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AHARONOV–BOHM EFFECT IN RESONANCES OF MAGNETIC SCHRÖDINGER OPERATORS IN TWO DIMENSIONS III

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Abstract. We study the Aharonov–Bohm effect (AB effect) in quantum resonances for magnetic scattering in two dimensions. The system consists of four scatters, two obstacles and two scalar potentials with compact support, which are largely separated from one another. The obstacles by which the magnetic fields are completely shielded are vertically placed between the supports of the two potentials. The system yields a two dimensional model of a toroidal scattering system in three dimensions. The resonances are shown to be generated near the real axis by the trajectories trapped between two supports of the scalar potentials as the distances between the scatterers go to infinity. We analyze how the AB effect influences the location of resonances. The result heavily depends on the width between the two obstacles as well as on the magnetic fluxes. The critical case is that the width is comparable to the square root of the distance between the supports of the two potentials.

1. Introduction

In quantum mechanics, a vector potential is said to have a direct significance to particles moving in a magnetic field. This quantum effect is known as the Aharonov–Bohm effect (AB effect) ([1]). This is the third paper on the AB effect in resonances of magnetic Schrödinger operators in two dimensions. In the first paper [7], we have considered a simple scattering system consisting of three scatterers, one bounded obstacle and two scalar potentials with compact supports at large separation, where the obstacle is placed between two supports and shields completely the support of a magnetic field. The field does not influence particles from a classical mechanical point of view, but quantum particles are influenced by the corresponding vector potential which does not necessarily vanish outside the obstacle. The resonances are shown to be generated near the real axis by the trajectories trapped between two supports of the scalar potentials as the distances between the three scatterers go to infinity. We there have shown that the location of the resonances is described in terms of the backward amplitudes for scattering by the two scalar potentials and it depends heavily on the

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magnetic flux of the field. In the second paper [9], we have discussed what happens in the case of two obstacles, where these obstacles are horizontally placed between the supports of the two scalar potentials. The obtained results depend on the ratios of the distances between the four scatters largely separated from one another as well as on the magnetic fluxes of the two fields. In the present paper, we study the case when the two obstacles are vertically placed between the supports of the two scalar potentials. This system yields a two dimensional model of toroidal scattering in three dimensions. In the vertical case, the width between the two obstacles plays an important role. For example, the AB effect is not observed in the system with the total flux vanishing, provided that the width is too large or too small in comparison with the distance between the supports of the two scalar potentials.

We set up our problem. We always work in the two dimensional space $\boldsymbol{R}^2$ with generic point $x = (x_1, x_2)$, and we write

$$H(A,V) = (-i\nabla - A)^2 + V = \sum_{j=1}^{2} (-i\partial_j - a_j)^2 + V, \quad \partial_j = \partial/\partial x_j,$$

for the magnetic Schrödinger operator with $A = (a_1, a_2) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ as a vector potential and $V : \boldsymbol{R}^2 \to \boldsymbol{R}$ as a scalar potential. Let $b_\pm \in C_0^\infty(\boldsymbol{R}^2)$ be two given magnetic fields with the fluxes

$$\alpha_\pm = (2\pi)^{-1} \int b_\pm(x)\, dx,$$

where the integral with no domain attached is taken over the whole space. We assume that the support of $b_\pm$ satisfies ,

$$\text{(1.1)} \qquad \operatorname{supp} b_\pm \subset \mathcal{O}_\pm \subset B = \{|x| < 1\}$$

for some simply connected bounded obstacle $\mathcal{O}_\pm$, where $\mathcal{O}_\pm$ has the smooth boundary $\partial\mathcal{O}_\pm$ and the origin as an interior point. We can take $A_\pm(x)$ to be

$$\text{(1.2)} \qquad A_\pm(x) = \alpha_\pm \Phi(x) \in C^\infty(\overline{\Omega}_\pm \to \boldsymbol{R}^2)$$

for the vector potential corresponding to $b_\pm$, where $\Omega_\pm = \boldsymbol{R}^2 \setminus \overline{\mathcal{O}}_\pm$ and $\Phi(x)$ is defined by

$$\text{(1.3)} \qquad \Phi = \left(-x_2/|x|^2, x_1/|x|^2\right) = \left(-\partial_2 \log|x|, \partial_1 \log|x|\right).$$

As is easily seen, Φ defines the δ–like magnetic field (solenoidal field)

$$\nabla \times \Phi = \Delta \log|x| = 2\pi\delta(x)$$

with center at the origin, when considered over the whole space $\boldsymbol{R}^2$. Assumption (1.1) means that the field $b_\pm$ is entirely shielded by the obstacle

$\mathcal{O}_\pm$, although the corresponding vector potential $A_\pm$ does not necessarily vanish outside $\mathcal{O}_\pm$.

Let $V_\pm \in C_0^\infty(\boldsymbol{R}^2)$ with $\operatorname{supp} V_\pm \subset B$. For $d \gg 1$, we set

$$d_- = (-\kappa_- d, 0), \qquad d_+ = (\kappa_+ d, 0), \quad \kappa_\pm > 0, \quad \kappa_- + \kappa_+ = 1, \tag{1.4}$$

so that $|d_+ - d_-| = d$ for the distance between the two centers d_- and d_+. Then we define

$$V_d(x) = V_-(x - d_-) + V_+(x - d_+) = V_{-d}(x) + V_{+d}(x). \tag{1.5}$$

We further set

$$\rho_\pm = (0, \pm\kappa d^{1/2}), \quad \kappa > 0, \tag{1.6}$$

and define

$$A_\rho(x) = A_-(x - \rho_-) + A_+(x - \rho_+) = A_{-\rho}(x) + A_{+\rho}(x) \tag{1.7}$$

over the exterior domain

$$\Omega_\rho = \boldsymbol{R}^2 \setminus \left(\overline{\mathcal{O}}_{-\rho} \cup \overline{\mathcal{O}}_{+\rho}\right), \quad \mathcal{O}_{\pm\rho} = \{x : x - \rho_\pm \in \mathcal{O}_\pm\}. \tag{1.8}$$

We now consider the self–adjoint operator

$$H_d = H(A_\rho, V_d), \qquad \mathcal{D}(H_d) = H^2(\Omega_\rho) \cap H_0^1(\Omega_\rho), \tag{1.9}$$

in $L^2(\Omega_\rho)$ under the zero boundary conditions, where $H_0^1(W)$ and $H^2(W)$ stand for the usual Sobolev spaces over a region W. We know that the resolvent

$$R(\zeta; H_d) = (H_d - \zeta)^{-1} : L^2(\Omega_\rho) \to L^2(\Omega_\rho), \quad \operatorname{Re}\zeta > 0,\ \operatorname{Im}\zeta > 0,$$

is meromorphically continued from the upper half plane of the complex plane to the lower half plane across the positive real axis where the continuous spectrum of H_d is located. Then $R(\zeta; H_d)$ with $\operatorname{Im}\zeta \le 0$ is well defined as an operator from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\Omega)$ in the sense that $\chi R(\zeta; H_d)\chi : L^2(\Omega_\rho) \to L^2(\Omega_\rho)$ is bounded for every $\chi \in C_0^\infty(\overline{\Omega}_\rho)$, where $L^2_{\mathrm{comp}}(W)$ denotes the space of square integrable functions with compact support in the closure $\overline{W}$ of a region $W \subset \boldsymbol{R}^2$ and $L^2_{\mathrm{loc}}(W)$ denotes the space of locally square integrable functions over $\overline{W}$. The resonances of H_d are defined as the poles of $R(\zeta; H_d)$ in the lower half plane (the unphysical sheet). This is shown by use of the complex scaling method [4, 5, 6]. Our aim is to study how the resonances are generated near the real axis by the trajectories trapped between the two centers d_- and d_+ as $d = |d_+ - d_-| \to \infty$ and how the AB effect influences the location of the resonances.

The obtained result is formulated in terms of the backward amplitudes by the potentials $V_\pm$. Let $K_0 = -\Delta$ be the free Hamiltonian and let $K_\pm$ be the Schrödinger operator defined by

$$K_\pm = K_0 + V_\pm = -\Delta + V_\pm, \quad \mathcal{D}(K_0) = \mathcal{D}(K_\pm) = H^2(\boldsymbol{R}^2). \tag{1.10}$$

We denote by $f_\pm(\omega \to \theta; E)$ the amplitude for scattering from the incident direction $\omega \in S^1$ to the final one θ at energy $E > 0$ for the pair $(K_0, K_\pm)$. These amplitudes admit the analytic extensions $f_\pm(\omega \to \theta; \zeta)$ in a complex neighborhood of the positive real axis as a function of E.

We now fix $E_0 > 0$ and take a complex neighborhood

$$D_d = \left\{ \zeta : |\mathrm{Re}\, \zeta - E_0| < \delta_0 E_0,\ |\mathrm{Im}\, \zeta| < (1 + 2\delta_0)\, E_0^{1/2} \left(\frac{\log\, d}{d} \right) \right\} \tag{1.11}$$

for δ_0, $0 < \delta_0 \ll 1$, small enough. We define

$$\pi_\pm(\zeta) = (1 - I_0(\zeta)) \cos((\alpha_+ + \alpha_-)\pi) + I_0(\zeta) \exp(\pm i(\alpha_+ - \alpha_-)\pi) \tag{1.12}$$

over D_d, where

$$I_0(\zeta) = (2/\pi)^{1/2}\, e^{-i\pi/4} \int_0^\tau e^{it^2/2}\, dt \tag{1.13}$$

with $\tau = \tau(\zeta) = \kappa\, (1/\kappa_- + 1/\kappa_+)^{1/2}\, \zeta^{1/4}$, while the branch $\zeta^{1/2}$ is taken in such a way that $\mathrm{Re}\, \zeta^{1/2} > 0$ for $\mathrm{Re}\, \zeta > 0$, and the contour is taken to be the segment from 0 to τ (although the integral does not depend on the contour). For the direction $\omega_1 = (1, 0)$, we further define

$$h(\zeta; d) = \left(\frac{e^{2ikd}}{d} \right) f_-(-\omega_1 \to \omega_1; \zeta) f_+(\omega_1 \to -\omega_1; \zeta) \pi_+(\zeta) \pi_-(\zeta) \tag{1.14}$$

with $k = \zeta^{1/2}$. We always use the notation k with the meaning ascribed here. Loosely speaking, the resonances in D_d of H_d are approximately determined by the solutions to the equation $h(\zeta; d) = 1$.

We shall formulate the obtained result more precisely. If $\zeta \in D_d$, then

$$2\, \mathrm{Im}\, k = 2\, \mathrm{Im}\, (\mathrm{Re}\, \zeta + i \mathrm{Im}\, \zeta)^{1/2} = \mathrm{Im}\, \zeta / (\mathrm{Re}\, \zeta)^{1/2} + O\left(|\mathrm{Im}\, \zeta|^3 \right) \tag{1.15}$$

and also we have

$$(\mathrm{Re}\, \zeta)^{1/2} = E_0^{1/2} \big(1 + (\mathrm{Re}\, \zeta - E_0) / (2E_0) + O(\delta_0^2) \big) \tag{1.16}$$

with $|\mathrm{Re}\, \zeta - E_0| < \delta_0 E_0$. Thus we can take $\delta_0 > 0$ in (1.11) so small that

$$d^{\delta_0} < \Big| \exp(2ikd) \Big| / d < d^{3\delta_0}, \qquad d \gg 1, \tag{1.17}$$

on the bottom of D_d ($\mathrm{Im}\, \zeta = -(1 + 2\delta_0)\, E_0^{1/2}\, ((\log\, d) / d)$). This implies that the curve defined by $|h(\zeta; d)| = 1$ with $|\mathrm{Re}\, \zeta - E_0| < \delta_0 E_0$ is completely contained in D_d, provided that $f_\pm(\pm\omega_1 \to \mp\omega_1; E_0) \neq 0$ and $\pi_\pm(E_0) \neq 0$. We denote by

$$\left\{\zeta_j(d)\right\}, \quad \zeta_j(d) \in D_d, \quad \mathrm{Re}\,\zeta_1 < \mathrm{Re}\,\zeta_2 < \cdots < \mathrm{Re}\,\zeta_{N_d},$$

the solutions to the equation

$$h(\zeta; d) = 1. \tag{1.18}$$

We know (see Lemma 3.3) that $\zeta_j(d)$ behaves like

$$\mathrm{Im}\,\zeta_j(d) \sim -E_0^{1/2}\left((\log\, d)/d\right), \quad \mathrm{Re}\left(\zeta_{j+1}(d) - \zeta_j(d)\right) \sim 2\pi E_0^{1/2}/d$$

for $d \gg 1$. With the notation above, we are now in a position to state the main theorem.

