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# PRIMARY DECOMPOSITIONS IN ABELIAN $R$-CATEGORIES

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Abstract. We shall generalize the theory of primary decomposition and associated prime ideals of finitely generated modules over a noetherian ring to general objects in an abelian $R$-category where $R$ is a noetherian commutative ring.

## 1. Introduction

The primary decomposition theorem is a well-known theorem in commutative algebra, which asserts that every ideal in a noetherian commutative ring is an intersection of finitely many primary ideals. The theorem was proved by E.Lasker for polynomial rings and convergent power series rings in 1905, and was proved for general noetherian rings by E.Noether in 1921. This has been extended to modules, namely, every submodule of a finitely generated module over a noetherian commutative ring is a finite intersection of primary submodule. See [7, Section 6], for example.

In addition, the analogous decomposition theorem has been studied by many authors. Firstly D.Kirby [5] proved it for artinian modules, and later R.Y.Sharp [9] showed it for injective modules. One can also see some of their applications in [6] and [10].

The main purpose of this paper is to show how we can extend such primary decomposition theorem for general objects in an abelian $R$-category, where $R$ is a commutative noetherian ring.

In Section 2 we shall recall some definitions and general facts from the theory of abelian categories, which are to prepare for later usage.

The first task to do is to define the 'associated prime ideals' for objects in an abelian $R$-category, which will be done in Section 3. For an object $X$ in an abelian $R$-category $\mathcal{C}$, we will be able to define the set $\mathfrak{Ass}_R(X)$ as a subset of $\mathrm{Spec}(R)$, whose element we call an associated prime to $X$. Of course, if $X$ is an $R$-module regarded as an object in the abelian category of all $R$-modules, $\mathfrak{Ass}_R(X)$ agrees with the set $\mathrm{Ass}_R(X)$ of ordinary associated primes. The basic properties for $\mathfrak{Ass}_R(X)$ are discussed in Section 3.

We say that a noetherian object $Y$ is primary if $\mathfrak{Ass}_R(Y)$ consists of a single element. In Section 5 we prove that, given a noetherian object $X$ in an abelian $R$-category, the zero subobject of $X$ is a finite intersection of

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subobjects $Y$ where $X/Y$ is primary. In such a sense the primary decomposition theorem holds true in wider context. See Theorem 4.7 for more detail.

Let $\mathcal{S}$ be a Serre subcategory of an abelian $R$-category $\mathcal{C}$. Then we can consider the quotient category $\mathcal{C}/\mathcal{S}$ with natural quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{S}$ sending an object $X$ to $X_{\mathcal{S}}$. It is natural and interesting as well to ask how the associated primes $\mathfrak{Ass}_R(X)$ or the primary decomposition of $X \in \mathcal{C}$ behaves under this functor. We are able to give a complete answer to this question in Theorem 8.3 in Section 8. The proof of this final theorem is not hard but needs some preliminaries. First of all we have to clarify how objects in $\mathcal{C}/\mathcal{S}$ can be noetherian. This is partially answered in Section 5 and some examples are given in Section 6. Next we need to know how the associated primes behave under the localization by a prime ideal. If $\mathfrak{p}$ is a prime ideal of $R$, we can consider the localized abelian $R_{\mathfrak{p}}$-category $\mathcal{C}_{\mathfrak{p}}$ with natural localization functor $\mathcal{C} \to \mathcal{C}_{\mathfrak{p}}$ sending an object $X$ to $X_{\mathfrak{p}}$. We shall show in Theorem 7.3 the equality $\mathfrak{Ass}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) = \{\mathfrak{q}R_{\mathfrak{p}} \mid \mathfrak{q} \in \mathfrak{Ass}_R(X) \text{ and } \mathfrak{q} \subseteq \mathfrak{p}\}$. After these preliminaries, we show in Theorem 8.3 how we can get the primary decomposition of $X_{\mathcal{S}} \in \mathcal{C}/\mathcal{S}$ from that of $X \in \mathcal{C}$.

## 2. Preliminary from category theory

We collect, in this section, necessary definitions and some known facts from the theory of categories. The most part of this section is well known, but the use of terminology sometimes depends on literatures. Thus the aim of this section is to fix the notation and the terminology used in this paper. See, for example, [3], [4] or [8] as general references of this section.

Let $\mathcal{C}$ be a category, where we denote by $Ob(\mathcal{C})$ the object class and by $\mathcal{C}(X, Y)$ the set of morphisms for objects $X, Y \in Ob(\mathcal{C})$. By definition, the composition of morphisms in $\mathcal{C}$ satisfies the associative law; $(fg)h = f(gh)$, and there is the identity morphism $1_X$ for any $X \in Ob(\mathcal{C})$.

Recall that $\mathcal{C}$ is called a preadditive category provided $\mathcal{C}(X, Y)$ is an abelian group for $X, Y \in Ob(\mathcal{C})$ and the composition of morphisms is bilinear, i.e. $f(g+h) = fg+fh$, $(g+h)f' = gf'+hf'$ and moreover there exists the null object 0 in $\mathcal{C}$.

An additive category is, by definition, a preadditive category with finite coproducts.

An additive category $\mathcal{C}$ is called an abelian category if the kernel and the cokernel exist for any morphism $f$ and moreover the equality $\mathrm{Cok}(\ker(f)) = \mathrm{Ker}(\mathrm{cok}(f))$ holds.

We recall how to construct an ideal quotient of a category.

2.1. **Localization.** ([4, 7.1],[8, 2.1],[3, chapter 1.3]) Let $\mathcal{C}$ be an additive category and let $S$ be a collection of morphisms in $\mathcal{C}$. We say that $S$ is a (saturated) multiplicative system in $\mathcal{C}$ if the following conditions are satisfied:

(1) $1_X \in S$ for all $X \in Ob(\mathcal{C})$ and if $s, t \in S$ then $st \in S$ whenever the composition is possible.

(2) For morphisms $Y \xleftarrow{f} X \xrightarrow{s} X'$ with $s \in S$, there exist morphisms $Y \xrightarrow{t} Y' \xleftarrow{g} X'$ with $t \in S$ such that $tf = gs$.

(3) For morphisms $Y \xrightarrow{t} Y' \xleftarrow{g} X'$ with $t \in S$, there exist morphisms $Y \xleftarrow{f} X \xrightarrow{s} X'$ with $s \in S$ such that $tf = gs$.

(4) $fs = 0$ for some $s \in S$ if and only if $tf = 0$ for some $t \in S$.

(5) (the saturation condition) Let $st \in S$. Then $s \in S$ if and only if $t \in S$.

Now let $\mathcal{C}$ be an additive category and let $S$ be a multiplicative system. Then we define the localization $S^{-1}\mathcal{C}$ as follows : The object class is the same as $\mathcal{C}$, i.e. $Ob(S^{-1}\mathcal{C}) = Ob(\mathcal{C})$. For objects $X, Y$, we consider a diagram $X \xleftarrow{s} X' \xrightarrow{f} Y$ for some object $X'$ and we denote it by $fs^{-1}$. The morphism set in $S^{-1}\mathcal{C}$ is

$$S^{-1}\mathcal{C}(X, Y) = \{fs^{-1} |\ s \in S\}/\sim,$$

where the equivalence relation $\sim$ is generated by $fs^{-1} \sim (fu)(su)^{-1}$ for a morphism $u : X'' \to X'$ in $S$. The composition of morphisms in $S^{-1}\mathcal{C}$ is defined as $(gt^{-1})(fs^{-1}) = (gf')(st')^{-1}$ where $f'$ and $t' \in S$ are morphisms satisfying $ft' = tf'$.

For given $fs^{-1}, gt^{-1} \in S^{-1}\mathcal{C}(X, Y)$ we can make the fractions have a common denominator. In fact, taking a morphism $u$ with $u = sa = tb$ for some $a$ and $b$ in $S$, we have $fs^{-1} = (fa)u^{-1}$ and $gt^{-1} = (gb)u^{-1}$. Therefore we can define the addition in $S^{-1}\mathcal{C}(X, Y)$ so that $fs^{-1}+gt^{-1} = (fa+gb)u^{-1}$. It is easy to see that $S^{-1}\mathcal{C}$ is again an additive category.

Note that $fs^{-1} = 0$ in $S^{-1}\mathcal{C}$ if and only if $fu = 0$ in $\mathcal{C}$ for some $u \in S$ if and only if $vf = 0$ for some $v \in S$.

Note also that there is a natural functor $\iota : \mathcal{C} \to S^{-1}\mathcal{C}$ defined by $\iota(X) = X$ and $\iota(f) = f1^{-1}$. It is clear that $\iota(s)$ is an isomorphism if $s \in S$.

It is known that if $\mathcal{C}$ is an abelian category, then $S^{-1}\mathcal{C}$ is also an abelian category for any multiplicative system $S$, cf. [4, Exercise 8.11]. Furthermore the functor $\iota$ is exact.

2.2. **Verdier quotient.** Let $\mathcal{C}$ be an abelian category. Recall that a full subcategory $\mathcal{S}$ of $\mathcal{C}$ is called a Serre subcategory if the following condition $(\Sigma)$ is satisfied:

$(\Sigma)$ For any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}$, $Y \in Ob(\mathcal{S})$ if and only if $X, Z \in Ob(\mathcal{S})$.

