TOMITA-TAKESAKI THEORY AND ITS APPLICATION TO THE STRUCTURE THEORY OF FACTORS OF TYPE III

Toshihiko Masuda

ABSTRACT. We give a survey of Tomita-Takesaki theory and the development of analysis of structure of type III factors, which started from Tomita-Takesaki theory.

1. Introduction

The theory of operator algebras was initiated by Murray-von Neumann's series of papers [17]. In these works, they classified factors (simple von Neumann algebras) to three classes, i.e., type I, II, III factors. Type I factors are ones which are isomorphic to $B(\mathfrak{H})$ for some Hilbert space $\mathfrak{H}$. A significant discovery is the existence of non type I factors. Indeed, Murray-von Neumann exhibited examples of type II and type III factors. Type II factors can be characterized as factors which have traces and no minimal projections. A remarkable fact is that type II factors realize continuous dimension. Type III factors are ones without any traces. Thus these classes can be characterized by the existence of traces.

In early days, the central objects for research were type II von Neumann algebras, and type III von Neumann algebras were thought of rather pathological objects. The reason is the existence of traces. If a von Neumann algebra is of type II, then it has a trace and we can develop noncommutative integration theory, e.g., noncommutative L^p-theory, Radon-Nikodym derivative theorem, in satisfactory way by regarding a trace as an analogue of a measure. However absence of traces was an obstruction for study of type III von Neumann algebras. For example, commutation theorem for tensor product of type III von Neumann algebras had been unsolved, although it seems a fundamental theorem.

One of the motivation of operator algebra theory comes from mathematical physics. It became clear that factors appearing in mathematical physics are of type III in many cases. So the study of type III factors can not be avoidable.

Tomita's theory [25], [26] appeared in such situation. It changed the theory of operator algebras drastically. After the appearance of Tomita's theory,

and the unification of it with Haag-Hugenholtz-Winnink theory by Takesaki [19], operator algebraist obtained the powerful tool for study of type III factors. In fact, the structure theory of type III factors has been developed rapidly in mid 1970's starting from the Tomita-Takesaki theory. Among them, the classification theorem of injective factors is the most brilliant success. Maybe no one could imagine such development before Tomita-Takesaki theory.

In this survey, we first explain the main theorem of Tomita-Takesaki theory, and then explain its application to the structure theorem of factors of type III. In particular, we explain the classification of injective factors.

Tomita-Takesaki theory also provides a powerful tool for application of theory of operator algebras to mathematical physics. We do not treat this topic in this survey. See [2], for example.

In many places, we state only results, and omit proofs. We sometimes give sketches of proofs, and formal discussions. For detail, we refer Takesaki's book [23]. His monograph [21] is also quickly accessible. (Note that [21] was published before appearance of [7], [12], so it does not treat a topic on the uniqueness of the injective factor of type III_1.)

For historical background of Tomita-Takesaki theory and the development of theory of type III factors, see [22].

Acknowledgements. The author is grateful to Professor T. Kajiwara, who gives him the chance for writing this survey paper. He also thanks the referee for various comments on this paper.

2. Type classification of von Neumann algebras

Tomita-Takesaki theory is the main tool for analysis of factors of type III. So in this section, we quickly explain the type classification of factors.

Definition 2.1. Let $\mathfrak{H}$ be a Hilbert space.
(1) For $S \subset B(\mathfrak{H})$, the commutant S' of S is defined by $S' := \{a \in B(\mathfrak{H}) \mid ax = xa \text{ for all } x \in S\}$.
(2) A unital $*$-subalgebra $\mathcal{M} \subset B(\mathfrak{H})$ is said to be a von Neumann algebra if $\mathcal{M}'' = \mathcal{M}$.
(3) A von Neumann algebra $\mathcal{M}$ is said to be a factor if its center $Z(\mathcal{M}) := \mathcal{M}' \cap \mathcal{M}$ is trivial, i.e., $Z(\mathcal{M}) = \mathbb{C}1$.

Murray-von Neumann's double commutant theorem [23, Theorem II.3.9] says that $\mathcal{M}$ is a von Neumann algebra if and only if $\mathcal{M}$ is closed in the strong operator topology. It is shown that a von Neumann algebra can be expressed as a direct integral of factors. Thus factors are fundamental objects in the theory of von Neumann algebras.

To explain the type classification of factors, we introduce the notion of weights on von Neumann algebras.

Definition 2.2. Let $\mathcal{M}$ be a von Neumann algebra, and $\mathcal{M}_+ = \{a^*a \mid a \in \mathcal{M}\}$ its positive part.
(1) A map $\varphi : \mathcal{M}_+ \to [0, \infty]$ is called a weight on $\mathcal{M}$ if the following two conditions are satisfied;

$$\varphi(x + y) = \varphi(x) + \varphi(y), \ \varphi(cx) = c\varphi(x), \ x, y \in \mathcal{M}_+, \ c \geq 0,$$

here we use the convention $0 \times \infty = 0$.
(2) Let φ be a weight on $\mathcal{M}$. Define

$$n_\varphi := \{x \in \mathcal{M} \mid \varphi(x^*x) < \infty\}, \ \ m_\varphi := \left\{\sum_{i=1}^n x_i^* y_i \mid x_i, y_i \in n_\varphi\right\}.$$

We say φ is semifinite if $m_\varphi \subset \mathcal{M}$ is dense. Here φ becomes a bounded valued functional on m_φ.
(3) A weight φ is normal if $\sup_i \varphi(x_i) = \varphi(x)$ for any increasing net $\{x_i\} \subset \mathcal{M}_+$ with $\lim_i x_i = x$.
(4) A weight φ is faithful if $\varphi(x^*x) = 0$ implies $x = 0$.
(5) A weight φ is tracial if $\varphi(x^*x) = \varphi(xx^*)$.

When $\varphi(1) < 0$, we can normalize φ as $\varphi(1) = 1$. Such a positive functional is called a state.

We can now introduce the type classification of factors by Murray-von Neumann.

Definition 2.3. Let $\mathcal{M}$ be a factor.
(I_n) We say $\mathcal{M}$ is of type I_n, $n \in \mathbb{N}$, if $\mathcal{M} = B(\mathfrak{H})$ for some Hilbert space $\mathfrak{H}$ with $\dim \mathfrak{H} = n$.
(I_∞) We say $\mathcal{M}$ is of type I_∞, if $\mathcal{M} = B(\ell^2(\mathbb{N}))$.
(II_1) We say $\mathcal{M}$ is of type II_1 if $\mathcal{M}$ is infinite dimensional and possesses a faithful normal tracial state τ.
(II_∞) We say $\mathcal{M}$ is of type II_∞ if $\mathcal{M} = \mathcal{N} \otimes B(\ell^2(\mathbb{N}))$ for some type II_1 factor $\mathcal{N}$.
(III) We say $\mathcal{M}$ is of type III if $\mathcal{M}$ has no normal semifinite tracial weight.

When $\mathcal{M} = B(\mathfrak{H})$ is of type I_n, $\mathcal{M}$ has a trace defined by $\mathrm{Tr}(a^*a) := \sum_i \langle a\xi_i, a\xi_i \rangle$, where $\{\xi_i\} \subset \mathfrak{H}$ is a complete orthonormal basis. When $\mathcal{M} = \mathcal{N} \otimes B(\ell^2(\mathbb{N}))$ is of type II_∞, $\mathcal{M}$ has an unbounded tracial weight $\tau := \tau_\mathcal{N} \otimes \mathrm{Tr}$, where $\tau_\mathcal{N}$ is a tracial state on a type II_1 factor $\mathcal{N}$. Thus a factor of non type III has a (finite or infinite) faithful, normal, semifinite tracial weight. Factoriality of $\mathcal{M}$ implies that a tracial weight is unique up to scalars.

The original definition of type classification of Murray-von Neumann is different from the above one. Let $\mathrm{Proj}(\mathcal{M})$ be a set of projections in $\mathcal{M}$.

They introduced an equivalence relation on $\mathrm{Proj}(\mathcal{M})$ by $p \sim q$ if and only if there exists $v \in \mathcal{M}$ such that $p = v^*v$, $q = vv^*$. Then the type classification is given as follows;

$$\mathrm{Proj}(\mathcal{M})/\sim = \begin{cases} \{0, 1, 2, \cdots, n\}, & \text{if } \mathcal{M} \text{ is of type I}_n, n \in \mathbb{N}, \\ \{0, 1, 2, \cdots, \infty\}, & \text{if } \mathcal{M} \text{ is of type I}_\infty, \\ [0, 1], & \text{if } \mathcal{M} \text{ is of type II}_1, \\ [0, \infty], & \text{if } \mathcal{M} \text{ is of type II}_\infty, \\ \{0, \infty\}, & \text{if } \mathcal{M} \text{ is of type III}. \end{cases}$$

3. Fundamental theorem of Tomita-Takesaki theory

In this section, we explain Tomita-Takesaki theory, and its immediate consequences.

Tomita's theory is described by the notion of Hilbert algebras, whose definition is presented below.

Definition 3.1. Let $\mathfrak{A}$ be an involutive algebra whose $*$-operation is denoted by $\xi \mapsto \xi^\#$. We say $\mathfrak{A}$ is a left Hilbert algebra if $\mathfrak{A}$ has an inner product $\langle \xi, \eta \rangle$ satisfying the following properties.
(1) For $\xi \in \mathfrak{A}$, the linear map $\pi_l(\xi) : \eta \mapsto \xi\eta$ is continuous.
(2) $\langle \xi\eta, \zeta \rangle = \langle \eta, \xi^\#\zeta \rangle$.
(3) $\mathfrak{A}^2$ is dense in $\mathfrak{A}$.
(4) Let $\mathfrak{H}$ be the completion of $\mathfrak{A}$. Then $\xi \to \xi^\#$ is preclosed.

In similar way, we can define a right Hilbert algebra $\mathfrak{A}$ with involution $\xi \mapsto \xi^\flat$.