Theorem 1.1. *Let the notation be as above. Assume that $\pi_\pm(E_0) \neq 0$ and*

$$f_\pm(\pm\omega_1 \to \mp\omega_1; E_0) \neq 0, \quad \omega_1 = (1, 0),$$

at energy $E_0 > 0$. Then we can take $\delta_0 > 0$ so small that the neighborhood D_d defined by (1.11) has the following property: For any $\varepsilon > 0$ small enough, there exists $d_\varepsilon \gg 1$ such that for $d > d_\varepsilon$, H_d has the resonances

$$\left\{\zeta_{\mathrm{res},j}(d)\right\}, \quad \zeta_{\mathrm{res},j}(d) \in D_d, \quad \mathrm{Re}\,\zeta_{\mathrm{res},1}(d) < \cdots < \mathrm{Re}\,\zeta_{\mathrm{res},N_d}(d),$$

in the neighborhood

$$\left\{\zeta \in D_d : |\zeta - \zeta_j(d)| < \varepsilon/d\right\}$$

and the resolvent $R(\zeta; H_d)$ is analytic over $D_d \setminus \left\{\zeta_{\mathrm{res},1}(d), \cdots, \zeta_{\mathrm{res},N_d}(d)\right\}$ as a function with values in operators from $L^2_{\mathrm{comp}}(\Omega_\rho)$ to $L^2_{\mathrm{loc}}(\Omega_\rho)$.

We shall discuss the scattering system with the total flux $\alpha_+ + \alpha_- = 0$ vanishing as a particular case of the theorem above. Such a system is important from a practical point of view as well as from a theoretical point of view, because it can be considered to be the two dimensional model for the toroidal scattering in three dimensions. If we set $\alpha_+ = \alpha$ and $\alpha_- = -\alpha$, then $\pi_\pm(\zeta)$ takes the form

$$\pi_\pm(\zeta) = 1 \pm (8/\pi)^{1/2} e^{i\pi/4} e^{\pm i\alpha\pi} \sin(\alpha\pi) \int_0^\tau e^{is^2/2}\, ds$$

with $\tau = \tau(\zeta)$ as in (1.13), and the location of the resonances is approximately determined from the equation $h(\zeta; d) = 1$ as in Theorem 1.1.

The proof of Theorem 1.1 is done by analyzing the asymptotic properties along forward directions of the resolvent kernels of magnetic Schrödinger operators with two solenoids. The analysis uses the results obtained by

[8]. The resolvent kernel with one solenoid is represented in terms of an integral in the complex plane, but it grows exponentially at infinity for spectral parameters in the lower half plane. We also make use of the complex scaling method developed by [3]. This method makes the composition of one solenoid kernels convergent and enables us to construct the kernels for two solenoids. The main body of the paper is devoted to the construction of two solenoid kernels and to the analysis on the asymptotic properties along forward directions of such kernels.

We end the section by making a brief comment on the case that the centers $\rho_\pm$ are located at $\rho_\pm = (0, -\kappa d^p)$ with $p \neq 1/2$ in the scattering system with the total flux vanishing. If $0 < p < 1/2$, then the width $2\kappa d^p$ between the obstacles $\mathcal{O}_{-\rho}$ and $\mathcal{O}_{+\rho}$ is small in comparison with the distance $d = |d_+ - d_-|$ between the two centers, and a main contribution comes from the closed trajectories enclosing the two obstacles, so that the phase factor of the wave function along such trajectories is not changed in the scattering system with the total flux vanishing. In fact, the integral interval $[0, \tau]$ shrinks ($\kappa \to 0$), and $\pi_+(\zeta) = \pi_-(\zeta) = 1$. Thus the AB effect is not observed in the location of the resonances. If, conversely, $1/2 < p < 1$, then the width is large, and a main contribution comes from the closed trajectories passing between the two obstacles. In this case, the integral interval $[0, \tau]$ expands to $[0, \infty)$ ($\kappa \to \infty$), and $\pi_\pm(\zeta)$ is calculated as $\pi_\pm(\zeta) = e^{\pm 2i\alpha\pi}$ by making use of the formula

$$\int_0^\infty e^{is^2/2}\, ds = (\pi/2)^{1/2} e^{i\pi/4}.$$

As a result, $\pi_-(\zeta)\pi_+(\zeta) = 1$, and the AB effect is not observed in the case $1/2 < p < 1$ either. The AB effect for resonances is observed in the critical case $p = 1/2$ only.

2. Asymptotic properties of resolvent kernel

We write $R(\zeta; T)$ for the resolvent $(T - \zeta)^{-1}$ of the operator T acting on $L^2(W)$, W being a domain of $\boldsymbol{R}^2$. We use the same notation $R(\zeta; T)$ for the resolvent meromorphically extended from the upper half plane of the complex plane to the lower half one. We also take μ to be

$$2/5 < \mu < 1/2\Big(< 1 - \mu\Big) \tag{2.1}$$

close enough to $1/2$. We use the notation μ with the meanings ascribed here throughout the entire discussion. We now define

$$H_{0d} = H(A_\rho, 0), \quad \mathcal{D}(H_{0d}) = H^2(\Omega_\rho) \cap H_0^1(\Omega_\rho), \tag{2.2}$$

where A_ρ and Ω_ρ are defined by (1.7) and (1.8), respectively. The next theorem is obtained as an immediate consequence of [3, Theorem 1.1]. For completeness, we give its proof at the end of this section.

Theorem 2.1. *Let D_d be defined by (1.11). Then*

$$R(\zeta; H_{0d}) : L^2_{\mathrm{comp}}(\Omega_\rho) \to L^2_{\mathrm{loc}}(\Omega_\rho)$$

is analytic over D_d as a function with values in operators for $|d| \gg 1$.

We formulate the two propositions on the asymptotic properties of the resolvent kernel $R(\zeta; H_{0d})(x, y)$ of $R(\zeta; H_{0d})$ with $\zeta \in D_d$. These propositions play an important role in proving Theorem 1.1. In what follows, we denote by $H_0(z) = H_0^{(1)}(z)$ the Hankel function of the first kind and of order zero and by $\gamma(\omega; \theta)$ the azimuth angle from $\omega \in S^1$ to θ.

Proposition 2.1. *Let $d_\pm$ be as in (1.4). Denote by $j_{\pm d}$ the characteristic function of the unit disk*

$$B_{\pm d} = \Big\{ |x - d_\pm| < 1 \Big\} \tag{2.3}$$

and define $c_0(\zeta)$ by

$$c_0(\zeta) = (8\pi)^{-1/2}\, e^{i\pi/4} \zeta^{-1/4}. \tag{2.4}$$

Assume that $\pi_\pm(E_0) \neq 0$ for the function $\pi_\pm(\zeta)$ defined by (1.12). Then the operator $j_{\pm d} R(\zeta; H_{0d}) j_{\mp d}$ acting on $L^2(\Omega_\rho)$ admits the decomposition

$$j_{\pm d} R(\zeta; H_{0d}) j_{\mp d} = R_{\pm 0}(\zeta; d) + R_{\pm 1}(\zeta; d)$$

where $R_{\pm 0}(\zeta; d)$ is the integral operator with the kernel

$$R_{\pm 0}(\zeta, d)(x, y) = c_0(\zeta) \pi_\pm(\zeta) e^{ik|x_1 - y_1|} |x_1 - y_1|^{-1/2}$$

for $(x, y) \in B_{\pm d} \times B_{\mp d}$, and the operator $R_{\pm 1}(\zeta; d)$ satisfies $\|R_{\pm 1}(\zeta; d)\| = O(d^{-\nu})$ uniformly in $\zeta \in D_d$ for some $\nu > 0$.

Proposition 2.2. *Let $\tilde{j}_{\pm d}$ denote the characteristic function of*

$$\tilde{B}_{\pm d} = \Big\{ d^\delta < |x - d_\pm| < 2d^\delta \Big\} \tag{2.5}$$

for $0 < \delta \ll 1$ fixed small enough. Write $x_{\pm\rho} = x - \rho_\pm$ and $\hat{x}_{\pm\rho} = x_{\pm\rho}/|x_{\pm\rho}|$. Then the operator $j_+ R(\zeta; H_{0d}) \tilde{j}_+ : L^2(\Omega_\rho) \to L^2(\Omega_\rho)$ admits the decomposition

$$j_{+d} R(\zeta; H_{0d}) \tilde{j}_{+d} = \tilde{R}_0(\zeta; d) + \tilde{R}_1(\zeta; d),$$

where the kernel $\tilde{R}_0(\zeta; d)(x, y)$ of $\tilde{R}_0(\zeta; d)$ is defined by

$$\tilde{R}_0(\zeta; d)(x, y) = (i/4) H_0(k|x - y|) a_0(x, y; d), \quad (x, y) \in B_{+d} \times \tilde{B}_{+d},$$

with

$$a_0 = \exp\Big(i\alpha_+(\gamma(\hat{x}_{+\rho}; -\hat{y}_{+\rho}) - \pi) + i\alpha_-(\gamma(\hat{x}_{-\rho}; -\hat{y}_{-\rho}) - \pi)\Big) \tag{2.6}$$

and $\tilde{R}_1(\zeta; d)$ *obeys the bound* $\|\tilde{R}_1(\zeta; d)\| = O(d^{-\nu})$ *uniformly in* $\zeta \in D_d$ *for some* $\nu > 0$. *A similar decomposition remains true for* $j_{-d}R(\zeta; H_{0d})\tilde{j}_{-d}$.

We prove the above propositions in section 5. The proof is based on the gauge transformation and on the complex scaling method. We end the section by proving Theorem 2.1.

Proof of Theorem 2.1. Assume that $\zeta \in D_d$, so that

$$|\mathrm{Im}\, k| = \left|\mathrm{Im}\, \zeta^{1/2}\right| = O\left((\log\, d)/d\right).$$

Since $\rho = |\rho_+ - \rho_-| = 2\kappa d^{1/2}$ by (1.6), we have $\left|e^{2ik\rho}/\rho\right| = O\left(\rho^{-1}\right)$ uniformly in $\zeta \in D_d$. Hence it follows from [3, Theorem 1.1] that the resolvent $R(\zeta; H_{0d})$ is analytic over D_d as a function with values in operators from $L^2_{\mathrm{comp}}(\Omega_\rho)$ to $L^2_{\mathrm{loc}}(\Omega_\rho)$. Thus the proof is complete. □

3. Proof of Theorem 1.1

This section is devoted to proving the main theorem. Once the two propositions in the previous section are established, the theorem is verified in exactly the same way as [9, Theorem 1.1]. We give only a sketch for a proof.

We fix the new notation. Let $\varphi_0(x; \omega, E)$ be the plane wave defined by

$$\varphi_0(x; \omega, E) = \exp\left(iE^{1/2}x \cdot \omega\right)$$

with ω as an incident direction at energy $E > 0$. Let $K_0 = -\Delta$ and $K_\pm$ be as in (1.10). Then we define

$$\varphi_\pm(x; \omega, \overline{\zeta}) = \left[(Id - R(\zeta; K_\pm)^* V_\pm)\, \varphi_0(\cdot; \omega, \overline{\zeta})\right](x). \tag{3.1}$$

The function $\varphi_\pm(x; \omega, \overline{\zeta})$ solves the equation $\left(K_\pm - \overline{\zeta}\right) \varphi_\pm(x; \omega, \overline{\zeta}) = 0$. If, in particular, $\zeta = E > 0$, then $\varphi_\pm(x; \omega, E)$ turns out to be the incoming eigenfunction of $K_\pm$, and the conjugate function $\overline{\varphi}_\pm(x; \omega, \overline{\zeta})$ of $\varphi_\pm(x; \omega, \overline{\zeta})$ is analytic in ζ. It should be noted that $\varphi_\pm(x; \omega, \overline{\zeta})$ itself is not analytic. We also note that $\varphi_+(x; \omega, E)$ does not denote the outgoing eigenfunction at energy $E > 0$ but the incoming eigenfunction of the Schrödinger operator K_+.

Let $j(x)$ be the characteristic function of the unit disk $B = \{|x| < 1\}$. Then

$$j_{\pm d}(x) = j\left(x - d_\pm\right) = j\left(x_{\pm d}\right), \quad x_{\pm d} = x - d_\pm,$$

defines the characteristic function of $B_{\pm d}$ defined by (2.3). We introduce the auxiliary operator

$$(3.2) \qquad H_{\pm d} = H(A_\rho, V_{\pm d}), \qquad \mathcal{D}(H_{\pm d}) = H^2(\Omega_\rho) \cap H_0^1(\Omega_\rho),$$

where $V_{\pm d}(x) = V_\pm(x - d_\pm)$ is as in (1.5) and $V_\pm$ is assumed to have support in B. We recall that the notation $\gamma(x;\omega) = \gamma(\hat{x};\omega)$ denotes the azimuth angle from ω to $\hat{x} = x/|x|$. We take a function $\gamma_0 \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ such that

$$\gamma_0(x) = \alpha_-\gamma(\hat{x}_{-\rho}; -\omega_2) + \alpha_+\gamma(\hat{x}_{+\rho}; \omega_2), \quad \omega_2 = (0, 1),$$

on

$$B_d = \Big\{|x - d_-| < |d_-|/2\Big\} \cup \Big\{|x - d_+| < |d_+|/2\Big\}$$

and that it satisfies $\partial_x^n \gamma_0 = O(|x|^{-|n|})$ as $|x| \to \infty$, where $\hat{x}_{\pm\rho} = x_{\pm\rho}/|x_{\pm\rho}|$ with $x_{\pm\rho} = x - \rho_\pm$. Since $\nabla\gamma(x;\omega) = \Phi(x)$ for Φ defined by (1.3), it follows from (1.7) that

$$\nabla\gamma_0 = \alpha_-\Phi(x - \rho_-) + \alpha_+\Phi(x - \rho_+) = A_{-\rho} + A_{+\rho} = A_\rho$$

on B_d. We further introduce the auxiliary operator

$$\tilde{K}_{\pm d} = e^{i\gamma_0} K_{\pm d} e^{-i\gamma_0} = H(\nabla\gamma_0, V_{\pm d}),$$

where $K_{\pm d} = K_0 + V_{\pm d}$ with $\mathcal{D}(K_{\pm d}) = H^2(\boldsymbol{R}^2)$. The operator $\tilde{K}_{\pm d}$ coincides with $H_{\pm d}$ over B_d. If we make use of these auxiliary operators and of Proposition 2.2, then the lemma below is obtained by modifying slightly the argument in the proof of [9, Lemmas 3.1 and 3.2]. We skip its proof.

