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GAUSS MAPS OF CUSPIDAL EDGES IN HYPERBOLIC 3-SPACE, WITH APPLICATION TO FLAT FRONTS

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Abstract. We study singularities of de Sitter Gauss map images of cuspidal edges in hyperbolic 3-space. We show relations between singularities of de Sitter Gauss map images and differential geometric properties of cuspidal edges. Moreover, we apply this result to flat fronts in hyperbolic 3-space.

1. Introduction

The hyperbolic 3-space $\mathbb{H}^3$ is a 3-dimensional Riemannian spaceform with constant sectional curvature -1 and the de Sitter 3-space $\mathbb{S}^{2,1}$ is a 3-dimensional Lorentzian spaceform with constant sectional curvature 1 in Lorentz-Minkowski 4-space $\mathbb{R}^{3,1}$. There are several articles on the study of surfaces in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$ (for example, [4, 5, 6, 7, 10, 12, 13, 14, 15, 16]). Gálvez, Martínez and Milán [7] showed the representation formula for flat surfaces in $\mathbb{H}^3$. For flat fronts in $\mathbb{H}^3$, Kokubu, Rossman, Saji, Umehara and Yamada [14] showed that generic singularities on flat fronts are cuspidal edges and swallowtails (see also [8]). Moreover, the criteria for cuspidal edges and swallowtails were obtained. Recently differential geometric properties of fronts are studied (cf. [17, 18, 20, 24, 25]). In particular, normal form and isometric deformation of cuspidal edges are obtained in [17, 20].

On the other hand, the Gauss map plays important roles to investigate differential geometry of surfaces in the Euclidean 3-space $\mathbb{R}^3$. Singularities and stabilities of Gauss maps for surfaces in $\mathbb{R}^3$ were studied by [3, 2]. Bleecker and Wilson [3] showed that generic singularities of the Gauss map are fold and cusp singularities. Banchoff, Gaffney and McCrory [2] studied geometric meaning of cusp singularities of Gauss maps. The differential of Gauss map, that is, the Weingarten map, gives the Gaussian curvature, the mean curvature and the principal curvatures for the surface. Therefore, for surfaces in $\mathbb{H}^3$, such maps also play important roles. In [7, 15, 16], flat fronts in $\mathbb{H}^3$ and their hyperbolic Gauss maps were studied. Izumiya, Pei and Sano

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[10] studied singularities of hyperbolic Gauss maps and hyperbolic Gauss indicatrices for surfaces in $\mathbb{H}^3$. They regard hyperbolic Gauss indicatrices as wave fronts and show that generic singularities of hyperbolic Gauss map images are cuspidal edges and swallowtails to apply the Legendrian singularity theory. Moreover, Izumiya [9] introduced Legendrian duality theorem for pseudo-spheres in the Lorentz-Minkowski space. Using this duality theorem, the de Sitter Gauss map image and the lightcone Gauss map image of surfaces in $\mathbb{H}^3$ can be constructed. Thus one can study extrinsic differential geometry of surfaces in $\mathbb{H}^3$.

In this paper, we clarify relations between differential geometric properties called ridge points of cuspidal edges in $\mathbb{H}^3$ and singularities of de Sitter Gauss map images in $\mathbb{S}^{2,1}$. In Section 2, we consider local differential geometric properties of cuspidal edges in $\mathbb{H}^3$. We shall define the principal curvature, the principal direction and ridge points for cuspidal edges. In Section 3, we show relations between differential geometric properties of cuspidal edges and singularities of de Sitter Gauss map images (Theorem 3.3). In Section 4, we consider the normal form of cuspidal edges in $\mathbb{H}^3$. The normal form of cuspidal edges in $\mathbb{R}^3$ was introduced in [17]. We give condition that the origin is a ridge point in context of the coefficients of the normal form. In the last section, we apply results obtained previous sections to flat fronts in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$. We show conditions of singularities of de Sitter Gauss map images for flat fronts in the context of the data which appear in the representation theorem for flat fronts in $\mathbb{H}^3$ given by [7] (Theorem 5.4). Furthermore, we consider the Enneper-type flat fronts as global examples. We give the duality between of Enneper-type flat fronts and their de Sitter Gauss map images (Theorem 5.8).

2. Local differential geometry of cuspidal edges in $\mathbb{H}^3$

Let $\mathbb{R}^4 = \{(x_0, x_1, x_2, x_3) | x_i \in \mathbb{R},\ i = 0, 1, 2, 3\}$ be a 4-dimensional vector space. For $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$, $\boldsymbol{y} = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4$, the *pseudo scalar product* $\langle , \rangle$ of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$. We call this space $(\mathbb{R}^4, \langle , \rangle)$ the *Lorentz–Minkowski* 4*-space* or briefly the *Minkowski* 4*-space*. We write $\mathbb{R}^{3,1}$ instead of $(\mathbb{R}^4, \langle , \rangle)$. For $\boldsymbol{x} \in \mathbb{R}^{3,1} \setminus \{\boldsymbol{0}\}$, there are three kinds of vectors called *spacelike, lightlike* or *timelike* and defined by $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, = 0$ or < 0 respectively. The norm of $\boldsymbol{x} \in \mathbb{R}^{3,1}$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. Especially, for a spacelike vector $\boldsymbol{x}$, the norm of $\boldsymbol{x}$ is $\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$. We now define the *pseudo wedge product* $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$

as follows:

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x_0^1 & x_1^1 & x_2^1 & x_3^1 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{vmatrix},$$

where $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is the canonical basis of $\mathbb{R}^{3,1}$ and vectors $\boldsymbol{x}_i = (x_0^i, x_1^i, x_2^i, x_3^i) \in \mathbb{R}^{3,1}\,(i = 1,2,3)$. One can easily check that $\langle \boldsymbol{x}, \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 \rangle = \det(\boldsymbol{x}, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$ holds. Hence $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ is pseudo orthogonal to any $\boldsymbol{x}_i\,(i = 1,2,3)$.

There are three kinds of pseudo-spheres in $\mathbb{R}^{3,1}$: the *hyperbolic* 3-*space* $\mathbb{H}^3$ is defined by

$$\mathbb{H}^3 = \left\{ \boldsymbol{x} \in \mathbb{R}^{3,1} \,\middle|\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1,\, x_0 > 0 \right\},$$

the *de Sitter* 3-*space* $\mathbb{S}^{2,1}$ is defined by

$$\mathbb{S}^{2,1} = \left\{ \boldsymbol{x} \in \mathbb{R}^{3,1} \,\middle|\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1 \right\}$$

and *open lightcone* LC^* is defined by

$$LC^* = \left\{ \boldsymbol{x} \in \mathbb{R}^{3,1} \setminus \{\boldsymbol{0}\} \,\middle|\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0 \right\}.$$

We recall that wave fronts in $\mathbb{H}^3$. Let $f : U \to \mathbb{H}^3 \subset \mathbb{R}^{3,1}$ be a C^∞-map, where $U \subset \mathbb{R}^2$ is a simply-connected domain with local coordinates u, v. We call f a *wave front* (or *front*, for short) if there exists a unit vector field $\nu : U \to \mathbb{S}^{2,1}$ along f such that the following conditions hold:

(1) $\langle df(X_p), \nu(p) \rangle = 0$, for any $X_p \in T_pU$, $p \in U$, and
(2) the pair $L_f = (f, \nu) : U \to T_1\mathbb{H}^3$ is an immersion, where $T_1\mathbb{H}^3 = \left\{ (\boldsymbol{v}, \boldsymbol{w}) \in \mathbb{H}^3 \times \mathbb{S}^{2,1} \,\middle|\, \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0 \right\}$ is the unit tangent bundle over $\mathbb{H}^3$ equipped with the canonical contact structure.

