BLOWUP AND GLOBAL EXISTENCE OF A SOLUTION TO A SEMILINEAR REACTION-DIFFUSION SYSTEM WITH THE FRACTIONAL LAPLACIAN

Tomoyuki Kakehi and Yoshihito Oshita

Abstract. In this paper, we deal with the semilinear reaction diffusion system with the fractional Laplacian.

(0.1)
$$u_t + (-\Delta)^\alpha u = v^p, \quad v_t + (-\Delta)^\alpha v = u^q, \quad x \in \mathbb{R}^n, \, t > 0,$$
$$u(0, x) = u_0(x) \geq 0, \quad v(0, x) = v_0(x) \geq 0, \quad x \in \mathbb{R}^n,$$

where $p, q > 1$ and $0 < \alpha < 1$. We study the existence of a global in time solution, the blowup of a solution, and the life span of the blowup solution to the above reaction-diffusion system for sufficiently small initial data.

1. Introduction

Let us first start with the semilinear reaction diffusion system

(1.1)
$$\begin{cases} u_t - \Delta u = v^p, & x \in \mathbb{R}^n, \, t > 0, \\ v_t - \Delta v = u^q, & x \in \mathbb{R}^n, \, t > 0, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \\ v(0, x) = v_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $p, q > 1$, $n \geq 1$, and where u_0 and v_0 are nonnegative bounded functions.

As is easily seen by the contraction argument, the solution of (1.1) exists locally in time. Here we define a blowup time and a blowup solution as follows. Let us write the solution of (1.1) as $(u(t), v(t))$ and let
(1.2)
$$T^* = \sup\{\, T > 0 \mid \sup_{0 < t < T} ||u(t)||_{L^\infty} < \infty, \text{ and } \sup_{0 < t < T} ||v(t)||_{L^\infty} < \infty \,\}.$$

We call T^* a blowup time and $(u, v) = (u(t), v(t))$ a blowup solution if $T^* < \infty$. On the other hand, we call $(u, v) = (u(t), v(t))$ a global in time solution if $T^* = \infty$. The problem of blowup and global existence for the above system has been extensively studied by a lot of people. For example, Escobedo and Herrero gave the following result.

Theorem 1.1 (Escobedo and Herrero, [2])**.**

(I)   *Suppose that*
$$\frac{\max\{p,q\}+1}{pq-1} < \frac{n}{2}.$$

Let

(1.3) $u_0 \in L^{\alpha_1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \qquad v_0 \in L^{\alpha_2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),$

where $\alpha_1 = \dfrac{n(pq-1)}{2(p+1)}$, $\alpha_2 = \dfrac{n(pq-1)}{2(q+1)}$. *If both* $\|u_0\|_{L^{\alpha_1}}$ *and* $\|v_0\|_{L^{\alpha_2}}$ *are sufficiently small, then there exists a unique global in time solution of* (1.1).

(II)   *Suppose that*
$$\frac{\max\{p,q\}+1}{pq-1} \geq \frac{n}{2}.$$

Then the solution (u,v) *of* (1.1) *with any non-trivial inticial data blows up in a finite time.*

In this paper, we consider the semilinear reaction diffusion system with the fractional Laplacian

(1.4)
$$\begin{cases} u_t + (-\Delta)^\alpha u = v^p, & x \in \mathbb{R}^n,\ t > 0, \\ v_t + (-\Delta)^\alpha v = u^q, & x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ v(0,x) = v_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $p, q > 1$, $n \geq 1$, $0 < \alpha < 1$, and where u_0 and v_0 are nonnegative bounded functions. Here in (1.4), the fractional Laplacian $(-\Delta)^\alpha$ is defined by

(1.5)
$$(-\Delta)^\alpha u(x) = C_{n,\alpha} \text{ p.v.} \int_{\mathbb{R}^n} \frac{u(x+y) - u(x)}{|y|^{n+2\alpha}}\, dy, \qquad C_{n,\alpha} = \frac{\alpha 2^{2\alpha}\Gamma(\frac{n}{2}+\alpha)}{\pi^{\frac{n}{2}}\Gamma(1-\alpha)}.$$

for a bounded C^2 function u whose first and second order derivatives are also bounded. We note that if u belongs to the Sobolev space $H^{2\alpha}(\mathbb{R}^n)$ of order 2α, then $(-\Delta)^\alpha u$ is defined by

$$(-\Delta)^\alpha u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}|\xi|^{2\alpha}\widehat{u}(\xi)\, d\xi,$$

where $\widehat{u}$ denotes the Fourier transform of u. In the same manner as in the case of (1.1), we define a blowup time and a blowup solution of (1.4). The exponent α in (1.4) is expected to measure the effect of diffusion. So our interest lies in the following problem:

"How does the exponent α of the fractional Laplacian in (1.4) affect the blowup and the global existence for (1.4)?"

For example, if α is small, then the diffusion term $(-\Delta)^\alpha u$ is expected to become weak and therefore, the corresponding blowup time is expected to become shorter. Surprisingly, it turns out that the effect of diffusion becomes stronger. Let us first state our result on the global in time solution of (1.4).

Theorem 1.2 (Global existence). *We assume that p, $q > 1$ and that*

$$(1.6) \qquad \frac{\max\{p, q\} + 1}{pq - 1} < \frac{n}{2\alpha}.$$

In addition, we assume that the initial data (u_0, v_0) satisfies

$$(1.7) \qquad 0 \leq u_0(x),\, v_0(x) \leq \delta_0 \left(1 + |x|\right)^{-n-2\alpha},$$

where δ_0 is a positive constant. Then the following (I) and (II) hold.

(I) *The global in time solution to the semilinear system with the fractional Laplacian (1.4) exists if δ_0 is sufficiently small.*

(II) *Let us write the global in time solution of (1.4) as $(u(t), v(t))$. Then $(u(t), v(t))$ satisfies*

$$\|u(t)\|_{L^\infty} \leq C(1+t)^{-\frac{p+1}{pq-1}}, \qquad \|v(t)\|_{L^\infty} \leq C(1+t)^{-\frac{q+1}{pq-1}}$$

for some constant $C > 0$.

For the details, see Theorem 5.2 and Corollary 5.1 in Section 5.

In the above theorem, we assume a stronger condition on the initial data of (1.4) than that on the initial data of Escobedo and Herrero [2]. Instead, we obtain a time decay of the global in time solution of (1.4). Next, we will state our result on the blowup solution of (1.4).

Theorem 1.3 (Blowup). *We assume that p, $q > 1$ and that*

$$(1.8) \qquad \frac{\max\{p, q\} + 1}{pq - 1} \geq \frac{n}{2\alpha}.$$

Then the solution to the semilinear reaction diffusion system with the fractional Laplacian (1.4) blows up in a finite time for any nontrivial initial data.

For the details, see Theorem 6.1 in Section 6.

Thus Theorems 1.2 and 1.3 can be regarded as a generalization of the result by Escobedo and Herrero [2]. In addition, we would like to emphasize the following.

Remark 1.1. We assume that u_0 and v_0 satisfy the condition (1.7) for sufficiently small $\delta_0 > 0$. Let $(u^{(1)}(t), v^{(1)}(t))$ and $(u^{(\alpha)}(t), v^{(\alpha)}(t))$ be the solution of (1.1) and the solution of (1.4) with the same initial data (u_0, v_0),

respectively. Next, let us take any $p > 1$ and $q > 1$ such that

$$\frac{\max\{p, q\} + 1}{pq - 1} \geq \frac{n}{2}.$$

Then $(u^{(1)}(t), v^{(1)}(t))$ blows up in a finite time, due to Theorem 1.1. However, if we take sufficiently small $\alpha > 0$ such that

$$\frac{\max\{p, q\} + 1}{pq - 1} < \frac{n}{2\alpha},$$

then $(u^{(\alpha)}(t), v^{(\alpha)}(t))$ becomes a global in time solution, due to Theorem 1.2. As a result, we see that if α becomes smaller then the effect of diffusion becomes stronger.

Finally, we go into the problem of life span for the blowup solution of (1.1). For example, Huang, Mochizuki and Mukai proved the following.

Theorem 1.4 (Huang, Mochizuki and Mukai [6])**.** *Suppose that $q \geq p \geq 1$, and $pq > 1$. Let $(u_0, v_0) = (\lambda^{\frac{1}{q+1}} \varphi, \lambda^{\frac{1}{p+1}} \psi)$. Here φ and ψ satisfy*

$$(1.9) \qquad \begin{aligned} &\limsup_{|x| \to \infty} |x|^{\frac{a}{2(q+1)}} \varphi(x) < \infty, \\ &\limsup_{|x| \to \infty} |x|^{\frac{a}{2(p+1)}} \psi(x) < \infty, \qquad \liminf_{|x| \to \infty} |x|^{\frac{a}{2(p+1)}} \psi(x) > 0 \end{aligned}$$

for some $a \in \mathbb{R}^n$. Let

$$(1.10) \qquad\qquad a^* := \frac{2(p+1)(q+1)}{pq - 1}.$$

If $0 \leq a \leq \min\{a^*, n(p+1)\}$, then the blowup time T_λ^* satisfies

$$(1.11) \qquad\qquad T_\lambda^* \sim \lambda^{-\frac{2}{a^* - a}}, \qquad\qquad as \ \lambda \to 0.$$

In the above theorem, the authors assume that the initial data satisfies the *slow decay* condition. However, in the case where the initial data decays faster near $|x| = \infty$, few results are known about the life span. For example, Kobayashi [8] assumes that the initial data has exponential decay near $|x| = \infty$, and gives an optimal estimate for the life span. (For the details, see Theorem 1 (ii) of [8].)

In the case of (1.4), we give an optimal estimate of the life span under the assumption that the initial data has a certain polynomial decay. Our third main result is the following.

Theorem 1.5 (Life span)**.** *Let $1 \leq p < 1 + \frac{2\alpha}{n}$, $1 \leq q < 1 + \frac{2\alpha}{n}$, $pq > 1$, and put*

$$(1.12) \qquad\qquad \mu^* := \frac{p+1}{pq - 1} - \frac{n}{2\alpha}, \qquad \nu^* := \frac{q+1}{pq - 1} - \frac{n}{2\alpha}.$$

We also assume that

(1.13) $$u_0(x) = \lambda^{\mu^*} \varphi(x), \qquad v_0(x) = \lambda^{\nu^*} \psi(x),$$

where λ is a small positive parameter and φ, ψ are continuous functions on $\mathbb{R}^n$ such that

(1.14) $$0 \leq \varphi(x), \ \psi(x) \leq C(1 + |x|)^{-n-2\alpha}$$

for some constant $C > 0$. Then the blowup time T_λ^* of the solution to (1.4) with initial data (1.13) satisfies

(1.15) $$T_\lambda^* \sim \frac{1}{\lambda} \qquad \text{as } \lambda \to 0.$$

In the above theorem, we note that $\mu^* > 0$ and $\nu^* > 0$. So as $\lambda \to 0$, the initial data also converges to 0. For the details, see Theorem 7.1 in Section 7.

Our paper is organized as follows. In Section 2, we will give a brief summary of the asymptotic behavior of the fundamental solution $W^{(\alpha)}(t, x)$ to the linear parabolic equation $u_t + (-\Delta)^\alpha u = 0$. Section 3 is devoted to the comparison theorem. Here we point out that it is a nontrivial problem to prove a comparison theorem in the case of (1.4). It is basically due to the fact that $(-\Delta)^\alpha$ is no longer a local operator. In Section 4, we prove that the solution to the system of the integral equations arising from (1.4) becomes a strong solution to (1.4) under the assumption that $\frac{1}{2} < \alpha < 1$. In Section 5 and 6, we prove Theorem 1.2 and Theorem 1.3, respectively. In Section 7, we deal with the problem of life span for (1.4) for small initial data.

2. Asymptotic Property of $W^{(\alpha)}(t, x)$ revisited

In this section, we give a brief summary on asymptotic properties of the function $W^{(\alpha)}(t, x)$ defined by the following Fourier integral:

(2.1) $$W^{(\alpha)}(t, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-t|\xi|^{2\alpha}} e^{i\xi \cdot x} d\xi, \qquad t > 0, \ x \in \mathbb{R}^n.$$

We see easily that $W^{(\alpha)}(t, x)$ is the fundamental solution to the parabolic pseudodifferential equation

(2.2) $$\partial_t u(t, x) + (-\Delta)^\alpha u(t, x) = 0,$$

in the sense that the solution u of the Cauchy problem

(2.3) $$\begin{cases} \partial_t u(t, x) + (-\Delta)^\alpha u(t, x) = 0 \\ u(0, x) = u_0(x) \in L^\infty(\mathbb{R}^n) \end{cases}$$

is written as

$$(2.4) \qquad u(t,x) = \int_{\mathbb{R}^n} W^{(\alpha)}(t, x - y) u_0(y) \, dy.$$

The asymptotic behavior of $W^{(\alpha)}(t,x)$ is explicitly given in Kakehi and Sakai [11]. We will summarize the results of Section 2 in [11] here.

Let

$$(2.5) \qquad w^{(\alpha)}(x) = W^{(\alpha)}(1, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-|\xi|^{2\alpha}} e^{i\xi \cdot x} d\xi.$$

Then, we have

$$(2.6) \qquad W^{(\alpha)}(t, x) = t^{-\frac{n}{2\alpha}} w^{(\alpha)}(t^{-\frac{1}{2\alpha}} x).$$

So, it suffices to calculate the asymptotic expansion of $w^{(\alpha)}(x)$ as $|x| \to \infty$ in order to know the asymptotic behavior of $W^{(\alpha)}(t, x)$.

Let us first state several results on the asymptotic expansion of $w^{(\alpha)}(x)$ near $|x| = +\infty$. Note that the asymptotic expansion

$$f(x) \sim \sum_{j=1}^{\infty} a_j \, |x|^{-n-2j\alpha} \qquad \text{as} \quad |x| \to +\infty$$

means that for each $N \in \mathbb{N}$, there holds

$$f(x) = \sum_{j=1}^{N} a_j \, |x|^{-n-2j\alpha} + O(|x|^{-n-2(N+1)\alpha}) \qquad \text{as} \quad |x| \to +\infty.$$

Theorem 2.1. *We define constants a_j $(j = 1, 2, \cdots)$ by*

$$(2.7) \qquad a_j = \frac{(-1)^{j-1} 2^{2\alpha j}}{j! \pi^{\frac{n}{2}+1}} \, \sin(\pi \alpha j) \, \Gamma(1 + \alpha j) \, \Gamma(\frac{n}{2} + \alpha j).$$

(I) $w^{(\alpha)}(x)$ *has the asymptotic expansion:*

$$(2.8) \qquad w^{(\alpha)}(x) \sim \sum_{j=1}^{\infty} a_j \, |x|^{-n-2j\alpha} \qquad \text{as} \quad |x| \to +\infty.$$

(II) *For each multi-index γ, the derivative $\partial_x^\gamma w^{(\alpha)}$ of $w^{(\alpha)}$ has the asymptotic expansion:*

$$(2.9) \qquad \partial_x^\gamma w^{(\alpha)}(x) \sim \sum_{j=1}^{\infty} a_j \, \partial_x^\gamma(|x|^{-n-2\alpha j}) \qquad \text{as} \quad |x| \to +\infty.$$

In addition to the above theorem, we have the following.

Theorem 2.2. *For $\beta > 0$, $(-\Delta)^\beta w^{(\alpha)}$ has the asymptotic expansion:*

$$(2.10) \qquad (-\Delta)^\beta w^{(\alpha)}(x) \sim \sum_{j=1}^{\infty} b_j \, |x|^{-n-2\alpha j - 2\beta} \qquad as \ |x| \to +\infty.$$

Here the constants b_j $(j = 1, 2, \cdots)$ in (2.10) are given by

$$(2.11) \ \ b_j \ = \ \frac{(-1)^{j-1} 2^{2\alpha j + 2\beta}}{j! \, \pi^{\frac{n}{2}+1}} \, \sin(\pi\alpha j + \pi\beta) \, \Gamma(1 + \alpha j + \beta) \, \Gamma(\frac{n}{2} + \alpha j + \beta).$$

Remark 2.1. For the proofs of the above two theorems, see Kakehi and Sakai [11], Section 2.

The following two corollaries are a direct consequence of Theorem 2.1 and Theorem 2.2.

