lambda-L)f\|_p}\kappa\|(\lambda-L)f\|_p\Big\}\\ &\le (1+\varepsilon)\log_+^{p/2}\frac{|Df|}{\kappa\|(\lambda-L)f\|_p} + K\log_+^{p/2}\kappa\|(\lambda-L)f\|_p.\end{aligned}$$

Hence we have

$$\int_B \frac{|Df|^p}{\kappa^p\|(\lambda-L)f\|_p^p}\Big\{\frac{1}{1+\varepsilon}\log_+^{p/2}|Df| - \frac{K}{1+\varepsilon}\log_+^{p/2}\kappa\|(\lambda-L)f\|_p\Big\}d\mu \le 1.$$

Thus we have

$$\begin{aligned}\int_B |Df|^p\log_+^{p/2}|Df|d\mu &\le (1+\varepsilon)\kappa^p\|(\lambda-L)f\|_p^p\\ &\quad + K\|Df\|_p^p\{\log_+\|(\lambda-L)f\|_p + \log_+\kappa\}^{p/2}\end{aligned}$$

which implies the desired result. □

Now we turn to the proof of Theorem 3.1 when $p > 2$. We take any $\alpha \le 2e$. By the Hausdorff-Young inequality, we have

$$\begin{aligned}|\langle b, Df\rangle|^2 &\le |Df|^2|b|^2\\ &\le \frac{\alpha|Df|^2}{2}\log\frac{\alpha|Df|^2}{2} - \frac{\alpha|Df|^2}{2} + e^{2|b|^2/\alpha}\\ &\le \alpha|Df|^2\log|Df| + \Big\{\frac{\alpha}{2}\log(\alpha/2) - \frac{\alpha}{2}\Big\}|Df|^2 + e^{2|b|^2/\alpha}.\end{aligned}$$

Then we have

$$|\langle b, Df\rangle|^2 \le \alpha|Df|^2\log_+|Df| + e^{2|b|^2/\alpha}.$$

Using the inequality (3.10),

$$|\langle b, Df\rangle|^p \le (1+\varepsilon)\alpha^{p/2}|Df|^p\log_+^{p/2}|Df| + Ke^{p|b|^2/\alpha}.$$

Now by Proposition 3.2,

$$\int_B |\langle b, Df\rangle|^p d\mu \le (1+\varepsilon)\alpha^{p/2}\beta\|(\lambda - L)f\|_p^p + K_3\|Df\|_p^p \log_+^{p/2}\|(\lambda - L)f\|_p + K_4\|Df\|_p^p + K\|e^{p|b|^2/\alpha}\|_1.$$

Here β is the constant that appeared in (3.7). We can choose small $\delta > 0$ so that

$$K_3\delta^p \log_+^{p/2} t \le \varepsilon\alpha^{p/2}\beta t^p, \quad t \ge 0.$$

Therefore if $\|Df\|_p = \delta$, we have

$$\int_B |\langle b, Df\rangle|^p d\mu \le (1+2\varepsilon)\alpha^{p/2}\beta\|(\lambda - L)f\|_p^p + K_4\delta^p + K\|e^{p|b|^2/\alpha}\|_1.$$

For general f, taking $\delta f/\|Df\|_p$, we have

$$\int_B |\langle b, \delta Df/\|Df\|_p\rangle|^p d\mu \le (1+2\varepsilon)\alpha^{p/2}\beta\|(\lambda - L)\delta f/\|Df\|_p\|_p^p + K_4\delta^p + K\|e^{p|b|^2/\alpha}\|_1.$$

Thus we have,

$$\int_B |\langle b, Df\rangle|^p d\mu \le (1+2\varepsilon)\alpha^{p/2}\beta\|(\lambda - L)f\|_p^p + \|Df\|_p^p(K_4\delta^p + K\|e^{p|b|^2/\alpha}\|_1)/\delta^p.$$

If we take K_5 sufficiently large, the following inequality holds

$$\|Df\|_p^p(K_4\delta^p + K\|e^{p|b|^2/\alpha}\|_1)/\delta^p \le \varepsilon\alpha^{p/2}\beta\|(\lambda - L)f\|_p^p + K_5\|f\|_p^p.$$

This inequality is known as the moment inequality (see, e.g., [12, Chapter VIII, Theorem 6]). So eventually we have

$$\int_B |\langle b, Df\rangle|^p d\mu \le (1+3\varepsilon)\alpha^{p/2}\beta\|(\lambda - L)f\|_p^p + K_5\|f\|_p^p.$$

If $\alpha^{p/2}\beta < 1$, then we can see that (3.2) holds. To assure this, it is sufficient to assume that $e^{|b|^2} \in L^{p\beta^{2/p}+}$. This exponent depends on the constant in Proposition 3.2.