Here, we call this vector ν a *unit pseudo normal vector* of f and L_f a *Legendrian lift* (see [1, 13, 14, 24]). A map f is called a *frontal* if f satisfies (1) of the above condition. A front f might have singularities. Arnold and Zakalyukin showed that the generic singularities of fronts in $\mathbb{R}^3$ are cuspidal edges and swallowtails (for example, see [1, 27]). A cuspidal edge is a map-germ $f : (\mathbb{R}^2, \boldsymbol{0}) \to (\mathbb{R}^3, \boldsymbol{0})$ $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2, v^3)$ and a swallowtail is a map-germ $f : (\mathbb{R}^2, \boldsymbol{0}) \to (\mathbb{R}^3, \boldsymbol{0})$ $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, 4v^3 + 2uv, 3v^4 + uv^2)$ at the origin, where two map-germs f, $g : (\mathbb{R}^2, \boldsymbol{0}) \to (\mathbb{R}^3, \boldsymbol{0})$ are $\mathcal{A}$-equivalent if there exist diffeomorphism-germs $\Xi_s : (\mathbb{R}^2, \boldsymbol{0}) \to (\mathbb{R}^2, \boldsymbol{0})$ on the source and $\Xi_t : (\mathbb{R}^3, \boldsymbol{0}) \to (\mathbb{R}^3, \boldsymbol{0})$ on the target such that $f \circ \Xi_s = \Xi_t \circ g$ holds (see Fig. 1).

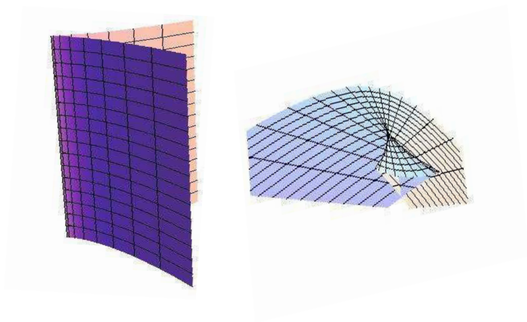

Fig. 1: Cuspidal edge (left) and swallowtail (right).

For a front f, we define a function called the *signed area density function* λ as follows: $\lambda = \det(f, f_u, f_v, \nu)$, where $f_u = \partial f/\partial u$ and $f_v = \partial f/\partial v$ respectively. We denote by $S(f)$ the singular set of f. By definition of the signed area density function, the relation $S(f) = \lambda^{-1}(0)$ holds. For a singular point $p \in S(f)$, we say that p is *non-degenerate* if the condition $d\lambda(p) \neq 0$ holds. Let $p \in S(f)$ be a non-degenerate singular point of f. Then, by the implicit function theorem, there exists a regular curve $\gamma : (-\varepsilon, \varepsilon) \ni t \mapsto \gamma(t) \in U\,(\varepsilon > 0)$ with $\gamma(0) = p$ such that γ locally parametrizes $S(f)$. Since non-degenerate singular points are corank one singular points, there exists a vector field η on $S(f)$ such that $df(\eta) = \mathbf{0}$ holds. We call such a vector field the *null vector field*. Under the above situation, the following criteria are known.

Theorem 2.1 ([14, Propositin 1.3])**.** *Let $f : (U, p) \to \mathbb{H}^3$ be a front-germ and $p \in U$ be a non-degenerate singular point of f. Then*

(1) *f at p is $\mathcal{A}$-equivalent to a cuspidal edge if and only if $\eta\lambda(p) \neq 0$ holds.*
(2) *f at p is $\mathcal{A}$-equivalent to a swallowtail if and only if $\eta\lambda(p) = 0$ and $\eta\eta\lambda(p) \neq 0$ hold.*

We consider (extrinsic) differential geometric properties of cuspidal edges in $\mathbb{H}^3$. Let $f : U \to \mathbb{H}^3$ be a front and $p \in U$ a cuspidal edge. In this case, we can take a special local coordinate system called the *adapted coordinate system* (see [17, 18], for example).

Definition 2.2. A coordinate system $(U; u, v)$ centered at p is called *adapted* if the following conditions hold:

1) the u-axis is the singular curve,
2) $\eta = \partial_v$ gives a null vector field along the u-axis, and

3) there are no singular points other than the u-axis.

We use this coordinate system in Sections 2, 3 and 4. With this coordinate, $df(\eta) = f_v = \mathbf{0}$ and $f_{vv} \neq \mathbf{0}$ hold along the u-axis. Thus there exists a smooth map $\varphi : U \to \mathbb{R}^{3,1} \setminus \{\mathbf{0}\}$ such that $f_v = v\varphi$ holds. We define a map

$$\nu = \frac{f \wedge f_u \wedge \varphi}{\|f \wedge f_u \wedge \varphi\|} : U \to \mathbb{S}^{2,1}.$$

From the definition of ν, we have

$$\langle f_u, \nu\rangle = \langle \varphi, \nu\rangle = \langle f, \nu\rangle = 0, \quad \langle \nu, \nu\rangle = 1.$$

We call this map ν the *de Sitter Gauss map image* or the *de Sitter Gauss image* of f.

The signed area density of f is given by

$$\lambda = \det(f, f_u, f_v, \nu) = v\tilde{\lambda} \qquad (\tilde{\lambda} = \det(f, f_u, \varphi, \nu)).$$

Since $\eta = \partial_v$, a point p is a cuspidal edge of f if and only if $\eta\lambda = \tilde{\lambda} \neq 0$ holds along the u-axis. Thus f, f_u, φ and ν are linearly independent.

We define the following functions:

$$\hat{E} = \langle f_u, f_u\rangle, \qquad \hat{F} = \langle f_u, \varphi\rangle, \qquad \hat{G} = \langle \varphi, \varphi\rangle, \tag{2.1}$$

$$\hat{L} = -\langle f_u, \nu_u\rangle, \qquad \hat{M} = -\langle \varphi, \nu_u\rangle, \qquad \hat{N} = -\langle \varphi, \nu_v\rangle. \tag{2.2}$$

We note that $\hat{E}\hat{G} - \hat{F}^2 \neq 0$ near p and $-\langle f_u, \nu_v\rangle = v\hat{M}$ holds. Using these functions, we have the following (cf. [26]).