Corollary 2.1. *The fundamental solution $W^{(\alpha)}(t, x)$ and its derivatives have the following asymptotic expansions:*

$$W^{(\alpha)}(t, x) \ \sim \ \sum_{j=1}^{\infty} a_j \, t^j |x|^{-n-2j\alpha} \qquad as \ |x| \to +\infty,$$

$$(2.12) \qquad \partial_x^\gamma W^{(\alpha)}(t, x) \ \sim \ \sum_{j=1}^{\infty} a_j \, t^j \, (\, \partial_x^\gamma |x|^{-n-2j\alpha} \,) \qquad as \ |x| \to +\infty,$$

$$(-\Delta)^\beta W^{(\alpha)}(t, x) \ \sim \ \sum_{j=1}^{\infty} b_j \, t^j |x|^{-n-2\alpha j - 2\beta} \qquad as \ |x| \to +\infty,$$

where the coefficients a_j $(j = 1, 2, \cdots)$ and b_j $(j = 1, 2, \cdots)$ are given respectively by (2.7) and (2.11).

Corollary 2.2. *There hold*

$$w^{(\alpha)}(x) = O(|x|^{-n-2\alpha}),$$

$$(2.13) \qquad \begin{aligned} \partial_{x_j} w^{(\alpha)}(x) &= O(|x|^{-n-1-2\alpha}) \qquad \text{for } j \ (1 \le j \le n), \\ \partial_{x_j} \partial_{x_k} w^{(\alpha)}(x) &= O(|x|^{-n-2-2\alpha}) \qquad \text{for } j, k \ (1 \le j, k \le n), \\ (-\Delta)^\beta w^{(\alpha)}(x) &= O(|x|^{-n-2\alpha-2\beta}) \qquad \text{for } \beta > 0, \end{aligned}$$

as $|x| \to +\infty$. In particular, $w^{(\alpha)}$, $\partial_{x_j} w^{(\alpha)}$, $\partial_{x_j} \partial_{x_k} w^{(\alpha)}$, and $(-\Delta)^\beta w^{(\alpha)}$ are all integrable on $\mathbb{R}^n$.

3. Comparison Theorem

In this section, we will show a comparison theorem for a semilinear reaction diffusion system with the fractional Laplacian.

Let us start with the following theorem.

Theorem 3.1. *The following* (i) *and* (ii) *hold.*

(i) $W^{(\alpha)}(t,x) > 0$ *for $t > 0$ and for $x \in \mathbb{R}^n$.*

(ii) $W^{(\alpha)}(t,x)$ *is monotone decreasing with respect to $|x|$, that is, $W^{(\alpha)}(t,x) > W^{(\alpha)}(t,y)$ if $|x| < |y|$.*

For the above theorem, see, for example, Kakehi and Sakai [11], Theorem 3.2.

By Corollary 2.2 combined with the above theorem, we have

Corollary 3.1. *For each fixed $t > 0$, there exist positive constants C_1 and C_2 such that*

$$(3.1) \quad C_1(1+|x|)^{-n-2\alpha} \;\leq\; W^{(\alpha)}(t,x) \;\leq\; C_2\,(1+|x|)^{-n-2\alpha} \qquad \text{for } x \in \mathbb{R}^n.$$

The positivity of the fundamental solution $W^{(\alpha)}(t,x)$ plays an essential role in the proof of our comparison theorem.

Let us consider the semilinear reaction diffusion system

$$(3.2) \quad \begin{cases} u_t + (-\Delta)^\alpha u = f(v), & x \in \mathbb{R}^n,\; t > 0, \\ v_t + (-\Delta)^\alpha v = g(u), & x \in \mathbb{R}^n,\; t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ v(0,x) = v_0(x), & x \in \mathbb{R}^n. \end{cases}$$

If $\alpha = 1$ and $(-\Delta)^\alpha = -\Delta$, then it is easy to show a comparison theorem for the above system (3.2) using the maximum principle for parabolic differential equations. (See Protter and Weinberger [18].) However, if $0 < \alpha < 1$, the usual maximum principle cannot be applied to (3.2), due to the fact that $(-\Delta)^\alpha$ is no longer a local operator. Instead, in order to obtain the corresponding comparison theorem, we use the positivity of the fundamental solution $W^{(\alpha)}(t,x)$.

Now, we go into our comparison theorem.

Theorem 3.2. *Let f and g be continuous functions on $[0, \infty)$ and $u_0, v_0, \overline{U}_0, \overline{V}_0, \underline{U}_0, \underline{V}_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Let $u, v \in C((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ be nonnegative solutions to the following system of integral equations:*
$$(3.3)$$
$$\begin{cases} u(t,x) = (W^{(\alpha)}(t,\cdot) * u_0)(x) + \int_0^t [W^{(\alpha)}(t-s,\cdot) * f(v(s,\cdot))](x)\,ds, \\ v(t,x) = (W^{(\alpha)}(t,\cdot) * v_0)(x) + \int_0^t [W^{(\alpha)}(t-s,\cdot) * g(u(s,\cdot))](x)\,ds. \end{cases}$$

In addition, let nonnegative functions $\overline{U}, \overline{V}, \underline{U}, \underline{V} \in C((0, T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ *satisfy the following inequalities:*

(3.4)
$$\begin{cases} \overline{U}(t, x) \geq (W^{(\alpha)}(t, \cdot) * \overline{U}_0)(x) + \int_0^t [W^{(\alpha)}(t - s, \cdot) * f(\overline{V}(s, \cdot))](x) \, ds, \\ \overline{V}(t, x) \geq (W^{(\alpha)}(t, \cdot) * \overline{V}_0)(x) + \int_0^t [W^{(\alpha)}(t - s, \cdot) * g(\overline{U}(s, \cdot))](x) \, ds, \end{cases}$$

(3.5)
$$\begin{cases} \underline{U}(t, x) \leq (W^{(\alpha)}(t, \cdot) * \underline{U}_0)(x) + \int_0^t [W^{(\alpha)}(t - s, \cdot) * f(\underline{V}(s, \cdot))](x) \, ds, \\ \underline{V}(t, x) \leq (W^{(\alpha)}(t, \cdot) * \underline{V}_0)(x) + \int_0^t [W^{(\alpha)}(t - s, \cdot) * g(\underline{U}(s, \cdot))](x) \, ds. \end{cases}$$

We assume the following conditions:

(A1)   *f and g are monotone increasing.*

(A2)   *For any $M > 0$, there exists a positive constant C_M such that*

(3.6) $\displaystyle \sup_{0 \leq v, \tilde{v} \leq M} \frac{f(v) - f(\tilde{v})}{v - \tilde{v}} \leq C_M$, $\displaystyle \sup_{0 \leq u, \tilde{u} \leq M} \frac{g(u) - g(\tilde{u})}{u - \tilde{u}} \leq C_M$.

(A3)

(3.7) $\underline{U}_0(x) \leq u_0(x) \leq \overline{U}_0(x)$, $\underline{V}_0(x) \leq v_0(x) \leq \overline{V}_0(x)$ *for $x \in \mathbb{R}^n$.*

(A4)   *For any T_0 $(0 < T_0 < T)$,*

$$\sup_{t \in (0, T_0]} ||u(t, \cdot)||_{L^\infty(\mathbb{R}^n)} < +\infty, \qquad \sup_{t \in (0, T_0]} ||v(t, \cdot)||_{L^\infty(\mathbb{R}^n)} < +\infty,$$

(3.8) $\displaystyle \sup_{t \in (0, T_0]} ||\overline{U}(t, \cdot)||_{L^\infty(\mathbb{R}^n)} < +\infty, \qquad \sup_{t \in (0, T_0]} ||\overline{V}(t, \cdot)||_{L^\infty(\mathbb{R}^n)} < +\infty,$

$$\sup_{t \in (0, T_0]} ||\underline{U}(t, \cdot)||_{L^\infty(\mathbb{R}^n)} < +\infty, \qquad \sup_{t \in (0, T_0]} ||\underline{V}(t, \cdot)||_{L^\infty(\mathbb{R}^n)} < +\infty.$$

Then we have

(3.9) $\underline{U}(t, x) \leq u(t, x) \leq \overline{U}(t, x)$, $\underline{V}(t, x) \leq v(t, x) \leq \overline{V}(t, x)$

for $(t, x) \in [0, T) \times \mathbb{R}^n$.

Proof. Let us take any $T_0 \in (0, T)$ and fix it. It suffices to show only two inequalities $u(t, x) \leq \overline{U}(t, x)$ and $v(t, x) \leq \overline{V}(t, x)$ for $(t, x) \in [0, T_0] \times \mathbb{R}^n$. In fact, the proof of the inequalities that $u(t, x) \geq \underline{U}(t, x)$ and $v(t, x) \geq \underline{V}(t, x)$ is similar.

Let $w = \overline{U} - u$, $w_0 = \overline{U}_0 - u_0$, $z = \overline{V} - v$, $z_0 = \overline{V}_0 - v_0$. Moreover let

(3.10) $G_1(t, x) = \begin{cases} \dfrac{f(\overline{V}(t, x)) - f(v(t, x))}{\overline{V}(t, x) - v(t, x)}, & \text{if } \overline{V}(t, x) \neq v(t, x) \\ 0, & \text{if } \overline{V}(t, x) = v(t, x) \end{cases}$,

(3.11) $G_2(t, x) = \begin{cases} \dfrac{g(\overline{U}(t, x)) - g(u(t, x))}{\overline{U}(t, x) - u(t, x)}, & \text{if } \overline{U}(t, x) \neq u(t, x) \\ 0, & \text{if } \overline{U}(t, x) = u(t, x) \end{cases}$.

Then by Theorem 3.1 (i), and the assumption that $w_0 = \overline{U}_0 - u_0 \geq 0$ and $z_0 = \overline{V}_0 - v_0 \geq 0$, we have

(3.12)
$$w(t,x) \geq W^{(\alpha)}(t,\cdot) * w_0(x)$$
$$+ \int_0^t \int_{\mathbb{R}^n} W^{(\alpha)}(t-s, x-y)\{f(\overline{V}(s,y)) - f(v(s,y))\}\, dyds$$
$$\geq \int_0^t \int_{\mathbb{R}^n} W^{(\alpha)}(t-s, x-y)G_1(s,y)z(s,y)\, dyds,$$
$$z(t,x) \geq W^{(\alpha)}(t,\cdot) * z_0(x)$$
$$+ \int_0^t \int_{\mathbb{R}^n} W^{(\alpha)}(t-s, x-y)\{g(\overline{U}(s,y)) - g(u(s,y))\}\, dyds$$
$$\geq \int_0^t \int_{\mathbb{R}^n} W^{(\alpha)}(t-s, x-y)G_2(s,y)w(s,y)\, dyds.$$

Let

(3.13)
$$M = \max\Big\{ \sup_{t\in(0,T_0]} ||u(t,\cdot)||_{L^\infty(\mathbb{R}^n)},\ \sup_{t\in(0,T_0]} ||\overline{U}(t,\cdot)||_{L^\infty(\mathbb{R}^n)},$$
$$\sup_{t\in(0,T_0]} ||v(t,\cdot)||_{L^\infty(\mathbb{R}^n)},\ \sup_{t\in(0,T_0]} ||\overline{V}(t,\cdot)||_{L^\infty(\mathbb{R}^n)} \Big\}.$$

We note that the above M is finite due to assumption (A4). Then, by (3.6),

(3.14) $\qquad 0 \leq G_j(t,x) \leq C_M \quad (j=1,2) \qquad$ for $(t,x) \in [0,T_0] \times \mathbb{R}^n$.

Now we introduce two linear operators $S_j : L^\infty([0,T_0]\times\mathbb{R}^n) \longrightarrow L^\infty([0,T_0]\times \mathbb{R}^n)$ $(j=1,2)$ as follows.

(3.15) $\qquad (S_j\phi)(t,x) = \int_0^t \int_{\mathbb{R}^n} W^{(\alpha)}(t-s, x-y)G_j(s,y)\phi(s,y)\, dyds$

for $\phi \in L^\infty([0,T_0] \times \mathbb{R}^n)$. Then by (3.12) we have

(3.16) $\qquad w(t,x) \geq (S_1 z)(t,x), \qquad z(t,x) \geq (S_2 w)(t,x).$

On the other hand, we see easily by induction that

(3.17)
$$|((S_1 S_2)^N \phi)(t,x)| \leq \frac{C_M^{2N}}{(2N)!} t^{2N} ||\phi||_{L^\infty([0,T_0]\times\mathbb{R}^n)}, \qquad N = 1,2,3,\cdots.$$

Hence

(3.18) $\qquad ||(S_1 S_2)^N|| \leq \frac{(C_M T_0)^{2N}}{(2N)!} \longrightarrow 0 \qquad$ as $N \to \infty.$

Since $W^{(\alpha)}(t,x) > 0$ (Theorem 3.1) and $G_j(t,x) \geq 0$, S_j maps a nonnegative function to a nonnegative function. So if $\phi(t,x) \geq \psi(t,x)$, then $S_j\phi(t,x) \geq S_j\psi(t,x)$. By making use of this property combined with (3.16), we have

$$(3.19) \quad w(t,x) \geq (S_1 z)(t,x) \geq (S_1 S_2 w)(t,x) \geq \cdots \geq ((S_1 S_2)^N w)(t,x)$$

for $N = 1, 2, 3, \cdots$. By letting $N \to \infty$ in (3.19) and using (3.18), we obtain

$$(3.20) \qquad w(t,x) \geq 0 \qquad \text{for } (t,x) \in [0, T_0] \times \mathbb{R}^n.$$

Similarly, we have

$$(3.21) \qquad z(t,x) \geq 0 \qquad \text{for } (t,x) \in [0, T_0] \times \mathbb{R}^n.$$

The proof is now completed. $\hfill\square$

As a direct consequence of Theorem 3.2, we have

Corollary 3.2. *Let u, v satisfy*

$$(3.22) \qquad \begin{cases} u_t + (-\Delta)^\alpha u = f(v), & x \in \mathbb{R}^n, \ t > 0, \\ v_t + (-\Delta)^\alpha v = g(u), & x \in \mathbb{R}^n, \ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ v(0,x) = v_0(x), & x \in \mathbb{R}^n. \end{cases}$$

In addition, let $\overline{U}, \overline{V}, \underline{U}$, and $\underline{V}$ satisfy

$$(3.23) \qquad \begin{cases} \overline{U}_t + (-\Delta)^\alpha \overline{U} \geq f(\overline{V}), & x \in \mathbb{R}^n, \ t > 0, \\ \overline{V}_t + (-\Delta)^\alpha \overline{V} \geq g(\overline{U}), & x \in \mathbb{R}^n, \ t > 0, \\ \overline{U}(0,x) = \overline{U}_0(x), & x \in \mathbb{R}^n, \\ \overline{V}(0,x) = \overline{V}_0(x), & x \in \mathbb{R}^n, \end{cases}$$

$$\begin{cases} \underline{U}_t + (-\Delta)^\alpha \underline{U} \leq f(\underline{V}), & x \in \mathbb{R}^n, \ t > 0, \\ \underline{V}_t + (-\Delta)^\alpha \underline{V} \leq g(\underline{U}), & x \in \mathbb{R}^n, \ t > 0, \\ \underline{U}(0,x) = \underline{U}_0(x), & x \in \mathbb{R}^n, \\ \underline{V}(0,x) = \underline{V}_0(x), & x \in \mathbb{R}^n. \end{cases}$$

We assume the same conditions (A1), (A2), and (A3) as in Theorem 3.2. Then we have

$$(3.24) \qquad \underline{U}(t,x) \leq u(t,x) \leq \overline{U}(t,x), \quad \underline{V}(t,x) \leq v(t,x) \leq \overline{V}(t,x).$$

4. EXISTENCE OF THE STRONG SOLUTION

The purpose of this section is to prove that the solution to the related system of integral equations satisfies (1.4).

We start with the system of integral equations arising from (1.4)

$$(4.1) \quad \begin{cases} u(t,x) & = (W^{(\alpha)}(t,\cdot) * u_0)(x) \; + \; \int_0^t [W^{(\alpha)}(t-s,\cdot) * v(s,\cdot)^p](x) \, ds, \\ v(t,x) & = (W^{(\alpha)}(t,\cdot) * v_0)(x) \; + \; \int_0^t [W^{(\alpha)}(t-s,\cdot) * u(s,\cdot)^q](x) \, ds. \end{cases}$$

Definition 4.1. A pair of functions (u,v) on $[0,T) \times \mathbb{R}^n$ is called *a mild solution* of (1.4) in $[0,T)$ if the following (i), (ii), (iii) and (iv) hold.

