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THE POSITIVITY OF THE TRANSMUTATION OPERATORS ASSOCIATED TO THE CHEREDNIK OPERATORS FOR THE ROOT SYSTEM BC_2

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Abstract. We consider the transmutation operators V_k, ${}^tV_k$ and V_k^W, ${}^tV_k^W$ associated respectively with the Cherednik operators and the Heckman-Opdam theory attached to the root system BC_2, called also in [8, 9, 10] the trigonometric Dunkl intertwining operators, and their dual. In this paper we prove that the operators V_k, ${}^tV_k$ and V_k^W, ${}^tV_k^W$ are positivity preserving and allows positive integral representations. In particular we deduce that the Opdam-Cherednik and the Heckman-Opdam kernels are positive definite.

1. Introduction

In [1] I. Cherednik introduced a family of differential-difference operators that nowadays bear his name. These operators play a crucial role in the theory of Heckman Opdam's hypergeometric functions, which generalizes the theory of Harish-Chandra's spherical functions on Riemannian symmetric spaces (see [3, 4, 6]).

To study in [9, 10] a harmonic analysis associated with the Cherednik operators, the author has introduced in [8, 10] the transmutation operators V_k, V_k^W called also the trigonometric Dunkl intertwining operators and their dual ${}^tV_k, {}^tV_k^W$. In many situations to solve problems of this harmonic analysis we need the positivity of the operators V_k, ${}^tV_k$ and $V_k^W, {}^tV_k^W$. This property is not yet proved in the general case, it is obtained only in the one dimensional case and for the root system of type A_2 (see [2, 11]).

This paper is a contribution towards this question in the case of the Cherednik operators attached to the root system of type BC_2.

In this paper we consider the Cherednik operators $T_j, j = 1, 2$ associated with the root system of type BC_2. We present definitions and properties of the trigonometric Dunkl intertwining operator V_k and of its dual ${}^tV_k$ (see [8, 9]), and we prove that they are positive integral transforms. To obtain this result we establish first that the function $V_k(p_s(u, .))(x)$ is positive, where $p_s(u, y), s > 0$, is the classical heat kernel on $\mathbb{R}^2$, and next we use the fact that the operators V_k and ${}^tV_k$ are transposes of each other, and that the family $\{p_s\}_{s>0}$ is an approximate of the identity.

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By using the relations between the operators V_k, ${}^tV_k$ and $V_k^W, {}^tV_k^W$, we deduce that the operators $V_k^W, {}^tV_k^W$ are also positive integral transforms.

The method used in this paper can be applied also to prove the positivity of the transmutation operators associated to the Cherednik operators and the Heckman-Opdam theory attached to the root system of type $BC_d, d \geq 3$. The results of this general case will be given in a forthcoming paper.

2. The Cherednik operators

We consider $\mathbb{R}^2$ with the standard basis $\{e_1, e_2\}$, and inner product $\langle ., .\rangle$ for which this basis is orthonormal. We extend this inner product to a complex bilinear form on $\mathbb{C}^2$.

2.1. The root system of type BC_2 and the Cherednik operators on $\mathbb{R}^2$. The root system of type BC_2 can be identified with the set $\mathcal{R}$ given by

$$\mathcal{R} = \{\pm e_1, \pm e_2, \pm 2e_1, \pm 2e_2\} \cup \{\pm e_1 \pm e_2\}, \tag{2.1}$$

which can also be written in the form

$$\mathcal{R} = \{\pm \alpha_i, i = 1, 2, ..., 6\},$$

with

$$\alpha_1 = e_1, \alpha_2 = e_2, \alpha_3 = 2e_1, \alpha_4 = 2e_2\ , \alpha_5 = (e_1 - e_2), \alpha_6 = (e_1 + e_2). \tag{2.2}$$

We denote by $\mathcal{R}_+$ the set of positive roots.

$$\mathcal{R}_+ = \{\alpha_i, i = 1, 2, ..., 6\}, \tag{2.3}$$

and by $\mathcal{R}_+^\circ$ the set of positive indivisible roots. For $\alpha \in \mathcal{R}$, we consider

$$r_\alpha(x) = x - \langle \breve{\alpha}, x\rangle \alpha, \text{ with } \breve{\alpha} = \frac{2\alpha}{\|\alpha\|^2}, \tag{2.4}$$

the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^2$ orthogonal to α.
The reflections $r_\alpha, \alpha \in \mathcal{R}$, generate a finite group W called the Weyl group associated with $\mathcal{R}$. In this case W is isomorphic to the hyperoctahedral group which is generated by permutations and sign changes of the $e_i, i = 1, 2$.

The multiplicity function $k : \mathcal{R} \to]0, +\infty[$ can be written in the form $k = (k_1, k_2, k_3)$ where k_1 and k_2 are the values on the roots α_1, α_2, and α_3, α_4 respectively, and k_3 is the value on the roots α_5, α_6.