Lemma 2.3. *The differentials ν_u and ν_v can be written as*

$$\nu_u = \frac{\hat{F}\hat{M} - \hat{G}\hat{L}}{\hat{E}\hat{G} - \hat{F}^2} f_u + \frac{\hat{F}\hat{L} - \hat{E}\hat{M}}{\hat{E}\hat{G} - \hat{F}^2}\varphi,$$
$$\nu_v = \frac{\hat{F}\hat{N} - v\hat{G}\hat{M}}{\hat{E}\hat{G} - \hat{F}^2} f_u + \frac{v\hat{F}\hat{M} - \hat{E}\hat{N}}{\hat{E}\hat{G} - \hat{F}^2}\varphi.$$

We set

$$\psi(t) = \det(\hat{\gamma}, \hat{\gamma}', D^f_\eta(\nu \circ \gamma), \nu \circ \gamma)(t),$$

where $\hat{\gamma} = f \circ \gamma$, D^f is the canonical covariant derivative along a map f induced from the Levi-Civita connection on $\mathbb{H}^3$ and $' = d/dt$. We note that $\psi(0) \neq 0$ if and only if (f, ν) is a Legendre immersion at p, that is, f is a front at p when $\hat{\gamma}'(0) \neq 0$ (see [6, 13]). Taking an adapted coordinate system $(U; u, v)$ around p, $D^f_\eta \nu = \nu_v$ holds. Thus we have

$$\begin{aligned}\psi(u) &= \det(f(u,0), f_u(u,0), \nu_v(u,0), \nu(u,0)) \\ &= -\frac{\hat{E}\hat{N}}{\hat{E}\hat{G} - \hat{F}^2} \det(f(u,0), f_u(u,0), \varphi(u,0), \nu(u,0))\end{aligned}$$

by Lemma 2.3, and we see that $\hat{N}$ does not vanish along the u-axis.

Remark 2.4. Let $f : U \to (M^3, g)$ be a front with non-degenerate singular points, where (M^3, g) is an oriented 3-dimensional Riemannian manifold. We set

$$f_\eta = df(\eta), \quad f_{\eta\eta} = \nabla_\eta f_\eta, \quad f_{\eta\eta\eta} = \nabla_\eta f_{\eta\eta},$$

where ∇ is the Levi-Civita connection of (M^3, g). In [18], a differential geometric invariant κ_c called the *cuspidal curvature* is defined by

$$\kappa_c(t) = \frac{\|\hat{\gamma}'(t)\|^{3/2} \det_g(\hat{\gamma}'(t), f_{\eta\eta}(\gamma(t)), f_{\eta\eta\eta}(\gamma(t)))}{\|\hat{\gamma}'(t) \times_g f_{\eta\eta}(\gamma(t))\|^{5/2}}$$

along the singular curve γ, where $\hat{\gamma} = f \circ \gamma$, $\det_g$ is the Riemannian volume element of (M^3, g) and $\langle \boldsymbol{a} \times_g \boldsymbol{b}, \boldsymbol{c} \rangle = \det_g(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ for each $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in T_q M^3$ ($q \in M^3$). If $M = \mathbb{R}^3$, then $\det_g$ can be identified with the usual determinant. By [24, Corollary 3.5], f at $p \in S(f)$ is a cuspidal edge if and only if $\kappa_c(p) \neq 0$ holds. For details about the cuspidal curvature κ_c, see [18].

Here the Gauss-Kronecker curvature function K_{ext} and the mean curvature function H of f are given by

$$K_{\mathrm{ext}} = \frac{\hat{L}\hat{N} - v\hat{M}^2}{v(\hat{E}\hat{G} - \hat{F}^2)}, \quad H = \frac{\hat{E}\hat{N} - 2v\hat{F}\hat{M} + v\hat{G}\hat{L}}{2v(\hat{E}\hat{G} - \hat{F}^2)}. \tag{2.3}$$

We define the matrix

$$S^d = -\begin{pmatrix} \hat{E} & \hat{F} \\ \hat{F} & \hat{G} \end{pmatrix}^{-1} \begin{pmatrix} \hat{L} & v\hat{M} \\ \hat{M} & \hat{N} \end{pmatrix},$$

and we call S^d the *modified shape operator* of f. Then the relation

$$\det S^d = vK_{\mathrm{ext}} = \hat{K}_{\mathrm{ext}}$$

holds. By definition of K_{ext} given by (2.3), $\hat{K}_{\mathrm{ext}}$ is C^∞-function of u, v. See [18, 24] for behavior of K_{ext} and H near non-degenerate singular points on fronts.

We consider the principal curvatures for cuspidal edges. We now define two functions as follows:

$$\kappa_1 = \frac{A+B}{2v(\hat{E}\hat{G}-\hat{F}^2)}, \quad \kappa_2 = \frac{A-B}{2v(\hat{E}\hat{G}-\hat{F}^2)}. \tag{2.4}$$

Here $A = \hat{E}\hat{N} - 2v\hat{F}\hat{M} + v\hat{G}\hat{L}$, $B = \sqrt{A^2 - 4v(\hat{E}\hat{G}-\hat{F}^2)(\hat{L}\hat{N}-v\hat{M}^2)}$. By definitions of κ_1 and κ_2, we have $K_{\text{ext}} = \kappa_1\kappa_2$ and $2H = \kappa_1 + \kappa_2$. Thus we may regard κ_1 and κ_2 as principal curvatures of f. However, we note that one of κ_i $(i = 1, 2)$ may not be well-defined along the singular curve. Two functions κ_1 and κ_2 in (2.4) can be rewritten as

$$\kappa_1 = \frac{2(\hat{L}\hat{N}-v\hat{M}^2)}{A-B}, \quad \kappa_2 = \frac{2(\hat{L}\hat{N}-v\hat{M}^2)}{A+B}. \tag{2.5}$$

Here, $A = \hat{E}\hat{N}$ and $B = \hat{E}|\hat{N}|$ hold on the singular set $\{v = 0\}$. Thus $A + B \neq 0$ (resp. $A - B \neq 0$) holds on $\{v = 0\}$ if $\hat{N} > 0$ (resp. $\hat{N} < 0$). By the above arguments, if $\hat{N}$ is positive (resp. negative) along the singular curve, κ_2 (resp. κ_1) is well-defined along the singular curve. So we can regard κ_2 as the *principal curvature* for the cuspidal edge if and only if $\hat{N}(u, 0)$ is a non-zero positive C^∞-function of u along the u-axis. We note that Murata and Umehara [19] introduced the notion of *principal curvature maps* for fronts in $\mathbb{R}^3$.

Let us consider the principal direction $\boldsymbol{v} = (\xi, \zeta)$ with respect to the principal curvature κ_2. In this case, $\boldsymbol{v}$ satisfies the following equation:

$$\begin{pmatrix} \hat{L} & v\hat{M} \\ v\hat{M} & v\hat{N} \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix} = \kappa_2 \begin{pmatrix} \hat{E} & v\hat{F} \\ v\hat{F} & v^2\hat{G} \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix}.$$

Thus the principal direction $\boldsymbol{v} = (\xi, \zeta)$ can be taken as

$$\boldsymbol{v} = (\hat{N} - v\kappa_2\hat{G}, -\hat{M} + \kappa_2\hat{F}). \tag{2.6}$$

Since $\hat{N} \neq 0$ holds on $\{v = 0\}$, the principal direction $\boldsymbol{v}$ is also well-defined along the u-axis.

Using the principal curvature κ_2 and the principal direction $\boldsymbol{v}$ corresponding to κ_2, we define a notion of ridge point.