   (i) $u, v \in C((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$.

   (ii) (u,v) satisfies the system of integral equations (4.1) for $(t,x) \in (0,T) \times \mathbb{R}^n$.

   (iii) $\lim\limits_{t \to 0+} u(t) = u_0$, $\lim\limits_{t \to 0+} v(t) = v_0$ in $L^1(\mathbb{R}^n)$.

   (iv) For any T_0 $(0 < T_0 < T)$, $\sup\limits_{t \in (0,T_0]} ||u(t,\cdot)||_{L^\infty(\mathbb{R}^n)} < +\infty$, and $\sup\limits_{t \in (0,T_0]} ||v(t,\cdot)||_{L^\infty(\mathbb{R}^n)} < +\infty$.

The following theorem is easily proved by the usual contraction argument.

Theorem 4.1. *Assume that $u_0(x), v_0(x) \geq 0$ and that $u_0, v_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. If $T > 0$ is sufficiently small, then the system of integral equations (4.1) has a unique mild solution (u,v) in $[0,T)$ in the sense of Definition 4.1.*

Now we define a strong solution of (1.4)

Definition 4.2. We assume that $u, v \in C((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$. A pair of functions (u,v) is called *a strong solution* of the semilinear reaction diffusion system (1.4) in $[0,T) \times \mathbb{R}^n$ with initial data (u_0, v_0) in $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ if u and v satisfy the following conditions.

   (i) $u, v \in C((0,T); H_1^{2\alpha}(\mathbb{R}^n) \cap H_\infty^{2\alpha}(\mathbb{R}^n))$.

   (ii) $u, v \in C^1((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$.

   (iii) As an equality in $C((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$, u and v satisfy the system of equations

$$(4.2) \quad \begin{cases} \partial_t u + (-\Delta)^\alpha u = v^p, \\ \partial_t v + (-\Delta)^\alpha v = u^q. \end{cases}$$

   (iv) $\lim\limits_{t \to +0} u(t,\cdot) = u_0$ and $\lim\limits_{t \to +0} v(t,\cdot) = v_0$ in $L^1(\mathbb{R}^n)$.

The main theorem in this section is stated as follows.

Theorem 4.2. *We assume that $\frac{1}{2} < \alpha < 1$. Let (u,v) be a mild solution of the semilinear reaction diffusion system (1.4) in $[0,T) \times \mathbb{R}^n$ with initial data (u_0, v_0) in $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$. Then the following (i), (ii), and (iii) hold.*

(i) u and v are of class C^1 in $t \in (0,T)$ and of class C^2 in $x \in \mathbb{R}^n$. Moreover, $u, v \in C((0,T), H_1^2(\mathbb{R}^n) \cap H_\infty^2(\mathbb{R}^n)) \cap C^1((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$

(ii) $(-\Delta)^\alpha u$, $(-\Delta)^\alpha v \in C((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$.

(iii) (u, v) is a unique strong solution of (1.4) in the sense of Definition 4.2.

The proof of Theorem 4.2 is split into 7 steps.

First, we will consider the regularity of u and v with respect to $x \in \mathbb{R}^n$.

<u>1st Step</u>. u and v are of class C^1 in $x \in \mathbb{R}^n$ for each $t \in (0,T)$. Moreover, for $1 \leq j \leq n$, $\partial_{x_j} u$, $\partial_{x_j} v \in C((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$.

Proof. We define an operator Φ on $C((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ by

$$(4.3) \qquad (\Phi f)(t,x) = \int_0^t \int_{\mathbb{R}^n} W^{(\alpha)}(t-s, x-y) \, f(s,y) \, dyds$$

for $f \in C((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$. We first note that if u and $v \in C((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ then so do u^q and v^p and thus $\Phi(v^p)$ and $\Phi(u^q)$ are well-defined. Then it suffices to show that $\partial_{x_j} \Phi(v^p)$, $\partial_{x_j} \Phi(u^q) \in C((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$.

By a straightforward computation, we have

(4.4)
$$\partial_{x_j}(\Phi(v^p))(t,x) = \int_0^t \int_{\mathbb{R}^n} (t-s)^{-\frac{1}{2\alpha}} (\partial_{x_j} w^{(\alpha)})(z) \, v(s, x-(t-s)^{\frac{1}{2\alpha}} z)^p \, dzds,$$

where $w^{(\alpha)}$ is the function given by (2.5). Thus by the assumption that $\frac{1}{2} < \alpha < 1$,

(4.5)
$$||\partial_{x_j}(\Phi(v^p))(t,\cdot)||_{L^\infty(\mathbb{R}^n)}$$
$$\leq ||\partial_{x_j} w^{(\alpha)}||_{L^1(\mathbb{R}^n)} ||v(t,\cdot)^p||_{L^\infty(\mathbb{R}^n)} \times \int_0^t (t-s)^{-\frac{1}{2\alpha}} \, ds \ < +\infty.$$

Similarly

(4.6)
$$||\partial_{x_j}(\Phi(v^p))(t,\cdot)||_{L^1(\mathbb{R}^n)}$$
$$\leq ||\partial_{x_j} w^{(\alpha)}||_{L^1(\mathbb{R}^n)} ||v(t,\cdot)^p||_{L^1(\mathbb{R}^n)} \times \int_0^t (t-s)^{-\frac{1}{2\alpha}} \, ds \ < +\infty.$$

The above two estimates show that $\partial_{x_j}(\Phi(v^p)) \in C((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$. Similarly we have $\partial_{x_j}(\Phi(u^q)) \in C((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$. $\square$

<u>2nd Step</u>. u and v are of class C^2 in $x \in \mathbb{R}^n$ for each $t \in (0,T)$. Moreover, for $1 \leq j, k \leq n$, $\partial_{x_j} \partial_{x_k} u$, $\partial_{x_j} \partial_{x_k} v \in C((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$.

Proof. As in the 1st step, it suffices to show that $\partial_{x_j}\partial_{x_k}\Phi(v^p)$, $\partial_{x_j}\partial_{x_k}\Phi(u^q) \in C((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$. Here we note that as a result of the 1st step $\partial_{x_j}(v^p)$, $\partial_{x_j}(u^q) \in C((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$. We also note that $\partial_{x_j}(v^p)(s,\cdot)$ and $\partial_{x_j}(u^q)(s,\cdot)$ may diverge as $s \to +0$.

By integral by parts with respect to x, we have

$$
\partial_{x_j}\partial_{x_k}(\Phi(v^p))(t,x)
$$

$$
(4.7) \quad = \int_0^{\frac{t}{2}} \int_{\mathbb{R}^n} (\partial_{x_j}\partial_{x_k}W^{(\alpha)})(t-s, x-y)\, v(s,y)^p\, dyds
$$

$$
+ \int_{\frac{t}{2}}^t \int_{\mathbb{R}^n} (t-s)^{-\frac{1}{2\alpha}} (\partial_{x_j}w^{(\alpha)})(z)\, \partial_{x_k}\{v(s, x - ((t-s)^{\frac{1}{2\alpha}}z)\}^p\, dzds.
$$

In the integrand of the first term of R.H.S of (4.7), $(\partial_{x_j}\partial_{x_k}W^{(\alpha)})(t-s, x-y)$ is integrable with respect to $(s,y) \in [0, \frac{t}{2}] \times \mathbb{R}^n$ due to Corollary 2.1 and Corollary 2.2. Moreover, in the integrand of the second term of R.H.S. of (4.7), $\partial_{x_k}\{v(s, x - ((t-s)^{\frac{1}{2\alpha}}z)\}^p$ is bounded and integrable with respect to $(s,x) \in [\frac{t}{2}, t] \times \mathbb{R}^n$ due to the 1st step. Therefore, we obtain similar estimates as in (4.5) and (4.6) for the first and the second term of (4.7), which proves that $\partial_{x_j}\partial_{x_k}\Phi(v^p) \in C((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$. Similarly we have $\partial_{x_j}\partial_{x_k}\Phi(u^q) \in C((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$. □

<u>3rd Step.</u> $u, v \in C((0,T); H_1^{2\alpha}(\mathbb{R}^n) \cap H_\infty^{2\alpha}(\mathbb{R}^n))$.   In particular, $(-\Delta)^\alpha u, (-\Delta)^\alpha v \in C((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$.

Proof. The assertion is obvious. In fact, as a direct consequence of the 2nd step, $u, v \in C((0,T); H_1^2(\mathbb{R}^n) \cap H_\infty^2(\mathbb{R}^n))$, and clearly $H_r^2(\mathbb{R}^n) \subset H_r^{2\alpha}(\mathbb{R}^n)$ for $1 \le r \le \infty$. □

Let us now consider the differentiability of u and v with respect to $t \in (0,T)$.

We start with the difference quotient
(4.8)
$$
\frac{1}{h}\{(\Phi(v^p))(t+h, x) - (\Phi(v^p))(t,x)\}
$$

$$
= \frac{1}{h}\int_t^{t+h} \int_{\mathbb{R}^n} W^{(\alpha)}(t+h-s, x-y)\, v(s,y)^p\, dyds
$$

$$
+ \frac{1}{h}\int_0^t \int_{\mathbb{R}^n} \{W^{(\alpha)}(t+h-s, x-y) - W^{(\alpha)}(t-s, x-y)\}\, v(s,y)^p\, dyds
$$

$$
\overset{\text{(we put)}}{=:} I(h) + J(h).
$$

Without loss of generality, we may assume that $h > 0$ when we take the limit $h \to 0$ in (4.8).

    <u>4th Step</u>. $I(h) \to v^p$ as $h \to 0$ in $C((0, T), L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$.

Proof.

$$I(h) = \frac{1}{h} \int_t^{t+h} \int_{\mathbb{R}^n} W^{(\alpha)}(t + h - s, x - y)\, v(s, y)^p\, dyds$$

$$(4.9) \qquad = \frac{1}{h} \int_t^{t+h} \int_{\mathbb{R}^n} w^{(\alpha)}(z)\, v(s, x + (t + h - s)^{\frac{1}{2\alpha}} z)^p\, dzds$$

$$= \int_{\mathbb{R}^n} w^{(\alpha)}(z) \left\{ \frac{1}{h} \int_t^{t+h} v(s, x + (t + h - s)^{\frac{1}{2\alpha}} z)^p\, ds \right\} dz.$$

Thus we have

$$|I(h) - v(t, x)^p|$$

$$(4.10) \quad \leq \int_{\mathbb{R}^n} w^{(\alpha)}(z) \left| \frac{1}{h} \int_t^{t+h} v(s, x + (t + h - s)^{\frac{1}{2\alpha}} z)^p\, ds - v(t, x)^p \right| dz$$

$$\to 0 \qquad \text{as } h \to 0.$$

Moreover, by the above computation, we see easily that $I(h) \to v^p$ as $h \to 0$ in the topology of $C((0, T), L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$. $\qquad\qquad \square$

    Next, let us consider the limit $\lim\limits_{h \to 0} J(h)$, namely,

(4.11)
$$\lim_{h \to 0} \frac{1}{h} \int_0^t \int_{\mathbb{R}^n} \left\{ W^{(\alpha)}(t + h - s, x - y) - W^{(\alpha)}(t - s, x - y) \right\} v(s, y)^p\, dyds.$$

Before computing the above limit, we have to rewrite $J(h)$.

    <u>5th Step</u>.

$$(4.12) \quad J(h) = -(-\Delta)^\alpha \int_0^1 \int_0^t \int_{\mathbb{R}^n} W^{(\alpha)}(t + h\tau - s, x - y)\, v(s, y)^p\, dydsd\tau.$$

Proof. We first note that the integrand of $J(h)$ has a singularity at $s = t$ as a function of s. In order to avoid this singularity, we introduce the following two integrals:

(4.13)
$$J^{(\varepsilon)}(h) := \frac{1}{h} \int_0^{t-\varepsilon} \int_{\mathbb{R}^n} \left\{ W^{(\alpha)}(t + h - s, x - y) - W^{(\alpha)}(t - s, x - y) \right\} v(s, y)^p\, dyds,$$

(4.14)
$$\Phi_{(h,\varepsilon)}(v^p)(t, x) := \int_0^1 \int_0^{t-\varepsilon} \int_{\mathbb{R}^n} W^{(\alpha)}(t + h\tau - s, x - y)\, v(s, y)^p\, dydsd\tau$$

for $\varepsilon \geq 0$. Since $\partial_t W^{(\alpha)} = -(-\Delta)^\alpha W^{(\alpha)}$, we have

(4.15)

$$
\begin{aligned}
J^{(\varepsilon)}(h) &= \int_0^{t-\varepsilon} \int_{\mathbb{R}^n} \int_0^1 (\partial_t W^{(\alpha)})(t+h\tau-s, x-y)\, v(s,y)^p\, d\tau dy ds \\
&= -\int_0^{t-\varepsilon} \int_{\mathbb{R}^n} \int_0^1 ((-\Delta)^\alpha W^{(\alpha)})(t+h\tau-s, x-y)\, v(s,y)^p\, d\tau dy ds \\
&= -(-\Delta)^\alpha \int_0^1 \int_0^{t-\varepsilon} \int_{\mathbb{R}^n} W^{(\alpha)}(t+h\tau-s, x-y)\, v(s,y)^p\, dy ds d\tau \\
&= -(-\Delta)^\alpha \Phi_{(h,\varepsilon)}(v^p)(t,x).
\end{aligned}
$$

Here we see that

(4.16)

$$
\begin{aligned}
\Phi_{(h,\varepsilon)}(v^p)(t,x) &\overset{\varepsilon \to 0}{\longrightarrow} \Phi_{(h,0)}(v^p)(t,x) \\
&= \int_0^1 \int_0^t \int_{\mathbb{R}^n} W^{(\alpha)}(t+h\tau-s, x-y)\, v(s,y)^p\, dy ds d\tau \\
&\qquad \text{in } C((0,T), H_1^2(\mathbb{R}^n) \cap H_\infty^2(\mathbb{R}^n)).
\end{aligned}
$$

As is well known, the fractional Laplacian $(-\Delta)^\alpha$ is a bounded linear operator from $H_q^{2\alpha}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 \leq q \leq \infty$. This fact is easily checked by the argument of Fourier multipliers. (See for example Bergh and Löfström [1], Chapter 6, Theorem 6.2.3.) Therefore, we have

(4.17)

$$
\begin{aligned}
J^{(\varepsilon)}(h) = -(-\Delta)^\alpha \Phi_{(h,\varepsilon)}(v^p) &\overset{\varepsilon \to 0}{\longrightarrow} -(-\Delta)^\alpha \Phi_{(h,0)}(v^p) \\
&\text{in } C((0,T), L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)).
\end{aligned}
$$

On the other hand, obviously $J^{(\varepsilon)}(h) \longrightarrow J(h)$ as $\varepsilon \to 0$ in $C((0,T), L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$, which proves the assertion of the 5th step. $\qquad \square$

We are now in a position to compute the limit $\lim_{h \to 0} J(h)$.

<u>6th Step.</u> $J(h) \to -(-\Delta)^\alpha \Phi(v^p)$ as $h \to 0$ in $C((0,T), L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$.

Proof. We see easily that

(4.18)

$$
\begin{aligned}
&\int_0^1 \int_0^t \int_{\mathbb{R}^n} W^{(\alpha)}(t+h\tau-s, x-y)\, v(s,y)^p\, dy ds d\tau, \\
&\overset{h \to 0}{\longrightarrow} \int_0^t \int_{\mathbb{R}^n} W^{(\alpha)}(t-s, x-y)\, v(s,y)^p\, dy ds \\
&= \Phi(v^p)(t,x) \qquad \text{in } C((0,T), H_1^2(\mathbb{R}^n) \cap H_\infty^2(\mathbb{R}^n)).
\end{aligned}
$$

Again, by making use of the fact that $(-\Delta)^\alpha$ is a bounded linear operator from $H_q^{2\alpha}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 \leq q \leq \infty$, we have

$$(4.19) \quad (-\Delta)^\alpha \int_0^1 \int_0^t \int_{\mathbb{R}^n} W^{(\alpha)}(t + h\tau - s, x - y)\, v(s,y)^p \, dy ds d\tau$$

$$\xrightarrow{h \to 0} (-\Delta)^\alpha \Phi(v^p)(t,x)$$

in $C((0,T), L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$. By (4.19) and (4.12), we obtain the assertion of the 6th step. $\qquad\square$

   7th Step. $u \in C^1((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$, and u satisfies $\partial_t u + (-\Delta)^\alpha u = v^p$ as an equality in $C((0,T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$.