Let $\mathcal{S}$ be a Serre subcategory of an abelian category $\mathcal{C}$. We say that a morphism $f$ in $\mathcal{C}$ is a pseudo-isomorphism (with respect to $\mathcal{S}$) if $\mathrm{Ker}(f)$ and $\mathrm{Cok}(f)$ belong to $\mathcal{S}$. Then the set $S$ of all the pseudo-isomorphisms forms a multiplicative system in $\mathcal{C}$. We define the Verdier quotient $\mathcal{C}/\mathcal{S}$ by

$$\mathcal{C}/\mathcal{S} = S^{-1}\mathcal{C}.$$

Note that $\mathcal{C}/\mathcal{S}$ is an abelian category as well, and there is a natural exact functor $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{S}$. The Verdier quotient satisfies the following universal property:

Let $F : \mathcal{C} \to \mathcal{C}'$ be an exact functor of abelian categories satisfying $F(X) = 0$ for all $X \in Ob(\mathcal{S})$. Then there is a unique exact functor $G : \mathcal{C}/\mathcal{S} \to \mathcal{C}'$ such that $F = G \cdot \pi$.

It is easy to see the following

**Remark 2.1.** *Let $fs^{-1}$ be a morphism in $\mathcal{C}/\mathcal{S}$, so that $f, s$ are morphisms in $\mathcal{C}$ with $s$ being a pseudo-isomorphism. Then, $fs^{-1}$ is a monomorphism in $\mathcal{C}/\mathcal{S}$ if and only if* $\mathrm{Ker}(f)$ *belong to $\mathcal{S}$. (In such a case we say that $f$ is a pseudo-monomorphism in $\mathcal{C}$.) Similarly, $fs^{-1}$ is a epimorphism in $\mathcal{C}/\mathcal{S}$ if and only if* $\mathrm{Cok}(f)$ *belong to $\mathcal{S}$ (i.e. $f$ is a pseudo-epimorphism in $\mathcal{C}$).*

2.3. **Abelian $R$-categories.** Let $R$ be a commutative ring and let $\mathcal{C}$ be an additive category. We say that $\mathcal{C}$ is an $R$-category, or more precisely an $R$-linear category, if the following conditions hold:

(1) $\mathcal{C}(X, Y)$ has a structure of $R$-module for each $X, Y \in Ob(\mathcal{C})$.
(2) The composition $\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$ is an $R$-bilinear, i.e. $(af)g = a(fg) = f(ag)$ for $f \in \mathcal{C}(X, Y)$, $g \in \mathcal{C}(Y, Z)$ and $a \in R$.

Note that if $\mathcal{C}$ is an abelian $R$-category and if $\mathcal{S}$ is a Serre subcategory of $\mathcal{C}$, then the Verdier quotient $\mathcal{C}/\mathcal{S}$ is an abelian $R$-category as well, and the canonical functor $\mathcal{C} \to \mathcal{C}/\mathcal{S}$ is $R$-linear, see [4, 8.5].

## 3. Associated Primes in abelian $R$-categories

In the rest of the paper $R$ always denotes a commutative noetherian ring and $\mathcal{C}$ is an abelian $R$-category.

Note that there is a unique identity morphism $1_X \in \mathcal{C}(X, X)$ for each $X \in Ob(\mathcal{C})$. Since $\mathcal{C}$ is an $R$-category, we may consider a morphism $a1_X \in \mathcal{C}(X, X)$ for any $a \in R$.

**Definition 1.** For an object $X \in Ob(\mathcal{C})$, set

$$\mathrm{ann}_R(X) = \{a \in R \mid a1_X = 0\},$$

and call it the annihilator of $X$.

It is quite easy to see the following.

**Remark 3.1.** (1) *The annihilator of an object in $\mathcal{C}$ is an ideal of $R$.*
(2) *For $X \in Ob(\mathcal{C})$, $\mathrm{ann}_R(X) = R$ if and only if $X = 0$ in $\mathcal{C}$.*
(3) *If $X \cong Y$ in $\mathcal{C}$, then $\mathrm{ann}_R(X) = \mathrm{ann}_R(Y)$.*

**Example 3.2.** *Let $\mathcal{S}$ be a Serre subcategory of $\mathrm{Mod}(R)$, where $\mathrm{Mod}(R)$ denotes the abelian $R$-category consisting of all $R$-modules and $R$-homomorphisms. We set $\mathcal{C} = \mathrm{Mod}(R)/\mathcal{S}$. In this case, for an $R$-module $M \in Ob(\mathcal{C})$, $\mathrm{ann}_R(M)$ is the set of elements $a$ of $R$ with $aM \in \mathcal{S}$ as an object in $\mathrm{Mod}(R)$.*

**Definition 2.** An object $X$ of $\mathcal{C}$ is called a noetherian object in $\mathcal{C}$ if any non-empty (small) set of subobjects of $X$ has a maximal object, or more precisely it satisfies the following condition :

For any class of monomorphisms $\{j_\lambda : Y_\lambda \hookrightarrow X \mid \lambda \in \Lambda\}$ indexed by a set $\Lambda$, there is a $\lambda_0 \in \Lambda$ such that if there is a morphism $f_\lambda : Y_{\lambda_0} \to Y_\lambda$ with $j_{\lambda_0} = j_\lambda f_\lambda$, then $f_\lambda$ is an isomorphism.

We should remark that this condition is equivalent to the ascending chain condition for subobjects of $X$.

Note that subobjects and quotient objects of a noetherian object are again noetherian. Hence the class of noetherian objects is a Serre subcategory of $\mathcal{C}$. Now we define the associated prime ideals for an object in an abelian $R$-category.

**Definition 3.** Let $X \in Ob(\mathcal{C})$. We say that a prime ideal $\mathfrak{p}$ of $R$ is an associated prime ideal of $X$ if there is a noetherian object $Y$ with a monomorphism $Y \hookrightarrow X$ such that $\mathfrak{p} = \mathrm{ann}_R(Y)$ holds. We denote by $\mathfrak{Ass}_R(X)$ the set of all associated prime ideals of $X$, that is,

$$\mathfrak{Ass}_R(X) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} = \mathrm{ann}_R(Y) \text{ where } Y \text{ is a noetherian object and there is a monomonorphism } Y \hookrightarrow X\}.$$

It is obvious from the definition that $\mathfrak{Ass}_R(X')$ is a subset of $\mathfrak{Ass}_R(X)$ whenever there is a monomorphism $X' \hookrightarrow X$.

The following proposition is a generalization of [7, theorem 6.1].

**Proposition 3.3.** *For an object $X \in Ob(\mathcal{C})$, $\mathfrak{Ass}_R(X)$ is a non-empty set if and only if $X$ contains a non-zero noetherian subobject.*

*Proof.* The 'only if' part is trivial. To prove the 'if' part, it is enough to show that $\mathfrak{Ass}_R(X) \neq \emptyset$ if $X$ is a noetherian object. Let $X$ be a noetherian object and consider a set of ideals

$$\Lambda = \{I(\neq R) \mid I = \mathrm{ann}_R(Y) \text{ for a subobject } Y \hookrightarrow X\}.$$

Taking a maximal element $\mathfrak{p} = \mathrm{ann}_R(Y_0)$ among $\Lambda$, we have only to prove that $\mathfrak{p}$ is a prime ideal. Let $a, b \in R$ and suppose that $ab \in \mathfrak{p}$ and $a \notin \mathfrak{p}$.

Since $a1_{Y_0} \neq 0$, the image $L := \mathrm{Im}(a1_{Y_0})$ is a non-zero subobject of $Y_0$, hence of $X$. Thus $\mathrm{ann}_R(L)$ belongs to $\Lambda$, and since $\mathrm{ann}_R(Y_0) \subseteq \mathrm{ann}_R(L)$, we have the equality $\mathfrak{p} = \mathrm{ann}_R(L)$. On the other hand, since $b(a1_{Y_0}) = (ab)1_{Y_0} = 0$, it follows that $b1_L = 0$, thus $b \in \mathrm{ann}_R(L) = \mathfrak{p}$. □

**Proposition 3.4.** *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathcal{C}$. Then $\mathfrak{Ass}_R(X) \subseteq \mathfrak{Ass}_R(Y) \subseteq \mathfrak{Ass}_R(X) \cup \mathfrak{Ass}_R(Z)$.*

*Proof.* The first inclusion is trivial. To prove the second, assume $\mathfrak{p} \in \mathfrak{Ass}_R(Y)$, and thus there is a noetherian subobject $Y'$ of $Y$ with $\mathfrak{p} = \mathrm{ann}_R(Y')$. Taking the pull-back diagram of $Y' \hookrightarrow Y$ and $X \to Y$, we have the commutative diagram of the following form where the rows are exact and the vertical morphisms are monomorphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & 0 \end{array}$$

Note that $X'$ (resp. $Z'$) is a noetherian subobject of $X$ (resp. $Z$), and the following inclusion relations hold:

$$\mathfrak{p} = \mathrm{ann}_R(Y') \subseteq \mathrm{ann}_R(X') \cap \mathrm{ann}_R(Z'),$$
$$\mathrm{ann}_R(X') \cdot \mathrm{ann}_R(Z') \subseteq \mathfrak{p} = \mathrm{ann}_R(Y').$$

Since $\mathfrak{p}$ is prime, it forces either $\mathfrak{p} = \mathrm{ann}_R(X')$ or $\mathfrak{p} = \mathrm{ann}_R(Z')$. Hence $\mathfrak{p} \in \mathfrak{Ass}_R(X) \cup \mathfrak{Ass}_R(Z)$. □

**Corollary 3.5.** *Let $X_1, \ldots, X_n \in Ob(\mathcal{C})$. Then $\mathfrak{Ass}_R(\coprod_{i=1}^{n} X_i) = \bigcup_{i=1}^{n} \mathfrak{Ass}_R(X_i)$.*

The following proposition is a generalization of [7, theorem 6.5].