Let $\mathfrak{A}$ be a left Hilbert algebra, and $\mathfrak{H}$ be its completion. Then $\pi_l(\xi) \in B(\mathfrak{H})$ by Definition 3.1 (1), and π_l is a non degenerate $*$-representation of $\mathfrak{A}$ by Definition 3.1 (2), (3).

Definition 3.2. (1) We say $\eta \in \mathfrak{H}$ is right bounded if $\xi \to \pi_l(\xi)\eta$ is bounded on $\mathfrak{H}$. We denote by $\mathfrak{B}'$ the set of all right bounded vectors.
(2) Let S be a closure of $\#$, and F the adjoint operator of S. Namely,

$$D(F) := \{\eta \in \mathfrak{H} \mid \text{there exists } \zeta \in \mathfrak{H} \text{ such that } \langle S\xi, \eta \rangle = \langle \zeta, \xi \rangle \text{ for all } \xi\}$$

and $F\eta = \zeta$ for $\eta \in D(F)$ in the above notation. Let $\mathfrak{A}' = \mathfrak{B}' \cap D(F)$.

It is shown that $\mathfrak{A}'$ is a right Hilbert algebra. We denote $F\eta = \eta^\flat$ for an involution on $\mathfrak{A}'$. We can define an anti-representation π_r of $\mathfrak{A}'$ on $\mathfrak{H}$ by $\pi_r(\xi)\eta = \eta\xi$.

By switching "left" and "right" in Definition 3.2, we can obtain a new left Hilbert algebra $\mathcal{A}''$ from $\mathcal{A}'$. It is obvious $\mathcal{A} \subset \mathcal{A}''$. If we repeat this construction for $\mathcal{A}''$, we obtain a right Hilbert algebra $\mathcal{A}'''$. However we can easily see $\mathcal{A}''' = \mathcal{A}'$. Similarly we have $\mathcal{A}'' = (\mathcal{A}'')''$.

Let $\mathcal{R}_l(\mathfrak{A}) = \pi_l(\mathfrak{A})''$, $\mathcal{R}_r(\mathfrak{A}') = \pi_r(\mathfrak{A}')''$.

Definition 3.3. Let $\mathfrak{A}$ be a left Hilbert algebra. We say that $\mathfrak{A}$ is achieved if $\mathfrak{A} = \mathfrak{A}''$. Two left Hilbert algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are said to be equivalent if $\mathfrak{A}_1'' = \mathfrak{A}_2''$.

Since $\mathfrak{A}''$ is achieved, any left Hilbert algebra is equivalent to achieved one.

Definition 3.4. Let $\mathfrak{A}$ be a left Hilbert algebra. We say $\mathfrak{A}$ is a Tomita algebra if $\mathfrak{A}$ possesses a family of (not necessary $*$-preserving) automorphisms $\{\Delta(\alpha)\}_{\alpha \in \mathbb{C}}$ satisfying the following conditions.
(1) $\alpha \to \langle \Delta(\alpha)\xi, \eta \rangle$ is entire,
(2) $(\Delta(\alpha)\xi)^{\#} = \Delta(-\bar{\alpha})\xi^{\#}$,
(3) $\langle \Delta(\alpha)\xi, \eta \rangle = \langle \xi, \Delta(\bar{\alpha})\eta \rangle$.
(4) $\langle \xi^{\#}, \eta^{\#} \rangle = (\Delta(1)\eta, \xi)$.

Remark. The original definition of a Tomita algebra demands one more condition, that is, the density of $(1 + \Delta(t))\mathfrak{A}$ in $\mathfrak{A}$ for all $t \in \mathbb{R}$. Haagerup pointed out that this condition follows from other conditions [11, p.111]. Also see the proof of [23, Theorem VI. 2.2].

Let $\mathfrak{A}$ be a Tomita algebra. Define $\xi^{\flat} := \Delta(1)\xi^{\#}$, $J\xi := \Delta\left(\frac{1}{2}\right)\xi^{\#}$. Then it follows that both $\flat$ and J are involutions with $\langle \xi^{\#}, \eta \rangle = \langle \eta^{\flat}, \xi \rangle$, $\langle J\xi, J\eta \rangle = \langle \eta, \xi \rangle$. Thus $\mathfrak{A}$ possesses three involutions $\#$, $\flat$, J. In fact, $(\mathfrak{A}, \flat)$ is a right Hilbert algebra, and $\mathfrak{A}$ generates $\mathcal{R}_l(\mathfrak{A})$ and $\mathcal{R}_l(\mathfrak{A})' = \mathcal{R}_r(\mathfrak{A})$.

We present typical examples of left Hilbert algebras.

Example 3.5. Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$, and $\xi_0 \in \mathfrak{H}$ be a cyclic separating vector. Let $\mathfrak{A} = \mathcal{M}\xi_0$, and define a product and $\#$-operation by $x\xi_0 \cdot y\xi_0 = xy\xi_0$, $(x\xi_0)^{\#} = x^*\xi_0$. As an inner product of $\mathfrak{A}$, we consider a restriction of one of $\mathfrak{H}$. The condition (1), (2) in Definition 3.1 is obvious. Since $\mathfrak{A}$ is unital, $\mathfrak{A}^2 = \mathfrak{A}$, and hence Definition 3.1 (3) holds.

We verify (4), i.e, the preclosedness of $\#$-operation. Assume $\lim_n a_n\xi_0 = 0$, and $\lim_n a_n^*\xi_0 = \eta$. Then for any $b \in \mathcal{M}'$

$$\langle b\xi_0, \eta \rangle = \lim_n \langle b\xi_0, a_n^*\xi_0 \rangle = \lim_n \langle a_n\xi_0, b^*\xi_0 \rangle = 0.$$

Since $\mathcal{M}'\xi_0$ is dense in $\mathfrak{H}$, we have $\eta = 0$.

In a similar way, we can see that $\mathcal{M}'\xi_0$ has the canonical right Hilbert algebra structure. It is shown that $\mathfrak{A}' = \mathcal{M}'\xi_0$, and $\mathcal{M}\xi_0$ is achieved.

Example 3.6. Let G be a locally compact group, and μ the left invariant Haar measure. Let $\mathfrak{A} := K(G)$ be a set of all compact supported continuous functions on G. Let $\mu_r(A) := \mu(A^{-1})$, and $\Delta(s)$ be the modular function

defined by the Radon-Nikodym derivative $\Delta(s) = \dfrac{d\mu}{d\mu_r}(s)$. We define a multiplication, an inner product and two involutions $\#, \flat$ on $\mathfrak{A}$ as follows.

$$f * g(s) = \int_G f(t)g(t^{-1}s)d\mu(t), \ \langle f, g \rangle = \int_G f(s)\overline{g(s)}d\mu(s),$$

$$f^{\#}(s) = \Delta(s^{-1})\overline{f(s^{-1})}, \ f^{\flat}(s) = \overline{f(s^{-1})}.$$

Then we can see that $\langle f^{\#} * g, h \rangle = \langle g, f * h \rangle$, $\langle g * f^{\flat}, h \rangle = \langle g, h * f \rangle$. Thus $\mathfrak{A}$ has structures of both left Hilbert algebra, and right Hilbert algebra.

Define $\Delta(\alpha)$ by $(\Delta(\alpha)f)(s) := \Delta(s)^{\alpha} f(s) \in \mathfrak{A}$ for $\alpha \in \mathbb{C}$. Then $(\mathfrak{A}, \Delta(\alpha))$ becomes a Tomita algebra. Actually, $\Delta(1)f^{\#} = f^{\flat}$ holds. An involution J is given by

$$(Jf)(s) = \left(\Delta\left(\frac{1}{2}\right)f^{\#} \right)(s) = \Delta(s^{-1})^{\frac{1}{2}}\overline{f(s^{-1})}.$$

The first step of Tomita-Takesaki theory is the polar decomposition of $*$-operation. This important step was taken by Tomita in [24], where the absolute part of $*$-operation was called an intersection operator.

To honor him, we present the proof of the following theorem according to [24], [26] (also see [19]).

Theorem 3.7. *Let $\mathfrak{A}$ be a left Hilbert algebra, and S the closure of $\#$-operation on $\mathfrak{H}$. Then we have a polar decomposition $S = J\Delta^{\frac{1}{2}}$, where J is a conjugate linear unitary, and Δ is a nonsingular positive operator satisfying $J^2 = 1$, $J\Delta J = \Delta^{-1}$.*

Proof. Define a new inner product and a norm on $D(S)$ by

$$\langle \xi, \eta \rangle_S = \langle \xi, \eta \rangle + \langle S\eta, S\xi \rangle, \ \|\xi\|_S = \langle \xi, \xi \rangle_S^{\frac{1}{2}}.$$

Via this inner product, $D(S)$ is a Hilbert space. In what follows, we denote $D(S)$ by $\mathfrak{H}_S$ when we regard $D(S)$ as a Hilbert space by the above inner product. Since

$$|\langle \xi, \eta \rangle| \leq \|\xi\|\|\eta\| \leq \|\xi\|_S\|\eta\|_S,$$

there exists $H \in B(\mathfrak{H}_S)$ such that $\langle \xi, \eta \rangle = \langle H\xi, \eta \rangle_S$ with $0 \leq H \leq 1$.