Proof. By the assertions of the 4th step and 6th step,

$$\lim_{h \to 0} \frac{1}{h}\{\Phi(v^p)(t+h,x) - \Phi(v^p)(t,x)\}$$

$$(4.20) \quad = \lim_{h \to 0} I(h) + \lim_{h \to 0} J(h)$$

$$= v(t,x)^p - (-\Delta)^\alpha \Phi(v^p)(t,x) \quad \text{in } C((0,T), L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)).$$

Since u satisfies the first integral equation of (4.1), namely,

$$(4.21) \qquad u(t,x) = (W^{(\alpha)}(t,\cdot) * u_0)(x) + \Phi(v^p)(t,x),$$

we have
(4.22)

$$\lim_{h \to 0} \frac{1}{h}\{u(t+h,x) - u(t,x)\}$$

$$= \partial_t W^{(\alpha)}(t,\cdot) * u_0(x) + \lim_{h \to 0} \frac{1}{h}\{\Phi(v^p)(t+h,x) - \Phi(v^p)(t,x)\}$$

$$\text{(by (4.20))} \quad = -(-\Delta)^\alpha W^{(\alpha)}(t,\cdot) * u_0(x) + (v^p)(t,x)^p - (-\Delta)^\alpha \Phi(v^p)(t,x)$$

$$\text{(by (4.21))} \quad = -(-\Delta)^\alpha\{W^{(\alpha)}(t,\cdot) * u_0(x) + \Phi(v^p)(t,x)\} + v(t,x)^p$$

$$= -(-\Delta)^\alpha u(t,x) + v(t,x)^p.$$

$\qquad\square$

   Obviously, the same argument holds for the second equation of the system (4.1). The proof of Theorem 4.2 is now completed.

   From now on, let us consider the semilinear reaction diffusion system (1.4) in a different setting.

   We introduce function spaces $B^m(\mathbb{R}^n)$, $(m = 0, 1, 2, \cdots)$. First, we define a function space $B^0(\mathbb{R}^n)$ by the space of bounded continuous functions on $\mathbb{R}^n$. Next, for a positive integer m, we define a function space $B^m(\mathbb{R}^n)$ by

$$(4.23) \quad B^m(\mathbb{R}^n) = \{\, f \in C^m(\mathbb{R}^n) \,|\, \partial_x^\gamma f \in B^0(\mathbb{R}^n) \text{ for } \forall \gamma \text{ with } |\gamma| \leq m \,\}.$$

Here we remark that if $f \in B^2(\mathbb{R}^n)$, then $(-\Delta)^\alpha f(x)$ is written as

$$(4.24) \qquad (-\Delta)^\alpha f(x) = C_{n,\alpha} \text{ p.v.} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+2\alpha}} \, dy,$$

where the constant $C_{n,\alpha}$ is given by

$$(4.25) \qquad C_{n,\alpha} = \frac{\alpha 2^{2\alpha} \Gamma(\frac{n}{2} + \alpha)}{\pi^{\frac{n}{2}} \Gamma(1 - \alpha)}.$$

(See, for example, Stein [19], Chapter V.) It follows easily from (4.24) that $(-\Delta)^\alpha u, (-\Delta)^\alpha v \in C((0, T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$.

Now we give the second definitions of a mild solution and a strong solution.

Definition 4.3. A pair of functions (u, v) is called a $B^0(\mathbb{R}^n)$-valued mild solution of (1.4) in $[0, T)$ if the following (i), (ii) and (iii) hold.

(i) $u, v \in C((0, T); B^0(\mathbb{R}^n))$.

(ii) (u, v) satisfies the system of integral equations (4.1) for $(t, x) \in (0, T) \times \mathbb{R}^n$.

(iii) $\lim_{t \to 0+} u(t, x) = u_0(x)$, $\lim_{t \to 0+} v(t, x) = v_0(x)$ for $x \in \mathbb{R}^n$.

Definition 4.4. We assume that $u, v \in C((0, T); B^0(\mathbb{R}^n))$. A pair of functions (u, v) is called *a strong solution* of the semilinear reaction diffusion system (1.4) in $[0, T) \times \mathbb{R}^n$ with initial data $(u_0, v_0) \in B^0(\mathbb{R}^n))$ if u and v satisfy the following conditions.

(i) $u, v \in C((0, T); B^2(\mathbb{R}^n))$.

(ii) $u, v \in C^1((0, T); B^0(\mathbb{R}^n))$.

(iii) As an equality in $C((0, T); B^0(\mathbb{R}^n))$, u and v satisfy the system of equations

$$(4.26) \qquad \begin{cases} \partial_t u + (-\Delta)^\alpha u = v^p, \\ \partial_t v + (-\Delta)^\alpha v = u^q. \end{cases}$$

(iv) $\lim_{t \to +0} u(t, x) = u_0(x)$ and $\lim_{t \to +0} v(t, x) = v_0(x)$ for any fixed $x \in \mathbb{R}^n$.

The following theorem is proved in the same manner as in the case of Theorem 4.2. So we omit the proof.

Theorem 4.3. *We assume that $\frac{1}{2} < \alpha < 1$. Let $u, v \in C((0, T); B^0(\mathbb{R}^n))$ satisfy the system of integral equations (4.1) in $[0, T) \times \mathbb{R}^n$. Then the following* (i), (ii), *and* (iii) *hold.*

(i) *u and v are of class C^1 in $t \in (0, T)$ and of class C^2 in $x \in \mathbb{R}^n$. Moreover, $u, v \in C((0, T), B^2(\mathbb{R}^n)) \cap C^1((0, T); B^0(\mathbb{R}^n))$.*

(ii) *$(-\Delta)^\alpha u, (-\Delta)^\alpha v \in C((0, T); B^0(\mathbb{R}^n))$.*

(iii) (u, v) *is a unique strong solution of* (1.4) *in the sense of Definition 4.4.*

Remark 4.1. In the case of (1.1), most references study the global existence and the blowup for the *mild solution* of (1.1), namely, the solution to the system of integral equations arising from (1.1). In fact, in this case, it is not difficult to show that the mild solution satisfies (1.1). However, in the case of the semilinear reaction diffusion system with the fractional Laplacian (1.4), it seems to be nontrivial to show that the mild solution of (4.1) becomes the strong solution of (1.4). Unfortunately, in our proof, we are obliged to assume that $\frac{1}{2} < \alpha < 1$.

5. EXISTENCE OF GLOBAL SOLUTIONS

The purpose of this section is to prove the existence of the global in time solution to the system (1.4).

Throughout this section, we assume that

$$(5.1) \qquad p > 1, \quad q > 1, \quad \frac{n}{2\alpha} > \frac{\max\{p, q\} + 1}{pq - 1}.$$

In addition, we also assume that the initial data (u_0, v_0) satisfy

$$(5.2) \quad 0 \le u_0(x), v_0(x) \le \delta_0 \left(1 + |x|\right)^{-n-2\alpha} \qquad \text{for some constant } \delta_0 > 0.$$

Due to Corollary 3.1, the condition (5.2) is equivalent to

$$(5.3) \qquad 0 \le u_0(x), v_0(x) \le c_0 \, W^{(\alpha)}(1, x) \qquad \text{for some constant } c_0 > 0.$$

For $0 < T \le \infty$, and $\mu > 0$ we define a Banach space $\mathcal{V}_{(T,\mu)}$ by the space of all measurable functions v on $[0, T) \times \mathbb{R}^n$ satisfying

$$(5.4) \qquad \|v\|_{\mathcal{V}_{(T,\mu)}} \overset{\text{def}}{=} \operatorname*{ess.sup}_{(t,x) \in [0,T) \times \mathbb{R}^n} \frac{|v(t, x)|}{(1 + t)^\mu \rho(t, x)} < +\infty,$$

where

$$(5.5) \qquad \rho(t, x) = W^{(\alpha)}(1 + t, x).$$

Moreover, we define a subset $\mathcal{V}^+_{(T,\mu)}$ of $\mathcal{V}_{(T,\mu)}$ by

$$(5.6) \qquad \mathcal{V}^+_{(T,\mu)} = \{v \in \mathcal{V}_{(T,\mu)} \, ; \, v(t, x) \ge 0, \ \text{for } (t, x) \in [0, T) \times \mathbb{R}^n \, \}.$$

For p, q with $pq > 1$, we put

$$(5.7) \qquad \mu := \frac{n}{2\alpha} - \frac{p + 1}{pq - 1}, \quad \nu := \frac{n}{2\alpha} - \frac{q + 1}{pq - 1}.$$

Here we note that μ and ν satisfy the system of linear equations

$$(5.8) \qquad \begin{cases} p\nu + 1 - \frac{n}{2\alpha}(p - 1) = \mu, \\ q\mu + 1 - \frac{n}{2\alpha}(q - 1) = \nu. \end{cases}$$

By the assumption on p and q, the above μ and ν satisfy

$$\text{(5.9)} \qquad\qquad \mu > 0, \qquad \nu > 0.$$

In view of Theorem 4.2, it suffices to show the existence of the global in time solution to the system of integral equations (4.1). For the initial data (u_0, v_0), let us put

$$\text{(5.10)} \quad U_0(t, x) = (W^{(\alpha)}(t, \cdot) * u_0)(x), \quad V_0(t, x) = (W^{(\alpha)}(t, \cdot) * v_0)(x).$$

Let us now rewrite the system of integral equations (4.1) as

$$\text{(5.11)} \qquad\qquad \begin{cases} u(t, x) &= U_0(t, x) + \Phi(v^p)(t, x), \\ v(t, x) &= V_0(t, x) + \Phi(u^q)(t, x), \end{cases}$$

where Φ is the operator defined by (4.3). What we are going to do is to apply a contraction argument to (5.11) in a certain Banach space. So we need to give a suitable estimate for each term of the above system (5.11). The following two lemmas play a crucial rule in our contraction argument.

Lemma 5.1.   (i) *For any T $(0 < T \leq \infty)$, we have*

$$\text{(5.12)} \qquad 0 \leq \Phi(v^p)(t, x) \leq \frac{\{w^{(\alpha)}(0)\}^{p-1}}{\mu} \, ||v||^p_{\mathcal{V}_{(T, \nu)}} \, (1 + t)^\mu \rho(t, x)$$

for $(t, x) \in [0, T) \times \mathbb{R}^n$.
   (ii) *For any T $(0 < T \leq \infty)$, we have*

$$\text{(5.13)} \qquad 0 \leq \Phi(u^q)(t, x) \leq \frac{\{w^{(\alpha)}(0)\}^{q-1}}{\nu} \, ||u||^p_{\mathcal{V}_{(T, \mu)}} \, (1 + t)^\nu \rho(t, x)$$

for $(t, x) \in [0, T) \times \mathbb{R}^n$.

Proof. It suffices to prove the first inequality (5.12) in (i). For simplicity, let us put $A = ||v||_{\mathcal{V}_{(T, \nu)}}$. Then by the definition of the norm $|| \cdot ||_{\mathcal{V}_{(T, \nu)}}$ and the assumption that $v \in \mathcal{V}^+_{(T, \nu)}$,

$$\text{(5.14)} \qquad\qquad 0 \leq v(t, x) \leq A(1 + t)^\nu \rho(t, x).$$

Here we note that

$$\text{(5.15)} \qquad \begin{aligned} \rho(t, x)^p &= W^{(\alpha)}(1 + t, x)^{p-1} W^{(\alpha)}(1 + t, x) \\ &\leq \{w^{(\alpha)}(0)\}^{p-1}(1 + t)^{-\frac{n}{2\alpha}(p-1)} W^{(\alpha)}(1 + t, x). \end{aligned}$$

Let us put $c_1 = \{w^{(\alpha)}(0)\}^{p-1}$. Then we have

(5.16)
$$0 \le \Phi(v^p)(t,x)$$

$$\le A^p \int_0^t \int_{\mathbb{R}^n} W^{(\alpha)}(t-s, x-y)\{(1+s)^\nu \rho(s,y)\}^p \, dx ds$$

$$\le c_1 A^p \int_0^t (1+s)^{-\frac{n}{2\alpha}(p-1)-p\nu} \int_{\mathbb{R}^n} W^{(\alpha)}(t-s, x-y) W^{(\alpha)}(1+s, y) \, dy ds$$

(by semigroup property)

$$\le c_1 A^p \, W^{(\alpha)}(1+t, x) \int_0^t (1+s)^{-\frac{n}{2\alpha}(p-1)-p\nu} \, ds$$

$$\overset{(5.8)}{=} c_1 A^p \, W^{(\alpha)}(1+t, x) \times \frac{1}{\mu}\{(1+t)^\mu - 1\}$$

$$\le \frac{c_1}{\mu} A^p \, (1+t)^\mu \rho(t,x),$$

which proves the assertion of (i). $\square$

Lemma 5.2.   (i) *If $v, \widetilde{v} \in \mathcal{V}^+_{(T,\nu)}$, then*

(5.17)
$$|\, \Phi(v^p)(t,x) - \Phi(\widetilde{v}^p)(t,x)\,|$$
$$\le \frac{p\{w^{(\alpha)}(0)\}^{p-1}}{\mu} \max\{\|v\|^{p-1}_{\mathcal{V}_{(T,\nu)}}, \|\widetilde{v}\|^{p-1}_{\mathcal{V}_{(T,\nu)}}\}$$
$$\times |v(t,x) - \widetilde{v}(t,x)|\,(1+t)^\mu \rho(t,x)$$
$$\text{for } (t,x) \in [0,T) \times \mathbb{R}^n.$$

(ii) *If $u, \widetilde{u} \in \mathcal{V}^+_{(T,\mu)}$, then*

(5.18)
$$|\, \Phi(u^q)(t,x) - \Phi(\widetilde{u}^q)(t,x)\,|$$
$$\le \frac{q\{w^{(\alpha)}(0)\}^{q-1}}{\nu} \max\{\|u\|^{q-1}_{\mathcal{V}_{(T,\mu)}}, \|\widetilde{u}\|^{q-1}_{\mathcal{V}_{(T,\mu)}}\}$$
$$\times |u(t,x) - \widetilde{u}(t,x)|\,(1+t)^\nu \rho(t,x)$$
$$\text{for } (t,x) \in [0,T) \times \mathbb{R}^n.$$

Proof. It suffices to prove (i). For simplicity, we put

(5.19) $A = \|v\|_{\mathcal{V}_{(T,\nu)}}, \quad B = \|\widetilde{v}\|_{\mathcal{V}_{(T,\nu)}}, \quad C = \|v - \widetilde{v}\|_{\mathcal{V}_{(T,\nu)}}.$

Since $p > 1$, we have

$$|v(s,y)^p - \widetilde{v}(s,y)^p|$$

(5.20)
$$\leq p \max\{\, |v(s,y)|^{p-1}, |\widetilde{v}(s,y)|^{p-1} \,\}|v(s,y) - \widetilde{v}(s,y)|$$
$$\leq p \max\{\, A^{p-1}, B^{p-1} \,\}(1+s)^{(p-1)\nu}\rho(s,y)^{p-1} \times C(1+t)^\nu \rho(t,x)$$
$$= p \max\{\, A^{p-1}, B^{p-1} \,\} C \times (1+s)^{p\nu}\rho(s,y)^p.$$

Thus, similarly as in the proof of Lemma 5.1, we have

$$| \Phi(v^p)(t,x) - \Phi(\widetilde{v}^p)(t,x) |$$
$$\leq \Phi(|v^p - \widetilde{v}^p|)(t,x)$$

(5.21)
$$\leq p \max\{\, A^{p-1}, B^{p-1} \,\}$$

$$\times C \int_0^t \int_{\mathbb{R}^n} W^{(\alpha)}(t-s, x-y)\{(1+s)^\nu \rho(s,y)\}^p \, dx ds.$$

The rest part of the proof is almost the same as that of Lemma 5.1. So we omit it. $\square$

The above two lemmas, Lemma 5.1 and Lemma 5.2 yield the following.