**Theorem 3.6.** *Let $X$ be a noetherian object in $\mathcal{C}$. Then $\mathfrak{Ass}_R(X)$ is a finite set.*

To prove this theorem, we need a lemma.

**Lemma 3.7.** *Under the same assumption in the theorem, let $H$ be a subset of $\mathfrak{Ass}_R(X)$. Assume that any pair of elements in $H$ has no inclusion relation, i.e. if $\mathfrak{p}, \mathfrak{q} \in H$ and $\mathfrak{p} \neq \mathfrak{q}$, then $\mathfrak{p} \not\subset \mathfrak{q}$ and $\mathfrak{q} \not\subset \mathfrak{p}$. Then $H$ is a finite set.*

*Proof.* Assume $H$ is an infinite set and take a countably infinite elements $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n, \ldots$ in $H$. For each $n$ there is a subobject $Y_n$ of $X$ with $\mathfrak{p}_n = \mathrm{ann}_R(Y_n)$. Setting the subobject $Z_n$ of $X$ to be the sum of $Y_1, Y_2, \ldots, Y_n$, we have an ascending chain of subobjects $Z_1 \subseteq Z_2 = Y_1 + Y_2 \subseteq Z_3 =$

$Y_1 + Y_2 + Y_3 \subseteq \cdots \subseteq X$. Since $X$ is noetherian, this sequence is stationary after a large number $n_0$. Thus $Y_n \subseteq Z_{n_0}$ for any $n$. Therefore we have

$$\prod_{i=1}^{n_0} \mathfrak{p}_i \subseteq \operatorname{ann}_R(Z_{n_0}) \subseteq \operatorname{ann}_R(Y_n) = \mathfrak{p}_n,$$

for every $n$. Hence, for every integer $n$, there is an integer $i$ with $1 \leq i \leq n_0$ such that $\mathfrak{p}_i \subseteq \mathfrak{p}_n$. This contradicts the assumption on $H$. □

Now we prove Theorem 3.6. Let $H_{max}$ be a subset of $\mathfrak{Ass}_R(X)$ consisting of all maximal elements in $\mathfrak{Ass}_R(X)$ with respect to inclusion relation. Then it follows from the lemma that $H_{\max}$ is a finite set. In particular, there is an upper bound $h$ for the set $\{\operatorname{ht}\mathfrak{p} \mid \mathfrak{p} \in \mathfrak{Ass}_R(X)\}$. Thus setting $H_n = \{\mathfrak{p} \in \mathfrak{Ass}_R(X) \mid \operatorname{ht}\mathfrak{p} = n\}$, we have $\mathfrak{Ass}_R(X) = \bigcup_{n=0}^{h} H_n$, where each set $H_n$ is also a finite set by the lemma. □

**Theorem 3.8.** *Let $X$ be a noetherian object in $\mathcal{C}$. Setting $I = \operatorname{ann}_R(X)$, we have $\operatorname{Ass}_R(R/I) \subseteq \mathfrak{Ass}_R(X)$. In particular, if $\mathfrak{p}$ is a minimal prime ideal containing $I$, then $\mathfrak{p}$ belongs to $\mathfrak{Ass}_R(X)$.*

*Proof.* Suppose $\mathfrak{p} \in \operatorname{Ass}_R(R/I)$. Then, by definition, there is an element $c \in R \setminus I$ such that $\mathfrak{p} = (I :_R c)$. Now let $cX$ be the image of the morphism $c1_X : X \to X$. Since $c \notin I$, we have $cX \neq 0$ that is a noetherian subobject of $X$. Note that, since $c\,\mathfrak{p} \subseteq I$, we see that $\mathfrak{p}(cX) = 0$ hence $\mathfrak{p} \subseteq \operatorname{ann}_R(cX)$. Conversely, if $a \in \operatorname{ann}_R(cX)$ then $a(cX) = 0$, therefore $ac \in I = \operatorname{ann}_R(X)$. Thus we have $a \in (I :_R c) = \mathfrak{p}$. As a result, we have $\mathfrak{p} = \operatorname{ann}_R(cX)$, and hence $\mathfrak{p} \in \mathfrak{Ass}_R(X)$. □

Let $A = \bigoplus_{\lambda \in \mathbb{N}^r} A_\lambda$ be a finitely generated $\mathbb{N}^r$-graded commutative $R$-algebra with $A_{\underline{0}} = R$ where $\underline{0} = (0, \ldots, 0) \in \mathbb{N}^r$ and let $M = \bigoplus_{\lambda \in \mathbb{Z}^r} M_\lambda$ be a finitely generated $\mathbb{Z}^r$-graded $A$-module. In such a case it is known that $\bigcup_{\lambda \in \mathbb{Z}^r} \operatorname{Ass}_R(M_\lambda)$ is a finite set. Usually the proof of this fact goes as follows: If $\mathfrak{p} \in \operatorname{Ass}_R(M_\lambda)$, then there is a graded prime ideal $\mathfrak{P}$ of $A$ that is associated with the graded $A$-module $M$ such that $\mathfrak{p} = \mathfrak{P} \cap R$, and there are only a finite number of such associated graded prime ideals $\mathfrak{P}$.

If $A$ is a non-commutative graded $R$-algebra, such a proof does not work but Theorem 3.6 assures that the same is true even if $A$ is non-commutative noetherian $R$-algebra.

In fact, we can prove the following

**Example 3.9.** *Let $A = \bigoplus_{\lambda \in \mathbb{N}^r} A_\lambda$ be a noetherian $\mathbb{N}^r$-graded (not necessarily commutative) $R$-algebra with $A_{\underline{0}} = R$ and let $M = \bigoplus_{\lambda \in \mathbb{Z}^r} M_\lambda$ be a finitely generated $\mathbb{Z}^r$-graded left $A$-module. Then $\bigcup_{\lambda \in \mathbb{Z}^r} \operatorname{Ass}_R(M_\lambda)$ is a finite set.*

*Proof.* Consider the category $\mathcal{C} = \mathrm{mod}^{\mathbb{Z}^r}(A)$ whose objects are finitely generated graded left $A$-modules and whose morphisms are degree-preserving graded $A$-module homomorphisms. Then $\mathcal{C}$ is an abelian $R$-category and every object of $\mathcal{C}$ is noetherian. Since $M \in Ob(\mathcal{C})$, we see from Theorem 3.6 that $\mathfrak{Ass}_R(M)$ is a finite set. Thus we have only to prove the equality $\mathfrak{Ass}_R(M) = \bigcup_{\lambda \in \mathbb{Z}^r} \mathrm{Ass}_R(M_\lambda)$.

To prove this equality we first note that the equality $\mathrm{ann}_R(Ax) = \mathrm{ann}_R(Rx)$ holds for any homogeneous element $x$ of $M$. In fact, since $Rx \subseteq Ax$, it follows $\mathrm{ann}_R(Ax) \subseteq \mathrm{ann}_R(Rx)$. On the other hand, if $a \in R$ and if $ax = 0$, then $a(Ax) = A(ax) = 0$ hence $a \in \mathrm{ann}_R(Ax)$.

Now let $\mathfrak{p} \in \mathfrak{Ass}_R(M)$. Then there is a graded $A$-submodule $Y$ of $M$ with $\mathfrak{p} = \mathrm{ann}_R(Y)$. Let $\{x_1, \dots, x_s\}$ be a set of homogeneous generators of $Y$. Since $\mathfrak{p} = \bigcap_{i=1}^{s} \mathrm{ann}_R(Ax_i)$ and since $\mathfrak{p}$ is prime, we have $\mathfrak{p} = \mathrm{ann}_R(Ax_i)$ for some $i$. Therefore, as remarked above, $\mathfrak{p} = \mathrm{ann}_R(Rx_i)$. Thus if $x_i$ belongs to $M_\lambda$ then $\mathfrak{p} \in \mathrm{Ass}_R(M_\lambda)$. This shows taht $\mathfrak{Ass}_R(M) \subseteq \bigcup_{\lambda \in \mathbb{Z}^r} \mathrm{Ass}_R(M_\lambda)$.

To show the reverse inclusion let $\mathfrak{p} \in \mathrm{Ass}_R(M_\lambda)$ for $\lambda \in \mathbb{Z}^r$. Then there is an element $x \in M_\lambda$ with $\mathfrak{p} = \mathrm{ann}_R(Rx)$. Hence $\mathfrak{p} = \mathrm{ann}_R(Ax)$ and thus $\mathfrak{p} \in \mathfrak{Ass}_R(M)$. $\square$

## 4. Primary decomposition

Recall that $R$ is a commutative noetherian ring, and $\mathcal{C}$ is an abelian $R$-category.

**Definition 4.** Let $X \in Ob(\mathcal{C})$ and let $a \in R$. We say that $a$ is an $X$-regular element if $a1_X : X \to X$ is a monomorphism.

