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ON A DUALITY OF GRAS BETWEEN TOTALLY POSITIVE AND PRIMARY CYCLOTOMIC UNITS

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Abstract. Let K be a real abelian field of odd degree over $\mathbb{Q}$, and C the group of cyclotomic units of K. We denote by C_+ and C_0 the totally positive and primary elements of C, respectively. G. Gras found a duality between the Galois modules C_+/C^2 and C_0/C^2 by some ingenious calculation on cyclotomic units. We give an alternative proof using a consequence (="Gras conjecture") of the Iwasawa main conjecture and the standard reflection argument. We also give some related topics.

1. Introduction

Let K be a real abelian field with $[K:\mathbb{Q}]$ odd, and let $\Delta = \mathrm{Gal}(K/\mathbb{Q})$. Let E be the group of units of K, E_+ the subgroup consisting of totally positive units, and E_0 the subgroup consisting of units ϵ satisfying $\epsilon \equiv u^2 \bmod 4$ for some $u \in K^\times$. (A unit satisfying this congruence is often called a "primary" unit.) Denote by C the group of cyclotomic units in the sense of Sinnott [11, page 209]. We put $C_+ = C \cap E_+$ and $C_0 = C \cap E_0$. Let χ be a nontrivial $\bar{\mathbb{Q}}_2$-valued character of Δ, and $\mathcal{O} = \mathcal{O}_\chi$ the subring of $\bar{\mathbb{Q}}_2$ generated over $\mathbb{Z}_2$ by the values of χ. Here, $\bar{\mathbb{Q}}_2$ is a fixed algebraic closure of the 2-adic rationals $\mathbb{Q}_2$, and $\mathbb{Z}_2$ is the ring of 2-adic integers. For a $\mathbb{Z}_2[\Delta]$-module M, we denote by $M(\chi) = M \otimes \mathcal{O}$ the χ-part of M, where the tensor product is taken over $\mathbb{Z}_2[\Delta]$ regarding $\mathcal{O}$ as a $\mathbb{Z}_2[\Delta]$-module via χ. We naturally regard $M(\chi)$ as an $\mathcal{O}$-module. We see that $(E/E^2)(\chi) \cong \mathcal{O}/2\mathcal{O}$ by a theorem of Minkowski on units of a Galois extension over $\mathbb{Q}$ (cf. Narkiewicz [9, Theorem 3.26]). Since the index $[E:C]$ is finite ([11, Theorem 4.1]), it follows that $(C/C^2)(\chi) \cong \mathcal{O}/2\mathcal{O}$. Therefore, each of the $\mathcal{O}$-modules $(E_+/E^2)(\chi)$, $(E_0/E^2)(\chi)$, $(C_+/C^2)(\chi)$ and $(C_0/C^2)(\chi)$ is either trivial or isomorphic to $\mathcal{O}/2\mathcal{O}$. In [4, Théorème III.2], Georges Gras found the following beautiful relation between the Galois modules C_+/C^2 and C_0/C^2.

Theorem 1 (Gras). *Under the above setting, we have $(C_+/C^2)(\chi) \cong \mathcal{O}/2\mathcal{O}$ if and only if $(C_0/C^2)(\chi^{-1}) \cong \mathcal{O}/2\mathcal{O}$.*

Gras proved this duality by some ingenious calculation on the "Fermat quotient" of certain cyclotomic units. The main purpose of this paper is to give a modern alternative proof using a consequence (="Gras conjecture") of the

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Iwasawa main conjecture and the standard reflection argument.

As for the groups E_+ and E_0, Taylor [12] proved the following

Theorem 2 (Taylor)**.** *Under the above setting, the following conditions are equivalent:*

(I) *At least one of* $(E_+/E^2)(\chi)$ *and* $(E_+/E^2)(\chi^{-1})$ *is nontrivial.*

(II) *At least one of* $(E_0/E^2)(\chi)$ *and* $(E_0/E^2)(\chi^{-1})$ *is nontrivial.*

Taylor proved this theorem by effectively using some properties of norm residue symbols including the product formula. We give an alternative proof using the reflection argument.