Lemma 3.1. *Let the notation be as above and let D_d be defined by (1.11). Define the operator $Q_\pm(\zeta; d)$ by*

$$Q_\pm(\zeta; d) = V_{\pm d} R(\zeta; H_{0d}) j_{\pm d} : L^2(B_{\pm d}) \to L^2(B_{\pm d})$$

for $\zeta \in D_d$, where the multiplication $j_{\pm d}$ is considered to be the extension to $L^2(\Omega_\rho)$ from $L^2(B_{\pm d})$. Then

$$Id + Q_\pm(\zeta; d) : L^2(B_{\pm d}) \to L^2(B_{\pm d})$$

has the inverse bounded uniformly in d and $\zeta \in D_d$. Moreover, we have the relation

$$R(\zeta; H_{\pm d}) j_{\pm d} = R(\zeta; H_{0d}) j_{\pm d} \left(Id + Q_\pm(\zeta; d)\right)^{-1} : L^2(B_{\pm d}) \to L^2_{\mathrm{loc}}(\Omega_\rho)$$

for $\zeta \in D_d$.

By the resolvent identity, it follows from Theorem 2.1 and Lemma 3.1 that the resolvent $R(\zeta; H_{\pm d})$ is represented as

$$(3.3) \quad R(\zeta; H_{\pm d}) = \Big(Id - R(\zeta; H_{\pm d}) V_{\pm d}\Big) R(\zeta; H_{0d}) : L^2_{\mathrm{comp}}(\Omega_\rho) \to L^2_{\mathrm{loc}}(\Omega_\rho)$$

for $\zeta \in D_d$ and is analytic there. The next lemma is also shown in almost the same way as in the proof of [9, Lemma 3.3]. The proof uses Proposition 2.1.

Lemma 3.2. *Define*

$$G(\zeta;d) = V_{+d}R(\zeta;H_{-d})V_{-d}R(\zeta;H_{+d})j_{+d} : L^2(B_{+d}) \to L^2(B_{+d}) \tag{3.4}$$

and write $x_{+d} = x - d_+$. Then $G(\zeta;d)$ admits the decomposition

$$G(\zeta;d) = G_0(\zeta;d) + G_1(\zeta;d),$$

where the kernel $G_0(\zeta,d)(x,y)$ of $G_0(\zeta;d)$ is defined by

$$\begin{aligned} G_0 = -c_0(\zeta)\pi_-(\zeta)\pi_+(\zeta) \left(e^{2ikd}/d\right) f_-(-\omega_1 \to \omega_1;\zeta) \\ \times\ V_+(x_{+d})\varphi_0(x_{+d};\omega_1,\zeta)\overline{\varphi}_+(y_{+d};-\omega_1,\overline{\zeta})j(y_{+d}) \end{aligned}$$

for $(x,y) \in B_{+d} \times B_{+d}$, and $G_1(\zeta;d)$ is analytic in $\zeta \in D_d$ with values in bounded operators acting on $L^2(B_{+d})$ and obeys $\|G_1(\zeta;d)\| = O\left(d^{-\nu}\right)$ uniformly in $\zeta \in D_d$ for some $\nu > 0$.

The lemma below is proved in the same way as in the proof of [7, Lemma 4.6].

Lemma 3.3. *Assume the same assumptions as in Theorem 1.1. Let $h(\zeta;d)$ be defined by (1.14). Then the equation $h(\zeta;d) = 1$ has a finite number of the solutions*

$$\left\{\zeta_j(d)\right\}_{1\le j\le N_d}, \quad \zeta_j(d) \in D_d, \quad \operatorname{Re}\zeta_1(d) < \cdots < \operatorname{Re}\zeta_{N_d}(d),$$

in D_d, and each solution $\zeta_j(d)$ has the properties

$$\begin{aligned} \left|\operatorname{Im}\zeta_j(d) + E_0^{1/2}(\log d)/d\right| &< \delta_0 E_0^{1/2}(\log d)/d, \\ \left|\operatorname{Re}\left(\zeta_{j+1}(d) - \zeta_j(d)\right) - 2\pi E_0^{1/2}/d\right| &< 2\pi\delta_0 E_0^{1/2}/d \end{aligned}$$

for $d \gg 1$.

Proof of Theorem 1.1. Recall the notation $H_{\pm d} = H(A_\rho, V_{\pm d})$ from (3.2). We know by (3.3) that $R(\zeta;H_{\pm d}) : L^2_{\mathrm{comp}}(\Omega_\rho) \to L^2_{\mathrm{loc}}(\Omega_\rho)$ is well defined for $\zeta \in D_d$ and is analytic there. We start with the relation

$$(H_d - \zeta)R(\zeta;H_{-d}) = Id + V_{+d}R(\zeta;H_{-d}).$$

We regard the operator on the right side as an operator acting on $L^2(B_{+d})$. By the resolvent identity, the operator on the right side equals

$$\begin{aligned} Id + V_{+d}R(\zeta;H_{-d})j_{+d} = \\ Id + V_{+d}R(\zeta;H_{0d})j_{+d} - V_{+d}R(\zeta;H_{-d})V_{-d}R(\zeta;H_{0d})j_{+d}. \end{aligned}$$

By Lemma 3.1, it is further equal to

$$Id + V_{+d}R(\zeta;H_{-d})j_{+d} = (Id - G(\zeta;d))\,(Id + Q_+(\zeta;d))\,, \tag{3.5}$$

where $G(\zeta;d)$ is again defined by

$$G(\zeta;d) = V_{+d}R(\zeta;H_{-d})V_{-d}R(\zeta;H_{+d})j_{+d} : L^2(B_{+d}) \to L^2(B_{+d})$$

as in Lemma 3.2. If one is not the eigenvalue of $G(\zeta;d)$ at $\zeta = \zeta_0(d) \in D_d$, then the resolvent $R(\zeta;H_d)$ in question turns out to be analytic in a neighborhood of ζ_0 as a function with values in operators from $L^2_{\mathrm{comp}}(\Omega_\rho)$ to $L^2_{\mathrm{loc}}(\Omega_\rho)$. In fact, $R(\zeta;H_d)$ is represented as

$$\begin{aligned} R(\zeta;H_d) &= R(\zeta;H_{-d}) \\ &\quad - \; R(\zeta;H_{-d})j_{+d}\,(Id + V_{+d}R(\zeta;H_{-d})j_{+d})^{-1}\,V_{+d}R(\zeta;H_{-d}). \end{aligned}$$

Thus the problem is reduced to specifying $\zeta \in D_d$ at which $G(\zeta;d)$ has one as an eigenvalue and to showing that this point is really the pole of $R(\zeta;H_d)$ in D_d.

Lemma 3.2 enables us to write $Id - G(\zeta;d)$ as

$$Id - G(\zeta;d) = (Id - \tilde{G}(\zeta;d))(Id - G_1(\zeta;d)) : L^2(B_{+d}) \to L^2(B_{+d}), \tag{3.6}$$

where $G_1(\zeta;d)$ is as in Lemma 3.2 and $\tilde{G}(\zeta;d)$ is defined by

$$\tilde{G}(\zeta;d) = G_0(\zeta;d)(Id - G_1(\zeta;d))^{-1} = G_0(\zeta;d)(Id + \tilde{G}_1(\zeta;d))$$

with $\tilde{G}_1(\zeta;d) = G_1(\zeta;d)(Id - G_1(\zeta;d))^{-1}$. We write $(\ ,\ )$ for the L^2 scalar product in $L^2(\boldsymbol{R}^2)$. We compute the integral

$$\begin{aligned} & c_0(\zeta)\int V_+(x_{+d})\varphi_0(x_{+d};\omega_1,\zeta)\overline{\varphi}_+(x_{+d};-\omega_1,\overline{\zeta})\,dx \\ &\quad = \; c_0(\zeta)\left(V_+\varphi_0(\cdot;\omega_1,\zeta), (Id - R(\zeta;K_+)^*V_+)\,\varphi_0(\cdot;-\omega_1,\overline{\zeta})\right) \\ &\quad = \; c_0(\zeta)\left(V_+\,(Id - R(\zeta;K_+)V_+)\,\varphi_0(\cdot;\omega_1,\zeta), \varphi_0(\cdot;-\omega_1,\overline{\zeta})\right) \\ &\quad = \; -f_+(\omega_1 \to -\omega_1;\zeta) \end{aligned} \tag{3.7}$$

with $x_{+d} = x - d_+$ again, and we set

$$\begin{aligned} h_1(\zeta;d) = -c_0(\zeta)\left(e^{2ikd}/d\right)\pi_-(\zeta)\pi_+(\zeta)f_-(-\omega_1 \to \omega_1;\zeta) \times \\ \times\; \left(\tilde{G}_1(\zeta;d)V_{+d}\varphi_0(\cdot - d_+;\omega_1,\zeta), j_{+d}\varphi_+(\cdot - d_+;-\omega_1,\overline{\zeta})\right). \end{aligned}$$

If we take $\delta_0 > 0$ small enough in D_d, then it follows from (1.17) and Lemma 3.2 that $h_1(\zeta;d)$ is analytic over D_d and obeys $|h_1(\zeta;d)| = O\left(d^{-\nu}\right)$ uniformly in ζ for some $\nu > 0$. The only nonzero eigenvalue of the operator $\tilde{G}(\zeta;d)$ of rank one is given by $h(\zeta;d) + h_1(\zeta;d)$, where $h(\zeta;d)$ is defined by (1.14).

We apply Rouché's theorem to the equation

$$h(\zeta;d)+h_1(\zeta;d)=1 \tag{3.8}$$

over D_d. Let $\left\{\zeta_j(d)\right\}_{1\le j\le N_d}$ be as in Lemma 3.3 and let

$$C_{j\varepsilon}=\left\{|\zeta-\zeta_j(d)|=\varepsilon/d\right\},\qquad D_{j\varepsilon}=\left\{|\zeta-\zeta_j(d)|<\varepsilon/d\right\}$$

for $\varepsilon>0$ fixed arbitrarily but sufficiently small. We may assume $D_{j\varepsilon}\subset D_d$ for $d\gg 1$ by expanding D_d slightly, if necessary. Since $h\left(\zeta_j(d);d\right)=1$, the derivative $h'(\zeta;d)$ behaves like

$$h'\left(\zeta_j(d);d\right)=i\zeta_j(d)^{-1/2}d\left(1+O(d^{-1})\right),$$

at $\zeta=\zeta_j(d)\in D_d$, so that $\left|h'\left(\zeta_j(d);d\right)\right|\ge c_1 d$ for some $c_1>0$. Hence it follows that

$$|h(\zeta;d)-1|\ge c_2\varepsilon$$

on $C_{j\varepsilon}$ for some $c_2>0$. Thus equation (3.8) has a unique solution $\zeta_{\mathrm{res},j}(d)$ in $D_{j\varepsilon}$ for $d\gg 1$.

Once the location $\zeta_{\mathrm{res},j}(d)$ is determined as above, we can show in exactly the same way as in the proof of [7, Theorem 1.1] (see step (3) there) that it really turns out to be the resonance of $R(\zeta;H_d)$. We do not go into details. Thus the proof of the theorem is complete. □

4. Complex scaling method

The remaining sections are devoted to proving Propositions 2.1 and 2.2 which have remained unproved in section 2. The main body of the present work is occupied by the proof of these propositions. The proof is done by constructing the resolvent kernel $R(\zeta;H_{0d})(x,y)$ with the spectral parameter ζ in the lower half plane. To do this, we make use of the complex scaling method to compose the Green kernel constructed for each obstacle $\mathcal{O}_{\pm\rho}$. Here we explain a strategy based on this method.

We begin by fixing the new notation. We introduce a smooth non–negative cut–off function $\chi\in C_0^\infty[0,\infty)$ with the properties

$$0\le\chi\le 1,\qquad \operatorname{supp}\chi\subset[0,2],\qquad \chi=1\ \text{ on }\ [0,1], \tag{4.1}$$

and we take smooth cut–off functions χ_∞ and $\chi_\pm$ over $\boldsymbol{R}$ with the following properties : $0\le\chi_\infty,\ \chi_\pm\le 1$ and

$$\chi_\infty(t)=1-\chi(|t|),$$

$$\chi_+(t)=1\ \text{ for }\ t\ge 1,\quad \chi_+(t)=0\ \text{ for }\ t\le -1,\quad \chi_-(t)=1-\chi_+(t).$$

We often use these functions without further references throughout the future discussion. We again set $\rho = |\rho_+ - \rho_-| = 2\kappa d^{1/2}$ for $\rho_\pm = (0, \pm\kappa d^{1/2})$ in (1.6), so that $d = \left(\rho/2\kappa\right)^2 \sim \rho^2$.