4. Finite dimensional space

In this section, we consider a similar problem in the finite dimensional Euclidean space and we determine the domain of the generator. The generator that we consider is of the form $A = \Delta + b$ on the Euclidean space $\mathbb{R}^d$ with $d \ge 3$. Here, Δ is the Laplacian and b is a vector field, i.e., first order differential operator. The associated bilinear form $\mathcal{E}$ is given by

$$\mathcal{E}(f, g) = \int_{\mathbb{R}^d} \{(\nabla f, \nabla g) - (b, \nabla f)g\}dx. \tag{4.1}$$

We assume the following integrability condition for b:

(A.2) There exists $q \in [d, \infty]$ such that $b \in L^q(\mathbb{R}^d)$.

Under this condition, the form $\mathcal{E}$ defines a closed coercive semi-Dirichlet form (see Lyons-Zhang [3]).

In this section, we will determine the domain of A. As before, we denote the generator in L^p by $A_{(p)}$ ($p \geq 2$).

Theorem 4.1. *Under the condition* (A.2)*, we have that for any* $p < q$, $\mathrm{Dom}(A_{(p)}) = \mathrm{Dom}(\Delta_{(p)})$ *and further if* $\mathcal{D} \subseteq \mathrm{Dom}(\Delta_{(p)})$ *is dense (with respect to the graph norm of* $\Delta_{(p)}$*), then* $\mathcal{D}$ *is also dense in* $\mathrm{Dom}(A_{(p)})$.

To prove this theorem, we need the following Sobolev inequality in $\mathbb{R}^d$. Let exponents p and r satisfy

$$\frac{1}{p} - \frac{1}{d} \leq \frac{1}{r} \leq \frac{1}{p}.$$

Then there exists a constant C such that

$$\|u\|_r \leq C\{\|\nabla u\|_p^{\lambda}\|u\|_p^{1-\lambda} + \|u\|_p\} \tag{4.2}$$

where $\lambda = d\big(\frac{1}{p} - \frac{1}{r}\big)$. If $\frac{1}{r} = \frac{1}{p} - \frac{1}{d}$, then $\lambda = 1$ and (4.2) reads

$$\|u\|_r \leq C\{\|\nabla u\|_p + \|u\|_p\}. \tag{4.3}$$

If $\frac{1}{r} > \frac{1}{p} - \frac{1}{d}$, we have a stronger inequality: for any $\varepsilon > 0$ we can choose large K so that

$$\|u\|_r \leq \varepsilon\|\nabla u\|_p + K\|u\|_p. \tag{4.4}$$

Even though this case is rather tractable, we use the Sobolev inequality of the form (4.3) to include the case $\frac{1}{r} = \frac{1}{p} - \frac{1}{d}$. We use the Sobolev inequality in this form.

Take any $f \in H^{1,p}(\mathbb{R}^d)$, i.e., f, $\nabla f \in L^p$. Set

$$h = |f|^{p/2}\,\mathrm{sgn}(f).$$

Then $\nabla h = \frac{p}{2}|f|^{(p/2)-1}\nabla f$ and

$$\|h\|_2^2 = \int_{\mathbb{R}^d} |f|^p dx,$$

$$\|\nabla h\|_2^2 = \frac{p^2}{4}\int_{\mathbb{R}^d} |f|^{p-2}|\nabla f|^2 dx.$$

Define $r > 1$ so that

$$\frac{1}{r} = \frac{1}{2} - \frac{1}{q}. \tag{4.5}$$

Since $q \geq d$, it holds that $\frac{1}{2} \geq \frac{1}{r} \geq \frac{1}{2} - \frac{1}{d}$. Now, by the Sobolev inequality (4.3), we have

$$\|h\|_r^2 \leq C\{\|\nabla h\|_2^2 + \|h\|_2^2\}$$

which implies

$$\Big\{\int_{\mathbb{R}^d} |f|^{pr/2} dx\Big\}^{2/r} \leq C\Big\{\int_{\mathbb{R}^d} |\nabla f|^2 |f|^{p-2} dx + \int_{\mathbb{R}^d} |f|^p dx\Big\}. \tag{4.6}$$

Now we are ready to prove the dissipative property (P.1). For any $f \in H^{2,p}(\mathbb{R}^d)$, $g = |f|^{p-1}\operatorname{sgn}(f)$, we have

$$\begin{aligned}
-(Af, g) &= \int_{\mathbb{R}^d} \{(\nabla f, \nabla g) - (\nabla f, b)|f|^{p-1}\operatorname{sgn}(f)\} dx \\
&\geq (p-1)\int_{\mathbb{R}^d} |\nabla f|^2 |f|^{p-2} dx - \int_{\mathbb{R}^d} |\nabla f||b||f|^{p-1} dx \\
&\geq (p-1)\int_{\mathbb{R}^d} |\nabla f|^2 |f|^{p-2} dx \\
&\quad - \int_{\mathbb{R}^d} \frac{\sqrt{p-1}}{2} |\nabla f||f|^{(p-2)/2} \frac{2}{\sqrt{p-1}} |b||f|^{p/2} dx \\
&\geq (p-1)\int_{\mathbb{R}^d} |\nabla f|^2 |f|^{p-2} dx \\
&\quad - 2\Big\{\frac{p-1}{4}\int_{\mathbb{R}^d} |\nabla f|^2 |f|^{p-2} dx + \frac{4}{p-1}\int_{\mathbb{R}^d} |b|^2 |f|^p dx\Big\} \\
&\geq \frac{p-1}{2}\int_{\mathbb{R}^d} |\nabla f|^2 |f|^{p-2} dx - \frac{8}{p-1}\int_{\mathbb{R}^d} |b|^2 |f|^p dx \\
&\geq \frac{p-1}{2}\int_{\mathbb{R}^d} |\nabla f|^2 |f|^{p-2} dx - \frac{8N^2}{p-1}\int_{\mathbb{R}^d} |f|^p dx \\
&\quad - \frac{8}{p-1}\int_{\mathbb{R}^d} |b|^2 1_{\{|b|>N\}} |f|^p dx.
\end{aligned}$$