The positive Weyl chamber denoted by $\mathfrak{a}^+$ is given by

$$\mathfrak{a}^+ = \{x \in \mathbb{R}^2\ ; \quad \forall\, \alpha \in \mathcal{R}_+, \langle \alpha, x\rangle > 0\}. \tag{2.5}$$

it can also be written in the form

$$\mathfrak{a}^+ = \{(x_1, x_2) \in \mathbb{R}^2 \quad x_1 > x_2 > 0\}. \tag{2.6}$$

Moreover, let $\mathcal{A}_k$ be the weight function

$$\forall\, x \in \mathbb{R}^2,\ \ \mathcal{A}_k(x) = \prod_{\alpha\in\mathcal{R}_+} |\sinh\langle\frac{\alpha}{2}, x\rangle|^{2k(\alpha)}. \tag{2.7}$$

The Cherednik operators $T_j, j = 1, 2$, are defined for functions f of class C^1 on $\mathbb{R}^2$ by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha\in\mathcal{R}_+} \frac{k(\alpha)\langle\alpha, e_j\rangle}{1 - e^{-\langle\alpha,x\rangle}}\{f(x) - f(r_\alpha x)\} - \rho_j f(x), \tag{2.8}$$

with

$$\rho_j = \frac{1}{2}\sum_{\alpha\in\mathcal{R}_+} k(\alpha)\langle\alpha, e_j\rangle, \quad j = 1, 2. \tag{2.9}$$

These operators can also be written in the following form

$$T_1 f(x) = \frac{\partial}{\partial x_1} f(x) + k_1\, \frac{\{f(x) - f(r_{\alpha_1}x)\}}{1 - e^{-\langle\alpha_1,x\rangle}} + 2k_2\frac{\{f(x) - f(r_{\alpha_3}x)\}}{1 - e^{-\langle\alpha_3,x\rangle}}$$

$$+k_3\Big[\frac{f(x) - f(r_{\alpha_5}x)}{1 - e^{-\langle\alpha_5,x\rangle}} + \frac{f(x) - f(r_{\alpha_6}x)}{1 - e^{-\langle r_{\alpha_6},x\rangle}}\Big] - (\frac{1}{2}k_1 + k_2 + k_3)f(x), \tag{2.10}$$

$$\begin{aligned} T_2 f(x) \quad &= \frac{\partial}{\partial x_2} f(x) + \frac{\{f(x) - f(r_{\alpha_2}x)\}}{1 - e^{-\langle\alpha_2,x\rangle}} + 2k_2\frac{\{f(x) - f(r_{\alpha_4}x)\}}{1 - e^{-\langle\alpha_4 x)\rangle}} \\ &+k_3\Big[- (\frac{f(x) - f(r_{\alpha_5}x)}{1 - e^{-\langle\alpha_5,x\rangle}}) + (\frac{f(x) - f(r_{\alpha_6}x)}{1 - e^{-\langle\alpha_6,x\rangle}})\Big] - (\frac{1}{2}k_1 + k_2)f(x). \end{aligned} \tag{2.11}$$

2.2. **The Opdam-Cherednik and the Heckman-Opdam kernels (see** [3, 4, 6, 8]**).** We denote by $G_\lambda, \lambda \in \mathbb{C}^2$, the eigenfunction of the operators $T_j, j = 1, 2$. It is the unique analytic function on $\mathbb{R}^2$ which satisfies the differential-difference system

$$\begin{cases} T_j G_\lambda(x) & = -i\lambda_j G_\lambda(x), j = 1, 2, x \in \mathbb{R}^2, \\ G_\lambda(0) & = 1. \end{cases} \tag{2.12}$$

It is called the Opdam-Cherednik kernel.

We consider the function $F_\lambda, \lambda \in \mathbb{C}^2$, defined by

$$\forall\, x \in \mathbb{R}^2,\ \ F_\lambda(x) = \frac{1}{|W|}\sum_{w\in W} G_\lambda(wx). \tag{2.13}$$

It is called the Heckman-Opdam hypergeometric function.

The functions G_λ and F_λ possess the following properties .

i) For all $x \in \mathbb{R}^2$ the function $\lambda \to G_\lambda(x)$ is entire on $\mathbb{C}^2$.

ii) We have

$$\forall\, x \in \mathbb{R}^2,\ \ \forall\, \lambda \in \mathbb{C}^2, |G_\lambda(x)| \le G_{Im(\lambda)}(x). \tag{2.14}$$

iii) We have

$$\forall\, x \in \mathbb{R}^2,\ \forall\, \lambda \in \mathbb{R}^2,\ |G_\lambda(x)| \leq |W|^{1/2}. \tag{2.15}$$

$$\forall\, x \in \mathbb{R}^2,\ \forall\, \lambda \in \mathbb{R}^2,\ |F_\lambda(x)| \leq |W|^{1/2}. \tag{2.16}$$

iv) For $x \in \mathbb{R}^2$, we denote by x^+ the only point in the orbit $W.x$ which lies in $\overline{\mathfrak{a}^+}$. Then we have

$$\forall\, x \in \mathbb{R}^2, G_0(x) \asymp \prod_{\substack{\alpha\in\mathcal{R}_+^\circ \\ \langle\alpha,x\rangle\geq 0}} (1+\langle\alpha,x\rangle)e^{-\langle\rho,x^+\rangle}. \tag{2.17}$$

v) Let p and q be polynomials of degree m and n. Then there exists a positive constant M such that for all $\lambda \in \mathbb{C}^2$ and $x \in \mathbb{R}^2$, we have

$$|p(\frac{\partial}{\partial\lambda})q(\frac{\partial}{\partial x})G_\lambda(x)| \leq M(1+\|x\|)^m(1+\|\lambda\|)^n F_0(x)e^{\max_{w\in W} Im\langle w\lambda,x\rangle}. \tag{2.18}$$

vi) The function F_0 satisfies the estimate

$$\forall\, x \in \overline{\mathfrak{a}^+}, F_0(x) \asymp e^{-\langle\rho,x\rangle} \prod_{\alpha\in\mathcal{R}_+^\circ} (1+\langle\alpha,x\rangle). \tag{2.19}$$

vii) The function $G_\lambda, \lambda \in \mathbb{C}^2$, admits the following Laplace type representation

$$\forall\, x \in \mathbb{R}^2, G_\lambda(x) = \langle K_x, e^{-i\langle\lambda,.\rangle}\rangle, \tag{2.20}$$

where K_x is some distribution in $\mathcal{E}'(\mathbb{R}^2)$ (the space of distributions on $\mathbb{R}^2$ with compact support) with support in $\Gamma = \text{conv}\{wx, w \in W\}$ (the convex hull of the orbit of x under W).