Remark 1. Assertion $(*)$ of [12, page 157] reads that if *both* of $(E_+/E^2)(\chi)$ and $(E_+/E^2)(\chi^{-1})$ are nontrivial, then the condition (II) in Theorem 2 holds. However, what Taylor actually proved in his paper is the above stronger version.

In §2, we recall some fundamental facts on the reflection argument, which is a basic tool of this paper. In §3, we prove Theorem 1. In §4, we prove Theorem 2 and give an example which shows that a duality corresponding to Theorem 1 does not hold in general for E_+ and E_0. In §5, we give a slight generalization of a theorem of Cornacchia [2, Theorem 1] on the triviality of the minus and plus class groups of a cyclotomic field of prime conductor by using a reflection theorem and "genus theory",

2. Reflection

In this section, we recall some fundamental facts on the reflection argument for the 2-parts of ideal class groups. Typical literatures on this topic are [8, 10, 12]. We use the same notation as in §1. In particular, K denotes a real abelian field of odd degree over $\mathbb{Q}$. For a number field N, let A_N (resp. $\tilde{A}_N$) be the 2-part of the ideal class group of N in the ordinary (resp. narrow) sense. We often abbreviate as $A = A_K$ and $\tilde{A} = \tilde{A}_K$ in the following. For an ideal $\mathfrak{A}$ of K, we denote by $[\mathfrak{A}]_0$ (resp. $[\mathfrak{A}]_\infty$) the ideal class in the ordinary (resp. narrow) sense containing $\mathfrak{A}$. Let H/K and $\tilde{H}/K$ be the class fields corresponding to the quotients A/A^2 and $\tilde{A}/\tilde{A}^2$, respectively. Then the Galois group $\mathrm{Gal}(H/K)$ (resp. $\mathrm{Gal}(\tilde{H}/K)$) is canonically isomorphic to A/A^2 (resp. $\tilde{A}/\tilde{A}^2$) via the reciprocity law map which is compatible with the action of Δ. For a multipicative abelian group X and an element $x \in X$, we denote by $[x]$ the class in the quotient X/X^2 represented by x. Let V and $\tilde{V}$ be the subgroups of $K^\times/(K^\times)^2$ such that

$$H = K(v^{1/2} \mid [v] \in V) \quad \text{and} \quad \tilde{H} = K(v^{1/2} \mid [v] \in \tilde{V}),$$

respectively. Clearly, we have

$$V = \{[v] \in \tilde{V} \mid v \gg 0\}.$$

Here, we write $x \gg 0$ when an element $x \in K^\times$ is totally positive. We can naturally regard the groups V and $\tilde{V}$ as modules over $\mathbb{Z}_2[\Delta]$. It is well known that

$$(E(K^\times)^2/(K^\times)^2) \cap \tilde{V} = E_0(K^\times)^2/(K^\times)^2 \ (= E_0/E^2)$$

and that

$$(E(K^\times)^2/(K^\times)^2) \cap V = (E_+ \cap E_0)(K^\times)^2/(K^\times)^2 \ (= (E_+ \cap E_0)/E^2)$$

(cf. Washington [13, Excercise 9.3]). We have a nondegenerate pairing

$$\tilde{A}/\tilde{A}^2 \times \tilde{V} \to \{\pm 1\}; \quad ([c], [v]) \to \langle c, v\rangle = (v^{1/2})^{\rho_c - 1},$$

where ρ_c denotes the automorphism of $\tilde{H}/K$ corresponding to the ideal class c. It is well known and easy to show that the pairing satisfies $\langle c^\delta, v^\delta\rangle = \langle c, v\rangle$ for any $c \in \tilde{A}$, $[v] \in \tilde{V}$ and $\delta \in \Delta$. Because of this relation, the above pairing induces a nondegenerate subpairing

$$(\tilde{A}/\tilde{A}^2)(\chi) \times \tilde{V}(\chi^{-1}) \to \{\pm 1\} \tag{1}$$

for each $\bar{\mathbb{Q}}_2$-valued character χ of Δ. Similarly, we have a nondegenerate pairing