Definition 2.5. Let $f : U \to \mathbb{H}^3$ be a cuspidal edge, κ_2 the principal curvatures of f and $\boldsymbol{v}$ the principal directions with respect to κ_2. The point $f(p)$ is called a *ridge point* relative to $\boldsymbol{v}$ if $\boldsymbol{v}\kappa_2(p) = 0$, where $\boldsymbol{v}\kappa_2$ is the directional derivative of κ_2 in direction $\boldsymbol{v}$. Moreover, $f(p)$ is called a *k-th order ridge point* relative to $\boldsymbol{v}$ if $\boldsymbol{v}^{(m)}\kappa_2(p) = 0$ $(1 \leq m \leq k)$ and

$\boldsymbol{v}^{(k+1)}\kappa_2(p) \neq 0$, where $\boldsymbol{v}^{(m)}\kappa_2$ is the directional derivative of κ_2 with respect to $\boldsymbol{v}$ applied m times.

Properties of ridge points for regular surfaces in $\mathbb{R}^3$ were first studied by Porteous to investigate caustics of them. For details, see [21, 22].

3. Singularities of the de Sitter Gauss map image

In this section, we consider the de Sitter Gauss image $\nu : U \to \mathbb{S}^{2,1}$ of f. From arguments in Section 2, the pair $L_f = (f, \nu)$ gives a Legendrian immersion. Thus ν can be regarded as a front in $\mathbb{S}^{2,1}$ with unit normal vector f. We assume that the principal curvature κ_2 is well-defined on the source, namely, $\hat{N}$ is positive on the u-axis.

We consider the signed area density function λ^ν of ν. Using ν, ν_u, ν_v and f, λ^ν is given by

$$\lambda^\nu = \det(f, \nu_u, \nu_v, \nu). \tag{3.1}$$

By Lemma 2.3 and (2.3), it can be written as

$$\lambda^\nu = vK_{\mathrm{ext}} \det(f, f_u, \varphi, \nu). \tag{3.2}$$

We note that $\det(f, f_u, \varphi, \nu)$ is a non-zero function. By definitions of the Gauss-Kronecker curvature K_{ext}, κ_1 and κ_2, we have

$$vK_{\mathrm{ext}} = (v\kappa_1)\kappa_2. \tag{3.3}$$

In this case, $v\kappa_1$ is a non-zero function on U. From (3.1), (3.2) and (3.3), we can consider the signed area density as

$$\hat{\lambda}^\nu = \kappa_2. \tag{3.4}$$

Thus we have the following:

Proposition 3.1. *Under the above conditions, a point $p \in U$ is a singular point of the de Sitter Gauss image ν if and only if $\kappa_2(p) = 0$ holds.*

We remark that relationships between singular points of the de Sitter Gauss image ν of a front f in $\mathbb{H}^3$ and the limiting normal curvature of f are shown in [18]. We denote by $S(\nu) = \{q \in U \mid \kappa_2(q) = 0\}$ the set of singular points of ν and we call a point $p \in S(\nu)$ a *parabolic point* for cuspidal edges. For $p \in S(\nu)$, a point p is a non-degenerate singular point of ν if and only if $(\kappa_2)_u(p) \neq 0$ or $(\kappa_2)_v(p) \neq 0$ hold, that is, p is not a critical point of κ_2.

We consider the case that $p \in S(\nu)$ is non-degenarate. In this case, there exists a vector field η^ν on $S(\nu)$ such that $d\nu(\eta^\nu) = \mathbf{0}$. We will find a concrete form for η^ν. Let us take $\eta^\nu = \eta_1^\nu \partial_u + \eta_2^\nu \partial_v$. Then

$$\begin{aligned}(3.5)\quad d\nu(\eta^\nu) = &\left(\frac{\hat{F}\hat{M} - \hat{G}\hat{L}}{\hat{E}\hat{G} - \hat{F}^2} f_u + \frac{\hat{F}\hat{L} - \hat{E}\hat{M}}{\hat{E}\hat{G} - \hat{F}^2} \varphi \right) \eta_1^\nu \\ &+ \left(\frac{\hat{F}\hat{N} - v\hat{G}\hat{M}}{\hat{E}\hat{G} - \hat{F}^2} f_u + \frac{v\hat{F}\hat{M} - \hat{E}\hat{N}}{\hat{E}\hat{G} - \hat{F}^2} \varphi \right) \eta_2^\nu = \mathbf{0}\end{aligned}$$

holds on $S(\nu)$ by Lemma 2.3. Since f_u and φ are linearly independent, this equation is equivalent to the following:

$$(3.6)\qquad \begin{pmatrix} \hat{E} & \hat{F} \\ \hat{F} & \hat{G} \end{pmatrix}^{-1} \begin{pmatrix} \hat{L} & v\hat{M} \\ \hat{M} & \hat{N} \end{pmatrix} \begin{pmatrix} \eta_1^\nu \\ \eta_2^\nu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

holds on $S(\nu)$. Now $\hat{N} \neq 0$ along the u-axis, so we can take η^ν as

$$(3.7)\qquad \eta^\nu = \hat{N}\partial_u - \hat{M}\partial_v$$

on $S(\nu)$. Moreover, η^ν can be extended on U by the form

$$\eta^\nu = (\hat{N} - v\kappa_2\hat{G})\partial_u + (-\hat{M} + \kappa_2\hat{F})\partial_v = \boldsymbol{v},$$

where $\boldsymbol{v}$ is the principal direction with respect to κ_2. Thus we can regard the principal direction $\boldsymbol{v}$ as a null vector field η^ν.

Using Theorem 2.1, the signed area density $\hat{\lambda}^\nu$ and null vector field η^ν, we obtain conditions for singularities of ν.

Proposition 3.2. *Let $f : U \to \mathbb{H}^3$ be a front, $p \in U$ a cuspidal edge and $\nu : U \to \mathbb{S}^{2,1}$ the de Sitter Gauss image of f. Suppose that $p \in S(\nu)$ is a non-degenerate singular point of ν. Then the following assertions hold.*

(1) *ν at p is a cuspidal edge if and only if $\boldsymbol{v}\kappa_2(p) \neq 0$ holds.*
(2) *ν at p is a swallowtail if and only if $\boldsymbol{v}\kappa_2(p) = 0$ and $\boldsymbol{v}^{(2)}\kappa_2(p) \neq 0$ hold.*

Combining the results obtained in Sections 2 and 3, we have relations between singularities of ν and the differential geometric properties of f.

Theorem 3.3. *Let $f : U \to \mathbb{H}^3$ be a front, $p \in U$ a cuspidal edge and $\nu : U \to \mathbb{S}^{2,1}$ the de Sitter Gauss image of f. Then the following properties hold.*

(1) *A point $p \in U$ is a singular point of ν if and only if $\kappa_2(p) = 0$ holds.*

(2) *A point* $p \in S(\nu)$ *is non-degenerate if and only if* p *is not a critical point of* κ_2.
(3) *For a non-degenerate singular point* $p \in S(\nu)$, ν *at* p *is a cuspidal edge if and only if* p *is not a ridge point of* f.
(4) *For a non-degenerate singular point* $p \in S(\nu)$, ν *at* p *is a swallowtail if and only if* p *is a first order ridge point of* f.