We define the mapping $j_\rho(x) : \boldsymbol{R}^2 \to \boldsymbol{C} \times \boldsymbol{R}$ by

$$j_\rho(x_1, x_2) = (x_1 + i\eta_\rho(x_1)x_1, x_2)\,, \quad \eta_\rho(t) = L_0\big((\log\,\rho)/\rho^2\big)\chi_\infty(t/d), \tag{4.2}$$

for $L_0 \gg 1$ fixed large enough, and we consider the complex scaling mapping

$$(J_\rho f)(x) = \Big[\det(\partial j_\rho/\partial x)\Big]^{1/2} f(j_\rho(x)) \tag{4.3}$$

associated with $j_\rho(x)$. The Jacobian $\det(\partial j_\rho/\partial x)$ of $j_\rho(x)$ does not vanish for $\rho \gg 1$, and it is easily seen that J_ρ is a one-to-one mapping. For notational brevity, we now write

$$P_\rho = H(A_\rho, 0), \qquad \mathcal{D}(P_\rho) = H^2(\Omega_\rho) \cap H_0^1(\Omega_\rho)$$

for the operator H_{0d} under consideration. Since the coefficients of P_ρ are analytic over Ω_ρ, we can define the operator

$$Q_\rho = J_\rho P_\rho J_\rho^{-1}. \tag{4.4}$$

This becomes a closed operator in $L^2(\Omega_\rho)$ with the same domain as P_ρ, but it is not necessarily self–adjoint. The future discussion does not require the explicit form of Q_ρ. We construct the resolvent kernel $R(\zeta; Q_\rho)(x, y)$ with $\zeta \in D_d$ instead of $R(\zeta; P_\rho)(x, y)$. The mapping j_ρ acts as the identity over the strip $\Big\{x = (x_1, x_2) : |x_1| < d\Big\}$ and hence we have the relation

$$R(\zeta; P_\rho)(x, y) = R(\zeta; P_\rho)(j_\rho(x), j_\rho(y)) = R(\zeta; Q_\rho)(x, y)$$

for $(x, y) \in B_{\pm d} \times B_{\mp d}$ or for $(x, y) \in B_{\pm d} \times \tilde{B}_{\pm d}$. Thus the necessary information can be obtained through the kernel $R(\zeta; Q_\rho)(x, y)$.

We introduce the auxiliary operators

$$P_{\pm\rho} = H(A_{\pm\rho}, 0), \qquad \mathcal{D}(P_{\pm\rho}) = H^2(\Omega_{\pm\rho}) \cap H_0^1(\Omega_{\pm\rho}), \tag{4.5}$$

where $A_{\pm\rho}(x)$ is defined in (1.7), and $\Omega_{\pm\rho} = \boldsymbol{R}^2 \setminus \overline{\mathcal{O}}_{\pm\rho}$. We define the complex scaled operator as in (4.4) for these auxiliary operators $P_{\pm\rho}$. Recall that $\gamma(x; \omega)$ denotes the azimuth angle from $\omega \in S^1$ to $\hat{x} = x/|x|$. The potential $\Phi(x)$ defined by (1.3) satisfies $\Phi(x) = \nabla\gamma(x; \omega)$. Hence it follows that

$$A_{\pm\rho}(x) = \alpha_\pm \nabla\gamma(x - \rho_\pm; \pm\omega_2), \quad \omega_2 = (0, 1).$$

If we take $\arg z$, $0 \le \arg z < 2\pi$, to be a single valued function over the complex plane slit along the direction ω_2, then the angle function $\gamma(x; \omega_2)$

is represented as

$$\gamma(x;\omega_2) = -\frac{i}{2}\Big(\log(x_1+ix_2) - \log(-x_1+ix_2)\Big) + \pi,$$

so that it is well defined for the complex variables also. Thus we can define

$$\gamma(j_\rho(x);\omega_2) = \frac{1}{2}\big((\arg(b_{+\rho}(x)) - \arg(b_{-\rho}(x))\big) + \pi - \frac{i}{2}\log|b_\rho(x)|,$$

where

$$b_{+\rho}(x) = x_1 + i\eta_\rho(x_1)x_1 + ix_2, \qquad b_{-\rho}(x) = -x_1 - i\eta_\rho(x_1)x_1 + ix_2,$$

and $b_\rho(x) = b_{+\rho}(x)/b_{-\rho}(x)$. The function $\gamma(j_\rho(x);-\omega_2)$ is similarly defined by taking $\arg z$ to be a single valued function over the complex plane slit along the direction $-\omega_2$.

We define $g_{\pm\rho}(x)$ by

$$g_{\pm\rho}(x) = \alpha_\pm\chi_\mp\big((32x_2/\rho)\mp 13\big)\gamma(j_\rho(x)-\rho_\pm;\pm\omega_2) \tag{4.6}$$

and $g_{0\rho}(x)$ by

$$g_{0\rho}(x) = g_{-\rho}(x) + g_{+\rho}(x). \tag{4.7}$$

By definition, $\operatorname{supp} g_{-\rho} \subset \{x : x_2 > -7\rho/16\}$ and

$$g_{-\rho}(x) = \alpha_-\gamma(j_\rho(x)-\rho_-;-\omega_2) \quad \text{on } \Sigma_+ = \{x : x_2 > -3\rho/8\}.$$

Hence $\exp(ig_{-\rho})$ acts as

$$\exp(ig_{-\rho})f(x) = \left(J_\rho\exp(i\alpha_-\gamma(x-\rho_-;-\omega_2))J_\rho^{-1}f\right)(x)$$

on functions $f(x)$ with support in Σ_+. On the other hand, $g_{+\rho}(x)$ has support in $\{x : x_2 < 7\rho/16\}$ and

$$g_{+\rho}(x) = \alpha_+\gamma(j_\rho(x)-\rho_+;\omega_2) \quad \text{on } \Sigma_- = \{x : x_2 < 3\rho/8\},$$

so that $\exp(ig_{+\rho})$ acts as

$$\exp(ig_{+\rho})f(x) = \left(J_\rho\exp(i\alpha_+\gamma(x-\rho_+;\omega_2))J_\rho^{-1}f\right)(x)$$

on functions $f(x)$ with support in Σ_-. We take these relations into account to define the following complex scaled operator

$$Q_{\pm\rho} = \exp(ig_{\mp\rho})\left(J_\rho P_{\pm\rho}J_\rho^{-1}\right)\exp(-ig_{\mp\rho}) \tag{4.8}$$

for $P_{\pm\rho}$ defined by (4.5), where $Q_{\pm\rho}$ has the same domain as $P_{\pm\rho}$. Since

$$Q_{+\rho} = J_\rho H(\alpha_-\nabla\gamma(x-\rho_-;-\omega_2) + A_{+\rho})J_\rho^{-1}$$

on Σ_+, we have

$$Q_{+\rho} = Q_\rho \quad \text{on } \Sigma_+ = \{x : x_2 > -3\rho/8\}. \tag{4.9}$$

Similarly we have

$$Q_{-\rho} = Q_\rho \quad \text{on } \Sigma_- = \{x : x_2 < 3\rho/8\}. \tag{4.10}$$

The function $g_{0\rho}(x)$ defined by (4.7) satisfies

$$g_{0\rho} = \alpha_- \gamma(j_\rho(x) - \rho_-; -\omega_2) + \alpha_+ \gamma(j_\rho(x) - \rho_+; \omega_2)$$

on $\Sigma_0 = \{x : |x_2| \le \rho/4\}$. If we define the operator $Q_{0\rho}$ by

$$Q_{0\rho} = \exp(ig_{0\rho}) \left(J_\rho K_0 J_\rho^{-1}\right) \exp(-ig_{0\rho}), \quad K_0 = -\Delta, \tag{4.11}$$

as a closed operator with domain $\mathcal{D}(Q_{0\rho}) = H^2(\boldsymbol{R}^2)$, then we obtain

$$Q_{0\rho} = Q_{\pm\rho} = Q_\rho \quad \text{on } \Sigma_0 = \{x : |x_2| \le \rho/4\}\,. \tag{4.12}$$

We set $\chi_{\pm\rho}(x) = \chi_\pm(16x_2/\rho)$ and take $\tilde{\chi}_{\pm\rho} \in C^\infty(\boldsymbol{R}^2)$ in such a way that

$$\tilde{\chi}_{\pm\rho} \text{ has a slightly larger support than } \chi_{\pm\rho}, \quad \tilde{\chi}_{\pm\rho}\chi_{\pm\rho} = \chi_{\pm\rho}.$$

We may assume that

$$\tilde{\chi}_{\pm\rho} j_{+d} = j_{+d}, \qquad \tilde{\chi}_{\pm\rho} j_{-d} = j_{-d} \tag{4.13}$$

for the characteristic function $j_{\pm d}$ of $B_{\pm d} = \Big\{|x - d_\pm| < 1\Big\}$, and similarly for the characteristic function $\tilde{j}_{\pm d}$ of $\tilde{B}_{\pm d}$. For the exterior domain $\Omega_{\pm\rho} = \boldsymbol{R}^2 \setminus \overline{\mathcal{O}}_{\pm\rho}$, we regard $\tilde{\chi}_{\pm\rho}$ as the extension from $L^2(\Omega_\rho)$ to $L^2(\Omega_{\pm\rho})$ and $\chi_{\pm\rho}$ as the restriction to $L^2(\Omega_\rho)$ from $L^2(\Omega_{\pm\rho})$. Then we define

$$\Lambda(\zeta; \rho) = \chi_{-\rho} R(\zeta; Q_{-\rho})\tilde{\chi}_{-\rho} + \chi_{+\rho} R(\zeta; Q_{+\rho})\tilde{\chi}_{+\rho}, \quad \zeta \in D_d, \tag{4.14}$$

as an operator from $L^2_{\mathrm{comp}}(\Omega_\rho)$ to $L^2_{\mathrm{loc}}(\Omega_\rho)$. We know (see [3]) that $R(\zeta; Q_{\pm\rho})$ is well–defined as an operator from $L^2_{\mathrm{comp}}(\Omega_{\pm\rho})$ to $L^2_{\mathrm{loc}}(\Omega_{\pm\rho})$ for $\zeta \in D_d$. Since $Q_\rho = Q_{\pm\rho}$ on $\operatorname{supp} \chi_{\pm\rho}$ by (4.9) and (4.10), we compute

$$\begin{aligned}(Q_\rho - \zeta)\,\Lambda &= (Q_{-\rho} - \zeta)\,\chi_{-\rho} R(\zeta; Q_{-\rho})\tilde{\chi}_{-\rho} + (Q_{+\rho} - \zeta)\,\chi_{+\rho} R(\zeta; Q_{+\rho})\tilde{\chi}_{+\rho} \\ &= Id + [Q_{-\rho}, \chi_{-\rho}] R(\zeta; Q_{-\rho})\tilde{\chi}_{-\rho} + [Q_{+\rho}, \chi_{+\rho}] R(\zeta; Q_{+\rho})\tilde{\chi}_{+\rho}.\end{aligned}$$

The function $\chi_{\pm\rho}$ depends on x_2 only, and the derivative $\chi'_{\pm\rho}$ has support in

$$\Pi_0 = \Big\{x = (x_1, x_2) : |x_2| < \rho/16\Big\}. \tag{4.15}$$

By (4.12), $Q_{\pm\rho} = Q_{0\rho}$ on Π_0, so that both the commutators $[Q_{-\rho}, \chi_{-\rho}]$ and $[\chi_{+\rho}, Q_{+\rho}]$ on the right side equal $[Q_{0\rho}, \chi_{-\rho}]$. Hence we have

$$(Q_\rho - \zeta)\,\Lambda(\zeta; \rho) = Id + \Gamma_0 \left(R(\zeta; Q_{-\rho})\tilde{\chi}_{-\rho} - R(\zeta; Q_{+\rho})\tilde{\chi}_{+\rho}\right), \tag{4.16}$$

where

$$\Gamma_0 = [Q_{0\rho}, \chi_{-\rho}], \qquad \chi_{-\rho} = \chi_-(16x_2/d). \tag{4.17}$$

We define $T(\zeta; \rho)$ by

$$T(\zeta; \rho) = \Gamma_0 \Big(R(\zeta; Q_{-\rho}) - R(\zeta; Q_{+\rho})\Big) p_0 \tag{4.18}$$

as an operator acting on $L^2(\Pi_0)$, where the multiplication by the characteristic function $p_0(x)\Big(= p_0(x_2)\Big)$ of Π_0 is regarded as the extension from $L^2(\Pi_0)$ to $L^2(\Omega_{-\rho})$ or to $L^2(\Omega_{+\rho})$. If

$$Id + T(\zeta;\rho) : L^2(\Pi_0) \to L^2(\Pi_0) \tag{4.19}$$

is shown to have the inverse bounded uniformly in $\zeta \in D_d$, then it follows that

$$R(\zeta;Q_\rho) = \Lambda(\zeta;\rho) - \Lambda(\zeta;\rho)p_0\,(Id+T)^{-1}\,\Gamma_0\Big(R(\zeta;Q_{-\rho})\tilde{\chi}_{-\rho} - R(\zeta;Q_{+\rho})\tilde{\chi}_{+\rho}\Big). \tag{4.20}$$

The proof of Propositions 2.1 and 2.2 is based on this representation.