$$(A/A^2)(\chi) \times V(\chi^{-1}) \to \{\pm 1\}. \tag{2}$$

For each element $[v]$ of V or $\tilde{V}$, we have $v\mathcal{O}_K = \mathfrak{A}^2$ for some ideal $\mathfrak{A}$ of K where $\mathcal{O}_K$ is the ring of integers of K. By mapping $[v]$ to the ideal class $[\mathfrak{A}]_0 \in A$, we obtain exact sequences

$$\{0\} \to E_0/E^2 \to \tilde{V} \to {}_2A \tag{3}$$

and

$$\{0\} \to (E_+ \cap E_0)/E^2 \to V \to {}_2A \tag{4}$$

compatible with the action of Δ. Here, ${}_2A$ denotes the elements c of A with $c^2 = 1$. Let $K_{>0}$ be the subgroup of $K^\times$ consisting of totally positive elements. We also need the following natural exact sequence

$$\{0\} \to K^\times/EK_{>0} \xrightarrow{f} \tilde{A} \xrightarrow{g} A \to \{0\} \tag{5}$$

which is compatible with the action of Δ. For an element $\alpha \in K^\times$, let $[\alpha]_\infty$ be the class in $K^\times/EK_{>0}$ containing α. The maps f and g are defined by $f([\alpha]_\infty) = [\alpha\mathcal{O}_K]_\infty$ and $g([\mathfrak{A}]_\infty) = [\mathfrak{A}]_0$, respectively. It is known that the $\mathcal{O}$-module $(K^\times/EK_{>0})(\chi)$ is either trivial or isomorphic to $\mathcal{O}/2\mathcal{O}$ and that the following equivalence holds.

$$(K^\times/EK_{>0})(\chi) = \{0\} \iff (E_+/E^2)(\chi) = \{0\}. \tag{6}$$

This is because the Galois module $K^\times/K_{>0}$ is isomorphic to $\mathbb{F}_2[\Delta]$ via the sign map where $\mathbb{F}_2$ is the finite field of 2 elements.

Lemma 1. *Under the above setting, if $A(\chi)$ is trivial and $A(\chi^{-1})$ is nontrivial, then we have $(E_+/E^2)(\chi) \cong \mathcal{O}/2\mathcal{O}$.*

Proof. As $A(\chi^{-1})$ is nontrivial, $V(\chi)$ is nontrivial by (2). Then, since $A(\chi)$ is trivial, it follows from (4) that

$$((E_+ \cap E_0)/E^2)(\chi) = V(\chi) \cong \mathcal{O}/2\mathcal{O}.$$

Hence, we obtain $(E_+/E^2)(\chi) \cong \mathcal{O}/2\mathcal{O}$. □

Theorem 3. *Under the above setting, the following conditions are equivalent.*

(I) *At least one of $\tilde{A}(\chi)$ and $\tilde{A}(\chi^{-1})$ is trivial.*

(II) *The groups $A(\chi)$ and $A(\chi^{-1})$ are both trivial.*

Proof. The implication (II) $\Rightarrow$ (I) follows immediately from [10, Théorème 2]. To show (I) $\Rightarrow$ (II), assume that $\tilde{A}(\chi)$ is trivial. We see that $A(\chi)$ is also trivial, and that $(E_+/E^2)(\chi) = \{0\}$ from (5) and (6). Then it follows from Lemma 1 that $A(\chi^{-1})$ is trivial. □

Remark 2. Theorem 3 is a refinement of [10, Corollary 2c].

3. Proof of Theorem 1

We use the same notation as in the previous sections. In particular, χ denotes a nontrivial $\bar{\mathbb{Q}}_2$-valued character of Δ. The following consequence of the Iwasawa main conjecture was proved in Greither [5, Theorem 4.14]:

$$|(E/C)(\chi)| = |A(\chi)|, \tag{7}$$

which is called a conjecture of Gras. Here, we abbreviate the χ-part of the $\mathbb{Z}_2[\Delta]$-module $(E/C)\otimes\mathbb{Z}_2$ as $(E/C)(\chi)$, the tensor product being taken over $\mathbb{Z}$.

