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AHARONOV–BOHM EFFECT IN RESONANCES OF MAGNETIC SCHRÖDINGER OPERATORS IN TWO DIMENSIONS II

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Abstract. We study the Aharonov–Bohm effect (AB effect) in quantum resonances for magnetic scattering in two dimensions. The system consists of four scatters, two obstacles and two scalar potentials with compact support, which are largely separated from one another. The obstacles by which the magnetic fields are completely shielded are horizontally placed between the supports of the two potentials. The fields do not influence particles from a classical mechanical point of view, but quantum particles are influenced by the corresponding vector potential which does not necessarily vanish outside the obstacle. This quantum phenomenon is called the AB effect. The resonances are shown to be generated near the real axis by the trajectories trapped between two supports of the scalar potentials as the distances between the scatterers go to infinity. We analyze how the AB effect influences the location of resonances. The result is described in terms of the backward amplitudes for scattering by each of the scalar potentials, and it depends heavily on the ratios of the distances between the four scatterers as well as on the magnetic fluxes of the fields.

1. Introduction

In quantum mechanics, a vector potential is said to have a direct significance to particles moving in a magnetic field. This is called the Aharonov–Bohm effect (AB effect) and is known as one of the most remarkable quantum phenomena ([1]). In this work, we study the AB quantum effect in resonances of magnetic Schrödinger operators in two dimensions. The scattering system consists of four scatters, two obstacles and two scalar potentials with compact support, where the scatters are largely separated from one another, and the obstacles are placed between the supports of the two potentials. The magnetic fields are assumed to be completely shielded by the obstacles, so that the fields do not influence particles from a classical mechanical point of view. However, by the AB effect, quantum particles are influenced by the corresponding vector potential which does not necessarily vanish outside the obstacle. We can show that the resonances are generated near the real axis by the trajectories trapped between two supports of the

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scalar potentials as the distances between the scatterers go to infinity. The location of the resonances is described in terms of the backward amplitudes for scattering by each of the scalar potentials, and it depends heavily on the magnetic fluxes of the fields. Thus we see that the location of resonances is strongly influenced by the AB quantum effect.

The present work is a continuation to [7], where we have studied the case of one obstacle. A new difficulty which happens to the case of two obstacles appears from the trapping phenomenon between the obstacles. In addition, we have to take account of the difficulty which comes from the fact that even if the supports of the magnetic fields are largely separated from each other, the corresponding vector potentials are not expected to be well separated. To overcome these difficulties, we make use of the gauge transformation and the complex scaling method. The location of the resonances depends not only on the magnetic fluxes but also on the location of the obstacles. Here we deal with the case when the obstacles are horizontally placed between the supports of the two potentials. In [9], we will consider the case when the obstacles are vertically placed between the supports of the two potentials.

We set up our problem precisely. We always work in the two dimensional space $\boldsymbol{R}^2$ with generic point $x = (x_1, x_2)$ and write

$$H(A, V) = (-i\nabla - A)^2 + V = \sum_{j=1}^{2} (-i\partial_j - a_j)^2 + V, \quad \partial_j = \partial/\partial x_j,$$

for the magnetic Schrödinger operator with $A = (a_1, a_2) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ as a vector potential and $V : \boldsymbol{R}^2 \to \boldsymbol{R}$. as a scalar potential. The magnetic field $b : \boldsymbol{R}^2 \to \boldsymbol{R}$ associated with A is defined by

$$b(x) = \nabla \times A(x) = \partial_1 a_2 - \partial_2 a_1$$

and the quantity defined as the integral $\alpha = (2\pi)^{-1} \int b(x)\, dx$ is called the magnetic flux of b, where the integration with no domain attached is taken over the whole space. We often use this abbreviation.

Let $b_\pm \in C_0^\infty(\boldsymbol{R}^2)$ be two given magnetic fields with the fluxes

$$\alpha_\pm = (2\pi)^{-1} \int b_\pm(x)\, dx.$$

We make the assumption that the support of $b_\pm$ satisfies

$$\operatorname{supp} b_\pm \subset \mathcal{O}_\pm \subset B = \{|x| < 1\} \tag{1.1}$$

for some simply connected bounded obstacle $\mathcal{O}_\pm$, where $\mathcal{O}_\pm$ is assumed to have the origin as an interior point and the smooth boundary $\partial\mathcal{O}_\pm$. For the vector potential $A_\pm(x)$ corresponding to $b_\pm$, we can take $A_\pm(x)$ to fulfill

$$A_\pm(x) = \alpha_\pm \Phi(x) \tag{1.2}$$

over $\Omega_\pm = \boldsymbol{R}^2 \setminus \overline{\mathcal{O}}_\pm$, where $\Phi(x)$ is defined by

$$\Phi = \left(-x_2/|x|^2, x_1/|x|^2\right) = \left(-\partial_2 \log|x|, \partial_1 \log|x|\right). \tag{1.3}$$

In fact, Φ defines the δ–like magnetic field (solenoidal field)

$$\nabla \times \Phi = \left(\partial_1^2 + \partial_2^2\right) \log|x| = \Delta \log|x| = 2\pi\delta(x)$$

with center at the origin, when considered over the whole space. This vector potential is often called the Aharonov–Bohm potential in physics literatures. Assumption (1.1) means that the field $b_\pm$ is entirely shielded by the obstacle $\mathcal{O}_\pm$, although the corresponding vector potential $A_\pm$ does not necessarily vanish outside $\mathcal{O}_\pm$.

For $d \in \boldsymbol{R}^2$ with $|d| \gg 1$, we set

$$d_- = -\kappa_- d, \quad d_+ = \kappa_+ d, \quad \kappa_\pm > 0, \quad \kappa_- + \kappa_+ = 1,$$

so that $d_+ - d_- = d$. The distance $|d| \gg 1$ is regarded as a large parameter with the direction $\hat{d} = d/|d|$ fixed. Let $V_\pm \in C_0^\infty(\boldsymbol{R}^2)$ with $\operatorname{supp} V_\pm \subset B$. Then we define

$$V_d(x) = V_{-d}(x) + V_{+d}(x) = V_-(x - d_-) + V_+(x - d_+). \tag{1.4}$$

For $\rho_\pm = \pm\kappa_0 d$ with $0 < \kappa_0 < \min(\kappa_-, \kappa_+)$, we also define

$$A_\rho(x) = A_{-\rho}(x) + A_{+\rho}(x) = A_-(x - \rho_-) + A_+(x - \rho_+) \tag{1.5}$$

over the exterior domain

$$\Omega_\rho = \boldsymbol{R}^2 \setminus \left(\overline{\mathcal{O}}_{-\rho} \cup \overline{\mathcal{O}}_{+\rho}\right), \qquad \mathcal{O}_{\pm\rho} = \{x : x - \rho_\pm \in \mathcal{O}_\pm\}. \tag{1.6}$$

We now consider the self–adjoint operator

$$H_d = H(A_\rho, V_d), \qquad \mathcal{D}(H_d) = H^2(\Omega_\rho) \cap H_0^1(\Omega_\rho), \tag{1.7}$$

in $L^2(\Omega_\rho)$ under the zero boundary conditions, where $H_0^1(W)$ and $H^2(W)$ stand for the usual Sobolev spaces over a region W. We know that the resolvent

$$R(\zeta; H_d) = (H_d - \zeta)^{-1} : L^2(\Omega_\rho) \to L^2(\Omega_\rho), \quad \operatorname{Re}\zeta > 0, \ \operatorname{Im}\zeta > 0,$$

is meromorphically continued from the upper half plane of the complex plane to the lower half plane across the positive real axis where the continuous spectrum of H_d is located. Then $R(\zeta; H_d)$ with $\operatorname{Im}\zeta \le 0$ is well defined as an operator from $L^2_{\mathrm{comp}}(\Omega_\rho)$ to $L^2_{\mathrm{loc}}(\Omega_\rho)$ in the sense that $\chi R(\zeta; H_d)\chi : L^2(\Omega_\rho) \to L^2(\Omega_\rho)$ is bounded for every $\chi \in C_0^\infty(\overline{\Omega}_\rho)$, where $L^2_{\mathrm{comp}}(W)$ denotes the space of square integrable functions with compact support in the closure $\overline{W}$ of W and $L^2_{\mathrm{loc}}(W)$ denotes the space of locally square integrable functions over $\overline{W}$. The resonances of H_d are defined as the poles of $R(\zeta; H_d)$ in the lower half plane (unphysical sheet). This is shown by use of the complex scaling method [4, 5, 6]. Our aim is to study how the resonances are generated near the real axis by the trajectories trapped between the two centers d_- and d_+ as $|d| = |d_+ - d_-| \to \infty$ and how the AB effect influences the location of the resonances.

The obtained result is formulated in terms of the backward amplitudes by the potentials $V_\pm$. Let $K_0 = -\Delta$ be the free Hamiltonian and let $K_\pm$ be the Schrödinger operator defined by

$$K_\pm = K_0 + V_\pm = -\Delta + V_\pm, \quad \mathcal{D}(K_0) = \mathcal{D}(K_\pm) = H^2(\boldsymbol{R}^2). \tag{1.8}$$

We denote by $f_\pm(\omega \to \theta; E)$ the amplitude for scattering from the incident direction $\omega \in S^1$ to the final one θ at energy $E > 0$ for the pair $(K_0, K_\pm)$. These amplitudes admit the analytic extensions $f_\pm(\omega \to \theta; \zeta)$ in a complex neighborhood of the positive real axis as a function of E.

We now fix $E_0 > 0$ and take a complex neighborhood

$$D_d = \left\{ \zeta : |\mathrm{Re}\,\zeta - E_0| < \delta_0 E_0,\ |\mathrm{Im}\,\zeta| < (1 + 2\delta_0)\, E_0^{1/2} \left(\frac{\log |d|}{|d|} \right) \right\} \tag{1.9}$$

for δ_0, $0 < \delta_0 \ll 1$, small enough. We also define the angle ψ_0 through the relation

$$\cos \psi_0 = \left(\frac{\kappa_- - \kappa_0}{\kappa_- + \kappa_0} \right)^{1/2} \left(\frac{\kappa_+ - \kappa_0}{\kappa_+ + \kappa_0} \right)^{1/2} < 1, \quad 0 < \psi_0 < \pi/2, \tag{1.10}$$

and set

$$\pi_0 = \left(1 - \frac{\psi_0}{\pi} \right) \cos\left((\alpha_+ + \alpha_-)\pi \right) + \frac{\psi_0}{\pi} \cos\left((\alpha_+ - \alpha_-)\pi \right). \tag{1.11}$$

We further define

$$h(\zeta; d) = \left(\frac{e^{2ik|d|}}{|d|} \right) f_-(-\hat{d} \to \hat{d}; \zeta) f_+(\hat{d} \to -\hat{d}; \zeta) \pi_0^2, \quad k = \zeta^{1/2}, \tag{1.12}$$

over D_d, where the branch $k = \zeta^{1/2}$ is taken in such a way that $\mathrm{Re}\,k > 0$ for $\mathrm{Re}\,\zeta > 0$. We always use the notation k with the meaning ascribed here. Since

$$2\,\mathrm{Im}\,k = 2\,\mathrm{Im}\left(\mathrm{Re}\,\zeta + i\mathrm{Im}\,\zeta\right)^{1/2} = \mathrm{Im}\,\zeta / \left(\mathrm{Re}\,\zeta\right)^{1/2} + O\left(|\mathrm{Im}\,\zeta|^3 \right) \tag{1.13}$$

for $\zeta \in D_d$ and since

$$(\mathrm{Re}\,\zeta)^{1/2} = E_0^{1/2} \Big(1 + \left(\mathrm{Re}\,\zeta - E_0\right)/(2E_0) + O(\delta_0^2) \Big) \tag{1.14}$$

with $|\mathrm{Re}\,\zeta - E_0| < \delta_0 E_0$, we can take $\delta_0 > 0$ so small that

$$|d|^{\delta_0} < \Big|\exp(2ik|d|)\Big|/|d| < |d|^{3\delta_0}, \qquad |d| \gg 1, \tag{1.15}$$

on the bottom of D_d $\left(\mathrm{Im}\,\zeta = -\left(1+2\delta_0\right)E_0^{1/2}\left((\log|d|)/|d|\right)\right)$. This implies that the curve defined by $|h(\zeta;d)| = 1$ with $|\mathrm{Re}\,\zeta - E_0| < \delta_0 E_0$ is completely contained in D_d, provided that $f_\pm(\pm\hat{d} \to \mp\hat{d}; E_0) \neq 0$ and $\pi_0 \neq 0$. We denote by

$$\Big\{\zeta_j(d)\Big\}, \quad \zeta_j(d) \in D_d, \quad \mathrm{Re}\,\zeta_1 < \mathrm{Re}\,\zeta_2 < \cdots < \mathrm{Re}\,\zeta_{N_d},$$

the solutions to the equation

$$h(\zeta;d) = 1. \tag{1.16}$$

We know (see Lemma 3.5) that $\zeta_j(d)$ behaves like

$$\mathrm{Im}\,\zeta_j(d) \sim -E_0^{1/2}\left((\log|d|)/|d|\right), \quad \mathrm{Re}\left(\zeta_{j+1}(d) - \zeta_j(d)\right) \sim 2\pi E_0^{1/2}/|d|,$$

for $|d| \gg 1$. With the notation above, we are now in a position to state the main result.