**Definition 5.** An object $X$ of $\mathcal{C}$ is called a primary object if the following conditions are satisfied:

(1) $X$ is a non-zero noetherian object in $\mathcal{C}$.
(2) Let $\mathfrak{p} = \sqrt{\mathrm{ann}_R(X)}$. Then any element $a \in R \setminus \mathfrak{p}$ is an $X$-regular element.

**Lemma 4.1.** *Let $X$ be a primary object in $\mathcal{C}$ and set $\mathfrak{p} = \sqrt{\mathrm{ann}_R(X)}$. Then $\mathfrak{p}$ is a prime ideal. In this case we say that $X$ is a $\mathfrak{p}$-primary object in $\mathcal{C}$.*

*Proof.* Let $a, b \in R \setminus \mathfrak{p}$. Then, since $(ab)^n 1_X = (a^n 1_X)(b^n 1_X)$ is a monomorphism for any integer $n$, we have $(ab)^n 1_X \neq 0$ for any $n$, and hence $ab \notin \mathfrak{p}$. $\square$

**Proposition 4.2.** *Let $X$ be a noetherian object in $\mathcal{C}$. Then the set $R \setminus \bigcup_{\mathfrak{p} \in \mathfrak{Ass}_R(X)} \mathfrak{p}$ consists exactly of all the $X$-regular elements.*

*Proof.* Suppose $a \in \bigcup_{\mathfrak{p} \in \mathfrak{Ass}_R(X)} \mathfrak{p}$. Then there is a non-zero subobject $Y$ of $X$ with $a \in \mathfrak{p} = \mathrm{ann}_R(Y)$. Therefore $\mathrm{Ker}(a1_X)$ contains the object $Y$, hence $a$ is not an $X$-regular element.

Contrarily, assume that $a$ is not an $X$-regular element. Then $Z := \mathrm{Ker}(a1_X)$ is a non-zero subobject of $X$. Note that $Z$ is noetherian as it is a subobject of a noetherian object. Thus $\mathfrak{Ass}_R(Z)$ is non-empty by Proposition 3.3. Taking $\mathfrak{p} \in \mathfrak{Ass}_R(Z)$, we find a subobject $Y$ of $Z$ such that $\mathfrak{p} = \mathrm{ann}_R(Y)$. Since $a1_Z = 0$, we have $a1_Y = 0$ and hence $a \in \mathfrak{p}$. Note that $\mathfrak{Ass}_R(Z) \subseteq \mathfrak{Ass}_R(X)$ since $Z \hookrightarrow X$, and thus $\mathfrak{p} \in \mathfrak{Ass}_R(X)$. □

The following lemma will be necessary to prove the next proposition.

**Lemma 4.3.** *Let $X$ be an object in an abelian $R$-category $\mathcal{C}$ and let $a \in R$. For a positive integer $n$, we set $K_n$ (resp. $I_n$) to be the kernel (resp. the image) of the morphism $a^n 1_X : X \to X$. If $K_n = K_{n+1}$ for an integer $n$, then $a$ is an $I_n$-regular element.*

*Proof.* This is almost trivial from the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & K_n \arrow[r] \arrow[d, equal] & X \arrow[r, "a^n"] \arrow[d, equal] & I_n \arrow[r] \arrow[d, "a1_{I_n}"'] & 0 \\
0 \arrow[r] & K_{n+1} \arrow[r] & X \arrow[r, "a^{n+1}"] & I_n. &
\end{tikzcd}$$

□

The following proposition is a generalization of [7, theorem 6.6].

**Proposition 4.4.** *Let $X$ be a noetherian object in $\mathcal{C}$. Then the following conditions are equivalent:*

(1) *$X$ is a $\mathfrak{p}$-primary object.*
(2) *$\mathfrak{Ass}_R(X) = \{\mathfrak{p}\}$.*

*Proof.* (1) $\Rightarrow$ (2): Assume that $X$ is a $\mathfrak{p}$-primary object. Recall from Proposition 3.3 that $\mathfrak{Ass}_R(X) \neq \emptyset$. Take any $\mathfrak{q} \in \mathfrak{Ass}_R(X)$. Then there is a subobject $Y$ of $X$ such that $\mathfrak{q} = \mathrm{ann}_R(Y)$. Since $\mathrm{ann}_R(X) \subseteq \mathrm{ann}_R(Y) = \mathfrak{q}$ and since $\mathfrak{q}$ is prime, we have $\mathfrak{p} = \sqrt{\mathrm{ann}_R(X)} \subseteq \mathfrak{q}$. Any element $a \in R \setminus \mathfrak{p}$ is an $X$-regular element, hence such an element $a$ is $Y$-regular as well. This forces that $\mathfrak{q} \subseteq \mathfrak{p}$, hence $\mathfrak{Ass}_R(X) = \{\mathfrak{p}\}$.

(2) $\Rightarrow$ (1): Assume that $\mathfrak{Ass}_R(X) = \{\mathfrak{p}\}$ and let $I = \mathrm{ann}_R(X)$. There is a subobject $Y$ of $X$ with $\mathfrak{p} = \mathrm{ann}_R(Y)$. Hence $I = \mathrm{ann}_R(X) \subseteq \mathrm{ann}_R(Y) = \mathfrak{p}$. Since $\mathfrak{p}$ is prime, it follows that $\sqrt{I} \subseteq \mathfrak{p}$. By virtue of Proposition 4.2 it remains to prove the equality $\sqrt{I} = \mathfrak{p}$. To prove $\mathfrak{p} \subseteq \sqrt{I}$, let $a \in R \setminus \sqrt{I}$. Set $K_n := \mathrm{Ker}(a^n 1_X)$ and $I_n := \mathrm{Im}(a^n 1_X)$ for any non-negative integer $n$. Since $K_1 \hookrightarrow K_2 \hookrightarrow \cdots \hookrightarrow K_n \hookrightarrow \cdots \hookrightarrow X$ and since $X$ is noetherian, the equality $K_n = K_{n+1}$ holds for a large integer $n$. Then it follows from Lemma 4.3 that $a$ is an $I_n$-regular element.

Note that $I_n \neq 0$, since $a^n 1_X \neq 0$. Therefore $\mathfrak{Ass}_R(I_n) \neq \emptyset$. However, since $\mathfrak{Ass}_R(I_n) \subseteq \mathfrak{Ass}_R(X) = \{\mathfrak{p}\}$, we have $\mathfrak{Ass}_R(I_n) = \{\mathfrak{p}\}$. We have shown that $a$ is an $I_n$-regular element, hence Proposition 4.2 forces that $a \notin \mathfrak{p}$. □

Let $Y_1 \hookrightarrow X$ and $Y_2 \hookrightarrow X$ be subobjects of an object $X \in \mathcal{C}$. Then recall that the intersection $Y_1 \cap Y_2 \hookrightarrow X$ is well-defined as a subobject of $X$ by the pull-back diagram;

$$\begin{array}{ccc} Y_1 \cap Y_2 & \longrightarrow & Y_1 \\ \downarrow & & \downarrow \\ Y_2 & \longrightarrow & X. \end{array}$$

**Definition 6.** Let $X \in \mathcal{C}$. We say that $Y \hookrightarrow X$ is irreducible if it satisfies the following condition; If $Y_1 \cap Y_2 = Y$ for subobjects $Y_1, Y_2$ of $X$, then either $Y_1 = Y$ or $Y_2 = Y$.

The following lemma is easily proved by using the noetherian induction.

**Proposition 4.5.** *Assume that $X$ is a noetherian object in $\mathcal{C}$. Then there are a finite number of subobjects $Y_i \hookrightarrow X$ $(1 \leq i \leq n)$ satisfying $Y_1 \cap \cdots \cap Y_n = 0$ and that each $Y_i \hookrightarrow X$ is irreducible for $1 \leq i \leq n$.*

**Proposition 4.6.** *Let $X$ be a non-zero noetherian object in $\mathcal{C}$. Assume $0 \hookrightarrow X$ is irreducible. Then $X$ is primary.*

*Proof.* Assuming $\mathfrak{Ass}_R(X)$ contains two distinct primes $\mathfrak{p} \neq \mathfrak{q}$, we prove that $0 \hookrightarrow X$ is not irreducible. There are subobjects $Y, Z$ of $X$ satisfying $\mathfrak{p} = \mathrm{ann}_R(Y)$ and $\mathfrak{q} = \mathrm{ann}_R(Z)$. After exchanging $\mathfrak{p}$ and $\mathfrak{q}$ if necessary, we may assume that there is an element $a \in \mathfrak{q} \setminus \mathfrak{p}$. Setting $K_n = \mathrm{Ker}(a^n 1_Y)$ and $I_n = \mathrm{Im}(a^n 1_Y)$ for positive integers $n$, we have the equality $K_n = K_{n+1}$ for a large integer $n$, and hence $a$ is an $I_n$-regular element by Lemma 4.3. Note that $a^n \notin \mathfrak{p}$, hence we have $a^n Y \neq 0$. Since $a$ is an $I_n \cap Z$-regular element and since $a$ annihilates $I_n \cap Z$, we have $I_n \cap Z = 0$. Hence $0 \hookrightarrow X$ is not irreducible. □

The following theorem shows the existence of primary decompositions.