5. Resolvent kernels of distorted operators

The aim of this section is to study the behavior at infinity of the resolvent kernel $R(\zeta;Q_{\pm\rho})(x,y)$ for the operator $Q_{\pm\rho}$ defined by (4.8).

Let $\mathcal{O}$ be a simply connected bounded domain in $\boldsymbol{R}^2$. We assume that the origin is included in $\mathcal{O}$ as an interior point and that the boundary $\partial\mathcal{O}$ is smooth. We consider the self–adjoint operator

$$P = H(\alpha\Phi, 0), \quad \mathcal{D}(P) = H_0^1(\Omega) \cap H^2(\Omega), \tag{5.1}$$

over the outside domain $\Omega = \boldsymbol{R}^2 \setminus \overline{\mathcal{O}}$ under the zero Dirichlet boundary conditions, where $\Phi(x)$ is the Aharonov–Bohm potential defined by (1.3). We denote by $R(\zeta;P)(x,y)$ the kernel of the resolvent $R(\zeta;P)$ with $\zeta \in D_d$.

Proposition 5.1. *Assume that $\rho/c < x_2,\ y_2 < c\rho$ for some $c > 1$, and let*

$$\psi_\rho(x,y) = \psi(j_\rho(x), j_\rho(y)), \quad \psi(x,y) = \gamma(x;-\omega_2) - \gamma(y;-\omega_2),$$

for $\omega_2 = (0,1)$. Set

$$Q(x,y;\zeta) = R(\zeta;P)(j_\rho(x), j_\rho(y))$$

for $\zeta \in D_d$. Then $Q(x,y;\zeta)$ admits the decomposition

$$Q(x,y;\zeta) = \exp(i\alpha\psi_\rho(x,y))R(\zeta;K_0)(j_\rho(x), j_\rho(y)) + R_{\rm sc}(\zeta;P)(x,y)$$

and the analytic function $R_{\rm sc}(\zeta;P)(x,y)$ over D_d satisfies the following estimates uniformly in $\zeta \in D_d$.

(1) *If $|x_1| + |y_1| > Ld = L\rho^2$ for $L \gg 1$ fixed arbitrarily, then*

$$R_{\rm sc}(\zeta;P)(x,y) = O\left((|x|+|y|)^{-\sigma L}\right)$$

for some $\sigma > 0$ independent of L, and a similar bound remains true for the derivatives $\partial R_{\rm sc}(\zeta;P)(x,y)/\partial x_2$ and $\partial R_{\rm sc}(\zeta;P)(x,y)/\partial y_2$.

(2) *If* $|x_1|+|y_1|<2Ld$ *and* $\big||\psi(x,y)|-\pi\big|>1/L$ *for* $L\gg 1$ *fixed above, then* $R_{\mathrm{sc}}(\zeta;P)(x,y)$ *takes the form*

$$R_{\mathrm{sc}}(\zeta;P)(x,y)=\exp(ikr_\rho(x))|x|^{-1/2}q_0(x,y;\zeta)|y|^{-1/2}\exp(ikr_\rho(y))$$

and $q_0(x,y;\zeta)$ *satisfies* $\left|(\partial/\partial x)^j\,(\partial/\partial y)^l\,q_0\right|=O\left(\rho^{-(|j|+|l|)}\right)$, *where*

$$r_\rho(x)=r(j_\rho(x)),\quad r(x)=\left(x_1^2+x_2^2\right)^{1/2},$$

and similarly for $r_\rho(y)$.

(3) *If* $|x_1|+|y_1|<2Ld$ *and* $\big||\psi(x,y)|-\pi\big|<2/L$ *for* $L\gg 1$ *fixed again, then* $R_{\mathrm{sc}}(\zeta;P)(x,y)$ *takes the form*

$$R_{\mathrm{sc}}(\zeta;P)(x,y)=\exp(ikr_\rho(x))q_1(x,y;\zeta)\exp(ikr_\rho(y))$$

and $q_1(x,y;\zeta)$ *satisfies*

$$\left|(\partial/\partial x)^j\,(\partial/\partial y)^l\,q_1\right|=O\left(\rho^{\varepsilon/2-(|j|+|l|)(1-\varepsilon)}\right)$$

with ε, $0<\varepsilon\ll 1$, *fixed arbitrarily but small enough.*

Remark 5. If $-c\rho<x_2,\ y_2<-\rho/c$ for some $c>1$, then the same statements as above remain true for $\psi(x,y)$ replaced with

$$\tilde\psi(x,y)=\gamma(x;\omega_2)-\gamma(y;\omega_2).$$

Sketch of proof. The proposition is proved in almost the same way as Propositions 6.1, 6.2 and 6.3 in [3]. We give only a sketch for a proof. We fix $\varepsilon>0$ arbitrarily but small enough throughout the proof.

We skip the proof of statement (1). To prove statements (2) and (3), we see from the arguments used for proving the propositions above that the leading term of $R_{\mathrm{sc}}(\zeta;P)(x,y)$ takes the form

$$(r_\rho(x)+r_\rho(y))^{-1/2}\exp\Big(i[\alpha]\psi_\rho(x,y)\Big)\exp(ikr_\rho(x))q(x,y;\zeta)\exp(ikr_\rho(y))$$

with $k=\zeta^{1/2}$, where the Gauss notation $[\alpha]$ denote the greatest integer not exceeding α and $q(x,y;\zeta)$ takes the integral form

$$q=\int\exp\big(ik\varphi_\rho(p,x,y)\big)\chi\left(\rho^{(1-\varepsilon)/2}|p|\right)\left(e^p+e^{-i\psi_\rho(x,y)}\right)^{-1}dp$$

with

$$\varphi_\rho=\Big(r_\rho(x)r_\rho(y)/(r_\rho(x)+r_\rho(y))\Big)\left(\cosh p-1\right).$$

We note that

$$k=\zeta^{1/2}=(E+i\eta)^{1/2}=E^{1/2}\left(1+iO\left((\log\rho)/\rho^2\right)+O\left(\rho^{-3}\right)\right)$$

for $\zeta \in D_d$ and $\cosh p - 1 = O(|p|^2) = O\left(\rho^{-(1-\varepsilon)}\right)$. Since

$$\frac{r_\rho(x)r_\rho(y)}{r_\rho(x)+r_\rho(y)} = \left(\frac{|x||y|}{|x|+|y|}\right)\left(1+O\left((\log\,\rho)/\rho^2\right)\right),$$

the function $\exp\left(ik\varphi_\rho(p,x,y)\right)$ is uniformly bounded, and also it follow that

$$\left|r_\rho(x)r_\rho(y)/(r_\rho(x)+r_\rho(y))\right| > c\rho$$

for some $c > 0$. If (x, y) fulfills the assumption of statement (2), then $e^p + e^{-i\psi_\rho}(x, y)$ is away from 0 uniformly in x, y and p with $|p| < 2\rho^{-(1-\varepsilon)/2}$. Thus the stationary phase method (or the method of steepest descent) yields the desired form.

We move to statement (3). In this case, $e^p + e^{-i\psi_\rho}(x, y)$ takes values close to 0. It occurs when $\gamma(x; -\omega_2) \sim \pi/2$ and $\gamma(y; -\omega_2) \sim 3\pi/2$ or when $\gamma(x; -\omega_2) \sim 3\pi/2$ and $\gamma(y; -\omega_2) \sim \pi/2$. We consider only the former case. We set $\theta_{+\rho}(x) = \theta_+(j_\rho(x))$ and $\theta_{-\rho}(y) = \theta_-(j_\rho(y))$, where

$$\theta_+(x) = \gamma(x; -\omega_2) - \pi/2 > 0, \qquad \theta_-(y) = 3\pi/2 - \gamma(y; -\omega_2) > 0.$$

Then $\psi_\rho(x, y) = \theta_{+\rho}(x) + \theta_{-\rho}(y) - \pi$, so that

$$e^p + e^{-i\psi_\rho(x,y)} \sim p + i\left(\theta_{+\rho}(x) + \theta_{-\rho}(y)\right).$$

Since $\theta_+(x) \sim x_2/|x|$ and $\theta_-(y) \sim y_2/|y|$, we have

$$\left|(\partial/\partial x)^j\, \theta_{+\rho}(x)\right| = \left(x_2/|x|\right)O\left(\rho^{-|j|}\right)$$

and similarly for $\theta_{-\rho}(y)$. This implies that

$$\left|(\partial/\partial x)^j\,(\partial/\partial y)^n\left(e^p + e^{-i\psi_\rho(x,y)}\right)^{-1}\right| = O\left(\rho^{1-(|j|+|n|)}\right),$$

because $\left|e^p + e^{-i\psi_\rho(x,y)}\right| \ge c\rho^{-1}$ for some $c > 0$. We also have that

$$\left|(\partial/\partial x)^j\,(\partial/\partial y)^l \exp\left(ik\varphi_\rho(p,x,y)\right)\right| = O\left(\rho^{-(|j|+|l|)(1-\varepsilon)}\right).$$

Thus the desired form is obtained. □

We recall that $Q_{\pm\rho}$ is defined by (4.8). The kernel $R(\zeta; Q_{\pm\rho})(x, y)$ of the resolvent $R(\zeta; Q_{\pm\rho})$ has the representation

$$\left[\det(\partial j_\rho/\partial x)\right]^{1/2} e^{ig_{\mp\rho}(x)} R(\zeta; P_{\pm\rho})(x, y) e^{-ig_{\mp\rho}(y)} \left[\det(\partial j_\rho/\partial y)\right]^{1/2}$$

for $\zeta \in D_d$. We set $\psi_{\pm\rho}(x, y) = \psi_\pm(j_\rho(x), j_\rho(y))$, where

$$\psi_\pm(x, y) = \gamma(x - \rho_\pm; \pm\omega_2) - \gamma(y - \rho_\pm; \pm\omega_2).$$

Then it follows from (4.6) and (4.7) that

$$\alpha_\pm \psi_{\pm\rho}(x, y) + g_{\mp\rho}(j_\rho(x)) - g_{\mp\rho}(j_\rho(y)) = g_{0\rho}(j_\rho(x)) - g_{0\rho}(j_\rho(y))$$

over $\Pi_0 \times \Pi_0$. Thus the proposition below is obtained as a consequence of (4.11) and Proposition 5.1 (see Remark 5 also).

Proposition 5.2. *Let $(x, y) \in \Pi_0 \times \Pi_0$, so that $|x_2| < \rho/16$ and $|y_2| < \rho/16$. Let $\psi_{\pm\rho}(x, y)$ be as above. Write $x_\pm = x - \rho_\pm$ and $y_\pm = y - \rho_\pm$ for $\rho_\pm = (0, \pm\kappa\rho)$, and define*

$$r_{\pm\rho}(x) = r_\pm(j_\rho(x)), \quad r_\pm(x) = \left(x_1^2 + (x_2 \mp \kappa\rho)^2\right)^{1/2},$$

and similarly for $r_{\pm\rho}(y)$. Then $R(\zeta; Q_{\pm\rho})(x, y)$ admits the decomposition

$$R(\zeta; Q_{\pm\rho})(x, y) = R(\zeta; Q_{0\rho})(x, y) + R_{\mathrm{sc}}(\zeta; Q_{\pm\rho})(x, y)$$

over $\Pi_0 \times \Pi_0$, and the analytic function $R_{\mathrm{sc}}(\zeta; Q_{\pm\rho})(x, y)$ over D_d satisfies the following estimates uniformly in $\zeta \in D_d$.

(1) *If $|x_1| + |y_1| > Ld = L\rho^2$ for $L \gg 1$ fixed arbitrarily, then*

$$R_{\mathrm{sc}}(\zeta; Q_{\pm\rho})(x, y) = O\left((|x_\pm| + |y_\pm|)^{-\sigma L}\right)$$

for some $\sigma > 0$ independent of L together with the derivatives

$$\partial R_{\mathrm{sc}}(\zeta; Q_{\pm\rho})/\partial x_2, \quad \partial R_{\mathrm{sc}}(\zeta; Q_{\pm\rho})/\partial y_2.$$

(2) *If $|x_1|+|y_1| < 2Ld$ and $\big||\psi_\pm(x, y)| - \pi\big| > 1/L$ for $L \gg 1$ fixed above, then $R_{\mathrm{sc}}(\zeta; Q_{\pm\rho})(x, y)$ takes the form*

$$R_{\mathrm{sc}}(\zeta; Q_{\pm\rho})(x, y) = e^{ikr_{\pm\rho}(x)}|x_\pm|^{-1/2}p_{\pm 0}(x, y; \zeta)|y_\pm|^{-1/2}e^{ikr_{\pm\rho}(y)}$$

and $p_{\pm 0}(x, y; \zeta)$ satisfies $\left|(\partial/\partial x)^j\,(\partial/\partial y)^l\,p_{\pm 0}\right| = O\left(\rho^{-(|j|+|l|)}\right)$.