Theorem 1.1. *Let the notation be as above. Assume that $\pi_0 \neq 0$ and*

$$f_\pm(\pm\hat{d} \to \mp\hat{d}; E_0) \neq 0, \quad \hat{d} = d/|d|,$$

at energy $E_0 > 0$. Then we can take $\delta_0 > 0$ so small that the neighborhood D_d defined by (1.9) has the following property: For any $\varepsilon > 0$ small enough, there exists $d_\varepsilon \gg 1$ such that for $|d| > d_\varepsilon$, H_d has the resonances

$$\Big\{\zeta_{\mathrm{res},j}(d)\Big\}, \quad \zeta_{\mathrm{res},j}(d) \in D_d, \quad \mathrm{Re}\,\zeta_{\mathrm{res},1}(d) < \cdots < \mathrm{Re}\,\zeta_{\mathrm{res},N_d}(d)$$

in the neighborhood

$$\Big\{\zeta \in D_d : |\zeta - \zeta_j(d)| < \varepsilon/|d|\Big\}$$

and the resolvent $R(\zeta; H_d)$ is analytic over $D_d \setminus \Big\{\zeta_{\mathrm{res},1}(d), \cdots, \zeta_{\mathrm{res},N_d}(d)\Big\}$ as a function with values in operators from $L^2_{\mathrm{comp}}(\Omega_\rho)$ to $L^2_{\mathrm{loc}}(\Omega_\rho)$.

The theorem above is proved in section 3 after a series of preliminary lemmas are formulated. In section 2, we mention two basic propositions (Propositions 2.1 and 2.2) on the asymptotic properties of the resolvent kernel $R(\zeta; H_{0d})(x,y)$ with $\zeta \in D_d$ for the self–adjoint operator

$$H_{0d} = H(A_\rho, 0), \quad \mathcal{D}(H_{0d}) = H^2(\Omega_\rho) \cap H^1_0(\Omega_\rho), \tag{1.17}$$

where A_ρ is defined by (1.5). In particular, Proposition 2.1 which is concerned with the asymptotic behavior as $|x-y| \gg 1$ along the forward direction $(x,y) \sim (d_\pm, d_\mp)$ plays an important role in proving the theorem.

Sections 4 and 5 are devoted to proving the two propositions. The operator H_{0d} acts on the domain exterior to the union of the two obstacles $\mathcal{O}_{-\rho}$ and $\mathcal{O}_{+\rho}$ placed at large separation, and the resolvent kernel with a spectral parameter in the unphysical sheet grows exponentially at infinity. In order to construct the resolvent kernel $R(\zeta; H_{0d})(x, y)$ with $\zeta \in D_d$, we make use of the gauge transformation to separate the two obstacles from each other and of the complex scaling method to compose the resolvent kernel constructed for each obstacle $\mathcal{O}_{\pm\rho}$. We rely on the results obtained in the previous work [8] for the proof of the two propositions. We summarize these results in Appendix in the form adapted to our application.

2. Asymptotic properties of resolvent kernel

We write $R(\zeta; T)$ for the resolvent $(T - \zeta)^{-1}$ of the operator T acting on $L^2(W)$, W being a domain of $\boldsymbol{R}^2$. We use the same notation $R(\zeta; T)$ for the resolvent meromorphically extended from the upper half plane to the lower half one. For notational brevity, we take the four centers $d_\pm$ and $\rho_\pm$ as follows:

$$d_\pm = (\pm\kappa_\pm d, 0), \quad \rho_\pm = (\pm\kappa_0 d, 0), \quad d \gg 1, \tag{2.1}$$

where

$$-\kappa_- < -\kappa_0 < 0 < \kappa_0 < \kappa_+, \quad \kappa_- + \kappa_+ = 1.$$

We identify the distance $|d| = |d_+ - d_-|$ between the two centers d_+ and d_- with d, while the direction of $d_+ - d_-$ remains fixed as $\omega_1 = (1, 0)$. We further take μ to be

$$2/5 < \mu < 1/2\Big(< 1 - \mu\Big) \tag{2.2}$$

close enough to $1/2$. Throughout the whole exposition, we use notation $d_\pm$, $\rho_\pm$ and μ with the meanings ascribed in (2.1) and (2.2).

Let H_{0d} be defined by (1.17). The next theorem is obtained as an immediate consequence of [3, Theorem 1.1] (see also [2]). We give its proof at the end of this section.

Theorem 2.1. *Let D_d be defined by (1.9) with $|d|$ replaced by d. Then*

$$R(\zeta; H_{0d}) : L^2_{\mathrm{comp}}(\Omega_\rho) \to L^2_{\mathrm{loc}}(\Omega_\rho)$$

is analytic over D_d for $d \gg 1$.

Let $R(\zeta; H_{0d})(x, y)$ be the resolvent kernel of $R(\zeta; H_{0d})$ with $\zeta \in D_d$. We are now in a position to state the asymptotic properties of $R(\zeta; H_{0d})(x, y)$ as the two propositions, which play an important role in proving Theorem 1.1. In what follows, $H_0(z) = H_0^{(1)}(z)$ denotes the Hankel function of the first kind and of order zero, and $\gamma(\omega;\theta)$ denotes the azimuth angle from $\omega \in S^1$ to θ.

Proposition 2.1. *Let $j_{\pm d}$ be the characteristic function of*

$$B_{\pm d} = \Big\{x : |x - d_\pm| < 1\Big\} \tag{2.3}$$

and let $c_0(\zeta)$ be defined by

$$c_0(\zeta) = (8\pi)^{-1/2}\, e^{i\pi/4}\zeta^{-1/4} = (8\pi)^{-1/2}\, e^{i\pi/4}k^{-1/2}. \tag{2.4}$$

Assume that $\pi_0 \neq 0$ for the constant π_0 defined by (1.11). Then the operator $j_{+d}R(\zeta; H_{0d})j_{-d}$ acting on $L^2(\Omega_\rho)$ admits the decomposition

$$j_{+d}R(\zeta; H_{0d})j_{-d} = R_0(\zeta; d) + R_1(\zeta; d),$$

where $R_0(\zeta; d)$ is the integral operator with the kernel

$$R_0(\zeta, d)(x, y) = c_0(\zeta)\pi_0 e^{ik|x_1-y_1|}|x_1 - y_1|^{-1/2}$$

for $(x, y) \in B_{+d} \times B_{-d}$, and $R_1(\zeta; d)$ obeys the bound $\|R_1(\zeta; d)\| = O(d^{-\nu})$ uniformly in $\zeta \in D_d$ for some $\nu > 0$. A similar decomposition remains true for $j_{-d}R(\zeta; H_{0d})j_{+d}$.

Proposition 2.2. *Let $\tilde{j}_{\pm d}$ denote the characteristic function of*

$$\tilde{B}_{\pm d} = \Big\{x : d^\delta < |x - d_\pm| < 2d^\delta\Big\} \tag{2.5}$$

with $0 < \delta \ll 1$ fixed small enough. Write $x_{\pm\rho} = x - \rho_\pm$ and $\hat{x}_{\pm\rho} = x_{\pm\rho}/|x_{\pm\rho}|$. Then the operator $j_{+d}R(\zeta; H_{0d})\tilde{j}_{+d}$ admits the decomposition

$$j_{+d}R(\zeta; H_{0d})\tilde{j}_{+d} = \tilde{R}_0(\zeta; d) + \tilde{R}_1(\zeta; d),$$

where the kernel $\tilde{R}_0(\zeta; d)(x, y)$ of the integral operator $\tilde{R}_0(\zeta; d)$ is defined by

$$\tilde{R}_0(\zeta; d)(x, y) = (i/4)H_0(k|x - y|)a_0(x, y; d)$$

for $(x, y) \in B_{+d} \times \tilde{B}_{+d}$ with

$$a_0 = \exp\Big(i\alpha_+(\gamma(\hat{x}_{+\rho}; -\hat{y}_{+\rho}) - \pi) + i\alpha_-(\gamma(\hat{x}_{-\rho}; -\hat{y}_{-\rho}) - \pi)\Big), \tag{2.6}$$

while $\tilde{R}_1(\zeta; d)$ obeys the bound $\|\tilde{R}_1(\zeta; d)\| = O(d^{-\nu})$ uniformly in $\zeta \in D_d$ for some $\nu > 0$. A similar decomposition remains true for $j_{-d}R(\zeta; H_{0d})\tilde{j}_{-d}$.

We prove these basic propositions in sections 4 and 5. The proof is based on the gauge transformation and on the complex scaling method, as stated at the end of the previous section. We end the section by proving Theorem 2.1.

Proof of Theorem 2.1. By assumption, $\zeta \in D_d$. Since $\rho = |\rho_+ - \rho_-| = 2\kappa_0 d < d$, we have

$$\left| e^{2ik\rho}/\rho \right| < 1, \quad \zeta \in D_d,$$

strictly for $d \gg 1$. This, together with [3, Theorem 1.1], implies that the resolvent $R(\zeta; H_{0d})$ is analytic over D_d as a function with values in operators from $L^2_{\mathrm{comp}}(\Omega_\rho)$ to $L^2_{\mathrm{loc}}(\Omega_\rho)$. Thus the proof is complete. □

3. Proof of main theorem

In this section, we complete the proof of the main theorem, accepting a series of preliminary lemmas as proved. These lemmas are verified at the end of the section.

3.1. We fix the notation to formulate the lemmas. Let $\varphi_0(x;\omega,E)$ be the plane wave defined by

$$\varphi_0(x;\omega,E) = \exp\left(iE^{1/2}x\cdot\omega\right)$$

with ω as an incident direction at energy $E > 0$. Let $K_0 = -\Delta$ and $K_\pm$ be as in (1.8). Then we define

$$\varphi_\pm(x;\omega,\overline{\zeta}) = \left[(Id - R(\zeta;K_\pm)^* V_\pm)\,\varphi_0(\cdot;\omega,\overline{\zeta})\right](x). \tag{3.1}$$

The function $\varphi_\pm(x;\omega,\overline{\zeta})$ solves the equation $\left(K_\pm - \overline{\zeta}\right)\varphi_\pm(x;\omega,\overline{\zeta}) = 0$. In, particular, if $\zeta = E > 0$, then $\varphi_\pm(x;\omega,E)$ turns out to be the incoming eigenfunction of $K_\pm$, and the conjugate function $\overline{\varphi}_\pm(x;\omega,\overline{\zeta})$ of $\varphi_\pm(x;\omega,\overline{\zeta})$ is analytic in ζ. It should be noted that $\varphi_\pm(x;\omega,\overline{\zeta})$ itself is not analytic. We also note that $\varphi_+(x;\omega,E)$ does not denote the outgoing eigenfunction at energy $E > 0$ but the incoming eigenfunction of the Schrödinger operator K_+.