**Theorem 4.7.** *Let $X$ be a noetherian object in an abelian $R$-category $\mathcal{C}$. Then there are subobjects $Y_i \hookrightarrow X$ $(1 \leq i \leq n)$ satisfying $Y_1 \cap \cdots \cap Y_n = 0$ and $\mathfrak{Ass}_R(X/Y_i) = \{\mathfrak{p}_i\}$ $(1 \leq i \leq n)$. Moreover one can take such decomposition satisfying $\mathfrak{Ass}_R(X) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.*

*Proof.* By the previous lemmas, we can show that there are irreducible subobjects $Y_i \hookrightarrow X$ $(1 \leq i \leq n)$ satisfying $Y_1 \cap \cdots \cap Y_n = 0$ and $\mathfrak{Ass}_R(X/Y_i) = \{\mathfrak{p}_i\}$ $(1 \leq i \leq n)$. In this case we have a monomorphism $X \hookrightarrow \coprod_{i=1}^{n}(X/Y_i)$ and hence $\mathfrak{Ass}_R(X) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, by Proposition

3.4 and Corollary 3.5. We have to show that one can choose such $Y_i$ so that $\mathfrak{Ass}_R(X) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Assume $\mathfrak{p}_i \in \mathfrak{Ass}_R(X)$ for $1 \leq i \leq m$ and $\mathfrak{p}_i \notin \mathfrak{Ass}_R(X)$ for $m+1 \leq i \leq n$. Then $Y_1 \cap \cdots \cap Y_m$ can be embedded into the both $X$ and $X/(Y_{m+1} \cap \cdots \cap Y_n)$. Hence $\mathfrak{Ass}_R(Y_1 \cap \cdots \cap Y_m)$ is a subset of both $\mathfrak{Ass}_R(X)$ and $\mathfrak{Ass}_R(X/(Y_{m+1} \cap \cdots \cap Y_n))$, the latter of which is contained in $\{\mathfrak{p}_{m+1}, \dots, \mathfrak{p}_n\}$. Hence we have $\mathfrak{Ass}_R(Y_1 \cap \cdots \cap Y_m) = \emptyset$. Thus it follows from Proposition 3.3 that $Y_1 \cap \cdots \cap Y_m = 0$ and $\mathfrak{Ass}_R(X) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$. $\square$

Recall that for an $R$-module $M$, a prime ideal $\mathfrak{p}$ of $R$ is said to be an attached prime ideal of $M$ if $\mathfrak{p} = \mathrm{ann}_R(M/N)$ for some submodule $N$ of $M$. We denote the set of attached prime ideals of $M$ by $\mathrm{Att}_R(M)$. A secondary decomposition of $M$ is, by definition, an expression as a finite sum of secondary modules: $M = N_1 + N_2 + \cdots + N_n$, where we say that a module $N$ is secondary if $\mathrm{Att}(N)$ consists of a single prime ideal. It is well-known that if $M$ is an artinian module, then it has a secondary decomposition (see [7], [5] ).

We suppose that $\mathcal{C}^{op}$ is the opposite category of an abelian $R$-category $\mathcal{C}$. $\mathcal{C}^{op}$ is also an abelian $R$-category. In general $X$ is a noetherian object in $\mathcal{C}$ if and only if $X^{op}$ is an artinian object in $\mathcal{C}^{op}$, hence if $M$ is an artinian module in $\mathrm{Mod}(R)$ then $M^{op}$ is a noetherian object in $\mathrm{Mod}(R)^{op}$. We recognize that $\mathrm{Att}_R(M)$ coincides with $\mathfrak{Ass}_R(M^{op})$, and secondary decomposition $M = N_1 + N_2 + \cdots + N_n$ coincides with a primary decomposition in opposite category $\mathrm{Mod}(R)^{op}$.

There are several other studies on primary decomposition such as by Goldman [2] and Storrer [11]. In fact, Goldman considers the category $\mathrm{Mod}\Lambda$ where $\Lambda$ is any ring, in which he proves primary decompositions for noetherian modules, and Storrer develop Goldman's theory in terms of 'atoms'. If $\Lambda$ is an algebra over a commutative noetherian ring $R$, then $\mathrm{Mod}\Lambda$ is an $R$-linear category. Even in this case, our primary decomposition (Theorem 4.7) induces a distinct one from Goldman's. Actually, let $R$ be a field and $\Lambda$ be the ring of $2 \times 2$ upper triangular matrices over $R$. Then (0) is, as a submodule of $\Lambda$, decomposable into two primary components in $\mathrm{Mod}\Lambda$ in the sense of Goldman, but (0) is primary in our sense of Theorem 4.7 when we regard $\mathrm{Mod}\Lambda$ as an $R$-linear category.

## 5. Remarks on noetherian objects in quotient categories

As before, $R$ is always a noetherian commutative ring. Suppose we are given an abelian $R$-category $\mathcal{C}$ and a Serre subcategory $\mathcal{S}$ of $\mathcal{C}$. Then the Verdier quotient category $\mathcal{C}/\mathcal{S}$ is again an abelian $R$-category. Recall that the object classes of $\mathcal{C}$ and $\mathcal{C}/\mathcal{S}$ are identical, but, for an object $X$ in $\mathcal{C}$, we denote by $X_{\mathcal{S}}$ the corresponding object of $\mathcal{C}/\mathcal{S}$, to make it clear in which category we consider the object $X$.

Recall that a morphism $s$ in $\mathcal{C}$ is called a pseudo-isomorphism (with respect to $\mathcal{S}$) if $\mathrm{Ker}(s) \in \mathcal{S}$ and $\mathrm{Cok}(s) \in \mathcal{S}$. The morphisms in $\mathcal{C}/\mathcal{S}$ are the equivalent classes of $fs^{-1}$ where $s$ is a pseudo-isomorphism. Also recall that the morphism $fs^{-1} = 0$ in $\mathcal{C}/\mathcal{S}$ if and only if $ft = 0$ in $\mathcal{C}$ for some pseudo-isomorphism $t$. (The last is equivalent to that $uf = 0$ in $\mathcal{C}$ for some pseudo-isomorphism $u$.)

Note that for an object $X \in Ob(\mathcal{C})$, it holds $X_{\mathcal{S}} = 0$ in $\mathcal{C}/\mathcal{S}$ if and only if $X \in Ob(\mathcal{S})$. In fact, since $1_{X_{\mathcal{S}}} = 1_X 1_X^{-1}$ in $\mathcal{C}/\mathcal{S}$, it follows that $X_{\mathcal{S}} = 0 \Leftrightarrow 1_X 1_X^{-1} = 0 \Leftrightarrow$ there is a pseudo-isomorphism $t : Y \to X$ with $t = 0 \Leftrightarrow X \in Ob(\mathcal{S})$.

The following observation will be necessary to argue about subobjects in the quotient categories.

**Lemma 5.1.** *Let $X \in Ob(\mathcal{C})$ and assume that a subobject $fs^{-1} : Y_{\mathcal{S}} \hookrightarrow X_{\mathcal{S}}$ in $\mathcal{C}/\mathcal{S}$ is given for an object $X_{\mathcal{S}}$. Then there is a subobject $j : X' \hookrightarrow X$ in $\mathcal{C}$ with the following commutative diagram :*

$$\begin{array}{ccc} Y_{\mathcal{S}} & \xrightarrow{fs^{-1}} & X_{\mathcal{S}} \\ \Vert & & \uparrow{\scriptstyle j1^{-1}} \\ Y_{\mathcal{S}} & \xrightarrow[\simeq]{gs^{-1}} & X'_{\mathcal{S}} \end{array}$$

*In other words, every subobject of $X_{\mathcal{S}}$ in the quotient category $\mathcal{C}/\mathcal{S}$ comes from a subobject of $X$ in $\mathcal{C}$. The same is true as well for quotient objects.*

*Proof.* Let $fs^{-1} = [Y \xleftarrow{s} Y' \xrightarrow{f} X]$. Setting $X' = \mathrm{Im}(f)$ that is a subobject of $X$, it is easy to see that there is a pseudo-isomorphism $g : Y' \to X'$, hence $gs^{-1} : Y_{\mathcal{S}} \to X'_{\mathcal{S}}$ is an isomorphism in $\mathcal{C}/\mathcal{S}$. $\square$

As a result of this lemma we can show the following proposition holds.

**Proposition 5.2.** *If $X \in Ob(\mathcal{C})$ is a noetherian object in $\mathcal{C}$, then $X_{\mathcal{S}}$ is a noetherian object in $\mathcal{C}/\mathcal{S}$.*

*Proof.* Suppose there is an ascending sequence of subobjects of $X_{\mathcal{S}}$ in $\mathcal{C}/\mathcal{S}$:

$$(Y_1)_{\mathcal{S}} \hookrightarrow (Y_2)_{\mathcal{S}} \hookrightarrow \cdots \hookrightarrow (Y_n)_{\mathcal{S}} \hookrightarrow (Y_{n+1})_{\mathcal{S}} \hookrightarrow \cdots \hookrightarrow X_{\mathcal{S}}$$

Then each $X_{\mathcal{S}}/(Y_{n+1})_{\mathcal{S}}$ is a quotient object of $X_{\mathcal{S}}/(Y_n)_{\mathcal{S}}$. Thus it follows from the previous lemma that there is an ascending sequence of subobjects of $X$ in $\mathcal{C}$; $Y'_1 \hookrightarrow Y'_2 \hookrightarrow \cdots \hookrightarrow Y'_n \hookrightarrow Y'_{n+1} \hookrightarrow \cdots \hookrightarrow X$ such that there is an isomorphism $X_{\mathcal{S}}/(Y_n)_{\mathcal{S}} \cong X_{\mathcal{S}}/(Y'_n)_{\mathcal{S}}$ for each $n$ that makes the following diagram commutative:

$$\begin{array}{ccc} X_{\mathcal{S}}/(Y_n)_{\mathcal{S}} & \longrightarrow & X_{\mathcal{S}}/(Y_{n+1})_{\mathcal{S}} \\ {\scriptstyle\cong}\big\downarrow & & {\scriptstyle\cong}\big\downarrow \\ X_{\mathcal{S}}/(Y'_n)_{\mathcal{S}} & \longrightarrow & X_{\mathcal{S}}/(Y'_{n+1})_{\mathcal{S}} \end{array}$$

Since the sequence $\{Y'_n\}$ is stationary for large $n$, the same is true for the sequence $\{(Y_n)_{\mathcal{S}}\}$. □

## 6. Examples

There is an abelian $R$-category $\mathcal{C}$ having a non-zero object that contains no non-zero notherian subobjects.