Proposition 5.1.   (i) *If* $v \in \mathcal{V}_{(T,\nu)}^+$, *then* $\Phi(v^p) \in \mathcal{V}_{(T,\mu)}^+$. *Moreover, we have*

(5.22)
$$||\Phi(v^p)||_{\mathcal{V}_{(T,\mu)}} \leq \frac{\{w^{(\alpha)}(0)\}^{p-1}}{\mu}||v||_{\mathcal{V}_{(T,\nu)}}^p.$$

(ii) *If* $u \in \mathcal{V}_{(T,\mu)}^+$, *then* $\Phi(u^q) \in \mathcal{V}_{(T,\nu)}^+$. *Moreover, we have*

(5.23)
$$||\Phi(u^q)||_{\mathcal{V}_{(T,\nu)}} \leq \frac{\{w^{(\alpha)}(0)\}^{q-1}}{\nu}||u||_{\mathcal{V}_{(T,\mu)}}^q.$$

(iii) *If* $v, \widetilde{v} \in \mathcal{V}_{(T,\nu)}^+$, *then we have*

(5.24) $|| \Phi(v^p) - \Phi(\widetilde{v}^p) ||_{\mathcal{V}_{(T,\mu)}}$
$$\leq \frac{p\{w^{(\alpha)}(0)\}^{p-1}}{\mu} \max\{\, ||v||_{\mathcal{V}_{(T,\nu)}}^{p-1}, ||\widetilde{v}||_{\mathcal{V}_{(T,\nu)}}^{p-1} \,\}|| v - \widetilde{v} ||_{\mathcal{V}_{(T,\nu)}}.$$

(iv) *If* $u, \widetilde{u} \in \mathcal{V}_{(T,\mu)}^+$, *then we have*

(5.25) $|| \Phi(u^q) - \Phi(\widetilde{u}^q) ||_{\mathcal{V}_{(T,\nu)}}$
$$\leq \frac{q\{w^{(\alpha)}(0)\}^{q-1}}{\mu} \max\{\, ||u||_{\mathcal{V}_{(T,\mu)}}^{q-1}, ||\widetilde{u}||_{\mathcal{V}_{(T,\mu)}}^{q-1} \,\}|| u - \widetilde{u} ||_{\mathcal{V}_{(T,\mu)}}.$$

Next, we will estimate $U_0(t,x)$ and $V_0(t,x)$ in (5.11). Due to the assumption (5.3), we see easily the following.

Lemma 5.3. *Under the assumption* (5.3), *for any* T $(0 < T \leq \infty)$, *we have*

(5.26)
$$0 \leq U_0(t, x) \leq c_0 \rho(t, x) \qquad for \in (0, T) \times \mathbb{R}^n,$$
$$0 \leq V_0(t, x) \leq c_0 \rho(t, x) \qquad for \ (t, x) \in (0, T) \times \mathbb{R}^n.$$

As a direct consequence of Lemma 5.3, we obtain

Proposition 5.2. *For the constant* c_0 *in* (5.3) *and for any* T $(0 < T \leq \infty)$, *we have*

(5.27)
$$||U_0||_{\mathcal{V}_{(T,\mu)}} \leq c_0, \qquad ||V_0||_{\mathcal{V}_{(T,\nu)}} \leq c_0.$$

Let us define a Banach space $\mathcal{V}_\infty$ and its norm $|| \cdot ||_{\mathcal{V}_\infty}$ by

(5.28)
$$\mathcal{V}_\infty := \mathcal{V}_{(\infty,\mu)} \times \mathcal{V}_{(\infty,\nu)},$$

(5.29)
$$||(u, v)||_{\mathcal{V}_\infty} := \max\{\, ||u||_{\mathcal{V}_{(\infty,\mu)}}, ||v||_{\mathcal{V}_{(\infty,\nu)}} \,\}.$$

In addition, we define a closed and convex subset $\mathcal{V}_\infty^+$ of $\mathcal{V}_\infty$ by

$$\mathcal{V}_\infty^+ = \mathcal{V}_{(\infty,\mu)}^+ \times \mathcal{V}_{(\infty,\nu)}^+.$$

Now we introduce a mapping $\Psi : \mathcal{V}_\infty^+ \to \mathcal{V}_\infty^+$ by

(5.30)
$$\Psi(u, v) := (U_0, V_0) + (\Phi(v^p), \Phi(u^q)).$$

Taking account of Proposition 5.1 and Proposition 5.2, we see easily that the above mapping Ψ is well defined as a mapping from $\mathcal{V}_\infty$ to itself.

Proposition 5.3. *Let*
(5.31)
$$C = \max \left\{ \frac{\{w^{(\alpha)}(0)\}^{p-1}}{\mu}, \frac{\{w^{(\alpha)}(0)\}^{q-1}}{\nu}, \frac{p\{w^{(\alpha)}(0)\}^{p-1}}{\mu}, \frac{q\{w^{(\alpha)}(0)\}^{q-1}}{\nu} \right\}.$$

   (i) *If* $||(U_0, V_0)||_{\mathcal{V}_\infty} \leq c_0$ *and if* $(u, v) \in \mathcal{V}_\infty^+$ *satisfies* $||(u, v)||_{\mathcal{V}_\infty} \leq \eta$, *then*

(5.32)
$$|| \Psi(u, v) ||_{\mathcal{V}_\infty} \leq c_0 + C \max\{\eta^p, \eta^q\}.$$

   (ii) *If* $||(u, v)||_{\mathcal{V}_\infty} \leq \eta$ *and* $||(\widetilde{u}, \widetilde{v})||_{\mathcal{V}_\infty} \leq \eta$, *then*

(5.33) $$||\Psi(u, v) - \Psi(\widetilde{u}, \widetilde{v})||_{\mathcal{V}_\infty} \leq C \max\{\eta^{p-1}, \eta^{q-1}\} \, ||(u, v) - (\widetilde{u}, \widetilde{v})||_{\mathcal{V}_\infty}.$$

We are now in a position to prove the global existence of the solution.
Let us choose a small positive number ε_0 such that

(5.34)
$$C \max\{(2\varepsilon_0)^{p-1}, (2\varepsilon_0)^{q-1}\} \leq \frac{1}{2},$$

where the constant C is given by (5.31). Next, we define a closed convex set $\mathcal{B}^+(2\varepsilon_0)$ in $\mathcal{V}_\infty$ by

(5.35) $$\mathcal{B}^+(2\varepsilon_0) = \{\, (u, v) \in \mathcal{V}_{(\infty,\mu)}^+ \times \mathcal{V}_{(\infty,\nu)}^+ \subset \mathcal{V}_\infty \mid ||(u, v)||_{\mathcal{V}_\infty} \leq 2\varepsilon_0 \,\}.$$

Then we have the following:

Proposition 5.4. *We assume that* $||(U_0, V_0)||_{\mathcal{V}_\infty} \leq \varepsilon_0$. *Then* Ψ *satisfies the following* (i) *and* (ii).

    (i) Ψ *maps* $\mathcal{B}^+(2\varepsilon_0)$ *into* $\mathcal{B}^+(2\varepsilon_0)$.
    (ii) *If* (u, v), $(\widetilde{u}, \widetilde{v}) \in \mathcal{B}^+(2\varepsilon_0)$,

$$(5.36) \qquad ||\Psi(u, v) - \Psi(\widetilde{u}, \widetilde{v})||_{\mathcal{V}_\infty} \leq \frac{1}{2} \, ||(u, v) - (\widetilde{u}, \widetilde{v})||_{\mathcal{V}_\infty}.$$

Proof. Take $c_0 = \varepsilon_0$ and $\eta = 2\varepsilon_0$ in Proposition 5.3. Then the above (i) and (ii) follow easily from Proposition 5.3. $\qquad\square$

The above proposition shows that $\Psi : \mathcal{B}^+(2\varepsilon_0) \to \mathcal{B}^+(2\varepsilon_0)$ is a contraction mapping. Therefore, by the fixed point theorem, for any $(U_0, V_0) \in \mathcal{V}_\infty$ with $||(U_0, V_0)||_{\mathcal{V}_\infty} \leq \varepsilon_0$, there exists a unique element $(u, v) \in \mathcal{B}^+(2\varepsilon_0)$ such that $\Psi(u, v) = (u, v)$. Namely, we have the following:

Theorem 5.1. *For a positive constant* ε_0 *satisfying* (5.34), *we take initial data* (u_0, v_0) *such that*

$$(5.37) \qquad\qquad 0 \leq u_0(x), \, v_0(x) \leq \varepsilon_0 \, W^{(\alpha)}(1, x).$$

Then the system of integral equations (5.11) *(or* (4.1)*) with the initial data* (u_0, v_0) *has a unique global in time solution* (u, v) *in* $\mathcal{B}^+(2\varepsilon_0)$.

As is easily seen, the above solution (u, v) satisfies that $u, v \in C((0, \infty); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$. Thus, by Theorem 4.2, we obtain

Theorem 5.2. *We assume that* $\frac{1}{2} < \alpha < 1$ *and that* (5.1) *holds. We also assume that initial data* (u_0, v_0) *satisfies the condition* (5.37). *Then, in the sense of Definition 4.2, there exists a global in time strong solution to the reaction diffusion system* (1.4). *In addition, if initial data* (u_0, v_0) *are continuous, then the above global solution becomes a unique strong solution in the sense of Definition 4.4.*

If u_0 satisfies that $0 \leq u_0(x) \leq c_0 \, W^{(\alpha)}(1, x)$ for some constant c_0, the the solution $u(t, x) = (W^{(\alpha)}(t, \cdot) * u_0)(x)$ to the linear equation $\partial_t u + (-\Delta)^\alpha u = 0$ satisfies

$$(5.38) \qquad\qquad \sup_{x \in \mathbb{R}^n} |u(t, x)| \leq C(1 + t)^{-\frac{n}{2\alpha}},$$

for some other constant C. In other words, the decay rate of u as $t \to \infty$ is $O(t^{-\frac{n}{2\alpha}})$. However, even if initial data (u_0, v_0) satisfies the condition (5.3), the decay rate of the global in time solution (u, v) of (1.4) (or (4.1)) as $t \to \infty$ may become weak due to the nonlinear terms. More precisely, the following holds.

Corollary 5.1. *Let (u, v) be the global in time solution in Theorem 5.1 (or in Theorem 5.2). Then we have*

(5.39)
$$\sup_{x \in \mathbb{R}^n} |u(t, x)| = O\left(t^{-\frac{p+1}{pq-1}}\right), \quad \sup_{x \in \mathbb{R}^n} |v(t, x)| = O\left(t^{-\frac{q+1}{pq-1}}\right) \quad as\ t \to \infty.$$

Proof. Since $(u, v) \in \mathcal{B}^+(2\varepsilon_0) \subset \mathcal{V}_{(\infty,\mu)} \times \mathcal{V}_{(\infty,\nu)}$, u and v satisfy

$$0 \le u(t, x) \le \|u\|_{\mathcal{V}_{(\infty,\mu)}} (1 + t)^\mu \rho(t, x) \quad \text{for } (t, x) \in [0, \infty) \times \mathbb{R}^n,$$
$$0 \le v(t, x) \le \|u\|_{\mathcal{V}_{(\infty,\nu)}} (1 + t)^\nu \rho(t, x) \quad \text{for } (t, x) \in [0, \infty) \times \mathbb{R}^n,$$

respectively. In addition, by (5.7), we have

$$(1 + t)^\mu \rho(t, x) \le \text{Const.} (1 + t)^{\mu - \frac{n}{2\alpha}} = \text{Const.} (1 + t)^{-\frac{p+1}{pq-1}},$$

$$(1 + t)^\nu \rho(t, x) \le \text{Const.} (1 + t)^{\nu - \frac{n}{2\alpha}} = \text{Const.} (1 + t)^{-\frac{q+1}{pq-1}},$$

which finishes the proof. $\square$

Remark 5.1. Let us consider the semilinear parabolic equation.

(5.40)
$$\begin{cases} \partial_t u + (-\Delta)^\alpha u = u^p, & t \in (0, \infty),\ x \in \mathbb{R}^n, \\ u(0, x) = u_0(x) \ge 0, & x \in \mathbb{R}^n, \end{cases}$$

where $0 < \alpha < 1$ and $1 < p$. If $1 + \frac{2\alpha}{n} < p$, there exists a global in time solution for sufficiently small initial data u_0. In this case, the decay rate of the global solution is the same as that of the solution to the corresponding linear equation. Namely, we have

(5.41)
$$\sup_{x \in \mathbb{R}^n} |u(t, x)| = O\left(t^{-\frac{n}{2\alpha}}\right) \quad as\ t \to \infty.$$

For the detail, see Kakehi and Sakai [11].

6. Blow up

In this section, we prove the following blow up result:

Theorem 6.1. *Assume that $1 \le p$, $1 \le q$, $pq > 1$, $\frac{\max\{p,q\}+1}{pq-1} \ge \frac{n}{2\alpha}$. Assume also that*

(6.1)
$$u_0(x) \ge 0, \quad v_0(x) \ge 0,$$

where $u_0(x)$ and $v_0(x)$ are bounded continuous functions on $\mathbb{R}^n$. Then any nontrivial mild solution for the reaction-diffusion system (1.4) blows up in a finite time.

Proof. Assume by contrary that a mild solution (u, v) to (1.4) exists globally in time. Then $u(t, x)$ and $v(t, x)$ are both nonnegative, and satisfy

$$\sup_{0 \leq t \leq T_0} \|u(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} < \infty, \quad \sup_{0 \leq t \leq T_0} \|v(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} < \infty \quad \text{for } 0 \leq T_0 < \infty$$

and the system of integral equations (4.1). Set

$$\begin{cases} F^{(1)}(t) = \displaystyle\int_{\mathbb{R}^n} W^{(\alpha)}(t, x) u(t, x) \, dx, \\ F^{(2)}(t) = \displaystyle\int_{\mathbb{R}^n} W^{(\alpha)}(t, x) v(t, x) \, dx. \end{cases}$$

Note that

$$\int_{\mathbb{R}^n} W^{(\alpha)}(t, x) \, dx = 1 \quad (t > 0).$$

We may assume that there exists a positive constant c_0 such that

$$u_0(x), v_0(x) \geq c_0 W^{(\alpha)}(1, x) \quad \text{for } x \in \mathbb{R}^n.$$

In fact we have the following:

Lemma 6.1. *Assume that $u_0 \not\equiv 0$, $v_0 \not\equiv 0$. For each $t_0 > 0$,*

$$u(t_0, x) \geq c_0 W^{(\alpha)}(t_0, x) \text{ and } v(t_0, x) \geq c_0 W^{(\alpha)}(t_0, x)$$

hold for some constant $c_0 > 0$.