(3) *If $|x_1|+|y_1| < 2Ld$ and $\big||\psi_\pm(x, y)| - \pi\big| < 2/L$ for $L \gg 1$ fixed again, then $R_{\mathrm{sc}}(\zeta; Q_{\pm\rho})(x, y)$ takes the form*

$$R_{\mathrm{sc}}(\zeta; Q_{\pm\rho})(x, y) = \exp(ikr_{\pm\rho}(x))p_{\pm 1}(x, y; \zeta)\exp(ikr_{\pm\rho}(y))$$

and $p_{\pm 1}(x, y; \zeta)$ satisfies

$$\left|(\partial/\partial x)^j\,(\partial/\partial y)^l\,p_{\pm 1}\right| = O\left(\rho^{\varepsilon/2-(|j|+|l|)(1-\varepsilon)}\right)$$

for $\varepsilon > 0$ fixed arbitrarily but small enough.

Let $R_{\mathrm{sc}}(\zeta; Q_{\pm\rho})(x, y)$ be as in Proposition 5.2. We now denote by $R_{\mathrm{sc}}(\zeta; Q_{\pm\rho})$ the integral operator with this kernel. Then we have

$$R(\zeta; Q_{\pm\rho}) = R(\zeta; Q_{0\rho}) + R_{\mathrm{sc}}(\zeta; Q_{\pm\rho}), \quad \zeta \in D_d, \tag{5.2}$$

as an operator from $L^2_{\mathrm{comp}}(\Pi_0)$ to $L^2_{\mathrm{loc}}(\Pi_0)$, and also it follows that the operator $T(\zeta; d)$ defined by (4.18) takes the form

$$T(\zeta; d) = \Gamma_0\left(R_{\mathrm{sc}}(\zeta; Q_{-\rho}) - R_{\mathrm{sc}}(\zeta; Q_{+\rho})\right)p_0, \quad \zeta \in D_d. \tag{5.3}$$

By Proposition 5.2, we see that $T(\zeta; d)$ is analytic in $\zeta \in D_d$ as a function with values in bounded operators acting on $L^2(\Pi_0)$ and obeys $\|T(\zeta; d)\| = O(\rho^\nu)$ uniformly in $\zeta \in D_d$ for some $\nu > 0$.

6. Proof of Propositions 2.1 and 2.2

The last section is devoted to proving Propositions 2.1 and 2.2.

6.1. We prove Proposition 2.1 only in the case when $(x, y) \in B_{+d} \times B_{-d}$. We assert that the operator $j_{+d}R(\zeta; H_{0d})j_{-d}$ in question takes the form

$$(6.1)\quad \begin{aligned} j_{+d}R(\zeta; H_{0d})j_{-d} &= j_{+d}R(\zeta; Q_\rho)j_{-d} \\ &= j_{+d}\Big(R(\zeta; Q_{-\rho}) + R(\zeta; Q_{+\rho}) - R(\zeta; Q_{0\rho})\Big)j_{-d} + R_1(\zeta; d) \end{aligned}$$

where the remainder operator $R_1(\zeta; d)$ satisfies $\|R_1(\zeta; d)\| = O(d^{-\nu})$ uniformly in $\zeta \in D_d$ for some $\nu > 0$. The section is almost occupied by the proof of this assertion. We first complete the proof of Proposition 2.1, accepting (6.1) as established. The proof is based on the following proposition which has been proved as Proposition 6.3 in [8].

Proposition 6.1. *Let $\rho_\pm = (0, \pm\kappa d^{1/2}) = (0, \pm\kappa\rho)$ be as in (1.6) and let $P_{\pm\rho}$ be defined by (4.5). Assume that $\zeta \in D_d$. Then there exists a constant $\nu > 0$ such that $R(\zeta; P_{\pm\rho})(x, y)$ behaves as follows:*

$$\begin{aligned} R(\zeta; P_{-\rho})(x, y) = c_0(\zeta)\left(\frac{e^{ik|x_1-y_1|}}{|x_1 - y_1|^{1/2}}\right)&\big(\cos(\alpha_-\pi) \mp i\sin(\alpha_-\pi)I_0(\zeta)\big) \\ &+ e^{ik|x_1-y_1|}|x_1 - y_1|^{-1/2}O\left(d^{-\nu}\right) \end{aligned}$$

for $(x, y) \in B_{\pm d} \times B_{\mp d}$, and

$$\begin{aligned} R(\zeta; P_{+\rho})(x, y) = c_0(\zeta)\left(\frac{e^{ik|x_1-y_1|}}{|x_1 - y_1|^{1/2}}\right)&\big(\cos(\alpha_+\pi) \pm i\sin(\alpha_+\pi)I_0(\zeta)\big) \\ &+ e^{ik|x_1-y_1|}|x_1 - y_1|^{-1/2}O\left(d^{-\nu}\right) \end{aligned}$$

for $(x, y) \in B_{\pm d} \times B_{\mp d}$, where $I_0(\zeta)$ is the integral defined by (1.13).

Proof of Proposition 2.1. The mapping j_ρ defined by (4.2) acts as the identity over $B_{\pm d}$, and hence it follows from (4.8) and (4.11) that

$$\begin{aligned} R(\zeta; Q_{\pm\rho})(x, y) &= \exp(ig_{\mp\rho}(x))R(\zeta; P_{\pm\rho})(x, y)\exp(-ig_{\mp\rho}(y)), \\ R(\zeta; Q_{0\rho})(x, y) &= \exp(ig_{0\rho}(x))R(\zeta; K_0)(x, y)\exp(-ig_{0\rho}(y)). \end{aligned}$$

Recall the definitions of $g_{\pm\rho}$ and $g_{0\rho}$ from (4.6) and (4.7), respectively. If $(x, y) \in B_{+d} \times B_{-d}$, then

$$g_{+\rho}(x) - g_{+\rho}(y) = \alpha_+\left(\gamma(x - \rho_+; \omega_2) - \gamma(y - \rho_+; \omega_2)\right) = \alpha_+\pi + O(\rho^{-1}).$$

Thus we have

$$R(\zeta;Q_{-\rho})(x,y)=\left(e^{i\alpha_+\pi}+O(\rho^{-1})\right)R(\zeta;P_{-\rho})(x,y).$$

We now use Proposition 6.1 to obtain that

$$\begin{aligned}R(\zeta;Q_{-\rho})(x,y)&=c_0(\zeta)\left(\frac{e^{ik|x_1-y_1|}}{|x_1-y_1|^{1/2}}\right)e^{i\alpha_+\pi}\left(\cos(\alpha_-\pi)-i\sin(\alpha_-\pi)I_0(\zeta)\right)\\&\quad+\ O\left(\rho^{-2\nu}\right)e^{ik|x_1-y_1|}|x_1-y_1|^{-1/2}\end{aligned}$$

for some $\nu>0$. Similarly

$$\begin{aligned}R(\zeta;Q_{+\rho})(x,y)&=c_0(\zeta)\left(\frac{e^{ik|x_1-y_1|}}{|x_1-y_1|^{1/2}}\right)e^{-i\alpha_-\pi}\left(\cos(\alpha_+\pi)+i\sin(\alpha_+\pi)I_0(\zeta)\right)\\&\quad+\ O\left(\rho^{-2\nu}\right)e^{ik|x_1-y_1|}|x_1-y_1|^{-1/2}.\end{aligned}$$

The asymptotic formula for the Hankel function $H_0(z)$ yields

$$\begin{aligned}R(\zeta;Q_{0\rho})(x,y)&=\left(e^{i(\alpha_+-\alpha_-)\pi}+O(\rho^{-1})\right)R(\zeta;K_0)(x,y)\\&=\ \left(c_0(\zeta)e^{i(\alpha_+-\alpha_-)\pi}+O(\rho^{-1})\right)e^{ik|x_1-y_1|}|x_1-y_1|^{-1/2}.\end{aligned}$$

We combine these three asymptotic formulas to calculate the leading coefficient $\pi_+(\zeta)$ defined by (1.12) as follows:

$$e^{i\alpha_+\pi}\cos(\alpha_-\pi)+e^{-i\alpha_-\pi}\cos(\alpha_+\pi)-e^{i(\alpha_+-\alpha_-)\pi}=\cos\left((\alpha_++\alpha_-)\pi\right),$$

$$i\left(-e^{i\alpha_+\pi}\sin(\alpha_-\pi)+e^{-i\alpha_-\pi}\sin(\alpha_+\pi)\right)=-\cos\left((\alpha_++\alpha_-)\pi\right)+e^{i(\alpha_+-\alpha_-)\pi}.$$

We evaluate remainder operators with kernels satisfying

$$O\left(\rho^{-2\nu}\right)e^{ik|x_1-y_1|}|x_1-y_1|^{-1/2}$$

for some $\nu>0$. If $(x,y)\in B_{+d}\times B_{-d}$, then it follows from (1.17) that this kernel obeys the bound $O\left(\rho^{-2\nu+3\delta_0}\right)$ uniformly in $\zeta\in D_d$. Hence we can take δ_0 so small that the remainder operators are bounded by $O\left(\rho^{-\nu}\right)$. This proves the proposition. □

6.2. Let $T=T(\zeta;\rho)$ be defined by (4.18) and let $\chi\in C_0^\infty[0,\infty)$ be a cut–off function with properties in (4.1). We establish the basic relation (4.20) by showing that $Id+T$ is invertible. We recall that μ, $2/5<\mu<1/2$, is fixed close to $1/2$ as in (2.1). We define

$$v_0(x_1)=\chi\left(2|x_1|/\rho^{1-\mu}\right),\quad \tilde v_0(x_1)=\chi\left(|x_1|/\rho^{1-\mu}\right),\quad v_1(x_1)=1-v_0(x_1)$$

and $\tilde v_1(x_1)=1-\chi\left(4|x_1|/\rho^{1-\mu}\right)$. Then $v_j\tilde v_j=v_j$ for $j=0,1$. We further define

$$T_{jk}=T_{jk}(\zeta;\rho)=v_jT(\zeta;\rho)\tilde v_k,\quad 0\le j,\ k\le 1.$$

We formulate the two lemmas below for these operators. For the first lemma, a similar lemma has been already proved as Lemma 7.1 in [3] or Lemma 5.1 in [9], although the slightly different notation has been used there.

Lemma 6.1. *The norm of the operators*

$$p_0 R_{\rm sc}(\zeta; Q_{\pm\rho}) \tilde{v}_1 \Gamma_0 R_{\rm sc}(\zeta; Q_{\pm\rho}) p_0, \quad p_0 R_{\rm sc}(\zeta; Q_{\pm\rho}) \tilde{v}_1 \Gamma_0 R_{\rm sc}(\zeta; Q_{\mp\rho}) p_0$$

acting on $L^2(\Pi_0)$ *is of order* $O\left(\rho^{-N}\right)$ *for any* $N \gg 1$*, and the norm of the operators*

$$T_{11}T_{11},\ T_{11}T_{10},\ T_{01}T_{11},\ T_{01}T_{10} : L^2(\Pi_0) \to L^2(\Pi_0)$$

is also of order $O\left(\rho^{-N}\right)$ *for any* $N \gg 1$.

Proof. The second statement is an immediate consequence of the first one. We consider only the operator with pair $\{R_{\rm sc}(\zeta; Q_{+\rho}), R_{\rm sc}(\zeta; Q_{+\rho})\}$ in the first statement. We decompose it into the sum

$$p_0 R_{\rm sc}(\zeta; Q_{\pm\rho}) \Big(\chi_L + (1-\chi_L)\Big) \tilde{v}_1 \Gamma_0 R_{\rm sc}(\zeta; Q_{\pm\rho}) p_0,$$

where $\chi_L(x_1) = \chi\left(|x_1|/L\rho^2\right)$ for $L \gg 1$ fixed arbitrarily but large enough. Then it follows from Proposition 5.2 (1) that the second operator on the right side obeys the bound $O\left(\rho^{-\sigma L}\right)$. By Proposition 5.2 ((2),(3)), the integral kernel of the first operator takes a form such as

$$e^{ikr_{+\rho}(x)} \left(\int \exp\left(2ikr_{+\rho}(u)\right) a(u; x, y, \zeta, \rho)\, du \right) e^{ikr_{+\rho}(y)}$$

where the function a satisfies

$$\left(\partial/\partial u_1\right)^n a = O\left(\rho^{-3-(1-\mu)n}\right)$$

and has support in

$$\left\{u = (u_1, u_2) : \rho^{1-\mu}/4 < |u_1| < 2L\rho^2,\ \ |u_2| < \rho/16\right\}$$

as a function of u. If $|u_1| > \rho^{1-\mu}/4$, then $\left|(\partial/\partial u_1) r_{+\rho}(u)\right| > c\,\rho^{-\mu}$ for some $c > 0$. Since $1-\mu > \mu$ by choice, the above integral obeys the bound $O\left(\rho^{-N}\right)$ for any $N \gg 1$ by repeated use of partial integral. This proves the lemma. $\square$

Lemma 6.2. $\|T_{00}\| = O\left(\rho^{-\mu}\right) = O\left(d^{-\mu/2}\right)$.