**Example 6.1.** *Let $R = k$ be a field and let $\mathrm{Mod}(k)$ be the abelian $k$-category consisting of all $k$-modules and $k$-homomorphisms. Let $\mathcal{S} = \mathrm{mod}(k)$ be the Serre subcategory of $\mathrm{Mod}(k)$ consisting of all finite dimensional $k$-vector spaces, and set $\mathcal{C} = \mathrm{Mod}(k)/\mathcal{S}$. If $V$ is a $k$-vector space of countably infinite dimension. Since $V \notin \mathcal{S}$, we have $V_{\mathcal{S}} \neq 0$ in $\mathcal{C}$. Note that every subobject $U_{\mathcal{S}}$ of $V_{\mathcal{S}}$ comes from the subspace $U$ of $V$, and $U_{\mathcal{S}}$ is either $0$ or isomorphic to $V_{\mathcal{S}}$ (in non-natural way) according to that the dimension of $U$ is finite or not. Thus it is easy to see that any non-zero subobject of $V_{\mathcal{S}}$ is not noetherian.*

*This is an example in which $V_{\mathcal{S}} \neq 0$ but $\mathfrak{Ass}_k(V_{\mathcal{S}}) = \emptyset$.*

There is an abelian $R$-category $\mathcal{C}$ and its Serre subcategory $\mathcal{S}$ such that there is a noetherian object in the quotient category $\mathcal{C}/\mathcal{S}$ that never comes from a noetherian object of $\mathcal{C}$.

**Example 6.2.** *Let $(R, \mathfrak{m})$ be a commutative noetherian complete local domain of dimension one. Denote by $\mathrm{Mod}(R)$ the abelian $R$-category of all $R$-modules and $R$-homomorphisms. Let $\mathcal{S}$ be the Serre subcategory of $\mathrm{Mod}(R)$ consisting of all $R$-modules of finite length.*

*We claim the following hold.*

1. *The object $E_{\mathcal{S}}$ of $\mathrm{Mod}(R)/\mathcal{S}$ is simple, so that $E_{\mathcal{S}}$ contains no proper subobject other than $0$. In particular, $E_{\mathcal{S}}$ is a noetherian object in $\mathrm{Mod}(R)/\mathcal{S}$.*
2. *There is no noetherian $R$-module $M$ with $E_{\mathcal{S}} \cong M_{\mathcal{S}}$ in $\mathrm{Mod}(R)/\mathcal{S}$.*

*Therefore this example shows that the converse of Proposition 5.2 does not hold in general.*

*To prove (1), it is enough to prove that if $M \subsetneqq E$ is an $R$-submodule, then $M$ is of finite length over $R$. Denoting the Matlis dual $\mathrm{Hom}_R(\ , E)$ by $(\ )^{\vee}$, we have a non-isomorphic surjective $R$-homomorphism $R = E^{\vee} \to M^{\vee}$. Thus $M^{\vee} \cong R/I$ where $I \neq (0)$. Since $R$ is an integral domain of dimension one, we have that the length of $R/I$ is finite, hence so is $M \cong (R/I)^{\vee}$.*

*To prove (2), assume that there is a finitely generated $R$-module $M$ with the isomorphism $E_{\mathcal{S}} \to M_{\mathcal{S}}$. Then there are pseudo-isomorphism $f, g$ with $E \xleftarrow{f} X \xrightarrow{g} M$ for some $X \in \mathrm{Mod}(R)$. By the proof of (1), we see that $f$ is a surjective $R$-homomorphism. In particular, $X$ is not finitely generated $R$-module. On the other hand, in the exact sequence*

$$0 \to \mathrm{Ker}(g) \to X \xrightarrow{f} M \to \mathrm{Cok}(g) \to 0,$$

*we have that* $\mathrm{Ker}(g)$ *and* $\mathrm{Cok}(g)$ *belong to $\mathcal{S}$, hence finitely generated. It thus follows that $M$ is not a finitely generated $R$-module, hence it is not a noetherian object in* $\mathrm{Mod}(R)$.

*This is also an example that shows that Nakayama's lemma does not holds in our context. In fact, $E_{\mathcal{S}}$ is a non-zero noetherian object in the abelian $R$-category* $\mathrm{Mod}(R)/\mathcal{S}$ *and $(R, \mathfrak{m})$ is a local ring. However, the morphism $a1_{E_{\mathcal{S}}} : E_{\mathcal{S}} \to E_{\mathcal{S}}$ is an epimorphism for any non-zero element $a \in \mathfrak{m}$.*

## 7. Localization at a prime ideal

As before let $\mathcal{C}$ be an abelian $R$-category, where $R$ is a noetherian commutative ring. Given a prime ideal $\mathfrak{p}$ of $R$, we have the multiplicative system in $\mathcal{C}$ consisting of all morphisms $f$ satisfying that $sf = tg$ for some $s, t \in R \backslash \mathfrak{p}$ and an isomorphism $g$. Then we denote by $\mathcal{C}_{\mathfrak{p}}$ the localization of $\mathcal{C}$ by this multiplicative system. By definition, $Ob(\mathcal{C}_{\mathfrak{p}}) = Ob(\mathcal{C})$, while the morphisms from $X$ to $Y$ in $\mathcal{C}_{\mathfrak{p}}$ are the elements of $\mathcal{C}(X,Y)_{\mathfrak{p}} (:= \mathcal{C}(X,Y) \otimes_R R_{\mathfrak{p}})$. Thus it is easy to see that $\mathcal{C}_{\mathfrak{p}}$ is an abelian $R_{\mathfrak{p}}$-category. For an object $X \in Ob(\mathcal{C})$ we denote by $X_{\mathfrak{p}}$ the corresponding object of $\mathcal{C}_{\mathfrak{p}}$. Recall that there is a natural exact functor $\iota : \mathcal{C} \to \mathcal{C}_{\mathfrak{p}}$, by which $\iota(X) = X_{\mathfrak{p}}$ for an object $X \in Ob(\mathcal{C})$.

**Lemma 7.1.** *Let $X \in Ob(\mathcal{C})$.*

(1) *The equality* $\mathrm{ann}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) = \mathrm{ann}_R(X) R_{\mathfrak{p}}$ *holds as ideals of $R_{\mathfrak{p}}$.*
(2) $X_{\mathfrak{p}} \neq 0$ *if and only if* $\mathrm{ann}_R(X) \subseteq \mathfrak{p}$.

*Proof.* (1) For an element $a/s \in R_{\mathfrak{p}}$ $(a, s \in R,\ s \notin \mathfrak{p})$, $a/s \in \mathrm{ann}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) \Leftrightarrow (a/s)1_{X_{\mathfrak{p}}} = (a/s)1_X 1_X^{-1} = 0$ in $\mathcal{C}_{\mathfrak{p}}$ $\Leftrightarrow$ there is an element $t \in R \setminus \mathfrak{p}$ such that $ta1_X = 0$ in $\mathcal{C}$ $\Leftrightarrow$ $a/s \in \mathrm{ann}_R(X) R_{\mathfrak{p}}$.

(2) follows from (1). □

**Lemma 7.2.** *Let $X$ be an object in $\mathcal{C}$ and let $\mathfrak{p}$ be a prime ideal of $R$.*

(1) *If $X$ is a noetherian object in $\mathcal{C}$, then $X_{\mathfrak{p}}$ is a notherian object in $\mathcal{C}_{\mathfrak{p}}$.*
(2) *Let $\mathfrak{q}$ be a prime ideal of $R$ and suppose that $X$ is a $\mathfrak{q}$-primary object in $\mathcal{C}$. If $\mathfrak{q} \subseteq \mathfrak{p}$, then $X_{\mathfrak{p}}$ is a $\mathfrak{q}R_{\mathfrak{p}}$-primary object in $\mathcal{C}_{\mathfrak{p}}$. Otherwise, $X_{\mathfrak{p}} = 0$ in $\mathcal{C}_{\mathfrak{p}}$.*

*Proof.* (1) Similarly to the proof of Lemma 5.1, one can see that any subobjects and quotients of $X_{\mathfrak{p}}$ in $\mathcal{C}_{\mathfrak{p}}$ come from the subobjects and the quotients of $X$ in $\mathcal{C}$. Hence the proof goes through as in the same way of the proof of Proposition 5.2.