Lemma 2. *If $|(E/C)(\chi)| > 1$, then $((C_+ \cap C_0)/C^2)(\chi) \cong \mathcal{O}/2\mathcal{O}$.*

Proof. Assume that $|(E/C)(\chi)| > 1$. Then, as $(E/E^2)(\chi) \cong \mathcal{O}/2\mathcal{O}$, it follows that there exists a nontrivial class $[c] \in (C/C^2)(\chi)$ for which c is a square in E. Hence, $c \in C_+ \cap C_0$, and the assertion follows. □

Proof of Theorem 1. First we assume that $(C_+/C^2)(\chi) \cong \mathcal{O}/2\mathcal{O}$ and show the "only if" part. Namely, we show that $(C_0/C^2)(\chi^{-1}) \cong \mathcal{O}/2\mathcal{O}$. If $A(\chi^{-1})$ is nontrivial, then it follows from (7) and Lemma 2 that $(C_0/C^2)(\chi^{-1}) \cong \mathcal{O}/2\mathcal{O}$. Therefore, we may further assume that $A(\chi^{-1})$ is trivial. Let us show that $\tilde{A}(\chi)$ is nontrivial. For this, we choose a cyclotomic unit $\eta \in C_+$ for which the class $[\eta]$ generates $(C_+/C^2)(\chi)$ ($\cong \mathcal{O}/2\mathcal{O}$). If η is a square in K, then $|(E/C)(\chi)| > 1$. Therefore, it follows from (7) that $A(\chi)$ is nontrivial, and hence so is $\tilde{A}(\chi)$. If η is not a square in K, then, as $(E/E^2)(\chi) \cong \mathcal{O}/2\mathcal{O}$, $(E/C)(\chi)$ is trivial. As η is totally positive, this implies that $(E/E^2)(\chi) = (E_+/E^2)(\chi) \cong \mathcal{O}/2\mathcal{O}$. Then it follows from (5) and (6) that $\tilde{A}(\chi)$ is nontrivial also when η is nonsquare. Now, we see from (1) that $\tilde{V}(\chi^{-1})$ is nontrivial. However, since $A(\chi^{-1})$ is trivial, it follows from (3) that

$$(E_0/E^2)(\chi^{-1}) = \tilde{V}(\chi^{-1}) \cong \mathcal{O}/2\mathcal{O}.$$

Again as $A(\chi^{-1})$ is trivial, we obtain from this and (7) that $(C_0/C^2)(\chi^{-1}) \cong \mathcal{O}/2\mathcal{O}$.

We assume that $(C_0/C^2)(\chi) \cong \mathcal{O}/2\mathcal{O}$ and show that $(C_+/C^2)(\chi^{-1}) \cong \mathcal{O}/2\mathcal{O}$. Similarly as in the proof of the "only if" part, we may as well assume that $A(\chi^{-1})$ is trivial because of (7) and Lemma 2. We choose a cyclotomic unit $\eta \in C_0$ for which the class $[\eta]$ generates $(C_0/C^2)(\chi)$. First, we deal with the case where η is a square in K. In this case, we have $(E/C)(\chi) \neq \{0\}$ and hence $A(\chi)$ is nontrivial by (7). Then, by Lemma 1, we see that $(E_+/E^2)(\chi^{-1}) \cong \mathcal{O}/2\mathcal{O}$. Since $A(\chi^{-1})$ is trivial, it follows from this and (7) that $(C_+/C^2)(\chi^{-1}) \cong \mathcal{O}/2\mathcal{O}$. Next, we deal with the case where η is not a square. Then, $[\eta]$ is a nontrivial element of $\tilde{V}(\chi)$. If $\eta \gg 0$, then as $[\eta] \in V(\chi)$, it follows from (2) that $A(\chi^{-1})$ is nontrivial, a contradiction. If η is not totally positive, we have $[\eta] \in \tilde{V}(\chi) \setminus V(\chi)$. Hence, $|(\tilde{A}/\tilde{A}^2)(\chi^{-1})| > |(A/A^2)(\chi^{-1})|$ by (1) and (2). Then, by (5) and (6), we see that $(E/E^2)(\chi^{-1}) = (E_+/E^2)(\chi^{-1}) \cong \mathcal{O}/2\mathcal{O}$. Since $A(\chi^{-1})$ is trivial, it follows from this and (7) that $(C_+/C^2)(\chi^{-1}) \cong \mathcal{O}/2\mathcal{O}$. □

4. Proof of Theorem 2

In this section, we show Theorem 2 by using the reflection argument.