We recall that the notation $\gamma(x;\omega) = \gamma(\hat{x};\omega)$ denotes the azimuth angle from ω to $\hat{x} = x/|x|$. We take a function $\gamma_\pm \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ such that

$$\gamma_\pm(x) = \alpha_-\gamma(\hat{x}_{-\rho};\mp\omega_1) + \alpha_+\gamma(\hat{x}_{+\rho};\mp\omega_1) \tag{3.2}$$

on $\{|x - d_\pm| < |d_\pm - \rho_\pm|/2\}$ and $\partial_x^n\gamma_\pm = O(|x|^{-|n|})$ as $|x| \to \infty$, where $\hat{x}_{\pm\rho} = x_{\pm\rho}/|x_{\pm\rho}|$ with $x_{\pm\rho} = x - \rho_\pm$. Since $\nabla\gamma(x;\omega) = \Phi(x)$ for Φ defined by (1.3), it follows from (1.5) that

$$\nabla\gamma_\pm = \alpha_-\Phi(x - \rho_-) + \alpha_+\Phi(x - \rho_+) = A_{-\rho} + A_{+\rho} = A_\rho \tag{3.3}$$

on $\{|x - d_\pm| < |d_\pm - \rho_\pm|/2\}$. If we denote by $j(x)$ the characteristic function of the unit disk B, then

$$j_{\pm d}(x) = j\left(x - d_\pm\right) = j\left(x_{\pm d}\right)$$

defines the characteristic function of $B_{\pm d}$ defined by (2.3). We introduce the auxiliary operator

$$H_{\pm d} = H(A_\rho, V_{\pm d}), \quad \mathcal{D}(H_{\pm d}) = H^2(\Omega_\rho) \cap H_0^1(\Omega_\rho), \tag{3.4}$$

where $V_{\pm d}(x) = V_\pm(x - d_\pm)$ is defined by (1.4) and $V_\pm$ is assumed to have support in B. With the notation above, we are now to state a series of preliminary lemmas.

Lemma 3.1. *Let the notation be as above and let D_d be defined by (1.9). Define the operator $Q_\pm(\zeta; d)$ by*

$$Q_\pm(\zeta; d) = V_{\pm d} R(\zeta; H_{0d}) j_{\pm d} : L^2(B_{\pm d}) \to L^2(B_{\pm d})$$

for $\zeta \in D_d$, where the multiplication $j_{\pm d}$ is understood to be the extension from $L^2(B_{\pm d})$ to $L^2(\Omega_\rho)$. Then $Id + Q_\pm(\zeta; d)$ takes the form

$$Id + Q_\pm(\zeta; d) = e^{i\gamma_\pm} \left(Id + \tilde{Q}_\pm(\zeta; d) \right) (Id + V_{\pm d} R(\zeta; K_0) j_{\pm d})\, e^{-i\gamma_\pm},$$

where $\tilde{Q}_\pm(\zeta; d)$ is analytic in $\zeta \in D_d$ with values in bounded operators acting on $L^2(B_{\pm d})$ and obeys $\|\tilde{Q}_\pm(\zeta; d)\| = O\left(|d|^{-\nu}\right)$ uniformly in ζ for some $\nu > 0$.

Lemma 3.2. *Let $Q_\pm(\zeta; d)$ be as in Lemma 3.1. Then*

$$Id + Q_\pm(\zeta; d) : L^2(B_{\pm d}) \to L^2(B_{\pm d})$$

has the inverse bounded uniformly in d and $\zeta \in D_d$. Moreover, we have the relation

$$R(\zeta; H_{\pm d}) j_{\pm d} = R(\zeta; H_{0d}) j_{\pm d} \left(Id + Q_\pm(\zeta; d)\right)^{-1} : L^2(B_{\pm d}) \to L^2_{\mathrm{loc}}(\Omega_\rho)$$

for $\zeta \in D_d$.

By the resolvent identity, it follows from Theorem 2.1 and Lemma 3.2 that the resolvent $R(\zeta; H_{\pm d})$ is represented as

$$R(\zeta; H_{\pm d}) = \left(Id - R(\zeta; H_{\pm d}) V_{\pm d} \right) R(\zeta; H_{0d}) : L^2_{\mathrm{comp}}(\Omega_\rho) \to L^2_{\mathrm{loc}}(\Omega_\rho) \tag{3.5}$$

for $\zeta \in D_d$ and is analytic there.

Lemma 3.3. *Assume that the constant π_0 defined by (1.11) does not vanish. Recall that the incoming eigenfunction $\varphi_\pm(x; \omega, \zeta)$ of $K_\pm = K_0 + V_\pm$ is defined by (3.1). Then we have the following statements :*

(1) *Let $G_+(\zeta; d)$ be the operator defined by*

$$G_+(\zeta; d) = V_{-d} R(\zeta; H_{+d}) j_{+d} : L^2(B_{+d}) \to L^2(B_{-d})$$

for $\zeta \in D_d$. Then $G_+(\zeta; d)$ admits the decomposition

$$G_+(\zeta; d) = G_{+d}(\zeta; d) + \tilde{G}_{+d}(\zeta; d),$$

where $G_{+d}(\zeta; d)$ is the integral operator with the kernel $G_{+d}(x, y; \zeta, d)$ defined by

$$G_{+d} = c_0(\zeta)\pi_0 \left(e^{ikd}/d^{1/2}\right) V_-(x_{-d})\varphi_0(x_{-d}; -\omega_1, \zeta)\overline{\varphi}_+(y_{+d}; -\omega_1, \overline{\zeta})j(y_{+d})$$

with $x_{\pm d} = x - d_\pm$ and $y_{\pm d} = y - d_\pm$, and $\tilde{G}_{+d}(\zeta; d)$ is analytic in $\zeta \in D_d$ with values in bounded operators from $L^2(B_{+d})$ to $L^2(B_{-d})$ and obeys $\|\tilde{G}_{+d}(\zeta; d)\| = O(|d|^{-\nu})$ uniformly in ζ for some $\nu > 0$.

(2) *Let $G_-(\zeta; d)$ be the operator defined by*

$$G_-(\zeta; d) = V_{+d}R(\zeta; H_{-d})j_{-d} : L^2(B_{-d}) \to L^2(B_{+d})$$

for $\zeta \in D_d$. Then $G_-(\zeta; d)$ admits the decomposition

$$G_-(\zeta; d) = G_{-d}(\zeta; d) + \tilde{G}_{-d}(\zeta; d),$$

where $G_{-d}(\zeta; d)$ is the integral operator with the kernel $G_{-d}(x, y; \zeta, d)$ defined by

$$G_{-d} = c_0(\zeta)\pi_0 \left(e^{ikd}/d^{1/2}\right) V_+(x_{+d})\varphi_0(x_{+d}; \omega_1, \zeta)\overline{\varphi}_-(y_{-d}; \omega_1, \overline{\zeta})j(y_{-d})$$

and $\tilde{G}_{-d}(\zeta; d)$ is analytic in $\zeta \in D_d$ with values in bounded operators from $L^2(B_{-d})$ to $L^2(B_{+d})$ and obeys $\|\tilde{G}_{-d}(\zeta; d)\| = O(|d|^{-\nu})$ uniformly in ζ for some $\nu > 0$.

Lemma 3.4. *Let $G_\pm(\zeta; d)$ be as in Lemma 3.3. Define*

$$G(\zeta; d) = G_-(\zeta; d)G_+(\zeta; d) = V_{+d}R(\zeta; H_{-d})V_{-d}R(\zeta; H_{+d})j_{+d} \tag{3.6}$$

as an operator acting on $L^2(B_{+d})$. Then $G(\zeta; d)$ admits the decomposition

$$G(\zeta; d) = G_0(\zeta; d) + G_1(\zeta; d),$$

where $G_0(\zeta; d)$ is the integral operator with the kernel $G_0(x, y; \zeta, d)$ defined by

$$G_0 = -c_0(\zeta)\pi_0^2 \left(e^{2ikd}/d\right) f_-(-\omega_1 \to \omega_1; \zeta) \times V_+(x_{+d})\varphi_0(x_{+d}; \omega_1, \zeta)\overline{\varphi}_+(y_{+d}; -\omega_1, \overline{\zeta})j(y_{+d})$$

and $G_1(\zeta; d)$ is analytic in $\zeta \in D_d$ with values in bounded operators acting on $L^2(B_{+d})$ and obeys $\|G_1(\zeta; d)\| = O\left(|d|^{-\nu}\right)$ uniformly in $\zeta \in D_d$ for some $\nu > 0$.

3.2. We are now in a position to prove the main theorem. The lemma below is proved in almost the same way as in the proof of [7, Lemma 4.6]. We skip its proof.

Lemma 3.5. *Assume the same assumptions as in Theorem 1.1. Let* $h(\zeta; d)$ *be defined by (1.12) with* $|d| = d$ *and* $\hat{d} = \omega_1 = (1, 0)$. *Then the equation* $h(\zeta; d) = 1$ *has a finite number of the solutions*

$$\left\{\zeta_j(d)\right\}_{1\le j\le N_d}, \quad \zeta_j(d) \in D_d, \quad \operatorname{Re}\zeta_1(d) < \cdots < \operatorname{Re}\zeta_{N_d}(d),$$

in D_d, *and each solution* $\zeta_j(d)$ *has the properties*

$$\left|\operatorname{Im}\zeta_j(d) + E_0^{1/2}(\log d)/d\right| < \delta_0 E_0^{1/2}(\log d)/d,$$
$$\left|\operatorname{Re}(\zeta_{j+1}(d) - \zeta_j(d)) - 2\pi E_0^{1/2}/d\right| < 2\pi\delta_0 E_0^{1/2}/d$$

for $d \gg 1$.

Proof of Theorem 1.1. Recall the notation $H_{\pm d} = H(A_\rho, V_{\pm d})$ from (3.4). We know by (3.5) that $R(\zeta; H_{\pm d}) : L^2_{\mathrm{comp}}(\Omega_\rho) \to L^2_{\mathrm{loc}}(\Omega_\rho)$ is well defined for $\zeta \in D_d$ and is analytic there. We start with the relation

$$(H_d - \zeta)R(\zeta; H_{-d}) = Id + V_{+d}R(\zeta; H_{-d}).$$

We regard the operator on the right side as an operator acting on $L^2(B_{+d})$. By the resolvent identity, the operator on the right side equals

$$Id + V_{+d}R(\zeta; H_{-d})j_{+d} =$$
$$Id + V_{+d}R(\zeta; H_{0d})j_{+d} - V_{+d}R(\zeta; H_{-d})V_{-d}R(\zeta; H_{0d})j_{+d}.$$

By Lemma 3.2, it is further equal to

$$Id + V_{+d}R(\zeta; H_{-d})j_{+d} = (Id - G(\zeta; d))\,(Id + Q_+(\zeta; d))\,, \tag{3.7}$$

where $G(\zeta; d)$ is again defined by

$$G(\zeta; d) = V_{+d}R(\zeta; H_{-d})V_{-d}R(\zeta; H_{+d})j_{+d} : L^2(B_{+d}) \to L^2(B_{+d})$$

as in Lemma 3.4. If one is not an eigenvalue of $G(\zeta; d)$ at $\zeta = \zeta_0 \in D_d$, then the resolvent $R(\zeta; H_d)$ in question turns out to be analytic in a neighborhood of ζ_0 as a function with values in operators from $L^2_{\mathrm{comp}}(\Omega_\rho)$ to $L^2_{\mathrm{loc}}(\Omega_\rho)$. In fact, $R(\zeta; H_d)$ is represented as

$$R(\zeta; H_d) = R(\zeta; H_{-d})$$
$$-H(\zeta; H_{-d})j_{+d}\,(Id + V_{+d}R(\zeta; H_{-d})j_{+d})^{-1}\,V_{+d}R(\zeta; H_{-d}).$$

Thus the problem is reduced to specifying $\zeta \in D_d$ at which $G(\zeta; d)$ has one as an eigenvalue and to showing that this point is really the pole of $R(\zeta; H_d)(\zeta)$ in D_d.