(2) Assume $\mathfrak{q} \subseteq \mathfrak{p}$. It then follows from the previous lemma that $\sqrt{\mathrm{ann}_{R_\mathfrak{p}}(X_\mathfrak{p})} = \sqrt{\mathrm{ann}_R(X)}R_\mathfrak{p} = \mathfrak{q}R_\mathfrak{p}$. Let $a/s \in R_\mathfrak{p} \setminus \mathfrak{q}R_\mathfrak{p}$ where $a \in R \setminus \mathfrak{q}$. Then, since $a1_X : X \to X$ is a monomorphism, and since $\iota : \mathcal{C} \to \mathcal{C}_\mathfrak{p}$ is exact, we see that $a/1$, as well as $a/s$, is an $X_\mathfrak{p}$-regular element. Hence $X_\mathfrak{p}$ is a $\mathfrak{q}R_\mathfrak{p}$-primary object. $\square$

**Theorem 7.3.** *Let $X \in Ob(\mathcal{C})$ be a noetherian object in $\mathcal{C}$ and let $\mathfrak{p}$ be a prime ideal of $R$. Then $\mathfrak{Ass}_{R_\mathfrak{p}}(X_\mathfrak{p}) = \{\mathfrak{q}R_\mathfrak{p} \mid \mathfrak{q} \in \mathfrak{Ass}_R(X)$ and $\mathfrak{q} \subseteq \mathfrak{p}\}$*

*Proof.* Suppose that $Y_1 \cap \cdots \cap Y_n = 0$ is a primary decomposition of $0 \hookrightarrow X$ in $\mathcal{C}$, so that $\mathfrak{Ass}_R(X) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ and $\mathfrak{Ass}_R(X/Y_i) = \{\mathfrak{p}_i\}$ for $1 \le i \le n$. Assume that $\mathfrak{p}_i \subseteq \mathfrak{p}$ for $1 \le i \le m$ and $\mathfrak{p}_i \not\subseteq \mathfrak{p}$ for $m+1 \le i \le n$. Note from Lemma 7.1 that $(X/Y_i)_\mathfrak{p} = 0$ for $\mathfrak{m}+1 \le i \le n$. Since the natural functor $\iota : \mathcal{C} \to \mathcal{C}_\mathfrak{p}$ is exact, we have $(Y_1)_\mathfrak{p} \cap \ldots \cap (Y_m)_\mathfrak{p} = 0$ and it follows from Lemma 7.2 that $\mathfrak{Ass}_{R_\mathfrak{p}}(X_\mathfrak{p}/(Y_i)_\mathfrak{p}) = \{\mathfrak{p}_i R_\mathfrak{p}\}$ for $1 \le i \le m$. Note that there is a monomorphism $X_\mathfrak{p} \hookrightarrow \coprod_{i=1}^m (X/Y_i)_\mathfrak{p}$ and hence we have $\mathfrak{Ass}_{R_\mathfrak{p}}(X_\mathfrak{p}) \subseteq \{\mathfrak{p}_1 R_\mathfrak{p}, \ldots, \mathfrak{p}_m R_\mathfrak{p}\}$.

To show the reverse inclusion, take a subobject $Z_i$ of $X$ with $\mathfrak{p}_i = \mathrm{ann}_R(Z_i)$ for $1 \le i \le m$. Then, by Lemma 7.1, $(Z_i)_\mathfrak{p}$ is a non-zero subobject of $X_\mathfrak{p}$ and satisfies $\mathrm{ann}_{R_\mathfrak{p}}((Z_i)_\mathfrak{p}) = \mathfrak{p}_i R_\mathfrak{p}$. Hence $\mathfrak{p}_i R_\mathfrak{p} \in \mathfrak{Ass}_{R_\mathfrak{p}}(X_\mathfrak{p})$. $\square$

**Example 7.4.** *Let* $\mathrm{Mod}(R)$ *(resp.* $\mathrm{mod}(R)$*) be the category of all $R$-modules (resp. all finitely generated $R$-modules) and $R$-homomorphisms. For a prime ideal $\mathfrak{p}$ of $R$, we compare* $\mathrm{Mod}(R)_\mathfrak{p}$ *with* $\mathrm{Mod}(R_\mathfrak{p})$.

*Note that there is a functor $F : \mathrm{mod}(R) \to \mathrm{mod}(R_\mathfrak{p})$ defined by the localization at $\mathfrak{p}$, i.e. $F(-) = - \otimes_R R_\mathfrak{p}$. If $s \in R \setminus \mathfrak{p}$, then $F(s1_X)$ is an isomorphism in $\mathrm{mod}(R_\mathfrak{p})$, hence it follows from the universal property of the localization of categories that it induces an $R_\mathfrak{p}$-linear functor $F_\mathfrak{p}$ : $\mathrm{mod}(R)_\mathfrak{p} \to \mathrm{mod}(R_\mathfrak{p})$. Clearly $F_\mathfrak{p}$ is a dense functor, while since there is a natural isomorphism $\mathrm{Hom}_R(M,N)_\mathfrak{p} \cong \mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})$ for $M, N \in \mathrm{mod}(R)$, the functor $F_\mathfrak{p}$ is fully faithful as well. Hence we have an equivalence of $R_\mathfrak{p}$-categories; $\mathrm{mod}(R)_\mathfrak{p} \simeq \mathrm{mod}(R_\mathfrak{p})$.*

*Similarly we can define a functor $F_\mathfrak{p} : \mathrm{Mod}(R)_\mathfrak{p} \to \mathrm{Mod}(R_\mathfrak{p})$. However this is not an isomorphism in general. In fact, if $X = R^{(\mathbb{N})}$ is a free $R$-module of countably infinite rank, then the natural map $\mathrm{Hom}_R(X,X)_\mathfrak{p} \to \mathrm{Hom}_{R_\mathfrak{p}}(X_\mathfrak{p}, X_\mathfrak{p})$ is not an isomorphism.*

## 8. Comparison between $\mathcal{C}$ and $\mathcal{C}/\mathcal{S}$

As before $R$ is always a noetherian commutative ring. Let $\mathcal{C}$ be an abelian $R$-category and let $\mathcal{S}$ be a Serre subcategory of $\mathcal{C}$. Considering the Verdier quotient $\mathcal{C}/\mathcal{S}$, the aim of this section is to compare the associated prime ideals $\mathfrak{Ass}_R(X)$ for an object $X \in Ob(\mathcal{C})$ with $\mathfrak{Ass}_R(X_{\mathcal{S}})$ for the corresponding object $X_{\mathcal{S}} \in Ob(\mathcal{C}/\mathcal{S})$.

Let us first remark the following observation.

If $\mathfrak{p}$ is a prime ideal of $R$, then we can construct the subcategory of $\mathcal{C}_{\mathfrak{p}}$ as a class of objects that are isomorphic in $\mathcal{C}_{\mathfrak{p}}$ to the objects in the class $Ob(\mathcal{S})$. We denote this subcategory by $\mathcal{S}_{\mathfrak{p}}$. Recalling that every short exact sequence in $\mathcal{C}_{\mathfrak{p}}$ is equivalent to the one that comes from a short exact sequence in $\mathcal{C}$, it is easy to see that $\mathcal{S}_{\mathfrak{p}}$ is a Serre subcategory of $\mathcal{C}_{\mathfrak{p}}$. In such a case, localizing the natural functor $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{S}$ by $\mathfrak{p}$, we have the exact $R_{\mathfrak{p}}$-linear functor $\pi_{\mathfrak{p}} : \mathcal{C}_{\mathfrak{p}} \to (\mathcal{C}/\mathcal{S})_{\mathfrak{p}}$. Note that $\pi_{\mathfrak{p}}(X) = 0$ for all $X \in Ob(\mathcal{S}_{\mathfrak{p}})$. Hence it induces an exact $R_{\mathfrak{p}}$-linear functor $\overline{\pi}_{\mathfrak{p}} : \mathcal{C}_{\mathfrak{p}}/\mathcal{S}_{\mathfrak{p}} \to (\mathcal{C}/\mathcal{S})_{\mathfrak{p}}$.

**Theorem 8.1.** *Under the notation above, $\overline{\pi}_{\mathfrak{p}} : \mathcal{C}_{\mathfrak{p}}/\mathcal{S}_{\mathfrak{p}} \to (\mathcal{C}/\mathcal{S})_{\mathfrak{p}}$ is an equivalence of categories for all prime ideals $\mathfrak{p}$.*

*Proof.* From the definition, it holds that $Ob(\mathcal{C}_{\mathfrak{p}}/\mathcal{S}_{\mathfrak{p}}) = Ob(\mathcal{C}) = Ob((\mathcal{C}/\mathcal{S})_{\mathfrak{p}})$, hence $\overline{\pi}_{\mathfrak{p}}$ gives is a bijection between the object classes.