We omit the proof. Next we show the following:

Lemma 6.2. *For any $t > 0$,*

$$F^{(1)}(t) \geq c_0 (2t + 1)^{-\frac{n}{2\alpha}} w^{(\alpha)}(0)$$
$$+ (2t)^{-\frac{n}{2\alpha}} \int_0^t s^{\frac{n}{2\alpha}} \left\{ F^{(2)}(s) \right\}^p \, ds,$$

$$F^{(2)}(t) \geq c_0 (2t + 1)^{-\frac{n}{2\alpha}} w^{(\alpha)}(0)$$
$$+ (2t)^{-\frac{n}{2\alpha}} \int_0^t s^{\frac{n}{2\alpha}} \left\{ F^{(1)}(s) \right\}^q \, ds.$$

Proof. We have

$$F^{(1)}(t) \geq c_0 \int_{\mathbb{R}^n} W^{(\alpha)}(t, x) [W^{(\alpha)}(t, \cdot) * W^{(\alpha)}(1, \cdot)](x) \, dx$$
$$+ \int_0^t \int_{\mathbb{R}^n} W^{(\alpha)}(t, x) [W^{(\alpha)}(t - s, \cdot) * v(s)^p](x) \, dx ds.$$

Then by using the semigroup property

$$\int_{\mathbb{R}^n} W^{(\alpha)}(t, x - y) W^{(\alpha)}(s, y) \, dy = W^{(\alpha)}(t + s, x) \quad (t, s > 0, \, x \in \mathbb{R}^n),$$

we have

The first term

$$= c_0 \int_{\mathbb{R}_x^n} W^{(\alpha)}(t,x) \int_{\mathbb{R}_y^n} W^{(\alpha)}(t,x-y) W^{(\alpha)}(1,y) \, dy \, dx$$

$$= c_0 \int_{\mathbb{R}_x^n} W^{(\alpha)}(t,x) W^{(\alpha)}(t+1,x) \, dx$$

$$= c_0 W^{(\alpha)}(2t+1,0)$$

$$= c_0 (2t+1)^{-\frac{n}{2\alpha}} w^{(\alpha)}(0).$$

On the other hand, we have

The second term

$$= \int_0^t \int_{\mathbb{R}_x^n} \int_{\mathbb{R}_y^n} W^{(\alpha)}(t,x) W^{(\alpha)}(t-s,x-y) \left\{ v(s,y) \right\}^p \, dy dx ds$$

$$= \int_0^t \int_{\mathbb{R}_y^n} \left\{ \int_{\mathbb{R}_x^n} W^{(\alpha)}(t,x) W^{(\alpha)}(t-s,x-y) \, dx \right\} \left\{ v(s,y) \right\}^p \, dy ds$$

$$= \int_0^t \int_{\mathbb{R}_y^n} W^{(\alpha)}(2t-s,y) \left\{ v(s,y) \right\}^p \, dy ds.$$

Note that

$$W^{(\alpha)}(2t-s,y) = (2t-s)^{-\frac{n}{2\alpha}} w^{(\alpha)} \left(\frac{y}{(2t-s)^{1/2\alpha}} \right)$$

$$\geq (2t)^{-\frac{n}{2\alpha}} s^{\frac{n}{2\alpha}} \left(\frac{2t-s}{2t} \right)^{-\frac{n}{2\alpha}} s^{-\frac{n}{2\alpha}} w^{(\alpha)} \left(\frac{y}{s^{1/2\alpha}} \right)$$

$$\geq (2t)^{-\frac{n}{2\alpha}} s^{\frac{n}{2\alpha}} W^{(\alpha)}(s,y)$$

for $0 \leq s \leq t \leq 2t - s \leq 2t$, since $w^{(\alpha)}(\cdot) > 0$ and $w^{(\alpha)}(x) < w^{(\alpha)}(y)$ for $|x| > |y|$. Using this inequality, we get

The second term

$$\geq (2t)^{-\frac{n}{2\alpha}} \int_0^t s^{\frac{n}{2\alpha}} \int_{\mathbb{R}_y^n} W^{(\alpha)}(s,y) \left\{ v(s,y) \right\}^p \, dy ds.$$

Hence by Jensen's inequality ($p \geq 1$), we have

The second term

$$\geq (2t)^{-\frac{n}{2\alpha}} \int_0^t s^{\frac{n}{2\alpha}} \left\{ \int_{\mathbb{R}^n} W^{(\alpha)}(s,y)v(s,y)\,dy \right\}^p ds$$

$$= (2t)^{-\frac{n}{2\alpha}} \int_0^t s^{\frac{n}{2\alpha}} \left\{ F^{(2)}(s) \right\}^p ds.$$

Thus we get the first inequality. We can prove the second inequality in the same way. $\qquad\square$

Set

$$G^{(j)}(t) = t^{\frac{n}{2\alpha}} F^{(j)}(t) \qquad (j = 1, 2).$$

Then we have

$$G^{(1)}(t) = t^{\frac{n}{2\alpha}} F^{(1)}(t)$$

$$\geq c_0 w^{(\alpha)}(0) \left(\frac{t}{2t+1} \right)^{\frac{n}{2\alpha}}$$

$$\qquad + 2^{-\frac{n}{2\alpha}} \int_0^t s^{\frac{n}{2\alpha}} \left\{ F^{(2)}(s) \right\}^p ds$$

$$\geq c_1 + c_2 \int_0^t s^{\frac{n}{2\alpha}(1-p)} \left\{ G^{(2)}(s) \right\}^p ds.$$

Set

$$\begin{cases} \tilde{G}^{(1)}(t) = c_1^{-1} G^{(1)}(t), \\ \tilde{G}^{(2)}(t) = c_1^{-1} G^{(2)}(t). \end{cases}$$

Then we have

$$\tilde{G}^{(1)}(t) \geq 1 + c_2 c_1^{-1} \int_0^t s^{\frac{n}{2\alpha}(1-p)} \left\{ c_1 \tilde{G}^{(2)}(s) \right\}^p ds$$

$$= 1 + c_2 c_1^{p-1} \int_0^t s^{\frac{n}{2\alpha}(1-p)} \left\{ G^{(2)}(s) \right\}^p ds$$

$$\geq 1 + c_3 \int_0^t s^{\frac{n}{2\alpha}(1-p)} \left\{ \tilde{G}^{(2)}(t) \right\}^p ds$$

for $t > 0$. Thus we have the following:

Lemma 6.3. *There exists a positive constant c_4 such that*

$$\begin{cases} \tilde{G}^{(1)}(t) \geq 1 + c_4 \int_1^t s^{\frac{n}{2\alpha}(1-p)} \left\{ \tilde{G}^{(2)}(t) \right\}^p ds, \\ \tilde{G}^{(2)}(t) \geq 1 + c_4 \int_1^t s^{\frac{n}{2\alpha}(1-q)} \left\{ \tilde{G}^{(1)}(t) \right\}^q ds, \end{cases}$$

for $t \geq 1$.

Note that
$$F^{(1)}(t) \leq \sup_{x \in \mathbb{R}^n} u(t,x) < \infty, \quad F^{(2)}(t) \leq \sup_{x \in \mathbb{R}^n} v(t,x) \leq \infty$$

for all $1 \leq t < \infty$, and hence $\tilde{G}^{(j)}(t) < \infty$ for all $1 \leq t < \infty$.

We have the blowup results for the following ODE system:
$$\begin{cases} x'(t) = c_4 t^{\frac{n}{2\alpha}(1-p)} y(t)^p & \text{for } t \geq 1, \\ y'(t) = c_4 t^{\frac{n}{2\alpha}(1-q)} x(t)^q & \text{for } t \geq 1, \\ x(1) = y(1) = 1. \end{cases}$$

Proposition 6.1. *Assume that $a > 0$, $b + qa > 0$, $p, q > 0$, $pq > 1$. Then the solution to the following ODE system*

(6.2)
$$\begin{cases} x'(t) = t^{a-1} y(t)^p & \text{for } t \geq 1, \\ y'(t) = t^{b-1} x(t)^q & \text{for } t \geq 1, \\ x(1) = y(1) = 1, \end{cases}$$

blows up in a finite time.

Proof. Assume by contrary that the solution exists globally in time. We see that $x(t) \geq 1$ and $y(t) \geq 1$ $(t \geq 1)$. Then we have
$$x'(t) \geq t^{a-1} \quad (t \geq 1),$$

and hence
$$x(t) \geq \int_1^t s^{a-1} \, ds = t^a - 1 \geq \frac{1}{2} t^a \quad (t \geq 2^{1/a}).$$

Hence there exists a constant $c_1 > 0$ such that
$$x(t) \geq c_1 t^a \quad (t \geq 1).$$

Therefore by (6.2),
$$y'(t) \geq c_1^q t^{b+aq-1} \quad (t \geq 1).$$

Integrating this inequality gives
$$y(t) \geq c_1^q \int_1^t s^{b+aq-1} \, ds = c_1^q (t^{b+aq} - 1) \geq \frac{c_1^q}{2} t^{b+aq} \quad (t \geq 2^{1/(b+aq)}).$$

Hence there exists a constant $d_1 > 0$ such that
$$y(t) \geq d_1 t^{b+aq} \quad (t \geq 1).$$

Repeating this procedure, we see that

$$x(t) \geq c_n t^{a_n} \qquad (t \geq 1),$$
$$y(t) \geq d_n t^{b_n} \qquad (t \geq 1),$$

for some $c_n > 0$, $d_n > 0$,

$$a_n = \frac{-(pb + a) + (pq)^{n-1} p(qa + b)}{pq - 1},$$

$$b_n = \frac{(qa + b)[(pq)^n - 1]}{pq - 1}.$$

Note that a_n and b_n satisfy

$$b_n = qa_n + b,$$

$$a_{n+1} = pb_n + a = pqa_n + pb + a,$$

and $a_n \to \infty$, $b_n \to \infty$ as $n \to \infty$ since $a > 0$, $b + qa > 0$, $pq > 1$. Take $\varepsilon > 0$ such that $p - \varepsilon > 0$, $q - \varepsilon > 0$, $(p - \varepsilon)(q - \varepsilon) > 1$. Choose sufficiently large $N \in \mathbb{N}$ such that

$$a - 1 + \varepsilon b_N > 0,$$
$$b - 1 + \varepsilon a_N > 0.$$

Then it follows that

$$\begin{cases} x'(t) \geq y(t)^{p - \varepsilon} & \text{for } t \geq \tau, \\ y'(t) \geq x(t)^{q - \varepsilon} & \text{for } t \geq \tau \end{cases}$$

for some $\tau > 0$. It follows from a comparison argument that $x(t) \geq \overline{x}(t)$, $y(t) \geq \overline{y}(t)$, where $\overline{x}(t), \overline{y}(t)$ are the solution to the problem

$$\begin{cases} \overline{x}'(t) = \overline{y}(t)^{p - \varepsilon} & \text{for } t \geq \tau, \\ \overline{y}'(t) = \overline{x}(t)^{q - \varepsilon} & \text{for } t \geq \tau, \\ x(t_N) > \overline{x}(t_N) > 0, \\ y(t_N) > \overline{y}(t_N) > 0. \end{cases}$$

However this contradicts to the fact that the solution $(\overline{x}(t), \overline{y}(t))$ blows up in a finite time. $\square$

Proposition 6.2. *Assume that $a > 0$, $p, q > 0$, $pq > 1$. Then the solution to the following ODE system*

(6.3)
$$\begin{cases} x'(t) = t^{a-1} y(t)^p & \text{for } t \geq 1, \\ y'(t) = t^{-qa-1} x(t)^q & \text{for } t \geq 1, \\ x(1) = x_0 > 0, \\ y(1) = y_0 > 0, \end{cases}$$

blows up in a finite time.

Proof. Without loss of generality, we may assume that

(6.4)
$$\frac{y_0^{p+1}}{p+1} > \frac{x_0^{q+1}}{q+1}$$

holds. In fact, if (6.4) does not hold, we can replace x_0 by a smaller positive number so that (6.4) holds, and then we can apply the comparison argument.

Assume by contrary that the solution exists globally in time.

In the same way as in the proof of Proposition 6.1, we see that there exist positive constants c_1, d_1 such that $x(t) \geq c_1 t^a$, $y(t) \geq d_1 \log t$ for all $t \geq 1$.

Set $c = \left(\frac{p+1}{q+1}\right)^{\frac{p}{p+1}}$ and $c_2 = 2a \left(\frac{2a}{c}\right)^{\frac{p+1}{pq-1}}$, and let $t_1 > 1$ be a constant satisfying $d_1 \log t_1 = (c_2)^{1/p}$. Then we have

$$x'(t) \geq c_2 t^{a-1} \qquad (t \geq t_1).$$

Hence

$$x(t) \geq c_2 \int_{t_1}^{t} s^{a-1}\, ds = \frac{c_2}{a}(t^a - t_1^a) \geq \frac{c_2}{2a} t^a \qquad (t \geq 2^{1/a} t_1).$$

Set $z(t) = \frac{x(t)}{t^a}$. Then $z(t) \geq c_1 > 0$ for $t \geq 1$,

(6.5)
$$z(t) \geq \frac{c_2}{2a} = \left(\frac{2a}{c}\right)^{\frac{p+1}{pq-1}} \qquad (t \geq 2^{1/a} t_1 =: t_2),$$

and

(6.6)
$$\begin{cases} z'(t) = \dfrac{y(t)^p - a z(t)}{t} & \text{for } t \geq 1, \\[2mm] y'(t) = \dfrac{z(t)^q}{t} & \text{for } t \geq 1, \\[2mm] z(1) = x_0, \\[1mm] y(1) = y_0. \end{cases}$$

Then we have

$$\left(\frac{y(t)^{p+1}}{p+1} - \frac{z(t)^{q+1}}{q+1}\right)' = a\frac{z(t)^{q+1}}{t} > 0 \qquad (t \geq 1).$$

Therefore

$$\frac{y(t)^{p+1}}{p+1} - \frac{z(t)^{q+1}}{q+1} \geq \frac{y_0^{p+1}}{p+1} - \frac{x_0^{q+1}}{q+1} > 0 \qquad (t \geq 1),$$

and hence

$$z'(t) \geq \frac{cz(t)^{\frac{p+pq}{p+1}} - a z(t)}{t} \qquad (t \geq 1).$$

Since there holds

$$z(t)^{\frac{p+pq}{p+1}} = z(t)^{\frac{pq-1}{p+1}} z(t) \geq \frac{2a}{c} z(t) \qquad (t \geq t_2)$$

by (6.5), we have

$$z'(t) \geq \frac{c}{2t} z(t)^{\frac{p+pq}{p+1}} \qquad (t \geq t_2).$$

But this implies that

$$z(t) \geq \left[z(t_2)^{-\frac{pq-1}{p+1}} - \frac{c(pq-1)}{2(p+1)} \log \frac{t}{t_2} \right]^{-\frac{p+1}{pq-1}} \qquad (t \geq t_2),$$

which is a contradiction.   □

Proposition 6.3. *Assume that $p, q > 0$ and $pq > 1$. Then the solution $(x(t), y(t))$ to the following ODE system*

$$\begin{cases} x'(t) = t^{-1} y(t)^p & \text{for } t \geq 1, \\ y'(t) = t^{-1} x(t)^q & \text{for } t \geq 1, \\ x(1) = x_0 > 0, \\ y(1) = y_0 > 0, \end{cases}$$

blows up in a finite time.