Let ι be an inclusion map $\mathfrak{H}_S \hookrightarrow \mathfrak{H}$. Since ι is continuous, there exists an adjoint operator $\iota^* : \mathfrak{H} \to \mathfrak{H}_S$. For $\xi, \eta \in \mathfrak{H}_S$,

$$\langle \iota^*\iota(\xi), \eta \rangle_S = \langle \iota(\xi), \iota(\eta) \rangle = \langle \xi, \eta \rangle.$$

Thus we have $\iota^*\iota = H$. The following computation shows $SHS = 1 - H$;

$$\langle SHS\xi, \eta \rangle_S = \langle S\eta, HS\xi \rangle_S = \langle S\eta, S\xi \rangle = \langle \xi, \eta \rangle_S - \langle \xi, \eta \rangle = \langle (1 - H)\xi, \eta \rangle_S.$$

Let $K = \iota\iota^* \in B(\mathfrak{H})$, and $\iota = VH^{\frac{1}{2}} = K^{\frac{1}{2}}V$ be a polar decomposition. Since ι is injective and has a dense range in $\mathfrak{H}$, V is a unitary and K is nonsingular. We can regard that $K^{\frac{1}{2}}$ is an extension of $H^{\frac{1}{2}}$ on $\mathfrak{H}$, since

$$K^{\frac{1}{2}}\iota(\xi) = K^{\frac{1}{2}}VH^{\frac{1}{2}}\xi = \iota\left(H^{\frac{1}{2}}\xi\right).$$

For $\xi \in D(S)$, we have

$$\|K^{\frac{1}{2}}\iota(\xi)\|_S = \|H^{\frac{1}{2}}\xi\|_S = \|\xi\|.$$

This implies $K^{\frac{1}{2}}$ is an isometry in $B(\mathfrak{H}, \mathfrak{H}_S)$. Since $H^{\frac{1}{2}}D(S)$ is dense, $K^{\frac{1}{2}}\mathfrak{H} = \mathfrak{H}_S = D(S)$. Here define $\Delta := K^{-1}(1 - K)$. Clearly $D(\Delta^{\frac{1}{2}}) = \text{Im}(K^{\frac{1}{2}}) = D(S)$.

Define $J := VSV^*$, which is conjugate linear isometry with $J^2 = 1$. (Note that S is a self-adjoint unitary on $\mathfrak{H}_S$.) The relation $J\Delta J = \Delta^{-1}$ can be verified as follows;

$$J\Delta J = VSV^*K^{-1}(1 - K)VSV^* = VSH^{-1}(1 - H)SV = V(1 - H)^{-1}HV^*$$
$$= (1 - K)^{-1}K = \Delta^{-1}.$$

For $\zeta \in D(S)$, $\zeta = \iota(\zeta) = K^{\frac{1}{2}}V\zeta$. Hence we have $V^*K^{-\frac{1}{2}}\zeta = \zeta$, and

$$J\Delta^{\frac{1}{2}}\xi = VSV^*K^{-\frac{1}{2}}(1 - K)^{\frac{1}{2}}\xi = VS(1 - K)^{\frac{1}{2}}\xi$$
$$= VH^{\frac{1}{2}}S\xi = \iota(S\xi) = S\xi.$$

Thus we get the polar decomposition $J\Delta^{\frac{1}{2}} = S$. □

Definition 3.8. We say J and Δ a modular conjugation and a modular operator respectively.

We can now state the fundamental theorem of Tomita.

Theorem 3.9. *Let $\mathfrak{A}$ be an achieved left Hilbert algebra, and J, Δ its modular conjugation and modular operator. Then we have the following assertions.*
(1) $\Delta^{it}\pi_l(\mathfrak{A})\Delta^{-it} = \pi_l(\mathfrak{A})$ *and hence* $\Delta^{it}\mathcal{R}_l(\mathfrak{A})\Delta^{-it} = \mathcal{R}_l(\mathfrak{A})$.
(2) $J\mathfrak{A} = \mathfrak{A}'$ *and* $J\mathcal{R}_l(\mathfrak{A})J = \mathcal{R}_l(\mathfrak{A})' = \mathcal{R}_r(\mathfrak{A}')$.

The original approach of Tomita involves Tomita algebras, which were called modular Hilbert algebras in [26], [19]. Original statement of the fundamental theorem is different from above one, and it is stated as follows (see [19, Theorem 10.1]); any left Hilbert algebra is equivalent to some Tomita

algebra. If we admit the fundamental theorem, it is rather easy to find a Tomita algebra. Let $\mathfrak{A}$ be an achieved left Hilbert algebra, and define $\mathfrak{A}_0$ by

$$\mathfrak{A}_0 = \left\{ \xi \in \mathfrak{A} \mid \xi \in \bigcap_{n \in \mathbb{Z}} D(\Delta^n), \Delta^n \xi \in \mathfrak{A} \right\}.$$

has the structure of a Tomita algebra [23, Theorem VI.2.2].

After an appearance of [19], many proofs of Theorem 3.9 have been presented, e.g., [8], [9], [28], [15]. Among them, the proof by van Daele [8], which avoids use of Tomita algebras, is short one, and adopted in many textbook including [23]. Thus we omit the proof of Theorem 3.9, and refer [23, Theorem VI.1.19].

A quick application of Tomita's theorem is the commutation theorem for tensor product of von Neumann algebras.

Theorem 3.10. *Let $\mathcal{M}$, $\mathcal{N}$ be von Neumann algebras. Then $(\mathcal{M} \otimes \mathcal{N})' = \mathcal{M}' \otimes \mathcal{N}'$ holds.*

Proof. Let $\mathcal{M} \subset B(\mathfrak{H})$, $\mathcal{N} \subset B(\mathfrak{K})$, and assume that $\xi \in \mathfrak{H}$, $\eta \in \mathfrak{K}$ are cyclic, separating vector for $\mathcal{M}$, and $\mathcal{N}$, respectively. Let J_ξ, J_η, $J_{\xi \otimes \eta}$ be modular conjugations for ξ, η, $\xi \otimes \eta$, respectively. Then it is shown that $J_{\xi \otimes \eta} = J_\xi \otimes J_\eta$ holds. (In this case, $\Delta_{\xi \otimes \eta}$ is a closure of $\Delta_\xi \otimes \Delta_\eta$.) By the fundamental theorem,

$$(\mathcal{M} \otimes \mathcal{N})' = J_{\xi \otimes \eta}(\mathcal{M} \otimes \mathcal{N})J_{\xi \otimes \eta} = J_\xi \mathcal{M} J_\xi \otimes J_\eta \mathcal{N} J_\eta = \mathcal{M}' \otimes \mathcal{N}'.$$

A general case can be reduced to this case. So we have the commutation theorem. $\qquad\square$

In this proof, we used full theory of Tomita's theorem. However it is known that Theorem 3.10 can be proved by using only polar decomposition of $*$-operation [20, Appendix].

More important consequence of Theorem 3.9 is the existence of modular automorphism groups. To explain, we fix notations on GNS representations.

Let $\mathcal{M}$ be a von Neumann algebra. We denote by $\mathcal{W}(\mathcal{M})$ the set of all semifinite normal weights on $\mathcal{M}$, and $\mathcal{W}_0(\mathcal{M}) := \{\varphi \in \mathcal{W}(\mathcal{M}) \mid \varphi \text{ is faithful}\}$. Take $\varphi \in \mathcal{W}(\mathcal{M})$. Let n_φ be as in Definition 2.2, and $N_\varphi := \{x \in \mathcal{M} \mid \varphi(x^*x) = 0\}$. Define an inner product on n_φ by $\langle x, y \rangle = \varphi(y^*x)$. Let $\eta_\varphi : n_\varphi \to n_\varphi/N_\varphi$ be the quotient map, and $\mathfrak{H}_\varphi$ the completion of $\eta_\varphi(n_\varphi)$. Define an operator $\pi_\varphi(a)$, $a \in \mathcal{M}$, by $\pi_\varphi(a)\eta_\varphi(x) = \eta_\varphi(ax)$. This π_φ is indeed a continuous $*$-representation of $\mathcal{M}$ on $\mathfrak{H}_\varphi$, and if $\varphi \in \mathcal{W}_0(\mathcal{M})$, then π_φ is faithful. This triplet $(\mathfrak{H}_\varphi, \pi_\varphi, \eta_\varphi)$ is a GNS representation for φ. If $\varphi \in \mathcal{M}_*$ is a faithful state, then $\eta_\varphi(1)$ is a cyclic and separating vector.

Lemma 3.11. *For any von Neumann algebra $\mathcal{M}$, $\mathcal{W}_0(\mathcal{M}) \neq \emptyset$.*

Proof. Let $\{\psi_i\} \subset \mathcal{M}_*$ be a maximal family of positive functionals such that the set of support projections $\{s(\psi_i)\}$ is an orthogonal family. Then $\varphi(x) := \sum_i \psi_i(x)$ is indeed in $\mathcal{W}_0(\mathcal{M})$. $\square$

Theorem 3.12. (1) *Let $\mathcal{M}$ be a von Neumann algebra. Then there exists an achieved left Hilbert algebra $\mathfrak{A}$ such that $\mathcal{R}_\ell(\mathfrak{A}) = \mathcal{M}$.*
(2) *Let $\mathfrak{A}$ be a left Hilbert algebra. Then there exists a faithful semifinite normal weight φ on $\mathcal{R}_l(\mathfrak{A})$ such that*

$$\varphi(x^*x) = \begin{cases} \|\xi\|^2, & x = \pi_l(\xi) \text{ for some } \xi \in \mathfrak{A}'', \\ \infty, & \text{otherwise.} \end{cases}$$

Sketch of proof. We present only the proof of (1). (See [23, Chapter VII.2] for details.) By Lemma 3.11, we can take $\varphi \in \mathcal{W}_0(\mathcal{M})$. Let $(\mathfrak{H}_\varphi, \pi_\varphi, \eta_\varphi)$ be a GNS representation associated with φ. Then $\mathfrak{A} = \eta_\varphi(n_\varphi \cap n_\varphi^*)$ has the structure of a left Hilbert algebra. Indeed, (1), (2), (3) in Definition 3.1 is easy to see. If φ is a state, then the preclosedness of $*$-operation is verified as in Example 3.5. For a general weight, the proof of the preclosedness of $*$-operation is rather complicated, and see [23, Theorem 2.6] for detail. We only comment that we use the following fact for proof.

$$\varphi(x) = \sup\{\omega(x) \mid \omega \leq \varphi, \omega \in (\mathcal{M}_*)_+\}, \ x \in \mathcal{M}_+.$$

$\square$

By Theorem 3.12, there exists a correspondence between faithful normal semifinite weights and left achieved Hilbert algebras.

Definition 3.13. Let φ be a normal faithful semifinite weight on $\mathcal{M}$, and $\mathfrak{A}_\varphi$ a corresponding left Hilbert algebra. Let J_φ and Δ_φ be a modular conjugation and a modular operator associated with $\mathfrak{A}_\varphi$. Then the modular automorphism group σ_t^φ is defined by $\sigma_t^\varphi(x) = \Delta_\varphi^{it} x \Delta_\varphi^{-it}$ for $x \in \mathcal{M} = \mathcal{R}_l(\mathfrak{A}_\varphi)$.