Lemma 3.4 enables us to write $Id - G(\zeta;d)$ as

$$(3.8)\quad Id - G(\zeta;d) = (Id - \tilde{G}(\zeta;d))(Id - G_1(\zeta;d)) : L^2(B_{+d}) \to L^2(B_{+d}),$$

where $G_1(\zeta;d)$ is as in Lemma 3.4 and $\tilde{G}(\zeta;d)$ is defined by

$$\tilde{G}(\zeta;d) = G_0(\zeta;d)(Id - G_1(\zeta;d))^{-1} = G_0(\zeta;d)(Id + \tilde{G}_1(\zeta;d))$$

with $\tilde{G}_1(\zeta;d) = G_1(\zeta;d)(Id - G_1(\zeta;d))^{-1}$. We write $(\ ,\ )$ for the L^2 scalar product in $L^2(\boldsymbol{R}^2)$. We compute the integral

$$\begin{aligned}(3.9)\qquad c_0(\zeta)\int V_+(x_{+d})\varphi_0(x_{+d};\omega_1,\zeta)\overline{\varphi}_+(x_{+d};-\omega_1,\overline{\zeta})\,dx \\ &= c_0(\zeta)\left(V_+\varphi_0(\cdot;\omega_1,\zeta),(Id - R(\zeta;K_+)^*V_+)\,\varphi_0(\cdot;-\omega_1,\overline{\zeta})\right)\\ &= c_0(\zeta)\left(V_+\,(Id - R(\zeta;K_+)V_+)\,\varphi_0(\cdot;\omega_1,\zeta),\varphi_0(\cdot;-\omega_1,\overline{\zeta})\right)\\ &= -f_+(\omega_1\to-\omega_1;\zeta)\end{aligned}$$

and we set

$$\begin{aligned}h_1(\zeta;d) = -c_0(\zeta)\left(e^{2ikd}/d\right)\pi_0^2 f_-(-\omega_1\to\omega_1;\zeta)\,\times\\ \left(\tilde{G}_1(\zeta;d)V_{+d}\varphi_0(\cdot-d_+;\omega_1,\zeta),j_{+d}\varphi_+(\cdot-d_+;-\omega_1,\overline{\zeta})\right).\end{aligned}$$

Then it follows from Lemma 3.4 and (1.15) that $h_1(\zeta;d)$ is analytic over D_d and obeys $|h_1(\zeta;d)| = O\left(|d|^{-\nu}\right)$ uniformly in ζ for some $\nu > 0$. The only nonzero eigenvalue of the operator $\tilde{G}(\zeta;d)$ of rank one is given by $h(\zeta;d) + h_1(\zeta;d)$, where $h(\zeta;d)$ is defined by (1.12).

We apply Rouché's theorem to the equation

$$(3.10)\qquad h(\zeta;d) + h_1(\zeta;d) = 1$$

over D_d. Let $\left\{\zeta_j(d)\right\}_{1\le j\le N_d}$ be as in Lemma 3.5 and let

$$C_{j\varepsilon} = \left\{|\zeta - \zeta_j(d)| = \varepsilon/d\right\},\qquad D_{j\varepsilon} = \left\{|\zeta - \zeta_j(d)| < \varepsilon/d\right\}$$

for $\varepsilon > 0$ fixed arbitrarily but sufficiently small. We may assume $D_{j\varepsilon} \subset D_d$ for $d \gg 1$ by expanding D_d slightly, if necessary. Since $h\left(\zeta_j(d);d\right) = 1$, the derivative $h'(\zeta;d)$ behaves like

$$h'\left(\zeta_j(d);d\right) = i\zeta_j(d)^{-1/2}d\left(1 + O(d^{-1})\right)$$

at $\zeta = \zeta_j(d) \in D_d$, so that $\left|h'\left(\zeta_j(d);d\right)\right| \ge c_1 d$ for some $c_1 > 0$. Hence it follows that

$$|h(\zeta;d) - 1| \ge c_2\varepsilon$$

on $C_{j\varepsilon}$ for some $c_2 > 0$. Thus equation (3.10) has a unique solution $\zeta_{\mathrm{res},j}(d)$ in $D_{j\varepsilon}$ for $d \gg 1$.

Once the location $\zeta_{\mathrm{res},j}(d)$ is determined as above, we can show in exactly the same way as in the proof of [7, Theorem 1.1] (see step (3) there) that it really becomes the resonance of $R(\zeta; H_d)$. We do not go into the details. Thus the proof of the theorem is complete. □

3.3. We shall prove the preliminary lemmas which remain unproved. We use the new notation

$$K_{\pm d} = K_0 + V_{\pm d}, \quad \mathcal{D}(K_{\pm d}) = H^2(\boldsymbol{R}^2)$$

where $V_{\pm d}(x) = V_\pm(x_{\pm d}) = V_\pm(x - d_\pm)$ is as in (1.4). We introduce a smooth non–negative cut–off function $\chi \in C_0^\infty[0, \infty)$ with the properties

$$0 \le \chi \le 1, \qquad \operatorname{supp} \chi \subset [0, 2], \qquad \chi = 1 \text{ on } [0, 1]. \tag{3.11}$$

This function is often used in the future discussion without further references.

Proof of Lemma 3.1. We prove the lemma for the operator $Q_+(\zeta; d)$ only. A similar argument applies to $Q_-(\zeta; d)$ also. Let $\gamma_+(x)$ be as in (3.2). Then we introduce

$$\tilde{K}_0 = H(\nabla\gamma_+, 0) = e^{i\gamma_+} K_0 e^{-i\gamma_+}, \quad \tilde{K}_+ = H(\nabla\gamma_+, V_+) = \tilde{K}_0 + V_+$$

and $\tilde{K}_{+d} = \tilde{K}_0 + V_{+d}$ as auxiliary operators. These operator have the common domain $H^2(\boldsymbol{R}^2)$. By definition, we have

$$R(\zeta; \tilde{K}_0) = e^{i\gamma_+} R(\zeta; K_0) e^{-i\gamma_+}, \quad R(\zeta; \tilde{K}_+) = e^{i\gamma_+} R(\zeta; K_+) e^{-i\gamma_+} \tag{3.12}$$

and similarly for $R(\zeta; \tilde{K}_{+d})$.

We take $\delta > 0$ small enough as in Proposition 2.2 and define $w_d(x)$ by

$$w_d(x) = \chi\left(|x_{+d}|/d^\delta\right) = \chi\left(|x - d_+|/d^\delta\right), \quad \operatorname{supp} w_d \subset \left\{|x_{+d}| < 2d^\delta\right\},$$

where χ is the smooth cut–off function with properties in (3.11). Then, by (3.3), $\nabla\gamma_+ = A_\rho$ on $\operatorname{supp} w_d$, so that $\tilde{K}_0 = H(A_\rho, 0) = H_{0d}$ there. Making use of the relations $w_d V_{+d} = V_{+d}$ and $w_d j_{+d} = j_{+d}$, we compute

$$\begin{aligned} Id + Q_+(\zeta; d) &= Id + V_{+d} R(\zeta; \tilde{K}_0) j_{+d} + V_{+d}\left(R(\zeta; H_{0d}) - R(\zeta; \tilde{K}_0)\right) j_{+d} \\ &= Id + V_{+d} R(\zeta; \tilde{K}_0) j_{+d} + V_{+d} R(\zeta; H_{0d})\left(w_d \tilde{K}_0 - H_{0d} w_d\right) R(\zeta; \tilde{K}_0) j_{+d} \\ &= Id + V_{+d} R(\zeta; \tilde{K}_0) j_{+d} + V_{+d} R(\zeta; H_{0d})[w_d, \tilde{K}_0] R(\zeta; \tilde{K}_0) j_{+d}. \end{aligned}$$

It follows by the resolvent identity that

$$R(\zeta; \tilde{K}_{+d}) j_{+d} = R(\zeta; \tilde{K}_0) j_{+d} \left(Id + V_{+d} R(\zeta; \tilde{K}_0) j_{+d}\right)^{-1}$$

on $L^2(B_{+d})$, and hence we obtain the following representation for the operator $Id + Q_+(\zeta; d)$ in question:

$$
\begin{aligned}
& Id + Q_+(\zeta; d) \\
= & \left(Id + V_{+d}R(\zeta; H_{0d})[w_d, \tilde{K}_0]R(\zeta; \tilde{K}_{+d})j_{+d}\right)\left(Id + V_{+d}R(\zeta; \tilde{K}_0)j_{+d}\right) \\
= & \; e^{i\gamma_+}\left(Id + \tilde{Q}_+(\zeta; d)\right)\left(Id + V_{+d}R(\zeta; K_0)j_{+d}\right)e^{-i\gamma_+},
\end{aligned}
$$

where $\tilde{Q}_+(\zeta; d)$ is defined by

$$\tilde{Q}_+(\zeta; d) = V_{+d}e^{-i\gamma_+}R(\zeta; H_{0d})e^{i\gamma_+}[w_d, K_0]R(\zeta; K_{+d})j_{+d} \tag{3.13}$$

as an operator acting on $L^2(B_{+d})$. Thus the problem is reduced to evaluating the operator norm of $\tilde{Q}_+(\zeta; d)$.

We compute the commutator

$$[w_d, K_0] = w_d K_0 - K_0 w_d = 2\nabla w_d \cdot \nabla + (\Delta w_d) = 2\nabla w_d \cdot \nabla + O(d^{-2\delta}).$$

The coefficients ∇w_d and Δw_d have support in $\tilde{B}_{+d}$ defined by (2.5). We study the behavior of the resolvent kernel

$$R(\zeta; K_{+d})(z, y) = R(\zeta; K_+)(z_{+d}, y_{+d})$$

when $(z, y) \in \tilde{B}_{+d} \times B_{+d}$. By the resolvent identity, we have

$$R(\zeta; K_{+d}) = R(\zeta; K_0)(Id - V_{+d}R(\zeta; K_{+d})). \tag{3.14}$$

For the free Hamiltonian $K_0 = -\Delta$, the resolvent kernel $R(\zeta; K_0)(z, y)$ is given by

$$R(\zeta; K_0)(z, y) = (i/4)H_0(k|z - y|)$$

and behaves like

$$R(\zeta; K_0)(z, y) = c_0(\zeta)e^{ik|z-y|}|z - y|^{-1/2}\left(1 + O(|z - y|^{-1})\right)$$

when $|z - y| \gg 1$. If $z \in \tilde{B}_{+d}$ and $y \in B_{+d}$, then

$$|z_{+d} - y_{+d}| = |z_{+d}| - y_{+d} \cdot \hat{z}_{+d} + O(|d|^{-\delta})$$

and hence

$$e^{ik|z_{+d}-y_{+d}|} = e^{ik|z_{+d}|}\left(\overline{\varphi}_0(y_{+d}; \hat{z}_{+d}, \overline{\zeta}) + O(|d|^{-\delta})\right),$$

where $\hat{z}_{+d} = z_{+d}/|z_{+d}|$. Thus $R(\zeta; K_0)(z_{+d}, y_{+d})$ behaves like

$$
\begin{aligned}
& R(\zeta; K_0)(z_{+d}, y_{+d}) = \\
& \qquad c_0(\zeta)e^{ik|z_{+d}|}|z_{+d}|^{-1/2}\left(\overline{\varphi}_0(y_{+d}; \hat{z}_{+d}, \overline{\zeta}) + r(z_{+d}, y_{+d}; \zeta)\right),
\end{aligned}
$$

where the remainder term $r(z_{+d}, y_{+d}; \zeta)$ satisfies

$$|\partial_z^n r(z_{+d}, y_{+d}; \zeta)| = O(d^{-(1+|n|)\delta}) \tag{3.15}$$

uniformly in z, y and $\zeta \in D_d$. Since

$$\varphi_+(x; \hat{z}_{+d}, \overline{\zeta}) = \left[(Id - R(\zeta; K_+)^* V_+)\varphi_0(\cdot; \hat{z}_{+d}, \overline{\zeta})\right](x)$$

by definition, it follows from (3.14) that the kernel $R(\zeta; K_{+d})(z, y)$ under consideration takes the asymptotic form

$$R(\zeta; K_{+d})(z, y) = c_0(\zeta)e^{ik|z_{+d}|}|z_{+d}|^{-1/2}\left(\overline{\varphi}_+(y_{+d}; \hat{z}_{+d}, \overline{\zeta}) + r_0(z_{+d}, y_{+d}; \zeta)\right), \tag{3.16}$$

where

$$r_0(z_{+d}, y_{+d}; \zeta) = r(z_{+d}, y_{+d}; \zeta) - \int r(z_{+d}, u; \zeta)V_+(u)R(\zeta; K_+)(u, y_{+d})\, du$$

is analytic in $\zeta \in D_d$ and obeys the same bound as in (3.15).