Proof. The kernel of the operator in the lemma has support

$$\left\{(x, y) : |x_1| < \rho^{1-\mu},\ |x_2| < \rho/16,\ |y_1| < 2\rho^{1-\mu},\ |y_2| < \rho/16\right\}$$

and $\left|\exp(ik|x_{\pm\rho}|)\right|$ is bounded uniformly in $\zeta \in D_d$ (and similarly for $|y_{\pm\rho}|$). Thus it follows from Proposition 5.2 (2) that the kernel obeys the bound $O\left(\rho^{-2}\right)$, and hence the Hilbert–Schmidt norm is evaluated as $\|T_{00}\|_{\mathrm{HS}} = O\left(\rho^{-\mu}\right)$. This proves the lemma. $\square$

For two bounded operators $A(\zeta)$ and $B(\zeta)$ acting on $L^2(\Pi_0)$ or on $L^2(\Omega_\rho)$, we use the notation $A(\zeta) \sim B(\zeta)$ to denote that the norm of $A(\zeta) - B(\zeta)$ is bounded by $O\left(\rho^{-N}\right)$ uniformly in $\zeta \in D_d$ for any $N \gg 1$. The operator $Id + T(\zeta;\rho)$ has the matrix representation

$$X = X(\zeta;\rho) = \begin{pmatrix} Id + T_{00} & T_{01} \\ T_{10} & Id + T_{11} \end{pmatrix}$$

as an operator acting on $L^2(\Pi_0) \oplus L^2(\Pi_0)$, and all the components $T_{jk}(\zeta;\rho)$ are bounded by $O\left(\rho^{\nu}\right)$ for some $\nu > 0$, as stated at the end of section 4. By Lemma 6.1, we have

$$(6.2) \qquad (Id + T_{11})^{-1} = \left(Id - T_{11}^2\right)^{-1}(Id - T_{11}) \sim Id - T_{11}$$

and X admits the decomposition

$$X = \begin{pmatrix} Id & 0 \\ 0 & Id + T_{11} \end{pmatrix}\begin{pmatrix} Id & T_{01} \\ 0 & Id \end{pmatrix}\begin{pmatrix} Id + T_{00} + T_N & 0 \\ (Id + T_{11})^{-1} T_{10} & Id \end{pmatrix},$$

where

$$T_N = T_N(\zeta;\rho) = -T_{01}\left(Id + T_{11}\right)^{-1} T_{10} \sim 0$$

by Lemma 6.1. Since the operator

$$Y_0 = Y_0(\zeta;\rho) = Id + T_{00} + T_N \sim Id + T_{00} : L^2(\Pi_0) \to L^2(\Pi_0)$$

is invertible by Lemma 6.2, X^{-1} is calculated as $\begin{pmatrix} Y_0^{-1} & X_{01} \\ X_{10} & X_{11} \end{pmatrix}$, where

$$X_{01} = -Y_0^{-1}T_{01}\left(Id + T_{11}\right)^{-1}, \quad X_{10} = -\left(Id + T_{11}\right)^{-1} T_{10} Y_0^{-1}$$

and

$$X_{11} = \left(Id + T_{11}\right)^{-1} T_{10} Y_0^{-1} T_{01}\left(Id + T_{11}\right)^{-1} + \left(Id + T_{11}\right)^{-1}.$$

If we take (6.2) into account, then we see from Lemma 6.1 that $(Id + T)^{-1}$ takes the form

$$(6.3) \qquad (Id + T)^{-1} \sim (Id - T_{10})\, Y_0^{-1}\left(v_0 - T_{01} v_1\right) + \left(Id - T_{11}\right) v_1.$$

Thus we can obtain the basic representation (4.20) for $R(\zeta;Q_\rho)$.

By Lemma 6.2, Y_0^{-1} is expanded into the Neumann series

$$(6.4) \quad Y_0^{-1} \sim Id + Y(\zeta;\rho), \quad Y = \sum_{l=1}^{L-1}(-1)^l T_{00}^l + (-1)^L T_{00}^L\left(Id + T_{00}\right)^{-1},$$

for $L \gg 1$ fixed large enough. Hence we have

$$(Id+T)^{-1} \sim Id - T_{10}v_0 - (T_{01} - T_{10}T_{01} + T_{11})\, v_1 + (Id - T_{10})\, Y\, (v_0 - T_{01}v_1)\,.$$

We also have

$$T_{01}v_1Tj_{-d} \sim 0, \quad T_{10}T_{01}v_1Tj_{-d} \sim 0, \quad T_{11}v_1Tj_{-d} \sim 0$$

by Lemma 6.1. Thus it follows from (4.20) that

$$j_{+d}R(\zeta; Q_\rho)j_{-d} \sim j_{+d}\Lambda j_{-d} - Z_0 + Z_1 - Z, \tag{6.5}$$

where

$$Z_0 = Z_0(\zeta;\rho) = j_{+d}\Lambda T j_{-d}, \quad Z_1 = Z_1(\zeta;\rho) = j_{+d}\Lambda T_{10}v_0Tj_{-d}$$

and $Z = Z(\zeta;\rho) = j_{+d}\Lambda\,(Id - T_{10})\, Y v_0 T j_{-d}$.

6.3. In what follows, we denote by $Op(\rho^\nu)$ the class of bounded operators acting on $L^2(\Omega_\rho)$ such that their norms obey $O(\rho^\nu)$ uniformly in $\zeta \in D_d$. We analyze the three operators Z_0, Z_1 and Z on the right side of (6.5). The obtained results are formulated as the three lemmas below.

Lemma 6.3. *The operator Z_0 takes the form*

$$Z_0 = -j_{+d}\left(\chi_{+\rho}R_{\mathrm{sc}}(\zeta; Q_{-\rho}) + \chi_{-\rho}R_{\mathrm{sc}}(\zeta; Q_{+\rho})\right) j_{-d} + Op\left(\rho^{-1-\mu+3\delta_0}\right).$$

Lemma 6.4. *The operator Z_1 is of class $Op\left(\rho^{-1-\mu+3\delta_0}\right)$.*

Lemma 6.5. *The operator Z is of class $Op\left(\rho^{-2\mu+6\delta_0}\right)$.*

We are now in a position to show assertion (6.1), accepting these three lemmas as proved.

Completion of Proof of Proposition 2.1 (Proof of (6.1)). We may assume that $\tilde{\chi}_{\pm\rho}j_{-d} = j_{-d}$ (see (4.13)). Then we recall from (4.14) that $j_{+d}\Lambda j_{-d}$ is given by

$$j_{+d}\Lambda j_{-d} = j_{+d}\left(\chi_{-\rho}R(\zeta; Q_{-\rho}) + \chi_{+\rho}R(\zeta; Q_{+\rho})\right) j_{-d}.$$

By definition (see (5.2)),

$$R(\zeta; Q_{\pm\rho}) = R(\zeta; Q_{0\rho}) + R_{\mathrm{sc}}(\zeta; Q_{\pm\rho}).$$

Hence (6.5), together with the three lemmas above, implies that the operator $j_{+d}R(\zeta; Q_\rho)j_{-d}$ under consideration takes the form

$$j_{+d}\left(R_{\mathrm{sc}}(\zeta; Q_{-\rho}) + R_{\mathrm{sc}}(\zeta; Q_{+\rho}) + R(\zeta; Q_{0\rho})\right) j_{-d} + Op\left(\rho^{-2\mu+6\delta_0}\right).$$

This proves (6.1), so that Proposition 2.1 is obtained. $\square$

6.4. The proof of Lemma 6.3 is based on the following three lemmas.

Lemma 6.6. *The operator defined by*

$$j_{+d}R_{\rm sc}(\zeta;Q_{\pm\rho})v_0\Gamma_0R_{\rm sc}(\zeta;Q_{\pm\rho})j_{-d}:L^2(\Omega_\rho)\to L^2(\Omega_\rho)$$

is of class $Op\left(\rho^{-N}\right)$ *for any* $N\gg 1$.

Lemma 6.7. *The operator defined by*

$$j_{+d}R_{\rm sc}(\zeta;Q_{\mp\rho})v_0\Gamma_0R_{\rm sc}(\zeta;Q_{\pm\rho})j_{-d}:L^2(\Omega_\rho)\to L^2(\Omega_\rho)$$

is of class $Op\left(\rho^{-1-\mu+3\delta_0}\right)$.

Lemma 6.8. *One has the following two relations:*

$$j_{+d}R(\zeta;Q_{0\rho})\Gamma_0R_{\rm sc}(\zeta;Q_{-\rho})j_{-d}\sim -j_{+d}\chi_{+\rho}R_{\rm sc}(\zeta;Q_{-\rho})j_{-d},$$
$$j_{+d}R(\zeta;Q_{0\rho})\Gamma_0\left(-R_{\rm sc}(\zeta;Q_{+\rho})\right)j_{-d}\sim -j_{+d}\chi_{-\rho}R_{\rm sc}(\zeta;Q_{+\rho})j_{-d}.$$

Proof of Lemma 6.6. We prove the lemma for the pair $\{Q_{+\rho},Q_{+\rho}\}$ only. By Proposition 5.2 (2), the integral kernel $R(x,y;\zeta,\rho)$ of the operator in question takes the form

$$R(x,y;\zeta,\rho)=e^{ik|x_{+\rho}|}\left(\int e^{2ik|u_{+\rho}|}a(u;x,y,\zeta,\rho)\,du\right)e^{ik|y_{+\rho}|}$$

for $(x,y)\in B_{+d}\times B_{-d}$, where a has support in $\left\{|u_1|<\rho^{1-\mu},\ |u_2|<\rho/16\right\}$ as a function of $u=(u_1,u_2)$ and satisfies

$$\left(\partial/\partial u_2\right)^n a=O\left(\rho^{-4-n}\right)$$

uniformly in (x,y) and $\zeta\in D_d$. Since $\left|\partial|u_{+\rho}|/\partial u_2\right|>1/2$, we see by repeated use of partial integration that a obeys $O\left(\rho^{-N}\right)$. This proves the lemma. $\square$

Proof of Lemma 6.7. We consider the pair $\{Q_{+\rho},Q_{-\rho}\}$ only. We again make use of Proposition 5.2 (2). Then it follows from (1.17) that the integral kernel $R(x,y;\zeta,\rho)$ of the operator under consideration is bounded by $O\left(\rho^{-1-\mu+3\delta_0}\right)$ uniformly in $(x,y)\in B_{+d}\times B_{-d}$ and $\zeta\in D_d$. Hence the lemma is immediately obtained. $\square$

Proof of Lemma 6.8. We only prove the first relation in some details. The proof is based on the same idea as developed in section 4. Let $\Lambda_\mp(\zeta;\rho)$ be defined by

$$\Lambda_-=\chi_{-\rho}R(\zeta;Q_{-\rho})\tilde\chi_{-\rho}+\chi_{+\rho}R(\zeta;Q_{0\rho})\tilde\chi_{+\rho},$$

$$\Lambda_+ = \chi_{-\rho}R(\zeta;Q_{0\rho})\tilde{\chi}_{-\rho} + \chi_{+\rho}R(\zeta;Q_{+\rho})\tilde{\chi}_{+\rho}.$$

Then we have the relation

$$(Q_{-\rho}-\zeta)\,\Lambda_- = Id + \Gamma_0\left(R(\zeta;Q_{-\rho})\tilde{\chi}_{-\rho} - R(\zeta;Q_{0\rho})\tilde{\chi}_{+\rho}\right).$$

We define the operator

$$T_-(\zeta;\rho) = \Gamma_0 R_{\mathrm{sc}}(\zeta;Q_{-\rho})p_0 : L^2(\Pi_0) \to L^2(\Pi_0)$$

corresponding to $T(\zeta;\rho)$ defined by (4.18), where the multiplication by the characteristic function p_0 of Π_0 again denotes the extension from $L^2(\Pi_0)$ to $L^2(\Omega_{\pm\rho})$. Then we have

$$R(\zeta;Q_{-\rho}) = \Lambda_- - \Lambda_- p_0\left(Id+T_-\right)^{-1}\Gamma_0\left(R(\zeta;Q_{-\rho})\tilde{\chi}_{-\rho} - R(\zeta;Q_{0\rho})\tilde{\chi}_{+\rho}\right),$$

which corresponds to relation (4.20). By Lemmas 6.1 and 6.6, we may write

$$\left(Id+T_-\right)^{-1} = \left(Id-T_-^2\right)^{-1}\left(Id-T_-\right) \sim Id - T_-.$$

We now compute

$$\Gamma_0\left(R(\zeta;Q_{-\rho})\tilde{\chi}_{-\rho} - R(\zeta;Q_{0\rho})\tilde{\chi}_{+\rho}\right)j_{-d} = \Gamma_0 R_{\mathrm{sc}}(\zeta;Q_{-\rho})j_{-d}$$

and hence it follows from Lemmas 6.1 and 6.6 that

$$\left(Id+T_-\right)^{-1}\Gamma_0\left(R(\zeta;Q_{-\rho})\tilde{\chi}_{-\rho} - R(\zeta;Q_{0\rho})\tilde{\chi}_{+\rho}\right)j_{-d} \sim \Gamma_0 R_{\mathrm{sc}}(\zeta;Q_{-\rho})j_{-d}.$$

Thus we obtain

$$j_{+d}R(\zeta;Q_{-\rho})j_{-d} \sim j_{+d}\Lambda_- j_{-d} - j_{+d}\Lambda_-\Gamma_0 R_{\mathrm{sc}}(\zeta;Q_{-\rho})j_{-d}.$$