When Tomita's paper appeared, some people noticed that the similarity of Tomita's theory and Haag-Hugenholtz-Winnink theory [10], although they treated different mathematical objects. Takesaki clarified the reason of this similarity by unifying Tomita theory and Haag-Hugenholtz-Winnink theory. Namely, the modular automorphism group σ^φ is characterized by the Kubo-Martin-Schwinger condition (KMS condition).

Theorem 3.14. *Let $\varphi \in \mathcal{W}_0(\mathcal{M})$, and σ^φ be the modular automorphism group for φ. Let $\mathbf{D} := \{z \in \mathbb{C} \mid 0 < \mathrm{Im}(z) < 1\}$, and*

$\mathcal{A}(\mathbf{D}) := \{f(z) \mid f \text{ is bounded, continuous on } \bar{\mathbf{D}}, \text{ and holomorphic on } \mathbf{D}\}.$

Then
(1) $\varphi \circ \sigma_t^\varphi = \varphi.$

(2) *For any $x, y \in n_\varphi \cap n_\varphi^*$, there exists $F_{x,y} \in \mathcal{A}(\mathbf{D})$ such that*

$$F_{x,y}(t) = \varphi(\sigma_t^\varphi(x)y), \ \ F_{x,y}(t + i) = \varphi(y\sigma_t^\varphi(x)), \ \ t \in \mathbb{R}.$$

If a one-parameter automorphism group α_t satisfies the above two conditions, then $\alpha_t = \sigma_t^\varphi$ holds.

Remark. When φ is bounded, the condition (1) follows from the condition (2).

See [23, Theorem VIII.1.2] for proof. Here we only remark that $F_{x,y}(z)$ in Theorem 3.14 is given by

$$F_{x,y}(z) = \langle \Delta_\varphi^{-\frac{zi}{2}} \eta_\varphi(x), \Delta_\varphi^{\frac{\bar{z}i}{2}} \eta_\varphi(y) \rangle.$$

As an application of Theorem 3.14, we see the following results.

Corollary 3.15. *Assume $x \in n_\varphi \cap n_\varphi^*$ is analytic, i.e., $\sigma_t^\varphi(x)$ can be extended to an analytic function on whole $\mathbb{C}$. Then we have $\varphi(\sigma_i(x)y) = \varphi(yx)$ for $y \in n_\varphi \cap n_\varphi^*$. In particular, we have $\varphi(\sigma_i(x)y) = \varphi(\sigma_{\frac{i}{2}}(x)\sigma_{-\frac{i}{2}}(y)) = \varphi(yx)$ for analytic elements $x, y \in n_\varphi \cap n_\varphi^*$.*

Proof. Since $F(t) = \varphi(\sigma_t^\varphi(x)y)$ is analytic, we have $F(t+i) = \varphi(\sigma_{t+i}(x)y)$. By comparing with KMS condition, we have $\varphi(\sigma_{t+i}(x)y) = \varphi(y\sigma_t^\varphi(x))$. We get the conclusion by putting $t = 0$.   □

Theorem 3.16. *Let $\mathcal{M}_1$, $\mathcal{M}_2$ be two isomorphic von Neumann algebras with an isomorphism $\theta : \mathcal{M}_1 \to \mathcal{M}_2$. Then we have $\sigma_t^{\varphi \circ \theta^{-1}} = \theta \circ \sigma_t^\varphi \circ \theta^{-1}$ for $\varphi \in \mathcal{W}_0(\mathcal{M}_1)$.*

Proof. We can easily verify that $\theta \circ \sigma_t^\varphi \circ \theta^{-1}$ satisfies the KMS condition for $\varphi \circ \theta^{-1}$. By Theorem 3.14, we get the conclusion.   □

Theorem 3.17. *Let $\mathcal{M}$ be a von Neumann algebra, φ a faithful normal semifinite weight. Define $\mathcal{M}_\varphi := \{x \in \mathcal{M} \mid \sigma_t^\varphi(x) = x, t \in \mathbb{R}\}$. Then $a \in \mathcal{M}_\varphi$ if and only if a satisfies the following two conditions;*
(1) $am_\varphi \subset m_\varphi$, $m_\varphi a \subset m_\varphi$,
(2) $\varphi(ax) = \varphi(xa)$ for all $x \in m_\varphi$.

Proof. We only present a proof for the case $\varphi(1) < \infty$. In this case, the condition (1) is unnecessary ($m_\varphi = \mathcal{M}$).

First suppose the condition (2). By the KMS condition, there exists $F(z) \in \mathcal{A}(\mathbf{D})$, such that, $\varphi(\sigma_t^\varphi(a)x) = F(t)$, $\varphi(x\sigma_t^\varphi(a)) = F(t + i)$, $t \in \mathbb{R}$. Here we have

$$\varphi(\sigma_t^\varphi(a)x) = \varphi(a\sigma_{-t}^\varphi(x)) = \varphi(\sigma_{-t}^\varphi(x)a) = \varphi(x\sigma_t^\varphi(a)).$$

Hence $F(t) = F(t + i)$ for all t. Thus we can extend F to a holomorphic function on whole $\mathbb{C}$, and F is bounded. By Liouville's theorem, $F(z)$ is a constant function. Hence $\varphi(\sigma_t^\varphi(a)x) = \varphi(ax)$ for all $x \in \mathcal{M}$, and $\varphi_t^\varphi(a) = a$ holds.

Conversely, suppose $\sigma_t^\varphi(a) = a$ for all $t \in \mathbb{R}$. By the KMS condition, there exists $F(z) \in \mathcal{A}(\mathbf{D})$, such that, $\varphi(\sigma_t^\varphi(a)x) = F(t)$, $\varphi(x\sigma_t^\varphi(a)) = F(t + i)$. Since $\sigma_t^\varphi(a) = a$, F must be constant, and in particular $F(0) = F(i)$. Hence $\varphi(ax) = \varphi(xa)$. $\qquad\square$

Theorem 3.18. *Let $\mathcal{M}$ be a von Neumann algebra. Then σ_t^φ is inner in the sense that $\sigma_t^\varphi = \mathrm{Ad}(u(t))$ for some one-parameter unitary group $u(t) \in \mathcal{M}$ if and only if $\mathcal{M}$ has a tracial weight $\tau \in \mathcal{W}_0(\mathcal{M})$.*

We present a formal discussion of proof. (See [23, Theorem VIII3.14] for proof.) If $\mathcal{M}$ has a tracial weight $\tau \in \mathcal{W}_0(\mathcal{M})$, then any weight has the form $\varphi = \tau_h$ for a positive operator h which is affiliated with $\mathcal{M}$, i.e., the spectral projections of h are in $\mathcal{M}$. And hence $\sigma_t^\varphi = \mathrm{Ad}\, h^{it}$ is inner (see Example 4.1 below).

Conversely, assume $\sigma_t^\varphi = \mathrm{Ad}\, h^{it}$ for some positive operator h which is affiliated with $\mathcal{M}$. Define $\tau := \varphi(h^{-\frac{1}{2}} \cdot h^{-\frac{1}{2}})$. Take analytic elements $x, y \in \mathcal{M}$. By KMS condition, we have

$$\varphi(yx) = \varphi(\sigma_{\frac{i}{2}}^\varphi(x)\sigma_{-\frac{i}{2}}^\varphi(y)) = \varphi(h^{-\frac{1}{2}}xhyh^{-\frac{1}{2}})$$

Hence

$$\tau(yx) = \varphi(h^{-\frac{1}{2}}yxh^{-\frac{1}{2}}) = \varphi(h^{-\frac{1}{2}}xh^{-\frac{1}{2}}hh^{-\frac{1}{2}}yh^{-\frac{1}{2}}) = \varphi(h^{-\frac{1}{2}}xyh^{-\frac{1}{2}}) = \tau(xy).$$

We close this section with the following Connes-Radon-Nikodym cocycle theorem.

Theorem 3.19. *Let $\varphi, \psi \in \mathcal{W}_0(\mathcal{M})$. Then there exists a unitary $[D\varphi : D\psi]_t$, $t \in \mathbb{R}$, such that*
(1) $\mathrm{Ad}([D\varphi : D\psi]_t) \circ \sigma_t^\psi = \sigma_t^\varphi$,
(2) (1-cocycle property) $[D\varphi : D\psi]_t \sigma_t^\psi([D\varphi : D\psi]_s) = [D\varphi : D\psi]_{s+t}$,
(3) (chain rule) $[D\varphi_1 : D\varphi_2]_t[D\varphi_2 : D\varphi_3]_t = [D\varphi_1 : D\varphi_3]_t$.

Proof. The proof is given by 2×2 matrix trick. Let $\mathcal{N} = \mathcal{M} \otimes M_2(\mathbb{C})$, and $\{e_{ij}\}$ be a system of matrix units for $M_2(\mathbb{C})$. Define $\omega \in \mathcal{W}_0(\mathcal{N})$ by $\omega(x) = \varphi(x_{11}) + \psi(x_{22})$ for $x = \sum_{i,j} x_{ij} \otimes e_{ij}$. Since $\omega((1 \otimes e_{ii})x) = \omega(x(1 \otimes e_{ii}))$, $\sigma_t^\omega(1 \otimes e_{ii}) = 1 \otimes e_{ii}$ by Theorem 3.17. We can see $\sigma_t^\omega(x \otimes e_{11}) = \sigma_t^\varphi(x) \otimes e_{11}$, $\sigma_t^\omega(x \otimes e_{22}) = \sigma_t^\psi(x) \otimes e_{22}$. By $e_{12} = e_{11}e_{12}e_{22}$, $\sigma_t^\omega(1 \otimes e_{12}) = u_t \otimes e_{12}$ for some unitary $u_t \in U(\mathcal{M})$, and we have

$$\sigma_t^\varphi(x) \otimes e_{11} = \sigma_t^\omega((1 \otimes e_{12})(x \otimes e_{22})(1 \otimes e_{21})) = \mathrm{Ad}\, u_t \circ \sigma_t^\psi(x) \otimes e_{11}.$$

Hence u_t satisfies (1). The condition (2) follows from the following computation;

$$\sigma_{t+s}^\omega(1 \otimes e_{12}) = \sigma_t^\omega \sigma_s^\omega(1 \otimes e_{12}) = \sigma_t^\omega((1 \otimes e_{12})(u_s \otimes e_{22})) = u_t \sigma_t^\psi(u_s) \otimes e_{12}.$$

If we consider a weight $\chi((x_{ij})) := \sum_{i=1}^3 \varphi_i(x_{ii})$ on $\mathcal{M} \otimes M_3(\mathbb{C})$, then (3) can be verified.   □

In fact, $[D\varphi : D\psi]_t$ can be characterized by the relative KMS condition [23, Theorem VIII.3.3].