viii) From (2.13), (2.20) we deduce that the function $F_\lambda, \lambda \in \mathbb{C}^2$, possesses the Laplace type representation

$$\forall\, x \in \mathbb{R}^2, F_\lambda(x) = \langle K_x^W, e^{-i\langle\lambda,.\rangle}\rangle, \tag{2.21}$$

where K_x^W is the distribution given by

$$K_x^W = \frac{1}{|W|}\sum_{w\in W} K_{wx}. \tag{2.22}$$

3. The trigonometric Dunkl intertwining operators and its dual

Notations. We denote by

- $\mathcal{E}(\mathbb{R}^2)$ the space of C^∞-functions on $\mathbb{R}^2$. Its topology is defined by the semi-norms

$$q_{n,K}(\varphi) = \sup_{\substack{|\mu|\leq n \\ x\in K}} |D^\mu\varphi(x)|.$$

where K is a compact subset of $\mathbb{R}^2$, $n \in \mathbb{N}$ and

$$D^\mu = \frac{\partial^{|\mu|}}{\partial^{\mu_1} x_1 \partial^{\mu_2} x_2}\ , \mu = (\mu_1, \mu_2) \in \mathbb{N}^2, |\mu| = \mu_1 + \mu_2.$$

- $\mathcal{D}(\mathbb{R}^2)$ the space of C^∞-functions on $\mathbb{R}^2$ with compact support. We have

$$\mathcal{D}(\mathbb{R}^2) = \cup_{a>0} \mathcal{D}_a(\mathbb{R}^2),$$

where $\mathcal{D}_a(\mathbb{R}^2)$ is the space of C^∞-functions on $\mathbb{R}^2$ with support in the closed ball $B(0, a)$ of center 0 and radius a. The topology of $\mathcal{D}_a(\mathbb{R}^2)$ is defined by the semi-norms

$$P_n(\psi) = \sup_{\substack{|\mu| \leq n \\ x \in B(0,a)}} |D^\mu \psi(x)|, n \in \mathbb{N}.$$

The space $\mathcal{D}(\mathbb{R}^2)$ is equipped with the inductive limit topology.

By using the distribution K_x given by (2.20) we define by applying [8], the trigonometric Dunkl intertwining operator V_k on $\mathcal{E}(\mathbb{R}^2)$ relating to the root system BC_2 by

$$\forall\, x \in \mathbb{R}^2, V_k(g)(x) = \langle K_x, g \rangle. \tag{3.1}$$

The operator V_k is the unique linear topological isomorphism from $\mathcal{E}(\mathbb{R}^2)$ onto itself satisfying the transmutation relations

$$\forall\, x \in \mathbb{R}^2,\ \ T_j V_k(g)(x) = V_k(\frac{\partial}{\partial y_j} g)(x), \ \ j = 1, 2, \tag{3.2}$$

and the condition

$$V_k(g)(0) = g(0). \tag{3.3}$$

The dual ${}^tV_k$ of the operator V_k is defined by the following duality relation

$$\int_{\mathbb{R}^2} {}^tV_k(f)(y) g(y) dy = \int_{\mathbb{R}^2} V_k(g)(x) f(x) \mathcal{A}_k(x) dx, \tag{3.4}$$

with f in $\mathcal{D}(\mathbb{R}^2)$ and g in $\mathcal{E}(\mathbb{R}^2)$.

The operator ${}^tV_k$ is a linear topological isomorphism from $\mathcal{D}(\mathbb{R}^2)$ onto itself satisfying the transmutation relations

$$\forall\, y \in \mathbb{R}^2, {}^tV_k((T_j + S_j)f)(y) = \frac{\partial}{\partial y_j} {}^tV_k(f)(y), j = 1, 2, \tag{3.5}$$

where S_j is the operator on $\mathcal{D}(\mathbb{R}^2)$ given by

$$\forall\, x \in \mathbb{R}^2, S_j(h)(x) = \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\ \langle \alpha, e_j \rangle h(r_\alpha x).$$

Remark 1. By using the distribution K_x^W given by (2.22) we have defined and studied in [10] the trigonometric Dunkl intertwining operator V_k^W on $\mathcal{E}(\mathbb{R}^2)^W$ (the space of C^∞-functions on $\mathbb{R}^2$ which are W-invariant) and we have studied also its dual ${}^tV_k^W$ on $\mathcal{D}(\mathbb{R}^2)^W$ (the space of C^∞-functions on $\mathbb{R}^2$ which are of compact support and W-invariant). We have given some properties of these operators .

Proposition 3.1. *Let $s > 0$. For all $x, u \in \mathbb{R}^2$, we have*

$$V_k(p_s(u,.))(x) = \int_{\mathbb{R}^2} e^{-s\|\lambda\|^2} G_\lambda(x) e^{i\langle\lambda,u\rangle} d\lambda, \tag{3.6}$$

where $p_s(u,z)$ is the classical heat kernel given by

$$\forall\, u, z \in \mathbb{R}^2, p_s(u,z) = \int_{\mathbb{R}^2} e^{-s\|\lambda\|^2} e^{i\langle\lambda,u-z\rangle} d\lambda\ . \tag{3.7}$$

Proof. From (3.1) we have

$$\forall\, x \in \mathbb{R}^2\backslash\{0\}, u \in \mathbb{R}^2,\ V_k(p_s(u,.))(x) = \langle K_x, p_s(u,.)\rangle. \tag{3.8}$$