Proof of (II) ⇒ (I). Assume that $(E_0/E^2)(\chi) \cong \mathcal{O}/2\mathcal{O}$ but $(E_+/E^2)(\chi^{-1})$ is trivial. Then we see that $\tilde{A}(\chi^{-1}) = A(\chi^{-1})$ from (5) and (6), and hence $\tilde{V}(\chi) = V(\chi)$. Since $(E_0/E^2)(\chi)$ is contained in $\tilde{V}(\chi)$, this implies that

$$(E_0/E^2)(\chi) = ((E_+ \cap E_0)/E^2)(\chi) \subseteq (E_+/E^2)(\chi).$$

Thus we obtain $(E_+/E^2)(\chi) \cong \mathcal{O}/2\mathcal{O}$. □

Proof of (I) ⇒ (II). Assume that $(E_+/E^2)(\chi) \cong \mathcal{O}/2\mathcal{O}$ but that both of $(E_0/E^2)(\chi)$ and $(E_0/E^2)(\chi^{-1})$ are trivial. Then, from (3), we obtain inclusions

$$V(\chi) \subseteq \tilde{V}(\chi) \hookrightarrow {}_2A(\chi) \quad \text{and} \quad V(\chi^{-1}) \subseteq \tilde{V}(\chi^{-1}) \hookrightarrow {}_2A(\chi^{-1}). \tag{8}$$

Combining these inclusions with (1) and (2), we see that

$$\mathrm{rk}({}_2A(\chi)) = \mathrm{rk}(V(\chi^{-1})) \leq \mathrm{rk}({}_2A(\chi^{-1})) = \mathrm{rk}(V(\chi)) \leq \mathrm{rk}({}_2A(\chi)).$$

Here, for an $\mathcal{O}$-module M, $\mathrm{rk}(M)$ denotes the dimension of M/M^2 over the finite field $\mathcal{O}/2\mathcal{O}$. Thus, the inclusions (8) yield the isomorphisms

$$V(\chi) = \tilde{V}(\chi) \cong {}_2A(\chi) \quad \text{and} \quad V(\chi^{-1}) = \tilde{V}(\chi^{-1}) \cong {}_2A(\chi^{-1}) \tag{9}$$

and the equalities

$$\mathrm{rk}({}_2A(\chi)) = \mathrm{rk}({}_2\tilde{A}(\chi)) \quad \text{and} \quad \mathrm{rk}({}_2A(\chi^{-1})) = \mathrm{rk}({}_2\tilde{A}(\chi^{-1})). \tag{10}$$

Since we are assuming that $(E_+/E^2)(\chi) \cong \mathcal{O}/2\mathcal{O}$, we see from (6) that $(K^\times/EK_{>0})(\chi) \cong \mathcal{O}/2\mathcal{O}$. Hence, there exists a nontrivial class $[\alpha]_\infty$ in $(K^\times/EK_{>0})(\chi)$ with $\alpha \in K^\times$. By the exact sequence (5) and the first equality in (10), we see that $f([\alpha]_\infty) = [\alpha\mathcal{O}_K]_\infty = [\mathfrak{A}]_\infty^2$ for some ideal $\mathfrak{A}$ of K with $[\mathfrak{A}]_0 \in A(\chi)$. Hence, $\alpha x\mathcal{O}_K = \mathfrak{A}^2$ for some totally positive element $x \in K^\times$. By $[\mathfrak{A}]_0 \in {}_2A(\chi)$, it follows from the first isomorphism in (9) that there exists a unit ϵ for which $[\alpha x\epsilon] \in V(\chi)$. This implies that $\alpha x\epsilon$ is totally positive, and hence the class $[\alpha]_\infty \in (K^\times/EK_{>0})(\chi)$ is trivial. This is a contradiction. □