In [25], duality between A_{k+1}-inflection point ($k \leq n$) of immersed C^∞-hypersurfaces $f : M^n \to P(\mathbb{R}^{n+2})$ and A_k-singularity of dual front $g : M^n \to P((\mathbb{R}^{n+2})^*)$ were shown, where M^n is an n-dimensional C^∞-manifold and A_2-singularity and A_3-singularity correspond to a cuspidal edge and a swallowtail for fronts respectively. We also remark that cusp singularities of Gauss maps of regular surfaces in $\boldsymbol{R}^3$ are related to parabolic points and ridge points for surfaces (see [2, Theorem 3.1]).

4. Normal form of cuspidal edges in $\mathbb{H}^3$

In this section, we consider the normal form of cuspidal edges in $\mathbb{H}^3$. For cuspidal edges in $\mathbb{R}^3$, the following normal form obtained by Martins and Saji in [17] is known.

Proposition 4.1 ([17, Theorem 3.1]). *Let* $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ *be a smooth map-germ and* $\mathbf{0}$ *a cuspidal edge. Then there exist a diffeomorphism-germ* $\psi : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ *on the source and an isometry-germ* $\Psi : (\mathbb{R}^3, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ *on the target such that*

$$\begin{aligned} &\Psi \circ f \circ \psi(u,v) \\ &= \left(u, \frac{a_{20}}{2}u^2 + \frac{a_{30}}{6}u^3 + \frac{v^2}{2}, \frac{b_{20}}{2}u^2 + \frac{b_{30}}{6}u^3 + \frac{b_{12}}{2}uv^2 + \frac{b_{03}}{6}v^3\right) + h(u,v) \end{aligned} \tag{4.1}$$

with $b_{20} \geq 0$ *and* $b_{03} \neq 0$, *where*

$$h(u,v) = (0, u^4 h_1(u), u^4 h_2(u) + u^2 v^2 h_3(u) + uv^3 h_4(u) + v^4 h_5(u,v)),$$

with $h_i(u)$ $(1 \leq i \leq 4)$, $h_5(u,v)$ *smooth functions.*

See [17] for detailed descriptions and geometric properties of the coefficients in (4.1). We extend (4.1) to the case of $\mathbb{H}^3$ by analogy to the *hyperbolic-Monge form* (or the *H-Monge form*, for short) for regular surfaces which is introduced by Izumiya, Pei and Sano (see [10, Section 8]). Let $f : U \to \mathbb{H}^3$ be a cuspidal edge. Then we have H-Monge form $f =$

(f_0, f_1, f_2, f_3) for surfaces with cuspidal edge as follows:

$$\begin{aligned}
f_0 &= \sqrt{1 + f_1^2 + f_2^2 + f_3^2},\\
f_1 &= \frac{b_{20}}{2}u^2 + \frac{b_{30}}{6}u^3 + \frac{b_{12}}{2}uv^2 + \frac{b_{03}}{6}v^3\\
&\qquad + u^4h_2(u) + u^2v^2h_3(u) + uv^3h_4(u) + v^4h_5(u,v),\\
f_2 &= u,\\
f_3 &= \frac{a_{20}}{2}u^2 + \frac{a_{30}}{6}u^3 + \frac{v^2}{2} + u^4h_1(u).
\end{aligned}$$

Using H-Monge form of cuspidal edge, we consider the conditions of ridge for cuspidal edge in terms of the coefficients. Let us assume that $b_{03} > 0$ holds, that is, κ_2 is well-defined. Here f, f_u and φ are

$$f = (1,0,0,0),\ f_u = (0,0,1,0) \text{ and } \varphi = (0,0,0,1)$$

at the origin. We can take the de Sitter Gauss image ν of f with $\nu = (0,1,0,0)$ at the origin. In this case, the coefficients of the first and the second fundamental forms are $\hat{E} = 1$, $\hat{F} = 0$, $\tilde{G} = 1$, $\hat{L} = b_{20}$, $\hat{M} = b_{12}$, $\hat{N} = b_{03}/2$ at the origin. Moreover, the principal curvature satisfies $\kappa_2 = \hat{L}/\hat{E} = b_{20}$, $(\kappa_2)_u = b_{30} - a_{20}b_{12}$ and $(\kappa_2)_v = -(4b_{12}^2 + a_{20}b_{03}^2)/2b_{03}$ at the origin. Thus we have the following lemma.

Lemma 4.2. *Let $f : U \to \mathbb{H}^3$ be a H-Monge form of cuspidal edge. Then $\kappa_2(\mathbf{0}) = 0$ if and only if $b_{20} = 0$. Moreover, the origin $\mathbf{0}$ is not critical point of κ_2 if and only if $b_{30} - a_{20}b_{12} \neq 0$ or $4b_{12}^2 + a_{20}b_{03}^2 \neq 0$ hold.*

We assume that $b_{20} = 0$ holds in what follows.

Lemma 4.3. *Let $f : U \to \mathbb{H}^3$ be the H-Monge form of a cuspidal edge, κ_2 the principal curvature and $\boldsymbol{v}$ the principal direction corresponding to κ_2. Then the following assertions hold.*

(1) *The origin is not a ridge point if and only if $4b_{12}^3 + b_{30}b_{03}^2 \neq 0$ holds.*
(2) *The origin is a first order ridge point if and only if $4b_{12}^3 + b_{30}b_{03}^2 = 0$ and*

$$b_{03}^4h_2(0) + 4b_{12}^2b_{03}^2h_3(0) - 8b_{12}^3b_{03}h_4(0) + 16b_{12}^4h_5(0,0) \neq 0.$$

hold.

Proof. First we show the condition (1) of Lemma 4.3. By defunitions of the principal curvature and the principal direction, we have $(\kappa_2)_u = b_{30} -$

$a_{20}b_{12}$, $(\kappa_2)_v = -(4b_{12}^2 + a_{20}b_{03}^2)/2b_{03}$, $\xi = b_{03}/2$, $\zeta = -b_{12}$ at the origin. Using these conditions,

$$\boldsymbol{v}\kappa_2(\mathbf{0}) = \xi(\mathbf{0})(\kappa_2)_u(\mathbf{0}) + \zeta(\mathbf{0})(\kappa_2)_v(\mathbf{0}) = \frac{4b_{12}^3 + b_{30}b_{03}^2}{2b_{03}}$$

holds. On the other hand, the condition that the origin $\mathbf{0}$ is not a ridge point is $\boldsymbol{v}\kappa_2(\mathbf{0}) \neq 0$. Thus it follows that the first assertion holds.