We consider $[w_d, K_0]R(\zeta; K_{+d})(z, y)$. Since $\nabla w_d \cdot \nabla = O(d^{-\delta})\hat{z}_{+d} \cdot \nabla$ for $w_d = w_d(z)$ and since

$$\partial_z^n \varphi_+(y_{+d}; \hat{z}_{+d}, \zeta) = O\left(|z_{+d}|^{-|n|}\right) = O(d^{-|n|\delta})$$

on $\tilde{B}_{+d}$, $[w_d, K_0]R(\zeta; K_{+d})(z, y)$ takes the form

$$\Big([w_d, K_0]R(\zeta; K_{+d})\Big)(z, y) = e^{ik|z_{+d}|}\tilde{r}_0(z, y; \zeta) \tag{3.17}$$

by (3.16), where $\tilde{r}_0(z, y; \zeta)$ satisfies $|\partial_z^n \tilde{r}_0| = O(d^{-(3/2+|n|)\delta})$.

We now evaluate the norm of the operator $\tilde{Q}_+(\zeta; d) : L^2(B_{+d}) \to L^2(B_{+d})$ defined by (3.13). We note that $\left|e^{ik|z_{+d}|}\right|$ is uniformly bounded in d for $z \in \tilde{B}_{+d}$, because

$$|\operatorname{Im} k||z_{+d}| = O\left((\log d)/d\right) O(d^\delta) = O(1)$$

for $\zeta \in D_d$. Similarly $\left|e^{ik|x-z|}\right|$ is also uniformly bounded in d when $(x, z) \in B_{+d} \times \tilde{B}_{+d}$. Hence it follows from (3.17) that

$$\|[w_d, K_0]R(\zeta; K_{+d})\| = O(d^{-\delta/2}) \tag{3.18}$$

as a bounded operator from $L^2(B_{+d})$ to $L^2(\tilde{B}_{+d})$. This is obtained by evaluating the Hilbert–Schmidt norm of the operator. According to Proposition 2.2, the operator $R(\zeta; H_{0d})$ admits the decomposition

$$R(\zeta; H_{0d}) = \tilde{R}_0(\zeta; d) + \tilde{R}_1(\zeta; d)$$

as a bounded operator from $L^2(\tilde{B}_{+d})$ to $L^2(B_{+d})$. The operator $\tilde{R}_0(\zeta; d)$ has the integral kernel

$$\tilde{R}_0(\zeta; d) = (i/4)H_0(k|x - y|)a_0(x, y; d)$$

with $a_0(x, z; d)$ defined by (2.6), and $\tilde{R}_1(\zeta; d)$ obeys $\|\tilde{R}_1(\zeta; d)\| = O(d^{-\nu})$ uniformly in $\zeta \in D_d$ for some $\nu > 0$. If $x \in B_{+d}$ and $z \in \tilde{B}_{+d}$, then

$$|\nabla_z (|x - z| + |z_{+d}|)| = |\nabla_z (|z - x| + |z - d_+|)| \geq c > 0$$

for some c independent of d, and hence we have

$$\int H_0(k|x - z|) e^{i\gamma_+(z)} a_0(x, z; d)[w_d, K_0] R(\zeta; K_{+d})(z, y)\, dz = O(d^{-N})$$

for any $N \gg 1$ by repeated use of integration by parts. By (3.18), we also have

$$\|\tilde{R}_1(\zeta; d)[w_d, K_0] R(\zeta; K_{+d}) j_{+d}\| = O(d^{-\nu})$$

as a bounded operator on $L^2(B_{+d})$. Thus we see that the operator $\tilde{Q}_+(\zeta; d)$ defined by (3.13) satisfies $\|\tilde{Q}_+(\zeta; d)\| = O(d^{-\nu})$ as a bounded operator acting on $L^2(B_{+d})$, and the proof is complete. □

Proof of Lemma 3.2. We prove the lemma for $Q_+(\zeta; d)$ only. A similar argument applies to $Q_-(\zeta; d)$ also. By the resolvent identity, we have

$$(Id + V_{\pm d} R(\zeta; K_0) j_{\pm d})\,(Id - V_{\pm d} R(\zeta; K_{\pm d}) j_{\pm d}) = Id$$

on $L^2(B_{\pm d})$. This implies that $Id + V_{\pm d} R(\zeta; K_0) j_{\pm d}$ is invertible and

$$(3.19) \qquad (Id + V_{\pm d} R(\zeta; K_0) j_{\pm d})^{-1} = Id - V_{\pm d} R(\zeta; K_{\pm d}) j_{\pm d} : L^2(B_{\pm d}) \to L^2(B_{\pm d})$$

is bounded uniformly in $\zeta \in D_d$. By Lemma 3.1 and (3.19), it follows that $Id + Q_+(\zeta; d)$ is invertible as an operator acting on $L^2(B_{+d})$. Since

$$(H_{+d} - \zeta)\, R(\zeta; H_{0d}) j_{+d} = Id + V_{+d} R(\zeta; H_{0d}) j_{+d} = Id + Q_+(\zeta; d)$$

on $L^2(B_{+d})$, the desired relation follows at once. □

Proof of Lemma 3.3. We prove statement (1) only. We use the notation with the same meanings ascribed in the proof of Lemma 3.1 throughout the proof. By Lemma 3.2, the operator $G_+(\zeta; d)$ is represented as

$$G_+(\zeta; d) = V_{-d} R(\zeta; H_{0d}) j_{+d} (Id + Q_+(\zeta; d))^{-1},$$

and by Lemma 3.1, the relation

$$Id + Q_+(\zeta; d) = e^{i\gamma_+} (Id + \tilde{Q}_+(\zeta; d))(Id + V_{+d} R(\zeta; K_0) j_{+d}) e^{-i\gamma_+}$$

holds true as an operator acting on $L^2(B_{+d})$. By (3.2), we have

$$(3.20) \qquad e^{i\gamma_+(x)} = e^{i(\alpha_- + \alpha_+)\pi} + O(d^{-1})$$

for $x \in B_{+d}$. Since $\tilde{Q}_+(\zeta; d)$ satisfies $\|\tilde{Q}_+(\zeta; d)\| = O\left(d^{-\nu}\right)$ for some $\nu > 0$ as a bounded operator acting on $L^2(B_{+d})$, the inverse $(Id + \tilde{Q}(\zeta; d))^{-1}$ exists, and we have the relation

$$(Id + \tilde{Q}_+(\zeta; d))^{-1} = Id - \tilde{Q}_+(\zeta; d)(Id + \tilde{Q}_+(\zeta; d))^{-1}.$$

This, together with (3.19) and (3.20), enables us to decompose $G_+(\zeta; d)$ into

$$G_+(\zeta; d) = G_{+0}(\zeta; d) + V_{-d}R(\zeta; H_{0d})j_{+d}G_{+1}(\zeta; d),$$

where

$$G_{+0}(\zeta; d) = V_{-d}R(\zeta; H_{0d})j_{+d}(Id - V_{+d}R(\zeta; K_{+d})j_{+d})$$

and $G_{+1}(\zeta; d)$ is analytic in D_d and satisfies $\|G_{+1}(\zeta; d)\| = O(d^{-\nu})$ as an operator acting on $L^2(B_{+d})$. We apply Proposition 2.1 to $V_{-d}R(\zeta; H_{0d})j_{+d}$. If $(x, y) \in B_{-d} \times B_{+d}$, then

$$|x_1 - y_1| = d - (x_{-d} - y_{+d}) \cdot \omega_1 + O(d^{-1}).$$

Hence the kernel $R_0(\zeta; d)(x, y)$ of Proposition 2.1 behaves like

$$R_0(\zeta; d)(x, y) = \left(\frac{e^{ikd}}{d^{1/2}}\right) \left(c_0(\zeta)\pi_0\varphi_0(x_{-d}; -\omega_1, \zeta)\overline{\varphi}_0(y_{+d}; -\omega_1, \overline{\zeta}) + O(d^{-1})\right)$$

uniformly in $(x, y) \in B_{-d} \times B_{+d}$ and $\zeta \in D_d$. Since

$$\varphi_+(y_{+d}; -\omega_1, \overline{\zeta}) = \left[(Id - R(\zeta; K_{+d})^* V_{+d})\varphi_0(\cdot - d_+; -\omega_1, \overline{\zeta})\right](y_{+d})$$

by definition and since

$$\left|e^{ikd}/d^{1/2}\right| = O\left(d^{3\delta_0/2}\right)$$

over D_d (see (1.15)), the leading term $G_{+d}(x, y; \zeta, d)$ is obtained from the kernel of $G_{+0}(\zeta; d)$ and the remainder operator $\tilde{G}_{+d}(\zeta; d)$ satisfies $\|\tilde{G}_{+d}(\zeta; d)\| = O(|d|^{-\nu})$ for some $\nu > 0$ by taking $\delta_0 > 0$ small enough. This yields the desired decomposition and proves the lemma. □

Proof of Lemma 3.4. We compute the integral

$$c_0(\zeta) \int V_-(x_{-d})\varphi_0(x_{-d}; -\omega_1, \zeta)\overline{\varphi}_-(x_{-d}; \omega_1, \overline{\zeta})\, dx = -f_-(-\omega_1 \to \omega_1; \zeta).$$

in exactly the same way as used to establish relation (3.9). This, together with Lemma 3.3, yields the kernel of $G_0(\zeta; d)$. Since $\left|e^{ikd}/d^{1/2}\right| = O\left(d^{3\delta_0/2}\right)$ over D_d, we can take δ_0 so small that the remainder operator $G_1(\zeta; d)$ obeys $\|G_1(\zeta; d)\| = O\left(d^{-\nu}\right)$ for some $\nu > 0$. Thus the proof is complete. □

4. Complex scaling method

The remaining two sections are devoted to proving Propositions 2.1 and 2.2 which have remained unproved in section 2. These propositions are proved by constructing the resolvent kernel $R(\zeta; H_{0d})(x, y)$ with the spectral parameter ζ in the lower half plane. To do this, we compose the resolvent kernel constructed for each obstacle $\mathcal{O}_{\pm\rho}$ by making use of the complex scaling method. In this section, we introduce the new notation and explain a strategy based on this method. For notational brevity, we write

$$P_\rho = H(A_\rho, 0), \qquad \mathcal{D}(P_\rho) = H^2(\Omega_\rho) \cap H_0^1(\Omega_\rho)$$

for the operator H_{0d} under consideration (see (1.17)). We set

$$\rho = 2\kappa_0 d = |\rho_+ - \rho_-|$$

for $\rho_\pm = (\pm\kappa_0 d, 0)$ as in (2.1). Besides the cut–off function $\chi \in C_0^\infty[0,\infty)$ with properties in (3.11), we further introduce smooth cut–off functions χ_∞ and $\chi_\pm$ over $(-\infty,\infty)$ with the following properties : $0 \le \chi_\infty,\ \chi_\pm \le 1$ and

$$\chi_\infty(t) = 1 - \chi(|t|),$$

$$\chi_+(t) = 1 \text{ for } t \ge 1, \qquad \chi_+(t) = 0 \text{ for } t \le -1, \qquad \chi_-(t) = 1 - \chi_+(t).$$

We often use these functions without further references throughout the future discussion.