Thus from (3.8), for all $x \in \mathbb{R}^2\backslash\{0\}, u \in \mathbb{R}^2$, we have

$$V_k(p_s(u,.))(x) = \langle K_x(z), \int_{\mathbb{R}^2} e^{-s\|\lambda\|^2} e^{i\langle\lambda,u-z\rangle} d\lambda\rangle. \tag{3.9}$$

As the distribution K_x belongs to $\mathcal{E}'(\mathbb{R}^2)$, then the relation (3.9) can also be written in the form

$$V_k(p_s(u,.))(x) = \int_{\mathbb{R}^2} e^{-s\|\lambda\|^2} \langle K_x(z), e^{-i\langle\lambda,z\rangle}\rangle e^{i\langle\lambda,u\rangle} d\lambda\ . \tag{3.10}$$

Thus from (2.20), for all $x \in \mathbb{R}^2\backslash\{0\}, u \in \mathbb{R}^2$, we get

$$V_k(p_s(u,.))(x) = \int_{\mathbb{R}^2} e^{-s\|\lambda\|^2} G_\lambda(x) e^{i\langle\lambda,u\rangle} d\lambda. \tag{3.11}$$

On the other hand from (3.3), (3.7), for all $u \in \mathbb{R}^2$, we have

$$V_k(p_s(u,.))(0) = p_s(u,0) = \int_{\mathbb{R}^2} e^{-s\|\lambda\|^2} e^{i\langle\lambda,u\rangle} d\lambda. \tag{3.12}$$

We deduce (3.6) from (3.11), (3.12) and the continuity of the function $x \to V_k(p_s(u,.))(x)$ at $x = 0$. □

Proposition 3.2. *Let $s > 0$. The function $V_k(p_s(u,.))(x)$ is of class C^1 on $\mathbb{R}^2 \times \mathbb{R}^2$ with respect to the variables x and u, and satisfies the equations*

$$\forall\, x, u \in \mathbb{R}^2, (T_j + \frac{\partial}{\partial u_j}) V_k(p_s(u,.))(x) = 0, \quad j = 1, 2. \tag{3.13}$$

Proof. We obtain the results of this proposition by derivation under the integral sign with respect to the variables $x_j, u_j, j = 1, 2$, in the relation (3.6), and by using the relation (2.12). □

Proposition 3.3. i) *Let $s > 0$. There exists a positive function $C(s)$ such that*

$$\forall\, x; u \in \mathbb{R}^2, |V_k(p_s(u, .))(x)| \leq C(s) \prod_{\alpha\in\mathcal{R}_+^{\circ}} (1 + |\langle\alpha, x\rangle|)e^{-\langle\rho,x^+\rangle}, \tag{3.14}$$

where x^+ is the only point in the orbit $W.x$ which lies in $\overline{\mathfrak{a}^+}$.

ii) *Let $s > 0$. We have*

$$\forall\, x \in \mathbb{R}^2, \lim_{\|u\|\to+\infty} V_k(p_s(u, .))(x) = 0. \tag{3.15}$$

iii) *Let $s > 0$. The function $V_k(p_s(u, .))(x)$ is bounded on $\mathbb{R}^2 \times \mathbb{R}^2$ and we have*

$$\lim_{\|(x,u)\|\to+\infty} V_k(p_s(u, .))(x) = 0. \tag{3.16}$$

Proof. i) We deduce (3.14) from the relation (3.6), (2.14), (2.17).

ii) By using (3.6) and the fact that from (2.15) the function

$$\lambda \to e^{-s\|\lambda\|^2} G_\lambda(x)$$

is for all $x \in \mathbb{R}^2$, integrable with respect to the Lebesgue measure, we deduce (3.15) from the Riemann-Lebesgue Lemma. iii) We obtain (3.16) from (3.14), (3.15). □

4. Positivity of the operators V_k and ${}^tV_k$

In this section we prove first that for $s > 0$ the function $(x, u) \to V_k(p_s(u, .))(x)$ given by (3.6) is positive on $\mathbb{R}^2 \times \mathbb{R}^2$, and next we deduce the positivity of the operators V_k and ${}^tV_k$.

Proposition 4.1. i) *The Weyl chambers attached to the root system of type BC_2 are the following*

$$\begin{cases} \mathfrak{a}^+ &= \{x \in \mathbb{R}^2; \langle\alpha_i, x\rangle > 0, i = 1, 2, ..., 6\} \\ \mathfrak{a}^- &= -\mathfrak{a}^+ \end{cases} \tag{4.1}$$

$$\begin{cases} \mathfrak{a}_1^+ &= \{x \in \mathbb{R}^2; \langle\alpha_i, x\rangle > 0, i = 1, 2, 3, 4, 6; \langle\alpha_5, x\rangle < 0\} \\ \mathfrak{a}_1^- &= -\mathfrak{a}_1^+ \end{cases} \tag{4.2}$$

ii) *We denote by C_1, C_2, the Weyl chambers $\mathfrak{a}^+, \mathfrak{a}_1^+$, and by C_3, C_4, the Weyl chambers $\mathfrak{a}^-, \mathfrak{a}_1^-$. Then we have*

$$\mathbb{R}^2 = \bigcup_{\ell=1}^{4} \overline{C}_\ell\ . \tag{4.3}$$

where $\overline{C}_\ell$ is the closure of C_ℓ.