Since $\frac{1}{r/2} + \frac{1}{q/2} = 1$, we can apply the Hölder inequality. Hence, by (4.6)

$$\begin{aligned}
&\int_{\mathbb{R}^d} |b|^2 1_{\{|b|>N\}} |f|^p dx \\
&\quad \leq \Big\{\int_{\mathbb{R}^d} |b|^{2q/2} 1_{\{|b|>N\}} dx\Big\}^{2/q} \Big\{\int_{\mathbb{R}^d} |f|^{pr/2} dx\Big\}^{2/r} \\
&\quad \leq C\Big\{\int_{\mathbb{R}^d} |b|^q 1_{\{|b|>N\}} dx\Big\}^{2/q} \Big\{\int_{\mathbb{R}^d} |f|^{p-2}|\nabla f|^2 dx + \int_{\mathbb{R}^d} |f|^p dx\Big\}.
\end{aligned}$$

We take N to be large enough so that

$$\frac{8C}{p-1}\Big\{\int_{\mathbb{R}^d} |b|^q 1_{\{|b|>N\}} dx\Big\}^{2/q} \leq \frac{p-1}{4}.$$

Thus we have

$$-(Af,g) \geq \frac{p-1}{4}\int_{\mathbb{R}^d} |\nabla f|^2 |f|^{p-2} dx - \left\{\frac{8N^2}{p-1} + \frac{p-1}{4}\right\}\int_{\mathbb{R}^d} |f|^p dx.$$

Set $\gamma = \frac{8N^2}{p-1} + \frac{p-1}{4}$. Then we eventually obtain

$$-(Af,g) + \gamma(f,g) \geq \frac{p-1}{4}\int_{\mathbb{R}^d} |\nabla f|^2 |f|^{p-2} dx \geq 0$$

which implies that $A - \gamma$ is dissipative.

Next we will see (P.2). We set $\frac{1}{\alpha} = \frac{p}{q}$ and $\frac{1}{\beta} = 1 - \frac{p}{q}$. Since $p < q$, we have $1 < \alpha, \beta$. Note that

$$\frac{1}{p\beta} = \frac{1}{p} - \frac{1}{q} \geq \frac{1}{p} - \frac{1}{d}.$$

Hence, by the Sobolev inequality, we have

$$\begin{aligned}\left\{\int_{\mathbb{R}^d} |\nabla f|^{p\beta} dx\right\}^{1/\beta} &\leq C\{\|\nabla^2 f\|_p^p + \|\nabla f\|_p^p\} \\ &\leq C'\{\|\Delta f\|_p^p + \|\nabla f\|_p^p\}.\end{aligned}$$

Now, by the Hölder inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^d} |(b,\nabla f)|^p dx &\leq \int_{\mathbb{R}^d} |b|^p 1_{\{|b|>N\}} |\nabla f|^p dx + N^p \int_{\mathbb{R}^d} |\nabla f|^p dx \\ &\leq \left\{\int_{\mathbb{R}^d} |b|^{p\alpha} 1_{\{|b|>N\}} dx\right\}^{1/\alpha} \left\{\int_{\mathbb{R}^d} |\nabla f|^{p\beta} dx\right\}^{1/\beta} \\ &\quad + N^p \int_{\mathbb{R}^d} |\nabla f|^p dx \\ &\leq C'\left\{\int_{\mathbb{R}^d} |b|^q 1_{\{|b|>N\}} dx\right\}^{1/\alpha} \{\|\Delta f\|_p^p + \|\nabla f\|_p^p\} + N^p \|\nabla f\|_p^p.\end{aligned}$$

For any $\varepsilon > 0$, we take N to be large enough so that

$$C'\left\{\int_{\mathbb{R}^d} |b|^q 1_{\{|b|>N\}} dx\right\}^{1/\alpha} \leq \varepsilon$$

Moreover, by the interpolation inequality, there exists a constant K such that

$$(\varepsilon + N^p)\|\nabla f\|_p^p \leq \varepsilon \|\Delta f\|_p^p + K\|f\|_p^p.$$

Thus we have

$$\int_{\mathbb{R}^d} |(b,\nabla f)|^p dx \leq 2\varepsilon \|\Delta f\|_p^p + K\|f\|_p^p$$

which implies (P.2). This completes the proof.

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