We define the mapping $j_\rho(x) : \boldsymbol{R}^2 \to \boldsymbol{R} \times \boldsymbol{C}$ by

$$(4.1) \quad j_\rho(x_1, x_2) = (x_1, x_2 + i\eta_\rho(x_2)x_2)\,, \quad \eta_\rho(t) = L_0\Big((\log\,\rho)/\rho\Big)\chi_\infty(t/\rho),$$

where $L_0 \gg 1$ is fixed large enough, and we consider the complex scaling mapping

$$(4.2) \qquad (J_\rho f)(x) = \Big[\det(\partial j_\rho/\partial x)\Big]^{1/2} f(j_\rho(x))$$

associated with $j_\rho(x)$. The Jacobian $\det(\partial j_\rho/\partial x)$ of $j_\rho(x)$ does not vanish for $d \gg 1$, and it is easily seen that J_ρ is a one-to-one mapping. Since the coefficients of P_ρ are analytic over Ω_ρ, we can define the operator

$$(4.3) \qquad Q_\rho = J_\rho P_\rho J_\rho^{-1}.$$

This becomes a closed operator in $L^2(\Omega_\rho)$ with the same domain as P_ρ, but it is not necessarily self–adjoint. We do not require the explicit form of Q_ρ in the future discussion. We construct the resolvent kernel $R(\zeta; Q_\rho)(x,y)$ with $\zeta \in D_d$ without constructing $R(\zeta; P_\rho)(x,y)$ directly. The mapping j_ρ acts as the identity over the strip $\Big\{x = (x_1, x_2) : |x_2| < \rho = 2\kappa_0 d\Big\}$, and hence we have the relation

$$R(\zeta; P_\rho)(x,y) = R(\zeta; P_\rho)(j_\rho(x), j_\rho(y)) = R(\zeta; Q_\rho)(x,y)$$

for $(x,y) \in B_{\pm d} \times B_{\mp d}$ or for $(x,y) \in B_{\pm d} \times \tilde{B}_{\pm d}$. Thus the necessary information can be obtained through the kernel $R(\zeta; Q_\rho)(x,y)$.

Here we make a brief comment on the motivation to introduce the complex scaled operator Q_ρ above. The scattering system by one solenoid is known to be exactly solvable. We make a full use of of the information from such a system to construct $R(\zeta; Q_\rho)(x, y)$ by composing the two resolvent kernels associated with each obstacle $\mathcal{O}_\pm$. In doing this, a difficulty comes from the exponential growth of resolvent kernels with spectral parameters in the lower half plane. For magnetic fields compactly supported, the corresponding vector potentials can not be expected to fall off rapidly at infinity, because of the topological feature of the two dimensional space that $\boldsymbol{R}^2 \setminus \{0\}$ is not simply connected. In fact, vector potentials have the long range property. As already stated in section 1, the vector potentials can not be expected to be well separated, even if the supports of the two magnetic fields are largely separated from each other. In other words, cut–off functions used to separate the two obstacles do not have bounded supports. Thus the composition of the resolvent kernels growing exponentially can not be controlled simply by integration by parts using oscillatory properties. The mapping j_ρ defined by (4.1) makes the resolvent kernels of one obstacle fall off rapidly even for spectral parameters in the lower half plane, so that the composition converges (see Theorem A.4 in Appendix). This is the reason why we consider Q_ρ in place of P_ρ.

We introduce the auxiliary operators

$$P_{\pm\rho} = H(A_{\pm\rho}, 0), \qquad \mathcal{D}(P_{\pm\rho}) = H^2(\Omega_{\pm\rho}) \cap H_0^1(\Omega_{\pm\rho}), \tag{4.4}$$

where $A_{\pm\rho}(x)$ is defined in (1.5) and $\Omega_{\pm\rho} = \boldsymbol{R}^2 \setminus \overline{\mathcal{O}}_{\pm\rho}$. We define the complex scaled operator as in (4.3) for these auxiliary operators $P_{\pm\rho}$. Recall that $\gamma(x; \omega)$ denotes the azimuth angle from $\omega \in S^1$ to $\hat{x} = x/|x|$. The potential $\Phi(x)$ defined by (1.3) satisfies the relation $\Phi(x) = \nabla\gamma(x; \omega)$. Hence it follows that

$$A_{\pm\rho}(x) = \alpha_\pm \nabla\gamma(x - \rho_\pm; \pm\omega_1), \quad \omega_1 = (1, 0).$$

If we take $\arg z$, $0 \le \arg z < 2\pi$, to be a single valued function over the complex plane slit along the direction ω_1, then the angle function $\gamma(x; \omega_1)$ is represented as

$$\gamma(x; \omega_1) = -\frac{i}{2}\Big(\log(x_1 + ix_2) - \log(x_1 - ix_2)\Big) + \pi,$$

so that it is well defined for the complex variables also. Thus we can define

$$\gamma(j_\rho(x); \omega_1) = \Big(\arg(b_{+\rho}(x)) - \arg(b_{-\rho}(x))\Big)/2 + \pi - i\left(\log|b_\rho(x)|\right)/2,$$

where

$$b_{+\rho}(x) = x_1 - \eta_\rho(x_2)x_2 + ix_2, \qquad b_{-\rho}(x) = x_1 + \eta_\rho(x_2)x_2 - ix_2,$$

and $b_\rho(x) = b_{+\rho}(x)/b_{-\rho}(x)$. The function $\gamma(j_\rho(x); -\omega_1)$ is similarly defined by taking $\arg z$ to be a single valued function over the complex plane slit along the direction $-\omega_1$.

We define $g_{\pm\rho}(x)$ by

$$g_{\pm\rho}(x) = \alpha_\pm \chi_\mp\big((32x_1/\rho) \mp 13\big)\gamma(j_\rho(x) - \rho_\pm; \pm\omega_1) \tag{4.5}$$

and $g_{0\rho}(x)$ by

$$g_{0\rho}(x) = \chi\left(\frac{4|x_1|}{\rho}\right)\Big(\alpha_-\gamma(j_\rho(x) - \rho_-; -\omega_1) + \alpha_+\gamma(j_\rho(x) - \rho_+; \omega_1)\Big). \tag{4.6}$$

By definition, $\operatorname{supp} g_{-\rho} \subset \{x : x_1 > -7\rho/16\}$ and

$$g_{-\rho}(x) = \alpha_-\gamma(j_\rho(x) - \rho_-; -\omega_1) \quad \text{on } \Sigma_+ = \{x : x_1 > -3\rho/8\}.$$

Hence $\exp(ig_{-\rho})$ acts as

$$\exp(ig_{-\rho})f(x) = \left(J_\rho \exp(i\alpha_-\gamma(x - \rho_-; -\omega_1))J_\rho^{-1} f\right)(x)$$

on functions $f(x)$ with support in Σ_+. On the other hand, $g_{+\rho}(x)$ has support in $\{x : x_1 < 7\rho/16\}$ and

$$g_{+\rho}(x) = \alpha_+\gamma(j_\rho(x) - \rho_+; \omega_1) \quad \text{on } \Sigma_- = \{x : x_1 < 3\rho/8\},$$

so that $\exp(ig_{+\rho})$ acts as

$$\exp(ig_{+\rho})f(x) = \left(J_\rho \exp(i\alpha_+\gamma(x - \rho_+; \omega_1))J_\rho^{-1} f\right)(x)$$

on functions $f(x)$ with support in Σ_-. We take these relations into account to define the following complex scaled operator

$$Q_{\pm\rho} = \exp(ig_{\mp\rho})\left(J_\rho P_{\pm\rho} J_\rho^{-1}\right)\exp(-ig_{\mp\rho}) \tag{4.7}$$

for $P_{\pm\rho}$ defined by (4.4), where $Q_{\pm\rho}$ has the same domain as $P_{\pm\rho}$. Since

$$Q_{+\rho} = J_\rho H(\alpha_-\nabla\gamma(x - \rho_-; -\omega_1) + A_{+\rho})J_\rho^{-1}$$

on Σ_+, we have

$$Q_{+\rho} = Q_\rho \quad \text{on } \Sigma_+ = \Big\{x : x_1 > -3\rho/8\Big\}. \tag{4.8}$$

Similarly we have

$$Q_{-\rho} = Q_\rho \quad \text{on } \Sigma_- = \Big\{x : x_1 < 3\rho/8\Big\}. \tag{4.9}$$

The function $g_{0\rho}(x)$ defined by (4.6) has support in $\{x : |x_1| < \rho/2\}$ and satisfies

$$g_{0\rho} = \alpha_-\gamma(j_\rho(x) - \rho_-; -\omega_1) + \alpha_+\gamma(j_\rho(x) - \rho_+; \omega_1)$$

on $\Sigma_0 = \{x : |x_1| \le \rho/4\}$. If we define the operator $Q_{0\rho}$ by

$$Q_{0\rho} = \exp(ig_{0\rho})\left(J_\rho K_0 J_\rho^{-1}\right)\exp(-ig_{0\rho}), \quad K_0 = -\Delta, \tag{4.10}$$

as a closed operator with domain $\mathcal{D}(Q_{0\rho}) = H^2(\boldsymbol{R}^2)$, then we obtain

$$Q_{0\rho} = Q_{\pm\rho} = Q_\rho \quad \text{on } \Sigma_0 = \Big\{x : |x_1| \le \rho/4\Big\}. \tag{4.11}$$

We set $\chi_{\pm\rho}(x) = \chi_\pm(16x_1/\rho)$ and take $\tilde{\chi}_{\pm\rho} \in C^\infty(\boldsymbol{R}^2)$ in such a way that

(4.12) $\tilde{\chi}_{\pm\rho}$ has a slightly larger support than $\chi_{\pm\rho}$, $\tilde{\chi}_{\pm\rho}\chi_{\pm\rho} = \chi_{\pm\rho}$.

For the exterior domain $\Omega_{\pm\rho} = \boldsymbol{R}^2 \setminus \overline{\mathcal{O}}_{\pm\rho}$, we regard $\tilde{\chi}_{\pm\rho}$ as the extension from $L^2(\Omega_\rho)$ to $L^2(\Omega_{\pm\rho})$ and $\chi_{\pm\rho}$ as the restriction to $L^2(\Omega_\rho)$ from $L^2(\Omega_{\pm\rho})$. Then we define

$$\Lambda(\zeta;\rho) = \chi_{-\rho}R(\zeta;Q_{-\rho})\tilde{\chi}_{-\rho} + \chi_{+\rho}R(\zeta;Q_{+\rho})\tilde{\chi}_{+\rho}, \quad \zeta \in D_d, \tag{4.13}$$

as an operator from $L^2_{\mathrm{comp}}(\Omega_\rho)$ to $L^2_{\mathrm{loc}}(\Omega_\rho)$. We note (see [3]) that $R(\zeta;Q_{\pm\rho})$ is well–defined as an operator from $L^2_{\mathrm{comp}}(\Omega_{\pm\rho})$ to $L^2_{\mathrm{loc}}(\Omega_{\pm\rho})$ for $\zeta \in D_d$. Since $Q_\rho = Q_{\pm\rho}$ on $\operatorname{supp}\chi_{\pm\rho}$ by (4.8) and (4.9), we compute

$$\begin{aligned}(Q_\rho - \zeta)\,\Lambda &= (Q_{-\rho} - \zeta)\,\chi_{-\rho}R(\zeta;Q_{-\rho})\tilde{\chi}_{-\rho} + (Q_{+\rho} - \zeta)\,\chi_{+\rho}R(\zeta;Q_{+\rho})\tilde{\chi}_{+\rho} \\ &= Id + [Q_{-\rho},\chi_{-\rho}]R(\zeta;Q_{-\rho})\tilde{\chi}_{-\rho} + [Q_{+\rho},\chi_{+\rho}]R(\zeta;Q_{+\rho})\tilde{\chi}_{+\rho}.\end{aligned}$$