Let $\mathcal{M} = L^\infty(X,\mu)$, and ν be a measure equivalent to μ. A Connes cocycle in this case is given by $[D\mu : D\nu]_t = \left(\dfrac{d\mu}{d\nu}\right)^{it}$. In this sense, Theorem 3.19 is a generalization of Radon-Nikodym derivative theorem. See [23, Chapter VIII.3] for other versions of Radon-Nikodym type theorem.

4. Examples.

Example 4.1. Let us consider the case $\mathcal{M} = M_n(\mathbb{C})$, i.e., $\mathcal{M}$ is of type I_n. Let Tr be a usual (non-normalized) trace on $M_n(\mathbb{C})$. Any normal state on $\mathcal{M}$ is given by $\varphi(x) = \text{Tr}(ax)$ for a positive operator with $\text{Tr}(a) = 1$. Moreover, φ is faithful if and only if a is nonsingular

Let $\mathfrak{H}_{\text{Tr}}$ be a GNS Hilbert space for Tr, and denote $\eta_{\text{Tr}}(x)$ by $\eta(x)$. Since $\varphi(x^*x) = \text{Tr}(a^{\frac{1}{2}}x^*xa^{\frac{1}{2}})$, the GNS representation $(\mathfrak{H}_\varphi, \pi_\varphi, \eta_\varphi)$ is identified with $\mathfrak{H}_\varphi = \mathfrak{H}_{\text{Tr}}$, $\pi_\varphi(a) = \pi_{\text{Tr}}(a)$ by the identification $\eta_\varphi(x) = \eta(xa^{\frac{1}{2}})$. In this case, S-operator is bounded and given by $S(\eta(xa^{\frac{1}{2}})) = \eta(x^*a^{\frac{1}{2}})$. The adjoint operator $F = S^*$ is given by $F(\eta(a^{\frac{1}{2}}x)) = \eta(a^{\frac{1}{2}}x^*)$. Hence

$$\begin{aligned}
\Delta_\varphi(\eta(xa^{\frac{1}{2}})) &= FS(\eta(xa^{\frac{1}{2}})) = F(\eta(x^*a^{\frac{1}{2}})) \\
&= F\eta((a^{\frac{1}{2}}a^{-\frac{1}{2}}x^*a^{\frac{1}{2}})) = \eta(axa^{-\frac{1}{2}}) = \eta(a(xa^{\frac{1}{2}})a^{-1}).
\end{aligned}$$

Then we obtain

$$\Delta_\varphi^{\frac{1}{2}}(\eta(xa^{\frac{1}{2}})) = \eta(a^{\frac{1}{2}}(xa^{\frac{1}{2}})a^{-\frac{1}{2}}) = \eta(a^{\frac{1}{2}}x)$$

and

$$J_\varphi(\eta(xa^{\frac{1}{2}})) = \Delta_\varphi^{\frac{1}{2}}(S\eta(xa^{\frac{1}{2}})) = \Delta_\varphi^{\frac{1}{2}}(\eta(x^*a^{\frac{1}{2}})) = \eta(a^{\frac{1}{2}}x^*) = \eta((xa^{\frac{1}{2}})^*).$$

Hence $\Delta_\varphi^{it}(\eta(xa^{\frac{1}{2}})) = \eta(a^{it}xa^{-it}a^{\frac{1}{2}})$, $J_\varphi = J_{\text{Tr}}$. We can compute the modular automorphism σ_t^φ as follows;

$$\begin{aligned}
\pi_\varphi(\sigma_t^\varphi(x))\eta(ya^{\frac{1}{2}}) &= \Delta_\varphi^{it}\pi_\varphi(x)\Delta_\varphi^{-it}(\eta(ya^{\frac{1}{2}})) = \Delta_\varphi^{it}\pi_\varphi(x)(\eta(a^{-it}ya^{it}a^{\frac{1}{2}})) \\
&= \Delta_\varphi^{it}(\eta(xa^{-it}ya^{it}a^{\frac{1}{2}})) = \eta(a^{it}xa^{-it}ya^{\frac{1}{2}})
\end{aligned}$$

$$= \pi_\varphi(a^{it}xa^{-it})\eta(ya^{\frac{1}{2}}).$$

Hence the modular automorphism group for φ is given by $\sigma_t^\varphi(x) = a^{it}xa^{-it}$. (Here we identify $x \in \mathcal{M}$ and $\pi_\varphi(x)$.) Of course $J_\varphi x^* J_\varphi$ gives a usual right action.

The KMS-condition can be verified easily. Indeed we have

$$\varphi(yx) = \text{Tr}(ayx) = \text{Tr}(xay) = \text{Tr}(aa^{-1}xay) = \varphi(\sigma_i^\varphi(x)y).$$

In a similar way, if $\mathcal{M}$ has a tracial weight, then $\sigma_t^\varphi = \text{Ad}\,a^{it}$ holds.

Example 4.2. Let $\mathcal{M}_k := M_{n_k}(\mathbb{C})$ be a type I_{n_k}-factor, and fix a faithful state $\varphi_k = \text{Tr}(a_k\cdot)$. Let us consider an infinite tensor product $\mathcal{A} = \bigotimes_{k=1}^\infty \mathcal{M}_k$. Then we can define a product state φ on $\mathcal{A}$ by $\varphi(x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes 1 \otimes \cdots) = \prod_{k=1}^n \varphi_{a_k}(x_k)$. Let $(\pi_\varphi, \mathfrak{H}_\varphi, \eta_\varphi)$ be a GNS-representation of $\mathcal{A}$ by φ, and $\mathcal{M} = \pi_\varphi(\mathcal{A})''$. This $\mathcal{M}$ is denoted by $\bigotimes_{k=1}^\infty(\mathcal{M}_k, \varphi_k)$, and called an ITPFI factor [1].

The modular automorphism for φ is given by $\text{Ad}(a_1^{it} \otimes a_2^{it} \otimes a_3^{it} \otimes \cdots)$. Note that $a_1^{it} \otimes a_2^{it} \otimes a_3^{it} \otimes \cdots$ may fail to be in $\mathcal{M}$.

The isomorphism classes of $\mathcal{M}$ heavily depend on the choice of a_k's. If we take $\varphi_k = n_k^{-1}\text{Tr}$, then we get a factor of type II_1. Other choices of $\{a_k\}$ produce many non-isomorphic factors of type III [1].

Example 4.3. Let $\mathcal{M}$ be a von Neumann algebra, G a locally compact group and assume that an action α of G on $\mathcal{M}$ is given. We describe a left Hilbert algebra and a Tomita algebra for $\mathcal{M} \rtimes_\alpha G$. (See Appendix for crossed product construction.) For simplicity, we assume that G is unimodular, and the existence of a faithful normal state φ such that $\varphi \circ \alpha_s = \varphi$ for all $s \in G$. (For general case, see [11].) By Theorem 3.16, $\alpha_s \circ \sigma_t^\varphi = \sigma_t^\varphi \circ \alpha_s$

Let $\mathcal{M}_a$ be a set of all analytic elements via σ_t^φ. Let $\mathfrak{A} := K(G, \mathcal{M})$ be a set of all σ-strongly* continuous compact supported $\mathcal{M}$-valued functions, and $\mathfrak{A}_0 = \{f \in \mathfrak{A} \mid f(s) \in \mathcal{M}_a\}$. Define

$$f * g(s) = \int_G f(t)\alpha_t(g(t^{-1}s))d\mu(t), \quad \langle f, g \rangle = \int_G \varphi(g(s)^*f(s))\,d\mu(s),$$

$$f^\#(s) = \alpha_s(f(s^{-1})^*), \quad \Delta(\alpha)(f)(s) = \sigma_{-i\alpha}^\varphi(f(s)).$$

Then $\mathfrak{A}$ becomes a left Hilbert algebra [11], and $\mathfrak{A}_0$ a Tomita algebra. For example, we verify Definition 3.4 (4).

$$\langle f^\#, g^\# \rangle = \int_G \varphi((g^\#(s))^* f^\#(s))d\mu(s)$$

$$= \int_G \varphi(\alpha_s(g(s^{-1}))\alpha_s(f(s^{-1})^*))d\mu(s)$$

$$= \int_G \varphi(g(s^{-1})f(s^{-1})^*)d\mu(s)$$

$$= \int_G \varphi(f(s)^*\sigma_{-i}(g(s)))d\mu(s) \ (\text{by KMS condition })$$

$$= \langle \Delta(1)g, f \rangle.$$

In this case, we indeed have $\mathcal{R}_l(\mathfrak{A}) = \mathcal{M} \rtimes_\alpha G$. The weight $\tilde{\varphi}$ associated with $\mathfrak{A}$ is called the dual weight for φ. The dual weight $\tilde{\varphi}$ and the modular automorphism group $\sigma_t^{\tilde{\varphi}}$ are given by

$$\tilde{\varphi}\left(\int_G x(s)\lambda(s)d\mu(s)\right) = \varphi(x(e)), \ \sigma_t^{\tilde{\varphi}}\left(\int_G x(s)\lambda(s)d\mu(s)\right)$$

$$= \int_G \sigma_t^\varphi(x(s))\lambda(s)d\mu(s)$$

for $\int_G x(s)\lambda(s)d\mu(s) \in \mathcal{M} \rtimes_\alpha G$, $x(\cdot) \in K(G, \mathcal{M})$.