Proof. We determine the Weyl chambers corresponding to the six roots of $\mathcal{R}_+$, and next by applying the relations

$$\begin{cases} 2\alpha_1 = \alpha_3 \\ 2\alpha_2 = \alpha_4 \\ \alpha_1 - \alpha_2 = \alpha_5 \\ \alpha_1 + \alpha_2 = \alpha_6. \end{cases}$$

we obtain the Weyl chambers (4.1), (4,2) and the others are empty. □

Proposition 4.2. i*) For all $s > 0, x, u \in \mathbb{R}^2$, the function $V_k(p_s(u,.))(x)$ is real.* ii*) Let $s > 0$. The function $V_k(p_s(u,.))(x)$ is strictly positive on the set*

$$Y = \{(x,u) \in \mathbb{R}^2 \times \mathbb{R}^2; x = 0, u \in \mathbb{R}^2\}. \tag{4.4}$$

Proof. i) From (2.12) we deduce that

$$\forall\, x \in \mathbb{R}^2\ , \forall\, \lambda \in \mathbb{R}^2, \overline{G_\lambda(x)} = G_{-\lambda}(x).$$

We obtain the result from (3.6) by change of variables and by using the previous relation. ii) By using the fact that

$$\forall\, \lambda \in \mathbb{R}^2, G_\lambda(0) = 1,$$

we deduce from (3.6), (3.3), (3.7) that

$$\forall\, u \in \mathbb{R}^2, V_k(p_s(u,.))(0) = p_s(u,0) > 0.$$

Thus for all $s > 0$, the function $V_k(p_s(u,.))(x)$ is strictly positive on the set Y. □

Proposition 4.3. *We consider for $s > 0$, the function $U_s(x,u)$ defined by*

$$\forall\, (x,u) \in \mathbb{R}^2 \times \mathbb{R}^2,\ U_s(x,u) = V_k(p_s(u,.))(x). \tag{4.5}$$

Then for some $\alpha \in \mathcal{R}_+$ and $(x,u) \in \mathbb{R}^2 \times \mathbb{R}^2$, we have

$$\begin{aligned} U_s(r_\alpha x,u) - U_s(x,u) &= -\langle \breve{\alpha}, x\rangle \langle \nabla U_s(x,u), \alpha\rangle \\ &\quad + \tfrac{1}{2}(\langle \breve{\alpha}, x\rangle)^2 \alpha^t D^2 U_s(\xi,u)\alpha, \end{aligned} \tag{4.6}$$

with some ξ on the line segment between x and $r_\alpha x$.

Proof. We obtain (4.6) from the relation (2.4) and Taylor's formula. □

Theorem 4.4. *For all $s > 0$, we have*

$$\forall\, (x,u) \in \mathbb{R}^2 \times \mathbb{R}^2,\ V_k(p_s(u,.))(x) \geq 0. \tag{4.7}$$

Proof. The proof is made up in two steps.

In the first step we obtain some results concerning the positivity of the function $U_s(x,u)$ given by (4.5) on each of the sets $\overline{C_\ell}\times\mathbb{R}^2, \ell=1,2,3,4$, where $\overline{C_\ell}$ is the closure of the Weyl chamber C_ℓ given by Proposition 4.1 ii).

In the second step we use the fact that $\mathbb{R}^2\times\mathbb{R}^2=(\cup_{\ell=1}^4\overline{C_\ell})\times\mathbb{R}^2$ and the result of the first step to deduce the positivity of the function $U_s(x,u)$ on $\mathbb{R}^2\times\mathbb{R}^2$.

1st Step

We consider the set Y_ℓ, $\ell=1,2,3,4$, defined by

$$Y_\ell=\{(x,u)\in\mathbb{R}^2\times\mathbb{R}^2;\ x\in\overline{C}_\ell, u\in\mathbb{R}^2\}.$$

We denote by

$$\mathcal{V}_s^\ell(x,u)=U_s(x,u)1_{Y_\ell}(x,u),$$

where 1_{Y_ℓ} is the characteristic function of the set Y_ℓ. From Proposition 4.2 ii) the function $\mathcal{V}_s^\ell(x,u)$ is strictly positive on the set Y.

We shall prove that the function $\mathcal{V}_s^\ell(x,u)$ is positive on the set $Y_\ell\backslash Y$. If not we suppose by using Proposition 4.2 i) and Proposition 3.3 iii) that it attains a strictly negative absolute minimum at $(x^\ell,u^\ell)\in Y_\ell\backslash Y$ i.e.

$$\mathcal{V}_s^\ell(x^\ell,u^\ell)=\inf_{(x,u)\in Y_\ell}\mathcal{V}_s^\ell(x,u)<0. \tag{4.8}$$

There are two possibilities : the point (x^ℓ,u^ℓ) is in the open subset $(Y_\ell\backslash Y)^0$ of $Y_\ell\backslash Y$ or in the set

$$Y_\ell^0=\{(x,u)\in\mathbb{R}^2\times\mathbb{R}^2\,;x\in\partial\overline{C_\ell},u\in\mathbb{R}^2\}, \tag{4.9}$$

where $\partial\overline{C_\ell}$ is the border of the Weyl chamber $\overline{C_\ell}$.

We suppose that $(x^\ell,u^\ell)\in(Y_\ell\backslash Y)^0$. As the point (x^ℓ,u^ℓ) is an absolute minimum then we have

$$\frac{\partial}{\partial x_j}\mathcal{V}_s^\ell(x^\ell,u^\ell)=\frac{\partial}{\partial u_j}\mathcal{V}_s^\ell(x^\ell,u^\ell)=0,\quad j=1,2. \tag{4.10}$$