The function $\chi_{\pm\rho}$ depends on x_1 only, and the derivative $\chi'_{\pm\rho}$ has support in

$$\Pi_0 = \Big\{x = (x_1,x_2) : |x_1| < \rho/16\Big\}. \tag{4.14}$$

By (4.11), $Q_{\pm\rho} = Q_{0\rho}$ on Π_0, so that both the commutators $[Q_{-\rho},\chi_{-\rho}]$ and $[\chi_{+\rho},Q_{+\rho}]$ on the right side equal $[Q_{0\rho},\chi_{-\rho}]$. Hence we have

$$(Q_\rho - \zeta)\,\Lambda(\zeta;d) = Id + \Gamma_0\left(R(\zeta;Q_{-\rho})\tilde{\chi}_{-\rho} - R(\zeta;Q_{+\rho})\tilde{\chi}_{+\rho}\right), \tag{4.15}$$

where

$$\Gamma_0 = [Q_{0\rho},\chi_{-\rho}], \qquad \chi_{-\rho} = \chi_-(16x_1/\rho). \tag{4.16}$$

We define $T(\zeta;\rho)$ by

$$T(\zeta;\rho) = \Gamma_0(R(\zeta;Q_{-\rho}) - R(\zeta;Q_{+\rho}))p_0 \tag{4.17}$$

as an operator acting on $L^2(\Pi_0)$, where the multiplication by the characteristic function $p_0(x)\big(= p_0(x_1)\big)$ of Π_0 is regarded as the extension from $L^2(\Pi_0)$ to $L^2(\Omega_{-\rho})$ or to $L^2(\Omega_{+\rho})$. We have shown ([3, section 6]) that $T(\zeta;d)$ obeys the bound $\|T(\zeta;d)\| = O\left(d^\nu\right)$ for some $\nu > 0$ uniformly in $\zeta \in D_d$ and is analytic over D_d as a function with values in bounded operators acting on $L^2(\Pi_0)$. If

$$Id + T(\zeta;\rho) : L^2(\Pi_0) \to L^2(\Pi_0) \tag{4.18}$$

is shown to have the inverse bounded uniformly in $\zeta \in D_d$, then it follows that

$$\begin{aligned}R(\zeta;Q_\rho) &= \Lambda(\zeta;d) \\ &\quad - \Lambda(\zeta;d)p_0\,(Id+T)^{-1}\,\Gamma_0\big(R(\zeta;Q_{-\rho})\tilde{\chi}_{-\rho} - R(\zeta;Q_{+\rho})\tilde{\chi}_{+\rho}\big).\end{aligned} \tag{4.19}$$

The proof of Propositions 2.1 and 2.2 is based on this relation.

5. Proof of Propositions 2.1 and 2.2

In this section we prove Propositions 2.1 and 2.2.

5.1. We begin by showing that the operator in (4.18) is invertible and establish the basic representation (4.19) for $R(\zeta; Q_\rho)$ with $\zeta \in D_d$. Let $\chi \in C_0^\infty[0,\infty)$ be a cut–off function with properties in (3.11) and let μ ($2/5 < \mu < 1/2$) be as in (2.2). We define

$$v_0(x_2) = \chi\left(2|x_2|/\rho^{1-\mu}\right), \quad \tilde{v}_0 = \chi\left(|x_2|/\rho^{1-\mu}\right), \quad v_1(x_2) = 1 - v_0(x_2)$$

and $\tilde{v}_1(x_2) = 1 - \chi\left(4|x_2|/\rho^{1-\mu}\right)$. Then $v_j\tilde{v}_j = v_j$ for $0 \le j \le 1$. We further define

$$T_{jk} = T_{jk}(\zeta;\rho) = v_j T(\zeta;\rho)\tilde{v}_k, \quad 0 \le j,\ k \le 1,$$

for the operator $T(\zeta;\rho)$ defined by (4.17). Then the following lemma has been established as Lemma 7.1 in [3], although the slightly different notation has been used there.

Lemma 5.1. *Let* $\mathrm{Op}(d^{-N})$ *denote the class of bounded operators on* $L^2(\Pi_0)$ *with bound* $O(d^{-N})$ *for any* $N \gg 1$. *Then the operators*

$$T_{11}T_{11},\ T_{11}T_{10},\ T_{01}T_{11},\ T_{01}T_{10} : L^2(\Pi_0) \to L^2(\Pi_0)$$

are all of class $\mathrm{Op}(d^{-N})$.

We discuss the asymptotic form of $T_{00}(\zeta;\rho)$. We introduce the self–adjoint operator

$$P_\pm = H(A_\pm, 0), \qquad \mathcal{D}(P_\pm) = H^2(\Omega_\pm) \cap H_0^1(\Omega_\pm),$$

where $A_\pm(x)$ is defined by (1.2) over $\Omega_\pm = \boldsymbol{R}^2 \setminus \overline{\mathcal{O}}_\pm$. Let $\tilde{f}_\pm(\pm\omega_1 \to \mp\omega_1;\zeta)$ be the function obtained from the backward amplitude for the pair $(K_0, P_\pm)$ by analytic continuation over D_d. We further define $u_\pm = u_\pm(x;\zeta,\rho)$ and $w_\pm = w_\pm(x;\zeta,\rho)$ as

$$\begin{aligned} u_\pm &= -2ikc_0(\zeta)\tilde{f}_\pm(\pm\omega_1 \to \mp\omega_1;\zeta)\chi'_{-\rho}(x_1)v_0(x_2)e^{ik|x_{\pm\rho}|}|x_{\pm\rho}|^{-1/2}, \\ w_\pm &= p_0(x_1)\tilde{v}_0\overline{e^{ik|x_{\pm\rho}|}|x_{\pm\rho}|^{-1/2}} = p_0(x_1)\tilde{v}_0(x_2)e^{-i\bar{k}|x_{\pm\rho}|}|x_{\pm\rho}|^{-1/2}, \end{aligned} \tag{5.1}$$

where $c_0(\zeta)$ is the constant defined by (2.4) and $p_0(x_1)(= p_0(x))$ is the characteristic function of Π_0. We write $u \otimes w$ for the integral operator with the kernel $u(x)\overline{w}(y)$. The next lemma has also been verified as [3, Lemmas 7.2].

Lemma 5.2. *The operator* $T_{00}(\zeta;\rho)$ *admits the decomposition*

$$T_{00}(\zeta;\rho) = Z_0(\zeta;\rho) + Z_1(\zeta;\rho).$$

Here the two operators on the right side have the following properties.

(1) *The operator* $Z_0(\zeta;\rho)$ *is defined by*

$$Z_0(\zeta;\rho) = u_- \otimes w_- + u_+ \otimes w_+$$

with $u_\pm$ *and* $w_\pm$ *defined by (5.1).*

(2) *We can take* $\delta_0 > 0$ *in (1.9) and* $0 < 1/2 - \mu \ll 1$ *in (2.2) so small that*

$$\|Z_1(\zeta;\rho)\| = O(d^{-3/8})$$

uniformly in $\zeta \in D_d$ *as a bounded operator acting on* $L^2(\Pi_0)$.

We are now in a position to prove the basic lemma below.

Lemma 5.3. *Let* $T(\zeta;\rho)$ *be defined by (4.17). Then*

$$Id + T(\zeta;\rho) : L^2(\Pi_0) \to L^2(\Pi_0)$$

is invertible for $\zeta \in D_d$*, and the inverse is bounded uniformly in* ζ *and* $d \gg 1$.

Proof. The proof is divided into four steps.

(1) The operator $Id + T(\zeta;\rho)$ in question has the matrix representation

$$X = X(\zeta;\rho) = \begin{pmatrix} Id + T_{00} & T_{01} \\ T_{10} & Id + T_{11} \end{pmatrix}$$

as an operator acting on $L^2(\Pi_0) \oplus L^2(\Pi_0)$, and all the components $T_{jk}(\zeta;\rho)$ are bounded with $\|T_{jk}\| = O(d^\nu)$ uniformly in $\zeta \in D_d$, as stated at the end of the previous section. By Lemma 5.1, we have

$$(5.2) \qquad (Id + T_{11})^{-1} = \left(Id - T_{11}^2\right)^{-1}(Id - T_{11}) = Id - T_{11} + \mathrm{Op}\left(d^{-N}\right)$$

and X admits the decomposition

$$X = \begin{pmatrix} Id & 0 \\ 0 & Id + T_{11} \end{pmatrix} \begin{pmatrix} Id & T_{01} \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id + T_{00} + T_N & 0 \\ (Id + T_{11})^{-1} T_{10} & Id \end{pmatrix},$$

where

$$T_N = T_N(\zeta;\rho) = -T_{01}(Id + T_{11})^{-1} T_{10}$$

is of class $\mathrm{Op}\left(d^{-N}\right)$ by Lemma 5.1. We now consider the operator

$$Y_0 = Y_0(\zeta;\rho) = Id + T_{00} + T_N : L^2(\Pi_0) \to L^2(\Pi_0).$$

If Y_0 is invertible, then it follows that X is also invertible, so that $Id+T(\zeta;\rho)$ becomes invertible. The inverse X^{-1} is calculated as

$$X^{-1} = \begin{pmatrix} Y_0^{-1} & X_{01} \\ X_{10} & X_{11} \end{pmatrix},$$

where

$$X_{01} = -Y_0^{-1} T_{01}(Id + T_{11})^{-1}, \quad X_{10} = -(Id + T_{11})^{-1} T_{10} Y_0^{-1}$$

and

$$X_{11} = (Id + T_{11})^{-1} T_{10} Y_0^{-1} T_{01} (Id + T_{11})^{-1} + (Id + T_{11})^{-1}.$$

If we take (5.2) into account, then it follows by Lemma 5.2 that $(Id+T)^{-1}$ takes the form

$$(5.3) \quad (Id + T)^{-1} = \left(Id - T_{10} + \mathrm{Op}\left(d^{-N}\right)\right) Y_0^{-1} \left(v_0 - T_{01} v_1 + \mathrm{Op}\left(d^{-N}\right)\right) + (Id - T_{11})\, v_1 + \mathrm{Op}\left(d^{-N}\right).$$

(2) We denote by $\langle \ , \ \rangle$ and $\| \ \ \|_0$ the L^2 scalar product and norm in $L^2(\Pi_0)$, respectively. Lemma 5.2 allows us to write

$$(5.4) \quad \begin{aligned} Y_0(\zeta;\rho) &= Id + Z_0(\zeta;\rho) + Z_{\mathrm{rem}}(\zeta;\rho) \\ &= (Id + Z_{\mathrm{rem}}(\zeta;\rho)) \left(Id + \tilde{Z}_0(\zeta;\rho)\right), \end{aligned}$$

where

$$(5.5) \quad Z_{\mathrm{rem}}(\zeta;\rho) = Z_1(\zeta;\rho) + T_N(\zeta;\rho)$$

and

$$\tilde{Z}_0(\zeta;\rho) = (Id + Z_{\mathrm{rem}}(\zeta;\rho))^{-1} Z_0(\zeta;\rho) = \tilde{u}_- \otimes w_- + \tilde{u}_+ \otimes w_+$$

with $\tilde{u}_\pm = \tilde{u}_\pm(x;\zeta,d)$ defined by

$$(5.6) \quad \begin{aligned} \tilde{u}_\pm &= (Id + Z_{\mathrm{rem}}(\zeta;\rho))^{-1} u_\pm \\ &= u_\pm - (Id + Z_{\mathrm{rem}}(\zeta;\rho))^{-1} Z_{\mathrm{rem}}(\zeta;\rho) u_\pm. \end{aligned}$$