Proof. Assume by contrary that the solution exists globally in time. It suffices to consider the case $\frac{x_0^{q+1}}{q+1} - \frac{y_0^{p+1}}{p+1} \geq 0$. Then since

$$\left(\frac{x(t)^{q+1}}{q+1} - \frac{y(t)^{p+1}}{p+1} \right)' = x^q x' - y^p y' = 0,$$

we have $\frac{x(t)^{q+1}}{q+1} - \frac{y(t)^{p+1}}{p+1} \geq 0$ for $t \geq 1$. Thus we see that $y'(t) \geq t^{-1} y(t)^{\frac{q+pq}{q+1}}$ for $t \geq 1$, which gives a contradiction.   □

Completion of the proof of Theorem. In what follows, we assume that $1 \leq p \leq q$ for definiteness. Then by the assumption,

$$\frac{1}{p-1} \geq \frac{q+1}{pq-1} \geq \frac{n}{2\alpha}.$$

Thus

$$p \leq 1 + \frac{2\alpha}{n}.$$

Set

$$a = 1 - \frac{n(p-1)}{2\alpha}$$

and

$$b = 1 - \frac{n(q-1)}{2\alpha}.$$

Then
$$a \geq 0, \quad b + qa = q + 1 - \frac{n}{2\alpha}(pq - 1) \geq 0, \quad b \leq a.$$

Let $(x(t), y(t))$ be the solution to the following ODE system.
$$\begin{cases} x'(t) = c_4 t^{a-1} y(t)^p & \text{for } t \geq 1, \\ y'(t) = c_4 t^{b-1} x(t)^q & \text{for } t \geq 1, \\ x(1) = y(1) = 1. \end{cases}$$

Then it follows from a comparison argument that

(6.7) $$\tilde{G}_\lambda^{(1)}(t) \geq x(t), \quad \tilde{G}_\lambda^{(2)}(t) \geq y(t) \quad \text{for } 1 \leq t < \infty.$$

This means that $x(t)$ and $y(t)$ must exist globally. However we see that $(x(t), y(t))$ blows up in a finite time by Proposition 6.1 in the case $a > 0$, $-qa < b \leq a$, Proposition 6.2 in the case $a > 0$, $b = -qa$, and Proposition 6.3 in the case $a = 0$, $b = 0$. We obtain a desired contradiction. $\square$

7. Life span

In this section, we consider the life span of the solution and prove the following:

Theorem 7.1. *Assume* $1 \leq p < 1 + \frac{2\alpha}{n}$, $1 \leq q < 1 + \frac{2\alpha}{n}$, $pq > 1$. *Assume also that*
$$u_0(x) = \lambda^{\mu^*} \varphi(x) \geq 0, \quad v_0(x) = \lambda^{\nu^*} \psi(x) \geq 0$$
where φ *and* ψ *are continuous functions on* $\mathbb{R}^n$ *such that*

(7.1) $$0 \leq \varphi(x), \psi(x) \leq C(1 + |x|)^{-n-2\alpha}$$

for some constant $C > 0$, $\mu^* = \frac{p+1}{pq-1} - \frac{n}{2\alpha}$, $\nu^* = \frac{q+1}{pq-1} - \frac{n}{2\alpha}$, *and* $\lambda > 0$ *is a small parameter. Then the life span* T_λ^* *of the nontrivial mild solution to the reaction-diffusion system* (1.4) *satisfies*
$$T_\lambda^* \sim \frac{1}{\lambda} \quad \text{as } \lambda \to 0$$

Proof. Under the assumption, there hold
$$\mu^* = \frac{p+1}{pq-1} - \frac{n}{2\alpha} > 0,$$
$$\nu^* = \frac{q+1}{pq-1} - \frac{n}{2\alpha} > 0.$$

Let the nonnegative functions $u_\lambda(t, x)$, $v_\lambda(t, x)$, $t \in [0, T_\lambda^*)$, $x \in \mathbb{R}^n$ be the mild solution to (1.4). Here $T_\lambda^* \in (0, \infty)$ is the life span of the solution. Then the solution (u_λ, v_λ) satisfies
$$\sup_{0 \leq t \leq T} \|u_\lambda(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} < \infty, \quad \sup_{0 \leq t \leq T} \|v_\lambda(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} < \infty \quad \text{for } 0 \leq T < T_\lambda^*,$$

and the following system of integral equations.

$$
\begin{cases}
u_\lambda(t,x) = \lambda^{\mu^*}(W^{(\alpha)}(t,\cdot) * \varphi)(x) + \int_0^t [W^{(\alpha)}(t-s,\cdot) * \{v_\lambda(s,\cdot)\}^p](x)\,ds, \\
v_\lambda(t,x) = \lambda^{\nu^*}(W^{(\alpha)}(t,\cdot) * \psi)(x) + \int_0^t [W^{(\alpha)}(t-s,\cdot) * \{u_\lambda(s,\cdot)\}^q](x)\,ds.
\end{cases}
$$

(1) In order to obtain an upper bound for the life span, we reduce the problem into an ODE system in the same way as in Section 6.

Set

$$
\begin{cases}
F_\lambda^{(1)}(t) = \int_{\mathbb{R}^n} W^{(\alpha)}(t,x) u_\lambda(t,x)\,dx, \\
F_\lambda^{(2)}(t) = \int_{\mathbb{R}^n} W^{(\alpha)}(t,x) v_\lambda(t,x)\,dx.
\end{cases}
$$

Note that

$$
\int_{\mathbb{R}^n} W^{(\alpha)}(t,x)\,dx = 1 \quad (t > 0).
$$

We may assume that there exists a positive constant c_0 such that

$$
\varphi(x), \psi(x) \geq c_0 W^{(\alpha)}(1,x) \quad \text{for } x \in \mathbb{R}^n.
$$

In fact we have the following.

Lemma 7.1. *Assume that $\varphi \not\equiv 0$, $\psi \not\equiv 0$. For each $0 < t_0 < T_\lambda^*$,*

$$
u_\lambda(t_0,x) \geq c_0 \lambda^{\mu^*} W^{(\alpha)}(t_0,x) \quad \text{and} \quad v_\lambda(t_0,x) \geq c_0 \lambda^{\nu^*} W^{(\alpha)}(t_0,x)
$$

hold for some constant $c_0 > 0$.

Lemma 7.2. *For $0 < t < T_\lambda^*$,*

$$
F_\lambda^{(1)}(t) \geq c_0 \lambda^{\mu^*} (2t+1)^{-\frac{n}{2\alpha}} w^{(\alpha)}(0)
$$
$$
+ (2t)^{-\frac{n}{2\alpha}} \int_0^t s^{\frac{n}{2\alpha}} \left\{ F_\lambda^{(2)}(s) \right\}^p ds,
$$

$$
F_\lambda^{(2)}(t) \geq c_0 \lambda^{\nu^*} (2t+1)^{-\frac{n}{2\alpha}} w^{(\alpha)}(0)
$$
$$
+ (2t)^{-\frac{n}{2\alpha}} \int_0^t s^{\frac{n}{2\alpha}} \left\{ F_\lambda^{(1)}(s) \right\}^q ds.
$$

Proof. We have

$$
F_\lambda^{(1)}(t) \geq c_0 \lambda^{\mu^*} \int_{\mathbb{R}^n} W^{(\alpha)}(t,x)[W^{(\alpha)}(t,\cdot) * W^{(\alpha)}(1,\cdot)](x)\,dx
$$
$$
+ \int_0^t \int_{\mathbb{R}^n} W^{(\alpha)}(t,x)[W^{(\alpha)}(t-s,\cdot) * v_\lambda(s)^p](x)\,dxds.
$$

Then by using the semigroup property

$$\int_{\mathbb{R}^n} W^{(\alpha)}(t, x - y)W^{(\alpha)}(s, y)\, dy = W^{(\alpha)}(t + s, x) \quad (t, s > 0,\ x \in \mathbb{R}^n),$$

we have

The first term

$$= c_0\lambda^{\mu^*} \int_{\mathbb{R}^n_x} W^{(\alpha)}(t, x) \int_{\mathbb{R}^n_y} W^{(\alpha)}(t, x - y)W^{(\alpha)}(1, y)\, dy\, dx$$

$$= c_0\lambda^{\mu^*} \int_{\mathbb{R}^n_x} W^{(\alpha)}(t, x)W^{(\alpha)}(t + 1, x)\, dx$$

$$= c_0\lambda^{\mu^*} W^{(\alpha)}(2t + 1, 0)$$

$$= c_0\lambda^{\mu^*}(2t + 1)^{-\frac{n}{2\alpha}} w^{(\alpha)}(0).$$

On the other hand, we have

The second term

$$= \int_0^t \int_{\mathbb{R}^n_x} \int_{\mathbb{R}^n_y} W^{(\alpha)}(t, x)W^{(\alpha)}(t - s, x - y)\left\{v_\lambda(s, y)\right\}^p\, dydxds$$

$$= \int_0^t \int_{\mathbb{R}^n_y} \left\{\int_{\mathbb{R}^n_x} W^{(\alpha)}(t, x)W^{(\alpha)}(t - s, x - y)\, dx\right\} \left\{v_\lambda(s, y)\right\}^p\, dyds$$

$$= \int_0^t \int_{\mathbb{R}^n_y} W^{(\alpha)}(2t - s, y)\left\{v_\lambda(s, y)\right\}^p\, dyds.$$

Note that

$$W^{(\alpha)}(2t - s, y) = (2t - s)^{-\frac{n}{2\alpha}} w^{(\alpha)}\left(\frac{y}{(2t - s)^{1/2\alpha}}\right)$$

$$\geq (2t)^{-\frac{n}{2\alpha}} s^{\frac{n}{2\alpha}} \left(\frac{2t - s}{2t}\right)^{-\frac{n}{2\alpha}} s^{-\frac{n}{2\alpha}} w^{(\alpha)}\left(\frac{y}{s^{1/2\alpha}}\right)$$

$$\geq (2t)^{-\frac{n}{2\alpha}} s^{\frac{n}{2\alpha}} W^{(\alpha)}(s, y)$$

for $0 \leq s \leq t \leq 2t - s \leq 2t$, since $w^{(\alpha)}(\cdot) > 0$ and $w^{(\alpha)}(x) < w^{(\alpha)}(y)$ for $|x| > |y|$.

Using this inequality, we get

The second term

$$\geq (2t)^{-\frac{n}{2\alpha}} \int_0^t s^{\frac{n}{2\alpha}} \int_{\mathbb{R}^n_y} W^{(\alpha)}(s, y)\left\{v_\lambda(s, y)\right\}^p\, dyds.$$

Hence by Jensen's inequality ($p \geq 1$), we have

The second term

$$\geq (2t)^{-\frac{n}{2\alpha}} \int_0^t s^{\frac{n}{2\alpha}} \left\{ \int_{\mathbb{R}^n} W^{(\alpha)}(s,y) v_\lambda(s,y)\, dy \right\}^p ds$$

$$= (2t)^{-\frac{n}{2\alpha}} \int_0^t s^{\frac{n}{2\alpha}} \left\{ F_\lambda^{(2)}(s) \right\}^p ds.$$

Thus we get the first inequality. We can prove the second inequality in the same way. □

Setting

$$G_\lambda^{(j)}(t) = t^{\frac{n}{2\alpha}} F_\lambda^{(j)}(t) \qquad (j = 1, 2),$$

we have

$$G_\lambda^{(1)}(t) = t^{\frac{n}{2\alpha}} F_\lambda^{(1)}(t)$$

$$\geq c_0 \lambda^{\mu^*} w^{(\alpha)}(0) \left(\frac{t}{2t+1} \right)^{\frac{n}{2\alpha}}$$

$$+ 2^{-\frac{n}{2\alpha}} \int_0^t s^{\frac{n}{2\alpha}} \left\{ F_\lambda^{(2)}(s) \right\}^p ds$$

$$\geq c_1 \lambda^{\mu^*} + c_2 \int_0^t s^{\frac{n}{2\alpha}(1-p)} \left\{ G_\lambda^{(2)}(s) \right\}^p ds.$$

Define

$$\begin{cases} \tilde{G}_\lambda^{(1)}(t) = c_1^{-1} \lambda^{-\mu^*} G_\lambda^{(1)}(t), \\ \tilde{G}_\lambda^{(2)}(t) = c_1^{-1} \lambda^{-\nu^*} G_\lambda^{(2)}(t). \end{cases}$$

Then

$$\tilde{G}_\lambda^{(1)}(t) \geq 1 + c_2 c_1^{-1} \int_0^t s^{\frac{n}{2\alpha}(1-p)} \lambda^{-\mu^*} \left\{ c_1 \lambda^{\nu^*} \tilde{G}_\lambda^{(2)}(s) \right\}^p ds$$

$$= 1 + c_2 c_1^{p-1} \int_0^t s^{\frac{n}{2\alpha}(1-p)} \lambda^{p\nu^* - \mu^*} \left\{ G_\lambda^{(2)}(s) \right\}^p ds$$

$$\geq 1 + c_3 \lambda^{1 + \frac{n}{2\alpha}(1-p)} \int_0^t s^{\frac{n}{2\alpha}(1-p)} \left\{ \tilde{G}_\lambda^{(2)}(t) \right\}^p ds$$

for $0 < t < T_\lambda$. Assume that $T_\lambda^* > 1$ in what follows. Then we have the following:

Lemma 7.3. *There exists a positive constant c_4 such that*

$$\begin{cases} \tilde{G}_\lambda^{(1)}(t) \geq 1 + c_4 \lambda^{1+\frac{n}{2\alpha}(1-p)} \int_1^t s^{\frac{n}{2\alpha}(1-p)} \left\{\tilde{G}_\lambda^{(2)}(t)\right\}^p ds, \\[2mm] \tilde{G}_\lambda^{(2)}(t) \geq 1 + c_4 \lambda^{1+\frac{n}{2\alpha}(1-q)} \int_1^t s^{\frac{n}{2\alpha}(1-q)} \left\{\tilde{G}_\lambda^{(1)}(t)\right\}^q ds, \end{cases}$$

for $1 \leq t < T_\lambda^*$.

Thus we see that T_λ^* is smaller than or equal to the blowup time for the following ODE systems.

$$\begin{cases} x'(t) = c_4 \lambda^{1+\frac{n}{2\alpha}(1-p)} t^{\frac{n}{2\alpha}(1-p)} y(t)^p & \text{for } t \geq 1, \\ y'(t) = c_4 \lambda^{1+\frac{n}{2\alpha}(1-q)} t^{\frac{n}{2\alpha}(1-q)} x(t)^q & \text{for } t \geq 1, \\ x(1) = y(1) = 1. \end{cases}$$

(2) Next in order to obtain a lower bound for the life span, we shall construct a super solution as follows.

First let $H^j(t,x)$ $(j = 1, 2)$ be the solution to

$$(\partial_t + (-\Delta)^\alpha) H^j(t,x) = 0 \quad (j = 1, 2)$$

with initial conditions

$$\begin{cases} H^1(0,x) = u_0(x) = \lambda^{\mu^*} \varphi(x), \\ H^2(0,x) = v_0(x) = \lambda^{\nu^*} \psi(x). \end{cases}$$

Note that $r > 0$ and $s > 0$ by the assumption. We may assume that there exists a positive constant c_0 such that

$$\varphi(x), \psi(x) \geq c_0 W^{(\alpha)}(1,x) \quad \text{for } x \in \mathbb{R}^n.$$

Lemma 7.4. *There exist positive constants c_1, c_2, C_1, C_2 such that*

$$\begin{cases} c_1 \lambda^{\mu^*} W^{(\alpha)}(1+t,x) \leq H^1(t,x) \leq C_1 \lambda^{\mu^*} W^{(\alpha)}(1+t,x), \\ c_2 \lambda^{\nu^*} W^{(\alpha)}(1+t,x) \leq H^2(t,x) \leq C_2 \lambda^{\nu^*} W^{(\alpha)}(1+t,x), \end{cases}$$

for $x \in \mathbb{R}^n$, $t \geq 0$.

Then we have

$$\begin{aligned} \frac{H^2(t,x)^p}{H^1(t,x)} &\leq C_3 \lambda^{p\nu^* - \mu^*} \frac{W^{(\alpha)}(1+t,x)^p}{W^{(\alpha)}(1+t,x)} \\ &= C_3 \lambda^{1+\frac{n}{2\alpha}(1-p)} W^{(\alpha)}(1+t,x)^{p-1} \\ &\leq C_4 \lambda^{1+\frac{n}{2\alpha}(1-p)}(1+t)^{\frac{n}{2\alpha}(1-p)}. \end{aligned}$$

Define

$$\begin{cases} f(t) = C_4 \lambda^{1 + \frac{n}{2\alpha}(1-p)}(1+t)^{\frac{n}{2\alpha}(1-p)}, \\ g(t) = C_4 \lambda^{1 + \frac{n}{2\alpha}(1-q)}(1+t)^{\frac{n}{2\alpha}(1-q)}. \end{cases}$$

Then we have

$$\begin{cases} f(t) \geq \sup_{x \in \mathbb{R}^n} \dfrac{H^2(t,x)^p}{H^1(t,x)}, \\ g(t) \geq \sup_{x \in \mathbb{R}^n} \dfrac{H^1(t,x)^q}{H^2(t,x)}. \end{cases}$$

Let $(\xi(t), \eta(t))$ be the solution to the following ODE system

(7.2)
$$\begin{cases} \xi'(t) = f(t)\eta(t)^p & \text{for } t \geq 0, \\ \eta'(t) = g(t)\xi(t)^q & \text{for } t \geq 0, \\ \xi(0) = \eta(0) = 1. \end{cases}$$

Then

$$\begin{cases} \xi'(t) \geq \dfrac{H^2(t,x)^p}{H^1(t,x)}\eta(t)^p & \text{for } t \geq 0, x \in \mathbb{R}^n, \\ \eta'(t) \geq \dfrac{H^1(t,x)^q}{H^2(t,x)}\xi(t)^q & \text{for } t \geq 0, x \in \mathbb{R}^n. \end{cases}$$

Set

$$\begin{cases} U(t,x) = \xi(t)H^1(t,x), \\ V(t,x) = \eta(t)H^2(t,x). \end{cases}$$

Then (U,V) is a super solution, namely,

$$\begin{cases} U_t + (-\Delta)^\alpha U \geq V^p & \text{for } t \geq 0, x \in \mathbb{R}^n, \\ V_t + (-\Delta)^\alpha V \geq U^q & \text{for } t \geq 0, x \in \mathbb{R}^n. \end{cases}$$

It follows from the comparison theorem that

$$\begin{cases} u_\lambda(t,x) \leq U(t,x) & \text{for } t \geq 0, x \in \mathbb{R}^n, \\ v_\lambda(t,x) \leq V(t,x) & \text{for } t \geq 0, x \in \mathbb{R}^n. \end{cases}$$

Thus we get an lower bound for the life span. That is, T_λ^* is greater than or equal to the blowup time for (7.2).