5. STRUCTURE OF FACTORS OF TYPE III

In this section, we present the most important application of Theorem 3.9. Namely, any factor of type III can be described as the crossed product of a type II_∞ von Neumann algebra by a certain one-parameter automorphism group.

Let $\mathcal{M}$ be a factor of type III, and φ a faithful normal semifinite weight on $\mathcal{M}$. Consider the crossed product $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$, and denote an implementing unitary by $\lambda^\varphi(t)$. By Theorem 3.19, $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ and $\mathcal{M} \rtimes_{\sigma^\psi} \mathbb{R}$ are isomorphic for any $\varphi, \psi \in \mathcal{W}_0(\mathcal{M})$. Indeed, an isomorphism $\Phi^{\psi,\varphi} : \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R} \to \mathcal{M} \rtimes_{\sigma^\psi} \mathbb{R}$ is given by

$$\Phi^{\psi,\varphi}(x\lambda^\varphi(t)) = x[D\varphi : D\psi]_t \lambda^\psi(t).$$

By the chain rule of a Connes-cocycle, $\Phi^{\varphi_3,\varphi_2}\Phi^{\varphi_2,\varphi_1} = \Phi^{\varphi_3,\varphi_1}$ holds. This suggests that the association $\mathcal{M} \to \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ is a functor. Indeed, we can construct a canonical von Neumann algebra $\tilde{\mathcal{M}}$ generated by $\mathcal{M}$ and the set of symbols $\{\varphi^{it}\}_{\varphi \in \mathcal{W}_0(\mathcal{M})}$ satisfying the following relations;

$$\varphi^{it}x(\varphi^{it})^* = \sigma_t^\varphi(x), \ (\varphi^{it})^* = \varphi^{-it}, \ \varphi^{it} = [D\varphi : D\psi]_t\psi^{it}.$$

See [23, Chapter XII.6], [29] for details of the construction of $\tilde{\mathcal{M}}$. The von Neumann algebra $\tilde{\mathcal{M}}$ is isomorphic to $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ via an isomorphism $x\varphi^{it} \mapsto x\lambda^\varphi(t)$.

Let θ be the one-parameter automorphism group on $\tilde{\mathcal{M}}$ defined by $\theta_t(x\varphi^{is}) = e^{-its}x\varphi^{is}$. Via an identification of $\tilde{\mathcal{M}}$ with $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$, θ is the dual action, and $\tilde{\mathcal{M}}^\theta = \mathcal{M}$ holds, where $\tilde{\mathcal{M}}^\theta := \{x \in \tilde{\mathcal{M}} \mid \theta_t(x) = x, t \in \mathbb{R}\}$ is a fixed point algebra.

Definition 5.1. We say $\tilde{\mathcal{M}}$ the core for $\mathcal{M}$, and $(\mathcal{M}, \theta)$ the covariant core system, (or noncommutative flow of weights).

Let h be a generator of φ^{it}, i.e., $\varphi^{it} = h^{it}$. Then h is a positive operator affiliated with $\tilde{\mathcal{M}}$. Let $\tilde{\varphi}$ be the dual weight of φ, and set $\tau(x) := \tilde{\varphi}(h^{-\frac{1}{2}}xh^{-\frac{1}{2}})$.

The structure theorem of type III factors can be stated as follows.

Theorem 5.2. (1) *In the above notation, τ is a faithful semifinite normal tracial weight, and θ scales τ, i.e, $\tau\theta_t = e^{-t}\tau$. The von Neumann algebra $\tilde{\mathcal{M}}$ is of type II_∞.*
(2) *For a type III factor $\mathcal{M}$, there exists a type II_∞ von Neumann algebra $\mathcal{N} \subset \mathcal{M}$ with a trace $\tau \in \mathcal{W}_0(\mathcal{N})$, a one parameter automorphism group θ_t on $\mathcal{N}$ such that $\tau\theta_t = e^{-t}\tau$, and we have $\mathcal{M} = \mathcal{N} \rtimes_\theta \mathbb{R}$.*

Sketch of proof. It is shown that the modular automorphism $\tilde{\varphi}$ is given by $\sigma_s^{\tilde{\varphi}} = \operatorname{Ad} \varphi^{is}$ (see Example 4.3), and hence inner. It follows that τ defined above is indeed a tracial weight by Theorem 3.18. By the definition of θ_t and $\tilde{\varphi}$, $\tilde{\varphi} \circ \theta_t = \tilde{\varphi}$ and $\theta_t(h^{is}) = e^{-its}h^{is} = (e^{-t}h)^{is}$. Hence $\theta_t(h) = e^{-t}h$, and

$$\tau\theta_t(x) = \tilde{\varphi}(h^{-\frac{1}{2}}\theta_t(x)h^{-\frac{1}{2}}) = \tilde{\varphi}(\theta_{-t}(h^{-\frac{1}{2}})x\theta_{-t}(h^{-\frac{1}{2}})) = e^{-t}\tau.$$

By the Takesaki duality [23, Theorem Theorem X.2.3], we have $\tilde{\mathcal{M}} \rtimes_\theta \mathbb{R} = \mathcal{M} \otimes B(L^2(\mathbb{R})) \cong \mathcal{M}$. Thus (2) is an immediate consequence of (1). $\square$

By means of Theorem 5.2, we can divide the set of factors of type III finer.

Definition 5.3. Let $\mathcal{M}$ be a factor of type III, and $\mathcal{C}_\mathcal{M} := Z(\tilde{\mathcal{M}})$. Then $(\mathcal{C}_\mathcal{M}, \theta)$ is called the flow of weights for $\mathcal{M}$.

Since $\mathcal{M}$ is a factor, $(\mathcal{C}_\mathcal{M}, \theta)$ is an ergodic flow, i.e., the fixed point algebra $\{a \in \mathcal{C}_\mathcal{M} \mid \theta_t(a) = a, t \in \mathbb{R}\}$ is trivial. Thus we can divide the set of type III factors as follows.

Definition 5.4. Let $\mathcal{M}$ be a factor of type III.
(1) $\mathcal{M}$ is of type III_λ, $0 < \lambda < 1$, if $(\mathcal{C}_\mathcal{M}, \theta)$ has a period $-\log\lambda$.
(2) $\mathcal{M}$ is of type III_1 if $\mathcal{C}_\mathcal{M} = \mathbb{C}$, i.e., $\tilde{\mathcal{M}}$ is a factor.
(3) $\mathcal{M}$ is of type III_0 if $(\mathcal{C}_\mathcal{M}, \theta)$ is aperiodic and recurrent.

Remark. If $\mathcal{M}$ is of type III, then an ergodic flow $(\mathbb{R}, \text{translation})$ does not appear.

Example 5.5. (1) Take $0 < \lambda < 1$. In Example 4.2, let

$$n_k = 2, \ a_k = \frac{1}{\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}} \begin{pmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & \lambda^{-\frac{1}{2}} \end{pmatrix}.$$

Then $\displaystyle\bigotimes_{k=1}^{\infty}(M_{n_k}(\mathbb{C}),\varphi_k)$ is of type III$_\lambda$. This factor is called a Powers factor [18], and denoted by $\mathcal{R}_\lambda$.

(2) Take $0 < \lambda, \mu < 1$ with $\dfrac{\log\lambda}{\log\mu} \notin \mathbb{Q}$. Let $n_k = 2$, and take a_k as

$$a_{2k} = \frac{1}{\lambda^{\frac{1}{2}}+\lambda^{-\frac{1}{2}}}\begin{pmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & \lambda^{-\frac{1}{2}} \end{pmatrix}, \ a_{2k+1} = \frac{1}{\mu^{\frac{1}{2}}+\mu^{-\frac{1}{2}}}\begin{pmatrix} \mu^{\frac{1}{2}} & 0 \\ 0 & \mu^{-\frac{1}{2}} \end{pmatrix}.$$

Then $\displaystyle\bigotimes_{k=1}^{\infty}(M_{n_k}(\mathbb{C}),\varphi_k)$ is of type III$_1$. This factor is denoted by $\mathcal{R}_\infty$, and called an Araki-Woods factor of type III$_1$ [1].

If we take $\{a_k\}$ suitably, we can realize factors of type III$_0$ in Example 4.2.

For factors of type III$_\lambda$, $\lambda \neq 1$, we have the following discrete decomposition.

Theorem 5.6. (1) *Let $\mathcal{M}$ be a factor of type III$_\lambda$, $0 < \lambda < 1$. Then there exist a subfactor $\mathcal{N} \subset \mathcal{M}$ of type II$_\infty$ with a tracial weight τ, and an automorphism θ of $\mathcal{N}$ such that $\tau\theta = \lambda\tau$ and $\mathcal{M} = \mathcal{N} \rtimes_\theta \mathbb{Z}$.*
(2) *Let $\mathcal{M}$ be a factor of type III$_0$. Then there exist a von Neumann subalgebra $\mathcal{N} \subset \mathcal{M}$ of type II$_\infty$ with diffused center, a tracial weight $\tau \in \mathcal{W}_0(\mathcal{N})$, and an automorphism θ of $\mathcal{N}$ such that $\tau\theta \leq \lambda\tau$ for some $0 < \lambda < 1$ and $\mathcal{M} = \mathcal{N} \rtimes_\theta \mathbb{Z}$.*

We explain the relation between continuous decomposition and discrete decomposition. Let $\mathcal{M}$ be of a factor of type III$_0$, and $\mathcal{M} = \mathcal{N} \rtimes_{\theta^{(0)}} \mathbb{Z}$ its discrete decomposition as above. Let $L^\infty(X,\mu) = Z(\mathcal{N})$, and S be an ergodic transformation determined by $\theta^{(0)}(f)(x) = f(S^{-1}x)$. Let f be a positive operator affiliated with $Z(\mathcal{N})$ such that $\tau\theta^{(0)}(x) = \tau(fx)$. Then the flow of weights $(\mathcal{C}_\mathcal{M}, \theta)$ is given as a flow built under a ceiling function by $(X, S, -\log f)$.