Recall that the morphisms in $\mathcal{C}_{\mathfrak{p}}$ are of the form $fs^{-1}$ where $f$ is a morphism in $\mathcal{C}$ and $s \in R \setminus \mathfrak{p}$. It is easy to see that such a morphism $fs^{-1}$ is a pseudo-isomorphism with respect to $\mathcal{S}_{\mathfrak{p}}$ if and only if $f$ is a pseudo-isomorphism with respect to $\mathcal{S}$ in $\mathcal{C}$. Thus the morphisms in $\mathcal{C}_{\mathfrak{p}}/\mathcal{S}_{\mathfrak{p}}$ are of the form $(fs^{-1})(gt^{-1})^{-1}$ where $g$ is a pseudo-isomorphism with respect to $\mathcal{S}$ in $\mathcal{C}$ and $s, t \in R \setminus \mathfrak{p}$. By the construction, $\overline{\pi}_{\mathfrak{p}}$ maps this morphism to the morphism $(tfg^{-1})(s^{-1})$ in $(\mathcal{C}/\mathcal{S})_{\mathfrak{p}}$. By this observation it is easy to see that $\overline{\pi}_{\mathfrak{p}}$ is fully faithful. □

**Lemma 8.2.** *Under the circumstance above, let $X$ be a noetherian object in $\mathcal{C}$. Assume that $X$ is a $\mathfrak{p}$-primary object in $\mathcal{C}$ for a prime ideal $\mathfrak{p}$. If $X \notin Ob(\mathcal{S})$, then $X_{\mathcal{S}}$ is a $\mathfrak{p}$-primary object in $\mathcal{C}/\mathcal{S}$.*

*Proof.* Recall that $X_{\mathcal{S}} = 0$ in $\mathcal{C}/\mathcal{S}$ if and only if $X \in Ob(\mathcal{S})$. Assume that $X \notin Ob(\mathcal{S})$. Since $X_{\mathcal{S}} = \pi(X)$ for the natural exact functor $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{S}$, and since $\pi$ is $R$-linear, we should note that $\mathrm{ann}_R(X) \subseteq \mathrm{ann}_R(X_{\mathcal{S}})$ holds. Therefore $\mathfrak{p} = \sqrt{\mathrm{ann}_R(X)} \subseteq \sqrt{\mathrm{ann}_R(X_{\mathcal{S}})}$. First we prove that this inclusion is in fact an equality. To show this assume that there is an element $a \in \sqrt{\mathrm{ann}_R(X_{\mathcal{S}})} \setminus \mathfrak{p}$. Then there is an integer $n$ such that $a^n 1_{X_{\mathcal{S}}} = 0$. Setting the image of the morphism $a^n 1_X : X \to X$ in $\mathcal{C}$ as $J$, we must have $J \in Ob(\mathcal{S})$. However, since $a \notin \mathfrak{p}$, the morphism $a^n 1_X : X \to X$ is a monomorphism in $\mathcal{C}$, and hence $X \cong J$ in $\mathcal{C}$. Therefore we have $X \in Ob(\mathcal{S})$, a contradiction.

Since we have shown that $\mathfrak{p} = \sqrt{\mathrm{ann}_R(X_{\mathcal{S}})}$, it remains to prove that every element $a \in R \setminus \mathfrak{p}$ is an $X_{\mathcal{S}}$-regular element. But this is obvious, because $a$ is an $X$-regular element and the natural functor $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{S}$ is exact. □

Let $X$ be a noetherian object in an abelian $R$-category $\mathcal{C}$. Recall that there is a primary decomposition of $0 \hookrightarrow X$ in $\mathcal{C}$ as $Y_1 \cap \cdots \cap Y_n = 0$ so that $\mathfrak{Ass}_R(X) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ and $\mathfrak{Ass}_R(X/Y_i) = \{\mathfrak{p}_i\}$ for $1 \leq i \leq n$. In this case we remark that we can take such $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ as they are all distinct prime ideals. In fact, if $\mathfrak{p}_1 = \mathfrak{p}_2$, then putting $Y' = Y_1 \cap Y_2$, we see that $X/Y' \hookrightarrow X/Y_1 \times X/Y_2$ and thus $\mathfrak{Ass}_R(X/Y') \subseteq \{\mathfrak{p}_1\}$. Therefore $X/Y'$ is $\mathfrak{p}_1$-primary and we can replace $Y_1 \cap Y_2$ with $Y'$.

**Theorem 8.3.** *Let $\mathcal{C}$ be an abelian $R$-category and let $\mathcal{S}$ be a Serre subcategory of $\mathcal{C}$. Let $X$ be a noetherian object in $\mathcal{C}$ and assume that there is a primary decomposition of* $0 \hookrightarrow X$ *in $\mathcal{C}$ as $Y_1 \cap \cdots \cap Y_n = 0$ so that $\mathfrak{Ass}_R(X) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ and $\mathfrak{Ass}_R(X/Y_i) = \{\mathfrak{p}_i\}$ for $1 \leq i \leq n$. Moreover we assume that $\mathfrak{p}_i \neq \mathfrak{p}_j$ if $i \neq j$.*

*Under such circumstance, there is a unique minimal subset $\{i_1, \ldots, i_r\}$ of $\{1, 2, \ldots, n\}$ satisfying $(Y_{i_1})_{\mathcal{S}} \cap \cdots \cap (Y_{i_r})_{\mathcal{S}} = 0$ in $\mathcal{C}/\mathcal{S}$ and it holds that $\mathfrak{Ass}_R(X_{\mathcal{S}}) = \{\mathfrak{p}_{i_1}, \ldots, \mathfrak{p}_{i_r}\}$.*

*Proof.* Assume that there are two subsets $I, J$ of $\{1, 2, \ldots, n\}$ such that $\bigcap_{i \in I}(Y_i)_{\mathcal{S}} = 0$ and $\bigcap_{i \in J}(Y_i)_{\mathcal{S}} = 0$. (We understand that if $I = \emptyset$ then $\bigcap_{i \in I}(Y_i)_{\mathcal{S}} = X_{\mathcal{S}}$.) In this case we can show that $\bigcap_{i \in I \cap J}(Y_i)_{\mathcal{S}} = 0$. In fact, there is a monomorphism

$$\bigcap_{i \in I \cap J} (Y_i)_{\mathcal{S}} \hookrightarrow X_{\mathcal{S}} / \bigcap_{i \in I \setminus J} (Y_i)_{\mathcal{S}} \hookrightarrow \coprod_{i \in I \setminus J} (X/Y_i)_{\mathcal{S}},$$

hence $\mathfrak{Ass}_R(\bigcap_{i \in I \cap J}(Y_i)_{\mathcal{S}}) \subseteq \{\mathfrak{p}_i \mid i \in I \setminus J\}$. Similarly we have $\mathfrak{Ass}_R(\bigcap_{i \in I \cap J}(Y_i)_{\mathcal{S}}) \subseteq \{\mathfrak{p}_i \mid i \in J \setminus I\}$. Therefore $\mathfrak{Ass}_R(\bigcap_{i \in I \cap J}(Y_i)_{\mathcal{S}}) = \emptyset$, and hence $\bigcap_{i \in I \cap J}(Y_i)_{\mathcal{S}} = 0$.

This observation shows that there is a minimum subset $I$ of $\{1, 2, \ldots, n\}$ with $\bigcap_{i \in I}(Y_i)_{\mathcal{S}} = 0$. In this case, since there is a monomorphism $X_{\mathcal{S}} \hookrightarrow \coprod_{i \in I}(X/Y_i)_{\mathcal{S}}$, we have $\mathfrak{Ass}_R(X_{\mathcal{S}}) \subseteq \{\mathfrak{p}_i \mid i \in I\}$.

We show below that $\mathfrak{p}_i \in \mathfrak{Ass}_R(X_{\mathcal{S}})$ if $i \in I$. Suppose $\mathfrak{p}_i \notin \mathfrak{Ass}_R(X_{\mathcal{S}})$ for $i \in I$. Then $\bigcap_{j \in I,\ j \neq i}(Y_j)_{\mathcal{S}}$ is embedded into both of $X_{\mathcal{S}}$ and $(X/Y_i)_{\mathcal{S}}$. Thus it follows that $\mathfrak{Ass}_R(\bigcap_{i \in I,\ j \neq i}(Y_i)_{\mathcal{S}}) = \emptyset$, and thus $\bigcap_{i \in I,\ j \neq i}(Y_i)_{\mathcal{S}} = 0$. This contradicts the minimality of $I$. Thus we have shown the equality $\mathfrak{Ass}_R(X_{\mathcal{S}}) = \{\mathfrak{p}_i \mid i \in I\}$. □

**Example 8.4.** *In the theorem, $\mathfrak{Ass}_R(X_{\mathcal{S}})$ depends on the primary components of $X$ and it never be decided only by $\mathfrak{Ass}_R(X)$.*

*For example, let $(R, \mathfrak{m})$ be a commutative noetherian complete local domain of dimension one. As in Example 6.2, we set as $\mathcal{S}$ the Serre subcategory of* $\mathrm{Mod}(R)$ *consisting of all $R$-modules of finite length and we denote by $\mathcal{C}$ the quotient category* $\mathrm{Mod}(R)/\mathcal{S}$. *As we have shown in Example 6.2 that $X = E_{\mathcal{S}}$ is a simple (hence noetherian) object in $\mathcal{C}$, where $E$ is the injective* *hull of the residue field $R/\mathfrak{m}$. It is easy to see that $\mathfrak{Ass}_R(X) = \{(0)\}$, hence $X$ is a $(0)$-primary object in $\mathcal{C}$.*

*On the other hand, $X' = R_{\mathcal{S}}$ is also a $(0)$-primary object in $\mathcal{C}$.*

*Now let $\mathcal{S}'$ be the Serre subcategory of $\mathcal{C}$ consisting of all finitely generated $R$-modules. Then, since $X_{\mathcal{S}'} \neq 0$, we have $\mathfrak{Ass}_R(X_{\mathcal{S}'}) = \{(0)\}$, while $X'_{\mathcal{S}'} = 0$ hence $\mathfrak{Ass}_R(X'_{\mathcal{S}'}) = \emptyset$.*

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