(3) By (1) and (2), in order to obtain an upper and lower bound for the life span, we need only consider the ODE system of the same type.

(7.3)
$$\begin{cases} u'(t) = \lambda^a t^{a-1} v(t)^p & \text{for } t \geq 1, \\ v'(t) = \lambda^b t^{b-1} u(t)^q & \text{for } t \geq 1, \\ u(1) = v(1) = 1, \end{cases}$$

where

$$a = 1 - \frac{n(p-1)}{2\alpha},$$

$$b = 1 - \frac{n(q-1)}{2\alpha}.$$

Under the assumption, we have $a > 0$ and $b > 0$. If $p \leq q$, then $a \geq b$. It suffices to consider only the case $a \geq b$.

Hence we obtain the upper and lower bound of the life span by the following Proposition 7.1. $\square$

Let $T = T(\lambda)$ be the blowup time for (7.3).

Next let $T_1 = T_1(\lambda)$ be the blowup time for the following ODE stystems

(7.4)
$$\begin{cases} x'(t) = \lambda^a t^{a-1} y(t)^p & \text{for } t \geq 1, \\ y'(t) = \lambda^b t^{b-1} x(t)^q & \text{for } t \geq 1, \\ x(1) = (q+1)^{1/(q+1)} > 1, \\ y(1) = (p+1)^{1/(p+1)} > 1, \end{cases}$$

and $T_2 = T_2(\lambda)$ be the blowup time for the following ODE stystems

(7.5)
$$\begin{cases} x'(t) = \lambda^a t^{a-1} y(t)^p & \text{for } t \geq \frac{1}{\lambda}, \\ y'(t) = \lambda^b t^{b-1} x(t)^q & \text{for } t \geq \frac{1}{\lambda}, \\ x(\frac{1}{\lambda}) = (p+1)^{-1/(q+1)} < 1, \\ y(\frac{1}{\lambda}) = (q+1)^{-1/(p+1)} < 1. \end{cases}$$

Note that

$$1 < T(\lambda) < \infty, \quad 1 < T_1(\lambda) < \infty, \quad \frac{1}{\lambda} < T_2(\lambda) < \infty.$$

Proposition 7.1. *Assume that*

$$a > 0, \quad b > 0,$$

$$pq > 1, \quad p > 0, \quad q > 0.$$

There exist positive constants c and C, depending only on p, q, a and b, such that

$$\frac{c}{\lambda} \leq T(\lambda) \leq \frac{C}{\lambda}$$

for all $\lambda \in (0, 1)$.

Proof. It follows from a comparison argument that

$$T_1(\lambda) \leq T(\lambda).$$

If $T(\lambda) \leq \frac{1}{\lambda}$, then $T(\lambda) \leq \frac{1}{\lambda} < T_2(\lambda)$.

Assume that $T(\lambda) > \frac{1}{\lambda}$. Then we see that $u(\frac{1}{\lambda}) > 1$ and $v(\frac{1}{\lambda}) > 1$ since $u(t)$ and $v(t)$ are monotonically increasing. Thus it follows from the comparison argument that

$$T(\lambda) \leq T_2(\lambda).$$

Hence we get the desired estimate from the following Lemma 7.5 and 7.6. $\qquad\square$

Lemma 7.5. *Assume that $a \geq b$. Then*

$$T_1(\lambda) \geq \min \left\{ \frac{1}{\lambda}, \left[\frac{b(p+1)^{-\frac{pq-1}{(p+1)(q+1)}} + \frac{pq-1}{q+1}\left(\frac{q+1}{p+1}\right)^{\frac{q}{q+1}}\lambda^b}{\frac{pq-1}{q+1}\left(\frac{q+1}{p+1}\right)^{\frac{q}{q+1}}\lambda^b} \right]^{1/b} \right\}$$

for all $\lambda \in (0,1)$.

Proof. It suffices to show that either $T_1(\lambda) \geq \frac{1}{\lambda}$ or

$$T_1(\lambda) \geq \left[\frac{b(p+1)^{-\frac{pq-1}{(p+1)(q+1)}} + \frac{pq-1}{q+1}\left(\frac{q+1}{p+1}\right)^{\frac{q}{q+1}}\lambda^b}{\frac{pq-1}{q+1}\left(\frac{q+1}{p+1}\right)^{\frac{q}{q+1}}\lambda^b} \right]^{1/b}$$

holds.

Assume that $T_1(\lambda) < \frac{1}{\lambda}$. Let $(x(t), y(t))$, $1 \leq t < T_1(\lambda)$ be the solution to (7.4). Since

$$\begin{aligned}
\{(p+1)&x(t)^{q+1} - (q+1)y(t)^{p+1}\}' \\
&= (p+1)(q+1)(x^q x' - y^p y') \\
&= (p+1)(q+1)(\lambda^{a-b}t^{a-b} - 1)\lambda^b t^{-1+b}x^q y^p \\
&\leq 0 \quad (1 \leq t < T_1(\lambda) \leq \tfrac{1}{\lambda}),
\end{aligned}$$

we have

$$\begin{aligned}
(p+1)&x(t)^{q+1} - (q+1)y(t)^{p+1} \\
&\leq (p+1)x(1)^{q+1} - (q+1)y(1)^{p+1} = 0
\end{aligned}$$

for $1 \leq t < T_1(\lambda)$. Hence

$$(p+1)x(t)^{q+1} \leq (q+1)y(t)^{p+1}$$

for $1 \leq t < T_1(\lambda)$. Then

$$\begin{aligned}
y'(t) &= \lambda^b t^{b-1} x(t)^q \\
&\leq \lambda^b \left(\frac{q+1}{p+1}\right)^{\frac{q}{q+1}} t^{b-1} y^{\frac{q+pq}{q+1}}
\end{aligned}$$

for $1 \le t < T_1(\lambda)$. Note that $\frac{q+pq}{q+1} > 1$ and $b > 0$. Hence

$$(7.6) \qquad y(t) \le \left[y(1)^{-\frac{pq-1}{q+1}} - \frac{pq-1}{q+1} \left(\frac{q+1}{p+1} \right)^{\frac{q}{q+1}} \frac{1}{b} \lambda^b (t^b - 1) \right]^{-\frac{q+1}{pq-1}}$$

for $1 \le t < T_1(\lambda)$. Therefore

$$T_1(\lambda) \ge \left[\frac{b(p+1)^{-\frac{pq-1}{(p+1)(q+1)}} + \frac{pq-1}{q+1} \left(\frac{q+1}{p+1} \right)^{\frac{q}{q+1}} \lambda^b}{\frac{pq-1}{q+1} \left(\frac{q+1}{p+1} \right)^{\frac{q}{q+1}} \lambda^b} \right]^{1/b}.$$

The proof is now completed. $\qquad\square$

Lemma 7.6. *Assume that $a \ge b$.*

$$T_2(\lambda) \le \left[\frac{b(q+1)^{\frac{pq-1}{(p+1)(q+1)}} + \frac{pq-1}{q+1} \left(\frac{q+1}{p+1} \right)^{\frac{q}{q+1}}}{\frac{pq-1}{q+1} \left(\frac{q+1}{p+1} \right)^{\frac{q}{q+1}}} \right]^{1/b} \frac{1}{\lambda}$$

for all $\lambda \in (0, 1)$.

Proof. Let $(x(t), y(t))$, $\frac{1}{\lambda} \le t < T_2(\lambda)$ be the solution to (7.5). Since

$$\begin{aligned} \{(p+1)x(t)^{q+1} &- (q+1)y(t)^{p+1}\}' \\ &= (p+1)(q+1)(x^q x' - y^p y') \\ &= (p+1)(q+1)(\lambda^{a-b} t^{a-b} - 1)\lambda^b t^{-1+b} x^q y^p \\ &\ge 0 \quad (\tfrac{1}{\lambda} \le t < T_2(\lambda)), \end{aligned}$$

we have

$$\begin{aligned} (p+1)x(t)^{q+1} &- (q+1)y(t)^{p+1} \\ &\ge (p+1)x(\tfrac{1}{\lambda})^{q+1} - (q+1)y(\tfrac{1}{\lambda})^{p+1} = 0 \end{aligned}$$

for $\frac{1}{\lambda} \le t < T_2(\lambda)$. Hence

$$(p+1)x(t)^{q+1} \ge (q+1)y(t)^{p+1}$$

for $\frac{1}{\lambda} \le t < T_2(\lambda)$. Then

$$\begin{aligned} y'(t) &= \lambda^b t^{b-1} x(t)^q \\ &\ge \lambda^b \left(\frac{q+1}{p+1} \right)^{\frac{q}{q+1}} t^{b-1} y^{\frac{q+pq}{q+1}} \end{aligned}$$

for $\frac{1}{\lambda} \le t < T_2(\lambda)$. Note that $\frac{q+pq}{q+1} > 1$ and $b > 0$. Hence

$$(7.7) \quad y(t) \ge \left[\left\{ y(\tfrac{1}{\lambda}) \right\}^{-\frac{pq-1}{q+1}} - \frac{pq-1}{q+1} \left(\frac{q+1}{p+1} \right)^{\frac{q}{q+1}} \frac{1}{b} \lambda^b (t^b - \lambda^{-b}) \right]^{-\frac{q+1}{pq-1}}$$

for $\frac{1}{\lambda} \le t < T_2(\lambda)$. Therefore

$$T_2(\lambda) \le \left[\frac{b(q+1)^{\frac{pq-1}{(p+1)(q+1)}} + \frac{pq-1}{q+1} \left(\frac{q+1}{p+1} \right)^{\frac{q}{q+1}}}{\frac{pq-1}{q+1} \left(\frac{q+1}{p+1} \right)^{\frac{q}{q+1}}} \right]^{1/b} \frac{1}{\lambda}.$$

The proof is now completed. $\qquad \qquad \square$

Similarly we can prove the following.

Theorem 7.2. *Assume that* $p = q = 1 + \frac{2\alpha}{n}$. *Assume also that*

$$u_0(x) = \lambda \varphi(x) \ge 0, \quad v_0(x) = \lambda \psi(x) \ge 0$$

where φ *and* ψ *are continuous functions on* $\mathbb{R}^n$ *such that*

$$(7.8) \qquad\qquad 0 \le \varphi(x), \psi(x) \le C(1 + |x|)^{-n-2\alpha}$$

for some constant $C > 0$, *and* $\lambda > 0$ *is a small parameter. Then the life span* T_λ^* *of the mild solution to the reaction-diffusion system* (1.4) *satisfies*

$$\log T_\lambda^* \sim \frac{1}{\lambda^{n/2\alpha}} \quad \text{as } \lambda \to 0.$$

8. Final remarks

8.1. **Semilinear parabolic equations with the fractional Laplacian.** As is well known, the problem of blowup for the semilinear parabolic differential equation was first studied by Fujita [3]. Since [3], this problem and related problems have been studied by a lot of people. See, for example, [14], [15], [16], [17], [22] and [23]. On the other hand, fewer results are known about the semilinear parabolic equation with the fractional Laplacian $\partial_t u + (-\Delta)^\alpha u = u^p$. See, for example, [5], [7], [11] and [20]. This is due to the fact that the detailed properties of the fundamental solution of the linear equation $\partial_t u + (-\Delta)^\alpha u = 0$ is not well known. We also note that recent developments on nonlinear differential equations with the fractional Laplacian are written in [21].

8.2. **Semilinear reaction diffusion system with the fractional Laplacian.** Finally, we remark that there are several references on semilinear reaction diffusion systems with the fractional Laplacian. (See [4], [10] and [9].) For example, Kirane and Qafsaoui [10] study some semilinear reaction diffusion systems with the fractional Laplacian and determine the Fujita type critical exponent. We also note that Kirane, Laskri and Tatar [9] deals with the blowup and the global existence for a certain semilinear reaction diffusion system with the fractional Laplacian and fractional time derivatives. However, none of those deal with the problem of life span.

References

[1] J. Bergh and J. Löfström, *Interpolation Spaces, An Introduction*, Springer-Verlag Berlin Heidelberg NewYork (1976).

[2] M. Escobedo and M. A. Herrero, *oundedness and blow up for a semilinear reaction-diffusion system*, J. Diff. Eqns., **89** (1991), 176–202.

[3] H. Fujita, *On the blowing up of the solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$*, J. Fac. Sci. Univ. Tokyo (I), **13** (1966), 109–124.

[4] A. Fino and M. Kirane, *On certain time- and space-fractional evolution equations*, preprint, https://hal.archives-ouvertes.fr/hal-00398110v1.

[5] K. Hayakawa, *On nonexistence of global solutions of some semilinear parabolic equations*, Proc.Japan Acad., **49** (1973), 503–505.

[6] Q. Huang, K. Mochizuki and K. Mukai, *Life span and asymptotic behavior for a semilinear parabolic system with slowly decaying initial values*, Hokkaido Math, J., **27** (1998), 393–407.

[7] K. Ishige, T. Kawakami and K. Kobayashi, *Global solutions for a nonlinear integral equation with a generalized heat kernel*, Disc. Cont. Dyn. Sys. Ser. S, **7**, no. 4, (2014), 767–783.

[8] Y. Kobayashi, *Behavior of the life span for solutions to the system of reaction-diffusion equations*, Hiroshima Math. J., **33** (2003), 167–187.

[9] M. Kirane, Y. Laskri, and N.-e. Tatar, *Critical exponents of Fujita type for certain evolution equations and systems with spatio-temporal fractional derivatives*, J. Math. Anal. Appl., **312** (2005), 488–501 .

[10] M. Kirane and M. Qafsaoui, *Global Nonexistence for the Cauchy Problem of Some Nonlinear Reaction-diffusion Systems*, J. Math. Anal. Appl., **268** (2002), 217–243 .

[11] T. Kakehi and K. Sakai, *Behavior of solutions to linear and semilinear parabolic pseudo-differential equations*, Tsukuba J. Math., Vol. **34**, no 2, (2010), 221–253.

[12] K. Kobayashi, T. Sirao and H. Tanaka, *On the growing up problem for semilinear heat equations*, J. Math. Soc. Japan, Vol. **29**, no 3, (1977), 407–424.

[13] H. A. Levine, *The role of critical exponents in blowup theorems*, SIAM Rev., Vol. **32**, no. 2, (1990), 262–288.

[14] T-Y. Lee and W-M. Ni, *Global existance,large time behavior and life span of solutions of a semilinear parabolic Cauchy problem*, Trans. Amer. Math. Soc., **333** (1992), 365–378.

[15] K. Mochizuki and R. Suzuki, *Critical exponent and critical blow up for quasilinear parabolic equation*, Israel J. Math., **98** (1997),141–156.

[16] R. G. Pinsky, *Existence and nonexistencs of global solutions for $u_t = \Delta u + a(x)u^p$ in $\mathbb{R}^d$*, J. Differential Equations, **133** (1997), 152–177.

[17] R. G. Pinsky, *The Behavior of the Life Span for Solutions to $u_t = \Delta u + a(x)u^p$ in $\mathbb{R}^d$*, J. Differential Equations, **147** (1998), 30–57.

[18] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Springer-Verlag, NewYork (1984).

[19] E. M. Stein, *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., (1970).

[20] S. Sugitani, *On nonexistance of global solutions for some nonlinear integral equations*, Osaka. J. Math., **12** (1975), 45–51.

[21] J. L. Vázquez, *Nonlinear Diffusion with Fractional Laplacian Operators*, Nonlinear Partial Differential Equations, **7** (2012), 271–298.

[22] B. Wang, *The Cauchy problem for critical and subcritical semilinear parabolic equations in L^r (I)*, Nonlinear Analysis, **48** (2002) 747–764.

[23] F. B. Weissler, *Existance and nonexistance of global solutions for a semilinear heat equation*, Israel J. Math., **38** (1981), 29–40.

Tomoyuki Kakehi
Department of Mathematics
Okayama University
e-mail address: kakehi@math.okayama-u.ac.jp

Yoshihito Oshita
Department of Mathematics
Okayama University
e-mail address: oshita@okayama-u.ac.jp