6. Classification of injective factors

Theorem 5.2 and 5.6 suggest that importance of analysis of type II factors and their automorphism groups for the study of type III factors.

A. Connes succeeded the classification of automorphisms of the AFD factor of type II$_1$ [4], [6], and applied these results for classification of injective factors of type III$_\lambda$, $\lambda \neq 1$ [5]. The uniqueness of the injective factor of type III$_1$ was solved by [7], [12]. In this section, we explain this classification of injective factors.

Definition 6.1. (1) A von Neumann algebra $\mathcal{M}$ is said to be approximately finite dimensional (AFD), if there exists an increasing family of finite dimensional subalgebras $\{M_i\}_{i \in I}$ such that $(\bigcup_i M_i)'' = \mathcal{M}$.
(2) A von Neumann algebra $\mathcal{M} \subset B(\mathfrak{H})$ is said to be injective if there exists a linear map $E : B(\mathfrak{H}) \to \mathcal{M}$ such that $\|E(x)\| \leq \|x\|$ and $E(a) = a$ for all $a \in \mathcal{M}$.

Tomiyama's theorem [27] says that E appearing in (2) is a conditional expectation, i.e., E is a positive map and $E(axb) = aE(x)b$ for $a, b \in \mathcal{M}$, $x \in B(\mathfrak{H})$. We remark that injectivity is a property for isomorphism classes of von Neumann algebras, i.e., it is independent from a representation $\mathcal{M} \subset B(\mathfrak{H})$.

The following is a fundamental result due to Murray-von Neumann.

Theorem 6.2. *Any AFD factor of type* II_1 *is isomorphic to an ITPFI type* II_1 *factor* $\bigotimes_{i=1}^{\infty} (M_2(\mathbb{C}), 2^{-1} \operatorname{Tr})$. *Hence the AFD factor of type* II_1 *is unique up to isomorphism. We denote the AFD factor of type* II_1 *by* $\mathcal{R}_0$.

See [23, Chapter XIV.2] for proof.
It is easy to see that AFD property implies injectivity. In many cases, it is difficult to see AFD property. For example, let $\mathcal{M}$ be an AFD factor of type II_∞, and decompose it as a tensor product $\mathcal{M} = \mathcal{N} \otimes B(\ell^2(\mathbb{N}))$ with a factor of type II_1 and a factor of type I_∞. One may expect $\mathcal{N}$ is AFD, but this is not a so obvious fact. On the contrary, injectivity of $\mathcal{N}$ can be easily checked. Indeed, let $E : B(\mathfrak{H}) \to \mathcal{M}$ be as in Definition 6.1, and φ a faithful normal state on $B(\ell^2(\mathbb{N}))$. Then $(\operatorname{id} \otimes \varphi) \circ E : B(\mathfrak{H}) \to \mathcal{N}$ is a conditional expectation. The class of injective von Neumann algebras is closed under many operations, e.g., taking commutants, inductive limits, crossed product by amenable groups.

The Connes-Krieger-Haagerup classification of injective factors [5], [14], [7], [12] can be stated as follows.

Theorem 6.3. (1) *A von Neumann algebra* $\mathcal{M}$ *is injective if and only if* $\mathcal{M}$ *is AFD.*
(2) *The injective factors of type* II_1 *and type* II_∞ *are unique up to isomorphism.*
(3) *Isomorphism classes of injective factors of type III are completely determined by their flow of weights. In particular, the Powers factor* $\mathcal{R}_\lambda$ *is the only injective factor of type* III_λ, $0 < \lambda < 1$, *and the Araki-Woods factor* $\mathcal{R}_\infty$ *is the only the injective factor of type* III_1.
(4) *Any aperiodic and recurrent ergodic flow can be realized as a flow of weights for some injective factor of type* III_0.

We roughly explain the proof of Theorem 6.3. The proof is given separately based on type of factors.

(Type II_1 case.) This case is the most fundamental. Let $\mathcal{M}$ be an injective factor of type II_1 with a tracial state τ. Let $\varphi(x) := \tau \circ E(x)$, which is called a hypertrace on $\mathcal{M}$ since φ satisfies $\varphi(ax) = \varphi(xa)$, $a \in \mathcal{M}$, $x \in B(\mathfrak{H})$. A fundamental idea is that we regard a hypertrace φ as a noncommutative analogue of an invariant mean on a discrete amenable group. By the Day-Namioka type trick, one can deduce a Følner type condition, which is a key for proof of $\mathcal{M}$ being AFD. By the uniqueness of the AFD factor of type II_1, it follows $\mathcal{M} \cong \mathcal{R}_0$. See [5], [23, Chapter XVI] for details of proof.

(Type II_∞ case.) Let $\mathcal{M}$ be an injective factor of type II_∞, and decompose $\mathcal{M} = \mathcal{N} \otimes B(\ell^2(\mathbb{N}))$ as a tensor product of a type II_1 factor and a type I_∞ factor. Then $\mathcal{N}$ is injective, and hence $\mathcal{N} \cong \mathcal{R}_0$. So we have $\mathcal{M} \cong \mathcal{R}_0 \otimes B(\ell^2(\mathbb{N}))$. Denote by $\mathcal{R}_{0,1}$ the injective factor of type II_∞.

(Type III_λ, $0 < \lambda < 1$, case.) Let $\mathcal{M}$ be an injective factor of type III_λ, $0 < \lambda < 1$, and $\mathcal{M} = \mathcal{N} \rtimes_\theta \mathbb{Z}$ be its discrete decomposition. Then $\mathcal{N}$ is also injective, and hence isomorphic to $\mathcal{R}_{0,1}$. Thus classification is reduced to that of automorphisms on $\mathcal{R}_{0,1}$. Then the uniqueness is the immediate consequence of the following theorem due to Connes [4], which has its own interest.

Theorem 6.4. *Let θ_i, $i = 1, 2$, be automorphisms of $\mathcal{R}_{0,1}$ with $\tau\theta_i = \lambda\tau$ for some $0 < \lambda < 1$. Then θ_i are conjugate, i.e., there exists an automorphism σ such that $\theta_1 = \sigma\theta_2\sigma^{-1}$.*

(Type III_0 case.) We explain two proofs for classification.

Let $\mathcal{M}$ be an injective factor of type III_0, and $\mathcal{N} \rtimes_\theta \mathbb{Z}$ be its discrete decomposition. As in type III_λ case, $\mathcal{N}$ is injective, and hence $\mathcal{N} \cong L^\infty(X, \mu) \otimes \mathcal{R}_{0,1}$ for some measure space (X, μ).

First approach. In this case, $\mathcal{M}$ turns out to be isomorphic to a Krieger factor [3], i.e., $\mathcal{M} = L^\infty(Y, \nu) \rtimes_T \mathbb{Z}$ for some ergodic transformation (Y, ν, T). By Krieger [14], the isomorphism classes of such factors are completely determined by their flow of weights.

Second approach. By the disintegration theory, we can reduce the classification of $\mathcal{M}$ to that of actions of a groupoid $X \rtimes_S \mathbb{Z}$ on $\mathcal{R}_{0,1}$. Such actions can be classified by Krieger's cohomology Theorem [23, XIII.3.16].

We show Theorem 6.3 (4), when an ergodic flow $(Z, \mathcal{F})$ admits an invariant measure μ. (See [23, Chapter XVIII.2] for a general case.)

Let α_t be $\alpha_t(f)(x) = f(\mathcal{F}_{-t}x)$, $f \in L^\infty(Z, \mu)$. Existence of an AFD factor of type III_1, e.g., $\mathcal{R}_\infty$, implies the existence of a trace scaling automorphism

$\theta_t^{(0)}$ on $\mathcal{R}_{0,1}$, i.e., $\tau_0\theta_t^{(0)} = e^{-t}\tau_0$ for the tracial weight $\tau_0 \in \mathcal{W}_0(\mathcal{R}_{0,1})$. Let $\theta_t := \theta_t^{(0)} \otimes \alpha_t$ on $\mathcal{N} := \mathcal{R}_{0,1} \otimes L^\infty(Z, \mu)$, $\tau := \tau_0 \otimes \mu$. Then $\mathcal{R}_\mathcal{F} := \mathcal{N} \rtimes_\theta \mathbb{R}$ is an injective factor of type III$_0$ whose flow of weights is $(Z, \mathcal{F})$.

Krieger [14] showed that the classification of Krieger factors is equivalent to that of ergodic transformations by orbit equivalence. We can not treat this theorem in this survey, and refer [23, Chapter XVIII.2] for this topic.

(Type III$_1$ case.) This case is extremely difficult, because of the difficulty of analysis of one-parameter automorphism groups. Connes [7] found a sufficient condition for the uniqueness of the injective factor of type III$_1$, and Haagerup [12] finally solved Connes' problem, and the huge project of classification of injective factors has been completed. In their theory, the notion of bicentralizer plays a central role.

Definition 6.5. Let $\mathcal{M}$ be a von Neumann algebra, and φ a faithful normal state. Define a bicentralizer B_φ of φ by

$$C_\varphi := \{(x_n)_{n\in\mathbb{N}} \mid \sup_n \|x_n\| < \infty, \lim_n \|[x_n, \varphi]\| = 0\},$$

$$B_\varphi := \{a \in \mathcal{M} \mid \lim_n [a, x_n] = 0 \text{ strongly for all } (x_n) \in C_\varphi\}.$$

Here $[x_n, \varphi] \in \mathcal{M}_*$ is defined by $[x_n, \varphi](y) := \varphi(yx_n - x_ny)$.

Connes [7] proved that the triviality of B_φ is sufficient for uniqueness.