Example. Let K be the real abelian field of degree 7 and conductor 491. It is known that $A(\chi^{-1}) \cong \mathcal{O}/2\mathcal{O}$ but $A(\chi)$ is trivial for some nontrivial $\bar{\mathbb{Q}}_2$-valued character χ of $\Delta = \mathrm{Gal}(K/\mathbb{Q})$. (Note that χ and χ^{-1} are not conjugate over $\mathbb{Q}_2$.) For this, see Cornacchia [3, §5], or Koyama and Yoshino [7, §8] combined with [13, page 421]. It follows from (2) that $V(\chi)$ is nontrivial and $V(\chi^{-1})$ is trivial. From the latter, it follows that $((E_+ \cap E_0)/E^2)(\chi^{-1})$ is trivial. Since $V(\chi)$ is nontrivial and $A(\chi)$ is trivial, we see from (4) that $((E_+ \cap E_0)/E^2)(\chi) \cong \mathcal{O}/2\mathcal{O}$. Therefore, both of $(E_+/E^2)(\chi)$ and $(E_0/E^2)(\chi)$ are nontrivial, while at least one of $(E_+/E^2)(\chi^{-1})$ and $(E_0/E^2)(\chi^{-1})$ is trivial. Thus, a duality corresponding to Theorem 1 does not hold in general for E_+ and E_0.

5. Minus class group

We use the same notation as in the previous sections. Let $k/\mathbb{Q}$ be an imaginary abelian extension of 2-power degree, and put $L = Kk$. Let k^+ and L^+ be the maximal real subfields of k and L, respectively. Let A_L^- be the kernel of the norm map $A_L \to A_{L^+}$. We naturally identify the Galois groups $\mathrm{Gal}(L/k)$ and $\mathrm{Gal}(L^+/k^+)$ with $\Delta = \mathrm{Gal}(K/\mathbb{Q})$, and regard A_L^- and A_{L^+} as modules over Δ. When L coincides with a cyclotomic field of prime conductor, Cornacchia [2, Theorem 1] used Theorem 1 and (7) to obtain a relation between the triviality of A_L^- and A_K. We slightly generalize his result as follows. Let S be the set of prime numbers ℓ such that a prime ideal of k^+ over ℓ ramifies in k. Let χ be a nontrivial $\bar{\mathbb{Q}}_2$-valued character of Δ, which we also regard as a primitive Dirichlet character.

Theorem 4. *Under the above setting, assume further that at most one prime ideal of K ramifies in L^+. Then the following conditions are equivalent.*

(I) *At least one of $A_L^-(\chi)$ and $A_L^-(\chi^{-1})$ is trivial.*

(II) *The groups $A_K(\chi)$ and $A_K(\chi^{-1})$ are both trivial, and $\chi(\ell) \neq 1$ for all $\ell \in S$.*

The case $L = \mathbb{Q}(\zeta_p)$ is due to Cornacchia, where p is an odd prime number. The assumption in Theorem 4 on ramification in L^+/K is rather strong. Actually, when the assumption is satisfied, we see that the conductor of the real abelian field k^+ is an odd prime number or a power of 2, and in particular $k^+/\mathbb{Q}$ is cyclic. To show the general case, we use a reflection theorem (Theorem 3) and "genus theory", and do not use Theorem 1 nor (7).

Proof. In [6, Corollary 2], we showed that $A_L^-(\chi)$ is trivial if and only if the narrow class group $\tilde{A}_{L^+}(\chi)$ is trivial and $\chi(\ell) \neq 1$ for all $\ell \in S$. We obtained this by using the exact hexagon of Conner and Hurrelbrink [1, Theorem 2.3], which is a kind of genus theory. By the assumption on ramification, L^+/K is a cyclic 2-extension in which exactly one prime ideal of K is ramified. Therefore, we see that $\tilde{A}_{L^+}(\chi)$ is trivial if and only if so is $\tilde{A}_K(\chi)$. Now we obtain the assertion from Theorem 3. □

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