By using the fact that

$$\forall\,\alpha\in\mathcal{R}_+,(r_\alpha x^\ell,u^\ell)\notin Y_\ell,$$

and by applying the relations (4.5), (3.13), (2.10), (2.11), we get

$$\Big\{k_1\Big(\frac{1}{1-e^{-\langle\alpha_1,x^\ell\rangle}}-\frac{1}{2}\Big)+2k_2\Big(\frac{1}{1-e^{-\langle\alpha_3,x^\ell\rangle}}-\frac{1}{2}\Big)$$
$$+k_3\Big[\Big(\frac{1}{1-e^{-\langle\alpha_5,x^\ell\rangle}}-\frac{1}{2}\Big)+\Big(\frac{1}{1-e^{-\langle\alpha_6,x^\ell\rangle}}-\frac{1}{2}\Big)\Big]\Big\}\mathcal{V}_s^\ell(x^\ell,u^\ell)=0. \tag{4.11}$$

$$\Big\{k_1\Big(\frac{1}{1-e^{-\langle\alpha_2,x^\ell\rangle}}-\frac{1}{2}\Big)+2k_2\Big(\frac{1}{1-e^{-\langle\alpha_4,x^\ell\rangle}}-\frac{1}{2}\Big)$$

$$+k_3\Big[-\Big(\frac{1}{1-e^{-\langle\alpha_5,x^\ell\rangle}}-\frac{1}{2}\Big)+\Big(\frac{1}{1-e^{-\langle\alpha_6,x^\ell\rangle}}-\frac{1}{2}\Big)\Big]\Big\}\mathcal{V}_s^\ell(x^\ell,u^\ell)=0. \quad (4.12)$$

Using the fact that from (4.8) the function $\mathcal{V}_s^\ell(x^\ell,u^\ell)$ is different from zero and that $k_3>0$, the equations (4.11), (4.12) can also be written in the form

$$\frac{k_1}{k_3}X_1^\ell+\frac{2k_2}{k_3}X_3^\ell+X_5^\ell+X_6^\ell=0 \quad (4.13)$$

$$\frac{k_1}{k_3}X_2^\ell+\frac{2k_2}{k_3}X_4^\ell-X_5^\ell+X_6^\ell=0, \quad (4.14)$$

with

$$X_i^\ell=\frac{1+e^{-\langle\alpha_i,x^\ell\rangle}}{1-e^{-\langle\alpha_i,x^\ell\rangle}}\ ,i=1,2,...,6. \quad (4.15)$$

Then the $X_i^\ell, i=1,2,...6$, are solutions of the system of linear equations (S) on $\mathbb{R}^4$:

$$(S)\left\{\begin{array}{ll}\frac{k_1}{k_3}X_1+2\frac{k_2}{k_3}X_3+X_5+X_6 & =0,\\ \frac{k_1}{k_3}X_2+2\frac{k_2}{k_3}X_4-X_5+X_6 & =0.\end{array}\right. \quad (4.16)$$

On the other hand from (4.15) we obtain

$$e^{-\langle\alpha_i,x^\ell\rangle}=\frac{X_i^\ell-1}{X_i^\ell+1}\ ,i=1,2,...,6. \quad (4.17)$$

We consider the function f defined on $\mathbb{R}\backslash\{-1\}$ by

$$f(y)=\frac{y-1}{y+1}\ ,$$

we have

$$f(y)\leq 0\Leftrightarrow y\in]-1,1], \quad (4.18)$$

$$0<f(y)<1\Leftrightarrow y\in]1,+\infty[, \quad (4.19)$$

$$f(y)>1\Leftrightarrow y\in]-\infty,-1[. \quad (4.20)$$

From the relation (4.17), we have

$$e^{-\langle\alpha_i,x^\ell\rangle}=f(X_i^\ell),\quad i=1,2,...,6. \quad (4.21)$$

As the first member of (4.21) is strictly positive, then from (4.18) the $X_i^\ell, i=1,2,...,6$, are not in the interval $]-1,1]$. They are in the interval $]-\infty,-1[\cup]1,+\infty[$. We consider two cases .

1st Case

(1) If $x^\ell \in C_\ell, \ell = 1$.
From the relation (4.1) we have $\langle \alpha_i, x^\ell \rangle > 0$ for $i = 1, 3, 5, 6$. Then by using (4.21), (4.19) we obtain

$$X_i^\ell \in]1, +\infty[, \quad i = 1, 3, 5, 6. \tag{4.22}$$

By applying (4.22) we get

$$\frac{k_1}{k_3} X_1 + +2\frac{k_2}{k_3} X_3 + X_5 + X_6 > \frac{k_1}{k_3} + 2\frac{k_2}{k_3} + 2 > 0.$$

Thus from (4.13) we obtain an absurdity, and then the $X_i^\ell, i = 1, 2, ..., 6$, are not solutions of the system (S) given by (4.16).

(2) If $x^\ell \in C_\ell, \ell = 2$.
The same proof as for the previous case shows that we obtain also an absurdity, and then the $X_i^\ell, i = 1, 2, ..., 6$, are not solutions of the system (S) given by (4.16).

2nd Case

(1) If $x^\ell \in C_\ell, \ell = 3$.
From the relation (4.2) we have $\langle \alpha_i, x \rangle > 0, i = 2, 4, 6$ and $\langle \alpha_5, x \rangle < 0$. Then by using (4.21), (4.19), (4.20), we obtain

$$X_i^\ell \in]1, +\infty[, i = 2, 4, 6, \quad \text{and } X_5^\ell \in] - \infty, -1[. \tag{4.23}$$

By applying (4.23) we get

$$\frac{k_1}{k_3} X_2^\ell + 2\frac{k_2}{k_3} X_4^\ell - X_5^\ell + X_6^\ell > \frac{k_1}{k_3} + 2\frac{k_2}{k_3} + 2 > 0.$$

Thus from (4.14) we obtain an absurdity, and then the $X_i^\ell, i = 1, 2, ..., 6$, are not solutions of the system (S) given by (4.16).

(2) If $x^\ell \in C_\ell, \ell = 4$.
The same proof as for the previous case shows that we obtain also an absurdity, and then the $X_i^\ell, 1, 2, ..., 6$, are not solutions of the system (S) given by (4. 16).