Hence we have

$$\begin{aligned} j_{+d}\Lambda_-\Gamma_0 R_{\mathrm{sc}}(\zeta;Q_{-\rho})j_{-d} &\sim j_{+d}\Lambda_- j_{-d} - j_{+d}R(\zeta;Q_{-\rho})j_{-d} \\ &= -j_{+d}\left(R(\zeta;Q_{-\rho}) - \chi_{-\rho}R(\zeta;Q_{-\rho}) - \chi_{+\rho}R(\zeta;Q_{0\rho})\right)j_{-d} \\ &= -j_{+d}\left(\chi_{+\rho}R(\zeta;Q_{-\rho}) - \chi_{+\rho}R(\zeta;Q_{0\rho})\right)j_{-d} \\ &= -j_{+d}\chi_{+\rho}R_{\mathrm{sc}}(\zeta;Q_{-\rho})j_{-d}. \end{aligned}$$

Since

$$j_{+d}\Lambda_-\Gamma_0 R_{\mathrm{sc}}(\zeta;Q_{-\rho})j_{-d} \sim j_{+d}R(\zeta;Q_{0\rho})\Gamma_0 R_{\mathrm{sc}}(\zeta;Q_{-\rho})j_{-d},$$

the first relation is verified. To prove the second relation, we start with the relation

$$(Q_{+\rho}-\zeta)\,\Lambda_+ = Id + \Gamma_0\left(R(\zeta;Q_{0\rho})\tilde{\chi}_{-\rho} - R(\zeta;Q_{+\rho})\tilde{\chi}_{+\rho}\right)$$

and repeat the same argument as above. □

Proof of Lemma 6.3. According to (5.2), we write

$$\chi_{-\rho}R(\zeta;Q_{-\rho}) + \chi_{+\rho}R(\zeta;Q_{+\rho}) = R(\zeta;Q_{0\rho}) + \chi_{-\rho}R_{\mathrm{sc}}(\zeta;Q_{-\rho}) + \chi_{+\rho}R_{\mathrm{sc}}(\zeta;Q_{+\rho})$$

and use Lemmas 6.1, 6.6 and 6.7 to compute

$$Z_0 = j_{+d}R(\zeta; Q_{0\rho})\Gamma_0 \left(R_{\rm sc}(\zeta; Q_{-\rho}) - R_{\rm sc}(\zeta; Q_{+\rho})\right) j_{-d} + Op\left(\rho^{-1-\mu+3\delta_0}\right),$$

which, together with Lemma 6.8, proves the lemma. □

6.5. The proof of Lemma 6.4 requires the two preliminary lemmas below. We give only a sketch for the proof of the first lemma (Lemma 6.9), because it is done in almost the same way as Lemma 6.8.

Lemma 6.9. *One has the relation*

$$j_{+d}R(\zeta; Q_{0\rho})\Gamma_0 R_{\rm sc}(\zeta; Q_{\pm\rho})v_0p_0 \sim \pm j_{+d}\chi_{\mp\rho}R_{\rm sc}(\zeta; Q_{\pm\rho})v_0p_0.$$

Lemma 6.10. *One has the relation*

$$j_{+d}R(\zeta; Q_{0\rho})v_0\Gamma_0 R_{\rm sc}(\zeta; Q_{\pm\rho})v_0p_0 \sim 0.$$

Proof of Lemma 6.9. We consider only the operator with $R_{\rm sc}(\zeta; Q_{-\rho})$. As in the proof of Lemma 6.8, we get the relation

$$j_{+d}R(\zeta; Q_{-\rho})v_0p_0 \sim j_{+d}\Lambda_- v_0p_0 - j_{+d}\Lambda_-\Gamma_0 R_{\rm sc}(\zeta; Q_{-\rho})v_0p_0.$$

We also have the relation

$$\Lambda_-\Gamma_0 = \left(R(\zeta; Q_{0\rho}) + \chi_{-\rho}R_{\rm sc}(\zeta; Q_{-\rho})\right)\Gamma_0.$$

Hence it follows from Lemmas 6.1 and 6.6 that

$$\begin{aligned} & j_{+d}\Lambda_-\Gamma_0 R_{\rm sc}(\zeta; Q_{-\rho})v_0p_0 \\ &\quad = j_{+d}\left(R(\zeta; Q_{0\rho}) + \chi_{-\rho}R_{\rm sc}(\zeta; Q_{-\rho})\right)\Gamma_0 R_{\rm sc}(\zeta; Q_{-\rho})v_0p_0 \\ &\quad \sim j_{+d}R(\zeta; Q_{0\rho})\Gamma_0 R_{\rm sc}(\zeta; Q_{-\rho})v_0p_0. \end{aligned}$$

Thus we have the desired relation

$$\begin{aligned} & j_{+d}R(\zeta; Q_{0\rho})\Gamma_0 R_{\rm sc}(\zeta; Q_{-\rho})v_0p_0 \sim \\ &\qquad j_{+d}\left(\Lambda_- - R(\zeta; Q_{-\rho})\right)v_0p_0 = -j_{+d}\chi_{+\rho}R_{\rm sc}(\zeta; Q_{-\rho})v_0p_0. \end{aligned}$$

This proves the lemma. □

Proof of Lemma 6.10. The lemma is verified by repeated use of partial integral. The kernel of the operator in the lemma is given by composition. Let $u = (u_1, u_2)$ be such that $|u_1| < \rho^{1-\mu}$ and $|u_2| < \rho/16$. The phase function $ik|x-u|$ with $x \in B_{+d}$ comes from the operator $R(\zeta; Q_{0\rho})$, while $ik|u_{\pm\rho}|$ comes from $R_{\rm sc}(\zeta; Q_{\pm\rho})$. If we note that

$$\partial|x-u|/\partial u_1 \sim -1, \qquad \partial|u_{\pm\rho}|/\partial u_1 \sim 0$$

for $\rho \gg 1$, then this enables us to obtain the lemma by repeated use of partial integral. □

Proof of Lemma 6.4. We use Lemmas 6.1, 6.9 and 6.10 to compute

$$\begin{aligned} & j_{+d}\Lambda T_{10}v_0\Gamma_0 = \\ & \qquad j_{+d}\left(R(\zeta;Q_{0\rho})+\chi_{-\rho}R_{\rm sc}(\zeta;Q_{-\rho})+\chi_{+\rho}R_{\rm sc}(\zeta;Q_{+\rho})\right)T_{10}v_0\Gamma_0 \\ & \qquad \sim \ j_{+d}R(\zeta;Q_{0\rho})v_1\Gamma_0\left(R_{\rm sc}(\zeta;Q_{-\rho})-R_{\rm sc}(\zeta;Q_{+\rho})\right)v_0\Gamma_0 \\ & \qquad \sim \ j_{+d}R(\zeta;Q_{0\rho})\Gamma_0\left(R_{\rm sc}(\zeta;Q_{-\rho})-R_{\rm sc}(\zeta;Q_{+\rho})\right)v_0\Gamma_0 \\ & \qquad \sim \ -j_{+d}\left(\chi_{+\rho}R_{\rm sc}(\zeta;Q_{-\rho})+\chi_{-\rho}R_{\rm sc}(\zeta;Q_{+\rho})\right)v_0\Gamma_0. \end{aligned}$$

Hence the operator $Z_1 = j_{+d}\Lambda T_{10}v_0Tj_{-d}$ in the lemma behaves like

$$\begin{aligned} Z_1 \sim -j_{+d}\left(\chi_{+\rho}R_{\rm sc}(\zeta;Q_{-\rho})+\chi_{-\rho}R_{\rm sc}(\zeta;Q_{+\rho})\right) \\ \times \ v_0\Gamma_0\left(R_{\rm sc}(\zeta;Q_{-\rho})-R_{\rm sc}(\zeta;Q_{+\rho})\right)j_{-d}. \end{aligned}$$

This shows that Z_1 is of class $Op\left(\rho^{-1-\mu+3\delta_0}\right)$ and completes the proof. □

We proceed to proving Lemma 6.5.

Proof of Lemma 6.5. We recall the definition of the operator Z from (6.5). The lemma follows as an immediate consequence of the assertion

$$\|U_l(\zeta;\rho)\| = \left\|j_{+d}\Lambda\left(Id-T_{10}\right)T_{00}^l v_0Tj_{-d}\right\| = O\left(\rho^{-l\mu-\mu+6\delta_0}\right) \tag{6.6}$$

uniformly in $\zeta \in D_d$ for integers $l \geq 1$. For brevity, we prove this for the case $l = 1$ only. A similar argument applies to the other cases. We set

$$W_1 = j_{+d}\Lambda T_{00}v_0Tj_{-d}, \quad W_2 = j_{+d}\Lambda T_{10}T_{00}v_0Tj_{-d},$$

so that $U_1 = W_1 - W_2$. We use Lemmas 6.10 to compute

$$W_1 \sim j_{+d}\left(\chi_{-\rho}R_{\rm sc}(\zeta;Q_{-\rho})+\chi_{+\rho}R_{\rm sc}(\zeta;Q_{+\rho})\right)T_{00}v_0Tj_{-d}.$$

By (1.17) and Proposition 5.2 (2), we have

$$\|j_{+d}R_{\rm sc}(\zeta;Q_{\pm\rho})v_0p_0\| = O\left(\rho^{1/2-\mu/2+3\delta_0}\right)$$

and $\|v_0Tj_{-d}\| = O\left(\rho^{-1/2-\mu/2+3\delta_0}\right)$. Since $\|T_{00}\| = O\left(\rho^{-\mu}\right)$ as shown in the proof of Lemma 6.2, it follows that $\|W_1\| = O\left(\rho^{-\mu-\mu+6\delta_0}\right)$. We combine Lemmas 6.1, 6.9 and 6.10 to obtain that

$$\begin{aligned} W_2 &\sim j_{+d}R(\zeta;Q_{0\rho})\Gamma_0\left(R_{\rm sc}(\zeta;Q_{-\rho})-R_{\rm sc}(\zeta;Q_{+\rho})\right)T_{00}v_0Tj_{-d} \\ &\sim Op\left(\rho^{1/2-\mu/2+3\delta_0}\right)T_{00}v_0Tj_{-d} \end{aligned}$$

and hence $\|W_2\| = O\left(\rho^{-\mu-\mu+6\delta_0}\right)$. Thus (6.6) is established and the proof is complete. □

6.6. We end the paper by proving Proposition 2.2. The proof is based on the same arguments as used for proving Proposition 2.1. We give only a sketch for it.

Proof of Proposition 2.2. We can get a relation similar to (6.5):

$$j_{+d}R(\zeta;Q_\rho)\tilde{j}_{+d} \sim j_{+d}\Lambda\tilde{j}_{+d} - \tilde{Z}_0 + \tilde{Z}_1 - \tilde{Z},$$

where

$$\tilde{Z}_0 = j_{+d}\Lambda T\tilde{j}_{+d}, \qquad Z_1 = j_{+d}\Lambda T_{10}v_0T\tilde{j}_{+d}$$

and $Z = j_{+d}\Lambda\,(Id - T_{10})\,Yv_0T\tilde{j}_{+d}$. We may assume that $\tilde{\chi}_{\pm\rho}\tilde{j}_{+d} = \tilde{j}_{+d}$. The leading term comes from

$$j_{+d}\Lambda\tilde{j}_{+d} = j_{+d}\left(R(\zeta;Q_{0\rho}) + \chi_{-\rho}R_{\rm sc}(\zeta;Q_{-\rho}) + \chi_{+\rho}R_{\rm sc}(\zeta;Q_{+\rho})\right)\tilde{j}_{+d}.$$

By (4.11), we have

$$R(\zeta;Q_{0\rho})(x,y) = \exp\bigl(i(g_{0\rho}(x) - g_{0\rho}(y))\bigr)R(\zeta;K_0)(x,y)$$

for $(x,y) \in B_{+d}\times\tilde{B}_{+d}$, and $\exp\bigl(i(g_{0\rho}(x) - g_{0\rho}(y))\bigr)$ yields $a_0(x,y;d)$ defined by (2.6). Thus the leading operator $\tilde{R}_0(\zeta;d)$ in the proposition is obtained. We evaluate the operator $j_{+d}R_{\rm sc}(\zeta;Q_{\pm\rho})\tilde{j}_{+d}$ by making use of Proposition 5.1 (2). If $x \in B_{+d}$, then

$$|x_{\pm\rho}| = |x - \rho_\pm| = \left(|x_1|^2 + O(d)\right)^{1/2} = \kappa_+d + O(1), \quad d \gg 1,$$

and hence it follows from (1.17) that

$$\left|\exp(ik|x_{\pm\rho}|)/|x_{\pm\rho}|^{1/2}\right| = O\left(d^{3\delta_0/2}\right)\left|\exp(-ik(1-\kappa_+)d)\right|$$

on the bottom of the neighborhood D_d. A similar bound remains true for $y \in \tilde{B}_{+d}$. Note that $\kappa_+ < 1$ strictly. Thus we can take $\delta_0 > 0$ and $\delta > 0$ so small that

$$\|j_{+d}R_{\rm sc}(\zeta;Q_{\pm\rho})\tilde{j}_{+d}\| = O\left(d^{-\nu}\right)$$

for some $\nu > 0$. The other operators $\tilde{Z}_1$ and $\tilde{Z}$ are also shown to obey $O\left(d^{-\nu}\right)$ for another $\nu > 0$ and are dealt with as remainder operators. This proves the proposition. □

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