Next we show (2) of Lemma 4.3. Let us assume that

$$\boldsymbol{v}\kappa_2(\mathbf{0}) = \frac{4b_{12}^3 + b_{30}b_{03}^2}{2b_{03}} = 0$$

holds, that is, the origin is a ridge point. By direct computations, we have $\xi_u(\mathbf{0}) = 3h_4(0)$, $\xi_v(\mathbf{0}) = 8h_5(0,0)$, $\zeta_u(\mathbf{0}) = -4h_3(0)$, $\zeta_v(\mathbf{0}) = -3h_4(0)$ and

$$\begin{aligned}
(\kappa_2)_{uu}(\mathbf{0}) &= -2a_{30}b_{12} + 24h_2(0) - 4a_{20}h_3(0)),\\
(\kappa_2)_{uv}(\mathbf{0}) &= \frac{1}{2b_{03}^2}(-a_{30}b_{03}^3 - 32b_{12}b_{03}h_3(0) + 6(4b_{12}^2 - a_{20}b_{03}^2)h_4(0)),\\
(\kappa_2)_{vv}(\mathbf{0}) &= \frac{4}{b_{03}^2}(b_{03}^2h_3(0) - 6b_{12}b_{03}h_4(0) + 2(8b_{12}^2 - a_{20}b_{03}^2)h_5(0,0)).
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
\boldsymbol{v}^{(2)}\kappa_2(\mathbf{0}) = \frac{1}{4b_{03}^2}(6b_{03}(4b_{03}^3h_2(0) &+ 16b_{12}^2b_{03}h_3(0)\\
&- (28b_{12}^3 - b_{30}b_{03}^2)h_4(0)) + 32b_{12}(8b_{12}^3 - b_{30}b_{03}^2)h_5(0,0)).
\end{aligned}$$

Since the condition $b_{30} = -4b_{12}^3/b_{03}^2$ holds, we have

$$\boldsymbol{v}^{(2)}\kappa_2(\mathbf{0}) = \frac{6}{b_{03}^2}(b_{03}^4h_2(0) + 4b_{12}^2b_{03}^2h_3(0) - 8b_{12}^3b_{03}h_4(0) + 16b_{12}^4h_5(0,0))).$$

This shows that (2) of Lemma 4.3 holds. $\square$

We show some examples of cuspidal edge in $\mathbb{H}^3$ and corresponding de Sitter Gauss image in $\mathbb{S}^{2,1}$. To visualize surface in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$, we will use the *Poincaré model* $\mathcal{P}$ and the *hollow ball model* $\mathcal{H}$ in what follows. For details of the hollow ball model, see [4]. Here the Poincaré model $\mathcal{P}$ is given by a map

$$\mathbb{H}^3 \ni (x_0, x_1, x_2, x_3) \mapsto \left(\frac{x_1}{1+x_0}, \frac{x_2}{1+x_0}, \frac{x_3}{1+x_0}\right) \in \mathcal{P}$$

and the hollow ball model $\mathcal{H}$ is given by

$$\mathbb{S}^{2,1} \ni (x_0, x_1, x_2, x_3) \mapsto \left(\frac{e^{\arctan x_0}}{\sqrt{1+x_0^2}} x_1, \frac{e^{\arctan x_0}}{\sqrt{1+x_0^2}} x_2, \frac{e^{\arctan x_0}}{\sqrt{1+x_0^2}} x_3 \right) \in \mathcal{H}.$$

Then we can view the Poincaré model $\mathcal{P}$ as the Euclidean unit ball

$$B^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 < 1\}$$

in $\mathbb{R}^3$. Moreover the hollow ball model $\mathcal{H}$ can be viewed as

$$\mathcal{H} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid e^{-\pi} < x_1^2 + x_2^2 + x_3^2 < e^{\pi}\}.$$

Example 4.4. Let $f = (f_0, f_1, f_2, f_3) : U \to \mathbb{H}^3$ be a cuspidal edge with

$$f_0 = \sqrt{1 + f_1^2 + f_2^2 + f_3^2},\ f_1 = u^3 + \frac{v^3}{3},\ f_2 = u,\ f_3 = \frac{u^2}{2} + \frac{u^3}{3} + \frac{v^2}{2} + u^4.$$

This form satisfies the conditions of Lemma 4.2 and (1) of Lemma 4.3. Thus, by Theorem 3.3, de Sitter Gauss image of f has cuspidal edge at the origin.

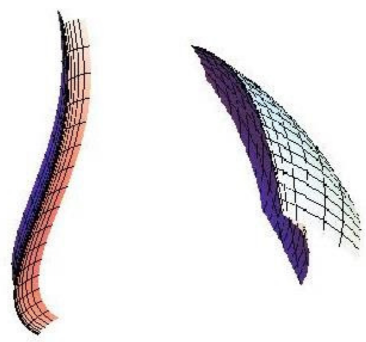

Fig. 2: Pictures of Example 4.4. Cuspidal edge in $\mathbb{H}^3$ (left) and its de Sitter Gauss image in $\mathbb{S}^{2,1}$ (right).

Example 4.5. Let $f = (f_0, f_1, f_2, f_3) : U \to \mathbb{H}^3$ be a cuspidal edge with

$$f_0 = \sqrt{1 + f_1^2 + f_2^2 + f_3^2},\ f_1 = \frac{v^3}{3} + u^4,\ f_2 = u,\ f_3 = \frac{u^2}{2} + \frac{u^3}{3} + \frac{v^2}{2}.$$

This form satisfies the conditions of Lemma 4.2 and (2) of Lemma 4.3. Thus, by Theorem 3.3, de Sitter Gauss image of f has swallowtail singularity at the origin.

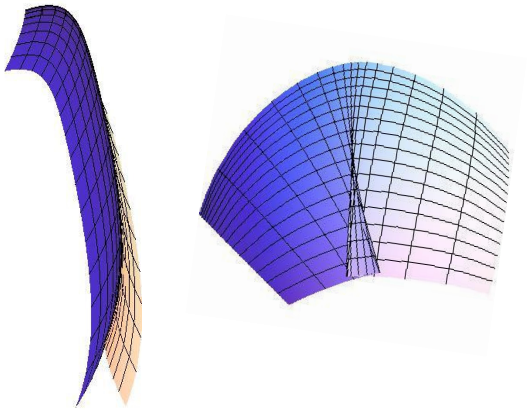

Fig. 3: Pictures of Example 4.5. Cuspidal edge in $\mathbb{H}^3$ (left) and its de Sitter Gauss image in $\mathbb{S}^{2,1}$ (right).

5. Application to flat fronts

In this section, we consider constant Gauss-Kronecker curvature $K_{\mathrm{ext}} = 1$ surfaces, called *flat fronts* in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$. First we introduce explicit formula for flat fronts in $\mathbb{H}^3$, called the Bryant-type representation, as in [7].

Proposition 5.1 ([7])**.** *Let U be a simply-connected domain in $\mathbb{C}$ with the usual complex coordinate $z = u + iv$. Then, any flat front $f : U \to \mathbb{H}^3$ is given by*

$$f = \mathcal{F}\overline{\mathcal{F}^t}, \quad \textit{where } d\mathcal{F} = \mathcal{F}\begin{pmatrix} 0 & h(z) \\ g(z) & 0 \end{pmatrix} dz \tag{5.1}$$

for some holomorphic functions g and h.

We also have the following well-known fact as in Proposition 5.2:

Proposition 5.2. *For flat fronts f in $\mathbb{H}^3$ as given in Proposition 5.1, the unit normal vector ν of f becomes* (*spacelike*) *flat fronts in $\mathbb{S}^{2,1}$, and ν can be described as*

$$\nu = \mathcal{F}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\overline{\mathcal{F}^t}. \tag{5.2}$$

5.1. **The singularity theory of flat fronts in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$.**

Here we introduce criteria for singularities of flat fronts f in $\mathbb{H}^3$ as in [14], and we give criteria for singularities of the de Sitter Gauss image ν (i.e. flat fronts in $\mathbb{S}^{2,1}$) of f in the same way.