From the first and second cases we deduce that our supposition that the function $\mathcal{V}_s^\ell(x^\ell, u^\ell)$ attains a strictly negative absolute minimum at (x^ℓ, u^ℓ) in $(Y_\ell \backslash Y)^0$ is absurd. Then the point (x^ℓ, u^ℓ) does not belong to $(Y_\ell \backslash Y)^0$, and it is in the set Y_ℓ^0.

2nd Step

From Proposition 4.2 ii) the function $U_s(x, u)$ is strictly positive on the set Y.
We shall prove that the function $U_s(x, u)$ is positive on the set $\mathbb{R}^2 \times \mathbb{R}^2 \backslash Y$. If not we suppose by using Proposition 4.2.i) and Proposition 3.3 iii) that it

attains a strictly negative absolute minimum at $(x_0,u_0)\in\mathbb{R}^2\times\mathbb{R}^2\backslash Y$. From the first step and the relation (4.3), the point (x_0,u_0) is in the set

$$Y^0=\bigcup_{\ell=1}^{4}Y^0_\ell,$$

with Y^0_ℓ given by (4.9). We have

$$Y^0=\{(x,u)\in\mathbb{R}^2\times\mathbb{R}^2\,;\forall\,\alpha\in\mathcal{R}_+,\ \langle\alpha,x\rangle=0,u\in\mathbb{R}^2\},$$

then

$$\forall\,\alpha\in\mathcal{R}_+,\ \langle\alpha,x_0\rangle=0.$$

We shall prove in the following that the point (x_0,u_0) is not in the set Y^0. As the point (x_0,u_0) is a strictly negative absolute minimum, then we have the following relations

$$U_s(x_0,u_0)=\inf_{(x,u)\in\mathbb{R}^2\times\mathbb{R}^2}U_s(x,u)<0, \tag{4.24}$$

and

$$\frac{\partial}{\partial x_1}U_s(x_0,u_0)=\frac{\partial}{\partial u_1}U_s(x_0,u_0)=0\ . \tag{4.25}$$

We write the relations (4.5), (3.13), (2.10) for x,u_0, and we get

$$\begin{aligned}&\frac{\partial}{\partial x_1}U_s(x,u_0)+\frac{\partial}{\partial u_1}U_s(x,u_0)+k_1\frac{\{U_s(x,u_0)-U_s(r_{\alpha_1}x,u_0)\}}{1-e^{-\langle\alpha_1,x\rangle}}\\&+2k_2\tfrac{\{U_s(x,u_0)-U_s(r_{\alpha_3}x,u_0)\}}{1-e^{-\langle\alpha_3,x\rangle}}\\&+k_3\Big[\frac{U_s(x,u_0)-U_s(r_{\alpha_5}x,u_0)}{1-e^{-\langle\alpha_5,x\rangle}}+\frac{U_s(x,u_0)-U_s(r_{\alpha_6}x,u_0)}{1-e^{-\langle\alpha_6,x\rangle}}\Big]\\&\qquad=(\frac{1}{2}k_1+k_2+k_3)U_s(x,u_0).\end{aligned} \tag{4.26}$$

Then by passing to the limit in (4.26), when $\langle\alpha,x\rangle$, for all $\alpha\in\mathcal{R}_+$, goes to $\langle\alpha,x_0\rangle=0$, and by using Proposition 4.3 and the relation (4.25), we obtain

$$(\frac{1}{2}k_1+k_2+k_3)U_s(x_0,u_0)=0.$$

As $k_i>0,i=1,2,3$, then

$$U_s(x_0,u_0)=0. \tag{4.27}$$

Thus (4.24) and (4.27) imply a contradiction, and the point (x_0,u_0) is not in the set Y^0.
Then the function $U_s(x,u)$ is positive on the set $\mathbb{R}^2\times\mathbb{R}^2\backslash Y$. We deduce the relation (4.7) from this result and the fact that the function $V_k(p_s(u,.))(x)$ is positive on the set Y. □

Theorem 4.5. *For all positive functions f in $\mathcal{D}(\mathbb{R}^2)$, we have*

$$\forall\, y \in \mathbb{R}^2, {}^tV_k(f)(y) \geq 0. \tag{4.28}$$

Proof. From the relations (3.4), (3.7), for all $s > 0$ and $y \in \mathbb{R}^2$ we have

$$\int_{\mathbb{R}^2} {}^tV_k(f)(x)p_s(y,x)dx = \int_{\mathbb{R}^2} f(z)V_k(p_s(y,.))(z)\mathcal{A}_k(z)dz.$$

But from Theorem 4.4, the second member of this relation is positive. Then

$$\int_{\mathbb{R}^2} {}^tV_k(f)(x)p_s(y,x)dx = {}^tV_k(f) * E_s(y) \geq 0,$$

with E_s the classical Gauss kernel given by

$$\forall\, u \in \mathbb{R}^2, E_s(u) = \int_{\mathbb{R}^2} e^{-s\|\lambda\|^2} e^{i\lambda u} d\lambda,$$

and $*$ the classical convolution product on $\mathbb{R}^2$.
Thus

$${}^tV_k(f)(y) = \lim_{s \to 0} {}^tV_k(f) * E_s(y) \geq 0.$$

□

Theorem 4.6. *There exists a σ-algebra $\mathfrak{m}$ in $\mathbb{R}^2$ which contains all Borel sets in $\mathbb{R}^2$, and for each $y \in \mathbb{R}^2$, there exists a unique positive measure ν_y on $\mathfrak{m}$ such that for every f in $\mathcal{D}(\mathbb{R}^2)$, we have*

$${}^tV_k(f)(y) = \int_{\mathbb{R}^2} f(x)d\nu_y(x). \tag{4.29}$$

The measure ν_y satisfies

$$\nu_y(K) < +\infty, \text{ for every compact } K \subset \mathbb{R}^2. \tag{4.30}$$