Theorem 6.6. *If an injective type III$_1$ factor $\mathcal{M}$ admits a faithful normal state φ with $B_\varphi = \mathbb{C}$, then $\mathcal{M}$ is isomorphic to $\mathcal{R}_\infty$.*

We explain the idea of Connes. Fix $T > 0$, and let $\alpha := \sigma_T^\varphi$. Then the crossed product $\mathcal{M} \rtimes_\alpha \mathbb{Z}$ is an injective factor of type III$_\lambda$, where $\lambda = e^{-T}$. Hence it is isomorphic to $\mathcal{R}_\lambda$. By the Takesaki duality, we only have to classify the dual action $\hat{\alpha}$ of $\mathbb{T}$ on $\mathcal{R}_\lambda$. The key for classification is approximate innerness of α. If this is the case, then $\hat{\alpha}$ is classifiable, and the uniqueness follows. He showed that α is indeed approximately inner under the condition $B_\varphi = \mathbb{C}$.

As explained above, Connes' approach avoids cleverly direct analysis of a trace scaling automorphism θ_t on $\mathcal{R}_{0,1}$. However we can classify θ_t and hence show the uniqueness based on Theorem 5.2 [16], if we admit approximate innerness of modular automorphisms.

We comment that Haagerup [13] obtained an alternative proof of Theorem 6.6.

The bicentralizer problem was finally solved by Haagerup, that is, he obtained the following theorem.

Theorem 6.7. *Any injective factor of type III$_1$ has a faithful normal state with $B_\varphi = \mathbb{C}$. Therefore $\mathcal{R}_\infty$ is the only injective factor of type III$_1$.*

APPENDIX A. CROSSED PRODUCT CONSTRUCTION

In this appendix, we summerize basic facts on crossed-product construction, which is necessary in this survey. See [23, Chapter X] for details.

Let $\mathcal{M}$ be a von Neumann algebra, and $\mathrm{Aut}(\mathcal{M})$ a set of automorphisms on $\mathcal{M}$. Let G be a locally compact group with a left invariant Haar measure μ, and α an action of G on $\mathcal{M}$, i.e., $\alpha : s \in G \mapsto \alpha_s \in \mathrm{Aut}(\mathcal{M})$ is a continuous homomorphism.

Assume $\mathcal{M} \subset B(\mathfrak{H})$, and

$$L^2(G, \mathfrak{H}) := \left\{ \xi : G \to \mathfrak{H}, \text{measurable} \ \Big| \int_G \|\xi(s)\|^2 d\mu(s) < \infty \right\}.$$

Define a $*$-representation $\pi_\alpha : \mathcal{M} \to B(L^2(G, \mathfrak{H}))$, and a unitary representation λ of G by

$$(\pi_\alpha(a)\xi)(s) := \alpha_{s^{-1}}(a)\xi(s), \ (\lambda(t)\xi)(s) := \xi(t^{-1}s).$$

We have a covariant relation $\lambda(t)\pi_\alpha(a)\lambda(t)^* = \pi_\alpha(\alpha_t(s))$.

Definition A.1. The crossed product $\mathcal{M} \rtimes_\alpha G$ is a von Neumann algebra generated by $\pi_\alpha(\mathcal{M})$ and $\{\lambda(t)\}$.

Remark. The algebraic class of $\mathcal{M} \rtimes_\alpha G$ is independent from $\mathfrak{H}$.

Let $K(G, \mathcal{M})$ be a set of compact supported continuous $\mathcal{M}$-valued functions. For $f \in K(G, \mathcal{M})$, let $\pi_\alpha(f)$ be

$$(\pi_\alpha(f)\xi)(s) := \int_G \alpha_{-t}(f(t))\xi(t^{-1}s)d\mu(t).$$

Then $\pi_\alpha(f) \in \mathcal{M} \rtimes_\alpha G$, and formally it is represented by

$$\pi_\alpha(f) = \int_G \pi_\alpha(f(t))\lambda(t)d\mu(t).$$

We can see that $\pi_\alpha(K(G, \mathcal{M}))$ is a dense $*$-subalgebra of $\mathcal{M} \rtimes_\alpha G$.

Let $v(s)$ be a 1-cocycle for α, i.e, $v(st) = v(s)\alpha_s(v(t))$ holds. Then $_v\alpha_s := \mathrm{Ad}\,v(s)\alpha_s$ is also an action of G. Let W be a unitary defined by

$$(W\xi)(s) := v(s^{-1})\xi(s).$$

Then we can see $W\pi_\alpha(a)W^* = \pi_{v\alpha}(a)$, and $W\lambda(t)W^* = \pi_{v\alpha}(v(t))\lambda(t)$. Hence $Ad(W)$ gives an isomorphism between $\mathcal{M} \rtimes_\alpha G$ and $\mathcal{M} \rtimes_{v\alpha} G$.

Further assume that G is abelian, and let $\hat{G}$ be its dual group. For $p \in \hat{G}$, define $\mu(p)$ by

$$(\mu(p)\xi)(s) := \overline{\langle s, p \rangle}\xi(s).$$

Then $\mathrm{Ad}(\mu(p))(\pi_\alpha(a)) = \pi_\alpha(a)$, and $\mathrm{Ad}(\mu(p))(\lambda(t)) := \overline{\langle s, p \rangle}\lambda(t)$ holds. Hence we have an action $\hat{\alpha}$ of $\hat{G}$ on $\mathcal{M} \rtimes_\alpha G$ by $\mathrm{Ad}(\mu(p))$, which is called the dual action of α. The Takesaki duality theorem can be stated as follows.

Theorem A.2. *There exists a canonical isomorphism* $\Phi : (\mathcal{M} \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G} \to \mathcal{M} \otimes B(L^2(G))$. *Via this isomorphism, the second dual action* $\hat{\hat{\alpha}}_s$ *is transformed to* $\alpha_s \otimes \operatorname{Ad} \lambda(s^{-1})$.

References

[1] Araki, H. and Woods, J., *A classification of factors*, Publ. Res. Inst. Math. Sci. **3** (1968), 51–130.

[2] Bratteli, O. and Robinson, D. W., *Operator Algebras and Quantum Statistical Mechanics*, second ed., vol. 1, (1987), vol. 2, ibid. (1997), Springer-verlag.

[3] Connes, A., *On hyperfinite factors of type III$_0$ and Krieger's factors*, J. Funct. Anal. **18** (1975), 318–327.

[4] Connes, A., *Outer conjugacy classes of automorphisms of factors*, Ann. Sci. Eco. Norm. Sup. **8** (1975), 383–419.

[5] Connes, A., *Classification of injective factors*, Ann. Math. **104** (1976), 73–115.

[6] Connes, A., *Periodic automorphisms of the hyperfinite factor of type II$_1$*, Acta Sci. Math **39** (1977), 39–66.

[7] Connes, A., *Factors of type III$_1$, property L'_λ and closure of inner automorphisms*, J. Operator Theory **14** (1985), 189–211.

[8] van Daele, A., *A new approach to the Tomita-Takesaki theory of generalized Hilbert algebras*, J. Funct. Anal. **15** (1974), 378–393.

[9] van Daele, A., and Rieffel, M. A., *A bounded operator approach to Tomita-Takesaki theory*, Pacific J. Math. **69** (1977), 187–221.

[10] Haag, R., Hugenholtz, N. M., and M., Winnink, *On the equilibrium states in quantum statistical mechanics*, Commun. Math. Phys. **5** (1967), 215–236.

[11] Haagerup, U., *On the dual weights for crossed products of von Neumann algebras. I.*, Math. Scand. **43** (1978), 99–118.

[12] Haagerup, U., *Connes' bicentralizer problem and uniqueness of the injective factor of type III$_1$*, Acta Math. **158** (1987), 95–148.

[13] Haagerup, U., *On the uniqueness of the injective type III$_1$ factor*, Doc. Math. **21** (2016), 1193–1226.

[14] Krieger, W., *On ergodic flows and the isomorphism of factors*, Math. Annal. **223** (1976), 19–70.

[15] Longo, R., *A simple proof of the existence of modular automorphisms in approximately finite-dimensional von Neumann algebras*, Pacific J. Math. **75** (1978), 199–205.

[16] Masuda, T. and Tomatsu, R., *Rohlin flows on von Neumann algebras*, Memoirs of Amer. Math. Soc **244** (2016), no. 1153.

[17] von Neumann, J., *Collected works. Vol. III: Rings of operators*, Pergamon Press, New York-Oxford-London-Paris, (1961).

[18] Powers, R. T., *Representations of uniformly hyperfinite algebras and their associated von Neumann rings*, Ann. Math. **86** (1967), 138–171.

[19] Takesaki, M., *Tomita's Theory of Modular Hilbert Algebras and its Applications*, vol. 128, Springer, Berlin, (1970).

[20] Takesaki, M., *Duality and von Neumann algebras*, Lectures on Operator Algebras (Berlin), Lecture Notes in Math., vol. 247, Springer, (1972), 665–786.

[21] Takesaki, M., *Structure of factors and automorphism groups*, no. 51, Regional Conference Series in Mathematics, Amer. Math. Soc., (1983).

[22] Takesaki, M., *Twenty-five years in the theory of type III von Neumann algebras*, C*-algebras, Contemp. Math., vol. 167, Province, RI, Amer. Math. Soc., (1994), 232–239.

[23] Takesaki, M., *Theory of Operator Algebras, I, II, III*, Springer-Verlag, Berlin-Heidelberg-New York, (2002).

[24] Tomita, M., *Spectral theory of operator algebras I.*, Math. J. Okayama Univ. **9** (1959), 63–98.

[25] Tomita, M., *Quasi-standard von Neumann algebras*, mimeographed note, (1967).

[26] Tomita, M., *Standard form of von Neumann algebras*, the Vth functional analysis symposium of the Math. Soc Japan, Sendai, (1967).

[27] Tomiyama, J., *On the projection of norm one in W*-algebras*, Proc. Japan Acad. **33** (1957), 608–612.

[28] Woronowicz, S. L., *Operator systems and their application to the Tomita-Takesaki theory*, J. Operator Theory **2** (1797), 169–209.

[29] Yamagami, S., *Algebraic aspects in modular theory*, Publ. Res. Inst. Math. Sci. **28** (1992), 1075–1106.

Toshihiko Masuda
Graduate School of Mathematics,
Kyushu University
744, Motooka, Nishi-ku, Fukuoka, 819-0395, JAPAN

e-mail address: masuda@math.kyushu-u.ac.jp