Proposition 5.3 ([14]). *Let U be a simply connected domain, and let $f : U \to \mathbb{H}^3$ be a flat front given as in Proposition 5.1.*

(1) *A point $p \in U$ is a non-degenerate singular point if and only if*

$$|g(p)| = |h(p)| \text{ and } (g_z h - g h_z)\big|_p \neq 0.$$

(2) *f has a cuspidal edge at a non-degenerate singular point $p \in U$ if and only if*

$$\operatorname{Im}\left(\frac{g_z h - g h_z}{(gh)^{\frac{3}{2}}}\right)\Bigg|_p \neq 0.$$

(3) *f has a swallowtail at a non-degenerate singular point $p \in U$ if and only if*

$$\operatorname{Im}\left(\frac{g_z h - g h_z}{(gh)^{\frac{3}{2}}}\right)\Bigg|_p = 0 \text{ and } \operatorname{Re}\left\{\left(\frac{g_z h - g h_z}{(gh)^{\frac{3}{2}}}\right)_z \overline{\left(\frac{g_z}{g} - \frac{h_z}{h}\right)}\right\}\Bigg|_p \neq 0.$$

Here we give criteria for singularities of the de Sitter Gauss image ν. The following theorem can be proven by the same way as in the proof of the above proposition. Thus, we omit that proof.

Theorem 5.4. *Let U be a simply connected domain, and let $\nu : U \to \mathbb{S}^{2,1}$ be a flat front given in Proposition 5.2.*

(1) *A point $p \in U$ is a non-degenerate singular point if and only if*

$$|g(p)| = |h(p)| \text{ and } (g_z h - g h_z)\big|_p \neq 0.$$

(2) *ν has a cuspidal edge at a non-degenerate singular point $p \in U$ if and only if*

$$\operatorname{Re}\left(\frac{g_z h - g h_z}{(gh)^{\frac{3}{2}}}\right)\Bigg|_p \neq 0.$$

(3) *ν has a swallowtail at a non-degenerate singular point $p \in U$ if and only if*

$$\operatorname{Re}\left(\frac{g_z h - g h_z}{(gh)^{\frac{3}{2}}}\right)\Bigg|_p = 0 \text{ and } \operatorname{Im}\left\{\left(\frac{g_z h - g h_z}{(gh)^{\frac{3}{2}}}\right)_z \overline{\left(\frac{g_z}{g} - \frac{h_z}{h}\right)}\right\}\Bigg|_p \neq 0.$$

We note that Fujimori, Noro, Saji, Sasaki and Yoshida [5] study fronts in $\mathbb{S}^{2,1}$ from the viewpoint of the de Sitter Schwarz map and they give criteria which correspond to Theorem 5.4 for singularities of flonts in $\mathbb{S}^{2,1}$ in terms of de Sitter Schwarz map (for details, see [5, Proposition 6]).

By the above Proposition 5.3 and Theorem 5.4, we get the following Corollary 5.5.

Corollary 5.5.

(1) *The singular set of f coincides with the singular set of ν, i.e.* $S(f) = S(\nu)$.
(2) *We define the set of non-degenerate singular points of f as $\Sigma(f)$, which is the subset of $S(f)$. We define $\Sigma(\nu)$ similarly. Then,* $\Sigma(f) = \Sigma(\nu)$.

We note that (1) of Corollary 5.5 is a special case of [18, Corollary C] (see also [5]).

5.2. **Global example: Enneper-type flat fronts and their singularities.** Here we introduce Enneper-type flat fronts in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$, which are flat fronts and have reflective symmetry (see Fig 4 and 5), as global application of Theorem 3.3
Define $g(z) = z^k$ and $h(z) = 1$ for $k \in \mathbb{N}$. Applying Proposition 5.1, we get $(k+2)$-legged Enneper-type flat fronts in $\mathbb{H}^3$. See Fig 4. We also have $(k+2)$-legged Enneper-type flat fronts in $\mathbb{S}^{2,1}$ applying Proposition 5.2. See Fig 5. Using polar coordinates $z = re^{i\theta}$ and applying Proposition 5.3, we get the following lemma:

Lemma 5.6. *Let f be a $(k+2)$-legged Enneper-type flat front in $\mathbb{H}^3$. Then:*

(1) $S(f) = \left\{(r_0, \theta) \,\middle|\, r_0^k = 1,\ 0 \leq \theta < 2\pi\right\}$.
(2) *f has $(k+2)$-swallowtails at (r_0, θ_0) such that $r_0^k = 1$ and $\theta_0 = \frac{2i\pi}{k+2}$ $(i = 0, 1, 2, \ldots, k+1)$.*
(3) *f has cuspidal edges at (r_0, θ_1) such that $r_0^k = 1$ and $\theta_1 \neq \frac{2i\pi}{k+2}$ $(i = 0, 1, 2, \ldots, k+1)$.*

Similarly we also get the following lemma by applying Theorem 5.4.

Lemma 5.7. *Let ν be a $(k+2)$-legged Enneper-type flat front in $\mathbb{S}^{2,1}$. Then:*

(1) $S(\nu) = S(f) = \left\{(r_0, \theta) \,\middle|\, r_0^k = 1,\ 0 \leq \theta < 2\pi\right\}$.
(2) *ν has $(k+2)$-swallowtails at (r_0, θ_2) such that $r_0^k = 1$ and $\theta_2 = \frac{(2i+1)\pi}{k+2}$ $(i = 0, 1, 2, \ldots, k+1)$.*
(3) *ν has cuspidal edges at (r_0, θ_3) such that $r_0^k = 1$ and $\theta_3 \neq \frac{(2i+1)\pi}{k+2}$ $(i = 0, 1, 2, \ldots, k+1)$.*

By the above two lemmas and Theorem 3.3, we get the following theorem:

Theorem 5.8 (Duality of Enneper-type flat fronts)**.** *Let f be a $(k+2)$-legged Enneper-type flat front in $\mathbb{H}^3$, and let ν be a de Sitter Gauss image of f. Then:*

(1) *Points* (r_0, θ_2) *such that* $r_0^k = 1$ *and* $\theta_2 = \frac{(2i+1)\pi}{k+2}$ *are first order ridge points of* f $(i = 0, 1, 2, \ldots, k+1)$.
(2) *Points* (r_0, θ_3) *such that* $r_0^k = 1$ *and* $\theta_3 \neq \frac{(2i+1)\pi}{k+2}$ *are not ridge points of* f $(i = 0, 1, 2, \ldots, k+1)$.