Proof. We deduce the results of this theorem from the relation (4.28) and Theorem 2.14 p. 42 of [5]. □

Theorem 4.7. *For all g in $\mathcal{E}(\mathbb{R}^2)$, positive, we have*

$$\forall\, x \in \mathbb{R}^2, \;\; V_k(g)(x) \geq 0. \tag{4.31}$$

Proof. From the relation (3.4), for all f in $\mathcal{D}(\mathbb{R}^2)$ positive and g in $\mathcal{E}(\mathbb{R}^2)$ positive, we have

$$\int_{\mathbb{R}^2} V_k(g)(x)f(x)\mathcal{A}_k(x)dx = \int_{\mathbb{R}^2} {}^tV_k(f)(y)g(y)dy.$$

By applying Theorem 4.5 to the second member, we deduce that

$$\int_{\mathbb{R}^2} V_k(g)(x)f(x)\mathcal{A}_k(x)dx \geq 0.$$

Thus

$$\int_{\mathbb{R}^2} f(x)V_k(g)(x)\mathcal{A}_k(x)dx = \langle T_{V_k(g)\mathcal{A}_k}, f\rangle \geq 0, \tag{4.32}$$

where $T_{V_k(g)\mathcal{A}_k}$ is the distribution of $\mathcal{D}'(\mathbb{R}^2)$ (the space of distributions on $\mathbb{R}^2$) given by the function $V_k(g)\mathcal{A}_k$. From (4.32) and Theorem V of [7] p. 29, this distribution is the positive measure of density $V_k(g)\mathcal{A}_k$ with respect to the Lebesgue measure on $\mathbb{R}^2$. Then by using the relation (2.7) and the continuity of the function $V_k(g)$ on $\mathbb{R}^2$, we obtain (4.31). □

Theorem 4.8. *There exists a σ-algebra $\mathfrak{m}$ in $\mathbb{R}^2$ which contains all Borel sets in $\mathbb{R}^2$, and for each $x \in \mathbb{R}^2$, there exists a unique positive measure μ_x on $\mathfrak{m}$ with support in $B(0, \|x\|)$ the closed ball of center 0 and radius $\|x\|$, such that for every g in $\mathcal{E}(\mathbb{R}^2)$, we have*

$$V_k(g)(x) = \int_{\mathbb{R}^2} g(y)d\mu_x(y). \tag{4.33}$$

Proof. From (3.1), (4.31) we have

$$V_k(g)(x) = \langle K_x, g\rangle \geq 0, \tag{4.34}$$

with K_x in $\mathcal{E}'(\mathbb{R}^2)$ such that supp $K_x \subset B(0, \|x\|)$.
Then from (4.34) and Theorem V of [7] p. 29, the distribution K_x is a positive measure on $\mathfrak{m}$ denoted by μ_x with support in $B(0, \|x\|)$. □

Corollary 4.9. *There exists a σ-algebra $\mathfrak{m}$ in $\mathbb{R}^2$ which contains all Borel sets in $\mathbb{R}^2$.*

(1) *For each $x \in \mathbb{R}^2$, there exists a unique positive measure μ_x^W on $\mathfrak{m}$ with support in $B(0, \|x\|)$ such that for every g in $\mathcal{E}(\mathbb{R}^2)^W$, we have*

$$V_k^W(g)(x) = \int_{\mathbb{R}^2} g(y)d\mu_x^W(y), \tag{4.35}$$

where

$$\mu_x^W = \frac{1}{|W|}\sum_{w\in W} \mu_{wx}. \tag{4.36}$$

(2) *For each $y \in \mathbb{R}^2$, there exists a unique positive measure ν_y^W on $\mathfrak{m}$ such that for every f in $\mathcal{D}(\mathbb{R}^2)^W$, we have*

$${}^tV_k^W(f)(y) = \int_{\mathbb{R}^2} f(x)d\nu_y^W(x),$$

where

$$\nu_y^W = \frac{1}{|W|}\sum_{w\in W} \nu_{wy}. \tag{4.37}$$

The measure ν_y^W satisfies

$$\nu_y^W(K) < +\infty, \text{ for every compact } K \subset \mathbb{R}^2. \tag{4.38}$$

Corollary 4.10. *We have the following* i) *The Opdam-Cherednik and the Heckman-Opdam kernels* $G_\lambda(x)$ *and* $F_\lambda(x), \lambda \in \mathbb{C}^2$*, possess the Laplace type integral representations*

$$G_\lambda(x) = \int_{\mathbb{R}^2} e^{-i\langle\lambda,y\rangle} d\mu_x(y), \quad \forall x \in \mathbb{R}^2, \tag{4.39}$$

$$F_\lambda(x) = \int_{\mathbb{R}^2} e^{-i\langle\lambda,y\rangle} d\mu_x^W(y), \quad \forall x \in \mathbb{R}^2. \tag{4.40}$$

ii) *We have*

$$\forall\, x \in \mathbb{R}^2,\ \forall\, \lambda \in \mathbb{R}^2, G_{i\lambda}(x) > 0. \tag{4.41}$$

$$\forall\, x \in \mathbb{R}^2, \forall\, \lambda \in \mathbb{R}^2,\ \ F_{i\lambda}(x) > 0. \tag{4.42}$$

iii) *For all* $x \in \mathbb{R}^2$*, the function* $\lambda \to G_\lambda(x)$ *and* $\lambda \to F_\lambda(x)$ *are positive definite on* $\mathbb{R}^2$*.*

Remark 2. We have studied in [12] the absolute continuity with respect to the Lebesgue measure of the measures μ_x, ν_y, μ_x^W, ν_y^W.

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