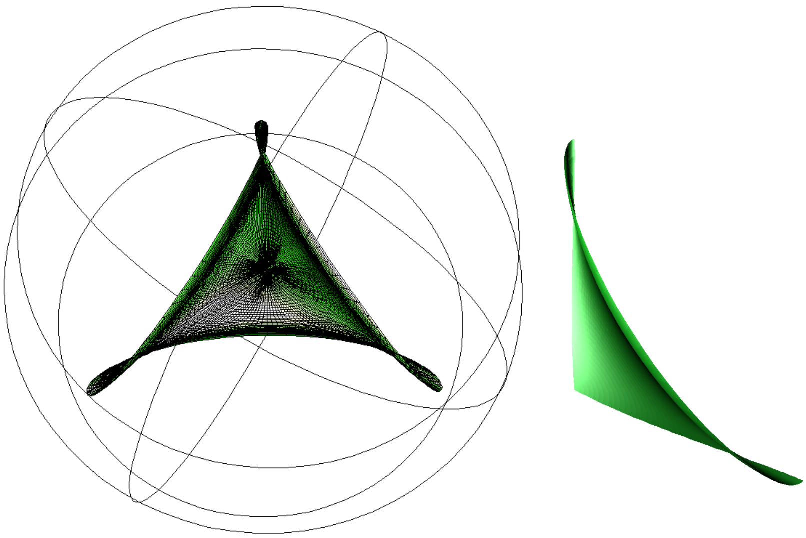

Fig. 4: The left image is a 3-legged Enneper-type flat front in $\mathbb{H}^3$, and the right is one portion of it between $0 \leq \theta \leq \frac{2\pi}{3}$.

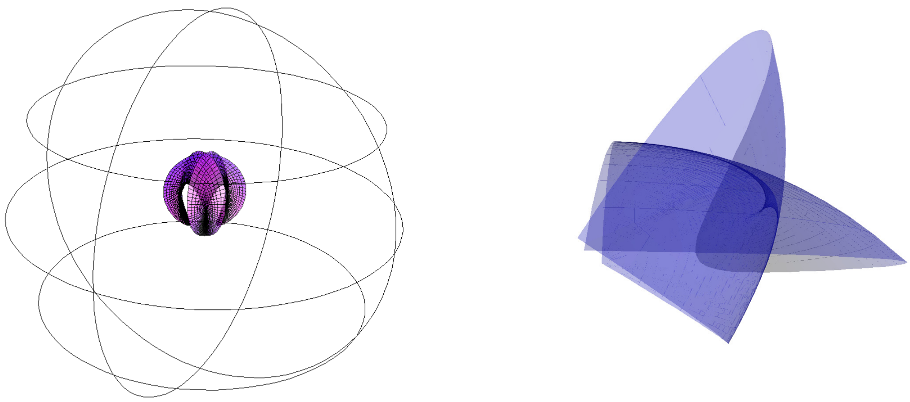

Fig. 5: The left image is the de Sitter Gauss image of a 3-legged Enneper-type flat front, and the right is one portion of it between $0 \leq \theta \leq \frac{2\pi}{3}$.

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References

[1] V.I. Arnol'd, S.M. Gusein-Zade and A.N. Varchenko, *Singularities of differentiable maps*, Vol.1, Monographs in Math. **82**, Birkhäuser, Boston, 1985.

[2] T. Banchoff, T. Gaffney and C. McCrory, *Cusps of Gauss mappings*, Research Notes in Math, vol. 55, Pitman (Advanced Publishing Program), Boston, Mass., 1982.

[3] D. Bleecker and L. Wilson, *Stability of Gauss maps*, Illinois J. Math. **22** (1978), 257–289.

[4] S. Fujimori, *Spacelike CMC 1 surfaces with elliptic ends in de Sitter 3-space*, Hokkaido Math. J. **35** (2006), 289–320

[5] S. Fujimori, M. Noro, K. Saji, T. Sasaki and M. Yoshida, *Schwarz maps for the hypergeometric differential equation*, Internat. J. Math. **26** (2015), 31pp.

[6] S. Fujimori, K. Saji, M. Umehara and K. Yamada, *Singularities of maximal surfaces*, Math. Z. **259** (2008), 827–848.

[7] J.A. Gálvez, A. Martínez and F. Milán, *Flat surfaces in the hyperbolic 3-space*, Math. Ann. **316**, no. 3 (2000), 419–435.

[8] G. Ishikawa and Y. Machida, *Singularities of improper affine spheres and surfaces of constant Gaussian curvature*, Internat. J. Math. **17** (2006), 269–293.

[9] S. Izumiya, *Legendrian dualities and spacelike hypersurfaces in the lightcone*, Mosc. Math. J. **9** (2009), 325–357

[10] S. Izumiya, D. Pei and T. Sano, *Singularities of hyperbolic Gauss maps*, Proc. London Math. Soc. **86** (2003), 485–512.

[11] S. Izumiya, M.C. Romero Fuster, M.A.S. Ruas and F. Tari, *Differential Geometry from a Singularity Theory Viewpoint*, World Scientific Publishing, 2015.

[12] S. Izumiya and K. Saji, *The mandala of Legendrian dualities for pseudo-spheres in Lorentz-Minkowski space and "flat" spacelike surfaces*, J. Singul. **2** (2010), 92–127.

[13] S. Izumiya, K. Saji and M. Takahashi, *Horospherical flat surfaces in hyperbolic 3-space*, J. Math. Soc. Japan **62** (2010), 789–849.

[14] M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada, *Singularities of flat fronts in hyperbolic space*, Pacific J. Math. **221** (2005), 303–351.

[15] M. Kokubu, W. Rossman, M. Umehara and K. Yamada, *Flat fronts in hyperbolic 3-space and their caustics*, J. Math, Soc. Japan **59** (2007), 265–299.

[16] M. Kokubu, M. Umehara and K. Yamada, *Flat fronts in hyperbolic 3-space*, Pacific. J. Math. **216** (2004), 149–175.

[17] L.F. Martins and K. Saji, *Geometric invariants of cuspidal edges*, to appear in Canad. J. Math., arXiv:1412.4099.

[18] L.F. Martins, K. Saji, M. Umehara and K. Yamada, *Behavior of Gaussian curvature and mean curvature near non-degenerate singular points on wave fronts*, arXiv:1308.2136.

[19] S. Murata and M. Umehara, *Flat surfaces with singuralities in Euclidean 3-space*, J. Differential Geom. **221** (2005), 303–351.

[20] K. Naokawa, M. Umehara and K. Yamada, *Isometric deformations of cuspidal edges*, to appare in Tohoku Math. J., arXiv:1408.4243.

[21] I.R. Porteous, *The normal singularities of submanifold*, J. Differential Geom. **5** (1971), 543–564.

[22] I.R. Porteous, *Geometric differentiation for the intelligence of curves and surfaces*, second edition, Cambridge University Press, 2001.

[23] K. Saji, *Koenderink type theorems for fronts*, J. Singul. **10** (2014), 264–270.

[24] K. Saji, M. Umehara and K. Yamada, *The geometry of fronts*, Ann. of Math. **169** (2009), 491–529.

[25] K. Saji, M. Umehara and K. Yamada, *The duality between singular points and inflection points on wave fronts*, Osaka J. Math. **47** (2010), 591–607.

[26] K. Teramoto, *Parallel and dual surfaces of cuspidal edges*, Differ. Geom. Appl. **44** (2016), 52–62.

[27] V.M. Zakalyukin, *Reconstructions of fronts and caustics depending on a parameter and versality of mappings*, J. Soviet Math. **27** (1984), 2713–2735.

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