We set

$$(5.7) \quad h_{\pm\pm}(\zeta;\rho) = \langle \tilde{u}_\pm, w_\pm \rangle, \qquad h_{\pm\mp}(\zeta;\rho) = \langle \tilde{u}_\pm, w_\mp \rangle$$

and define $h_0(\zeta;\rho)$ by

$$(5.8) \quad h_0(\zeta;\rho) = (1 + h_{--}(\zeta;\rho))(1 + h_{++}(\zeta;\rho)) - h_{-+}(\zeta;\rho) h_{+-}(\zeta;\rho)$$

for $\zeta \in D_d$. If $h_0(\zeta;\rho)$ does not vanish over D_d, then a direct computation yields

$$(5.9) \quad \left(Id + \tilde{Z}_0(\zeta;\rho)\right)^{-1} = Id - h_0^{-1} Z_2(\zeta;\rho),$$

where

$$Z_2 = Z_2(\zeta;\rho) = (1 + h_{++}) Z_{--} - h_{+-} Z_{-+} - h_{-+} Z_{+-} + (1 + h_{--}) Z_{++}$$

with $Z_{\pm\pm} = \tilde{u}_\pm \otimes w_\pm$ and $Z_{\mp\pm} = \tilde{u}_\mp \otimes w_\pm$. If we further write

$$(Id + Z_{\mathrm{rem}})^{-1} = Id - Z_{\mathrm{rem}} (Id + Z_{\mathrm{rem}})^{-1},$$

then it follows from (5.4) and (5.9) that

$$(5.10) \quad Y_0^{-1} = Id - Z_{\mathrm{rem}} (Id + Z_{\mathrm{rem}})^{-1} - h_0^{-1} Z_3,$$

where

$$Z_3 = Z_3(\zeta;\rho) = (1+h_{++})\tilde{Z}_{--} - h_{+-}\tilde{Z}_{-+} - h_{-+}\tilde{Z}_{+-} + (1+h_{--})\tilde{Z}_{++}$$

and $\tilde{Z}_{\pm\pm} = \tilde{u}_\pm \otimes \tilde{w}_\pm$ and $\tilde{Z}_{\mp\pm} = \tilde{u}_\mp \otimes \tilde{w}_\pm$ with

$$\tilde{w}_\pm = \tilde{w}_\pm(x;\zeta,d) = \left(Id + Z_{\rm rem}(\zeta;\rho)^*\right)^{-1} w_\pm. \tag{5.11}$$

(3) We claim that $h_0(\zeta;\rho)$ defined by (5.8) never vanishes over D_d. To see this, we evaluate the norms $\|u_\pm\|_0$ and $\|w_\pm\|_0$ in $L^2(\Pi_0)$. Recall that $\rho = |\rho_+ - \rho_-| = 2\kappa_0 d$. If $x \in \Pi_0 \cap \operatorname{supp}\tilde{v}_0$, then

$$7\rho/16 < |x_1 - \rho/2| < 9\rho/16 = 9\kappa_0 d/8$$

and $|x_2| \le 2\rho^{1-\mu}$. Hence we have

$$|x_{+\rho}| = |x-\rho_+| = |x_1-\rho/2|\left(1+O(\rho^{-2\mu})\right) \le (9\kappa_0 d/8)\left(1+O(d^{-2\mu})\right).$$

By (1.13) and (1.14), we have

$$|\operatorname{Im} k| \le (1/2 + 3\delta_0/2)\,((\log d)/d)$$

uniformly in $\zeta \in D_d$, so that

$$\left| e^{ik|x_{+\rho}|}/|x_{+\rho}|^{1/2} \right| = O(d^{-1/2})O\left(d^{9\kappa_0(1+3\delta_0)/16}\right)$$

and similarly for $e^{ik|x_{-\rho}|}/|x_{-\rho}|^{1/2}$ with $x_{-\rho} = x - \rho_-$. Note that $|\chi'_{-d}(x_1)| = O(d^{-1})$ and $\kappa_0 < 1/2$ strictly. Hence we can take $\delta_0 > 0$ so small that

$$-3/2 + 9\kappa_0/16 + 27\delta_0/32 + 1 - \mu/2 < -7/32 - \mu/2.$$

Thus we obtain the bounds

$$\|u_\pm\|_0 = O(d^{-7/32-\mu/2}), \qquad \|w_\pm\|_0 = O(d^{25/32-\mu/2}). \tag{5.12}$$

(4) We prove the claim in step (3) by analyzing the asymptotic behaviors of the L^2 scalar products $h_{\pm\pm}(\zeta;\rho)$ and $h_{\mp\pm}(\zeta;\rho)$ defined by (5.7). Recall that $u_\pm$ and $w_\pm$ are defined by (5.1). If we note that $|\partial_1|x_{\pm\rho}|| > c > 0$ for $x \in \Pi_0 \cap \operatorname{supp} v_0$, then we can easily show by repeated use of integration by parts that $\langle u_\pm, w_\pm\rangle = O(d^{-N})$ for any $N \gg 1$. We make use of the stationary phase method (or the method of steepest descent) to see the behavior of $\langle u_\pm, w_\mp\rangle$. For x_1 fixed, the phase function $|x_{-\rho}| + |x_{+\rho}|$ attains its minimum at $x_2 = 0$ as a function of x_2. Hence we have the relation

$$\langle u_\pm, w_\mp\rangle = e^{ik\rho}O(d^{-1/2}).$$

The operator $Z_{\rm rem}$ defined by (5.5) satisfies $\|Z_{\rm rem}\| = O(d^{-3/8})$ by Lemma 5.2. Thus we obtain

$$|\langle \tilde{u}_\pm - u_\pm, w_\pm\rangle| = O(1)\|Z_{\rm rem}\|\|u_\pm\|_0\|w_\pm\|_0 = O\left(d^{3/16-\mu}\right)$$

by (5.6) and (5.12). This implies that $h_{\pm\pm}(\zeta;\rho) = O(d^{3/16-\mu})$ uniformly in $\zeta \in D_d$, and also we have

$$h_{\mp\pm}(\zeta;\rho) = O(d^{-1/2})\left|e^{ik\rho}\right| = O(d^{-1/2})\left(d^{\kappa_0(1+3\delta_0)}\right).$$

Hence we can take $\delta_0 > 0$ so small that

$$|h_{++}(\zeta;\rho)| + |h_{--}(\zeta;\rho)| + |h_{+-}(\zeta;\rho)h_{-+}(\zeta;\rho)| = o(1), \quad d \to \infty,$$

so that the claim is verified. The proof of the lemma is complete. □

By definition, $v_0 + v_1 = 1$. We now combine (5.3) with (5.10) to see that $(Id + T(\zeta;\rho))^{-1}$ admits the following decomposition

$$(5.13) \qquad (Id+T)^{-1} = Id + \sum_{j=1}^{3} S_j(\zeta;\rho) + \mathrm{Op}\left(d^{-N}\right),$$

where

$$\begin{aligned} S_1 &= -T_{10}v_0 - (Id - T_{10})\,T_{01}v_1 - T_{11}v_1, \\ S_2 &= -\,(Id - T_{10})\,Z_{\mathrm{rem}}\,(Id + Z_{\mathrm{rem}})^{-1}\,(v_0 - T_{01}v_1)\,, \\ S_3 &= -h_0^{-1}\,(Id - T_{10})\,Z_3\,(v_0 - T_{01}v_1)\,, \end{aligned}$$

and Z_3 is defined as in (5.10). According to (5.13), it follows from (4.19) that the resolvent $R(\zeta;Q_\rho)$ in question is decomposed into the sum

$$(5.14) \qquad R(\zeta;Q_\rho) = \Lambda(\zeta;\rho) + \Lambda_0(\zeta;\rho) + \sum_{j=1}^{3}\Lambda_j(\zeta;\rho) + \Lambda_N(\zeta;\rho),$$

where Λ is defined by (4.13) and

$$\begin{aligned} \Lambda_0 &= -\Lambda\Gamma_0\Big(R(\zeta;Q_{-\rho})\tilde\chi_{-\rho} - R(\zeta;Q_{+\rho})\tilde\chi_{+\rho}\Big) \\ \Lambda_j &= -\Lambda p_0 S_j\Gamma_0\Big(R(\zeta;Q_{-\rho})\tilde\chi_{-\rho} - R(\zeta;Q_{+\rho})\tilde\chi_{+\rho}\Big), \quad 1 \le j \le 3, \\ \Lambda_N &= \Lambda p_0\mathrm{Op}\left(d^{-N}\right)\Gamma_0\Big(R(\zeta;Q_{-\rho})\tilde\chi_{-\rho} - R(\zeta;Q_{+\rho})\tilde\chi_{+\rho}\Big). \end{aligned}$$

5.2. We prove Propositions 2.1 and 2.2, accepting the preliminary lemmas below as proved. These lemmas are proved in subsection 5.3. We recall that $j_{\pm d}$ and $\tilde{j}_{\pm d}$ denote the characteristic functions of $B_{\pm d}$ and $\tilde{B}_{\pm d}$, respectively.

Lemma 5.4. *Let $\Lambda_1(\zeta;\rho)$ be as above. Then*

$$\|j_{+d}\Lambda_1(\zeta;\rho)j_{-d}\| + \|j_{+d}\Lambda_1(\zeta;\rho)\tilde{j}_{+d}\| = O(d^{-N})$$

uniformly in $\zeta \in D_d$ for any $N \gg 1$ as a bounded operator on $L^2(\Omega_\rho)$.

Lemma 5.5. *Let $\Lambda_2(\zeta;\rho)$ be as above. Then there exists $\nu > 0$ such that*

$$\|j_{+d}\Lambda_2(\zeta;\rho)j_{-d}\| + \|j_{+d}\Lambda_2(\zeta;\rho)\tilde{j}_{+d}\| = O(d^{-\nu})$$

uniformly in $\zeta \in D_d$ as a bounded operator on $L^2(\Omega_\rho)$.

Lemma 5.6. *Let $\Lambda_3(\zeta;\rho)$ be as above. Then there exists $\nu > 0$ such that*

$$\|j_{+d}\Lambda_3(\zeta;\rho)j_{-d}\| + \|j_{+d}\Lambda_3(\zeta;\rho)\tilde{j}_{+d}\| = O(d^{-\nu})$$

uniformly in $\zeta \in D_d$ as a bounded operator on $L^2(\Omega_\rho)$.

Lemma 5.7. *Let $\Lambda_N(\zeta;\rho)$ be as above. Then*

$$\|j_{+d}\Lambda_N(\zeta;\rho)j_{-d}\| + \|j_{+d}\Lambda_N(\zeta;\rho)\tilde{j}_{+d}\| = O(d^{-N})$$

uniformly in $\zeta \in D_d$ for any $N \gg 1$ as a bounded operator on $L^2(\Omega_\rho)$.

Proof of Proposition 2.1. We first note that $j_{+d}R(\zeta;H_{0d})j_{-d}$ in question is represented as

$$j_{+d}R(\zeta;H_{0d})j_{-d} = j_{+d}R(\zeta;P_\rho)j_{-d} = j_{+d}R(\zeta;Q_\rho)j_{-d}$$

according to the notation in this section. We recall the representation for $\Lambda(\zeta;\rho)$ from (4.13). If we take (4.12) into account, then $j_{+d}\Lambda(\zeta;\rho)j_{-d} = 0$. Thus it follows from Lemmas 5.4 $\sim$ 5.7 that the leading term $R_0(\zeta;d)$ comes from the operator

$$\begin{aligned} j_{+d}\Lambda_0(\zeta;\rho)j_{-d} &= -j_{+d}\Lambda\Gamma_0 R(\zeta;Q_{-\rho})j_{-d} \\ &= -j_{+d}R(\zeta;Q_{+\rho})(v_0+v_1)\Gamma_0 R(\zeta;Q_{-\rho})j_{-d}. \end{aligned}$$

The coefficients of the commutator Γ_0 defined by (4.16) have supports in Π_0. We can show in almost the same way as used to prove Lemma 7.1 in [3] (see also Lemma 5.1) that the second operator on the right sides obeys

$$\|j_{+d}R(\zeta;Q_{+\rho})v_1\Gamma_0 R(\zeta;Q_{-\rho})j_{-d}\| = O(d^{-N})$$

for any $N \gg 1$ as a bounded operator acting on $L^2(\Omega_\rho)$. This is intuitively clear. In fact, the particle which starts from B_{-d} and passes over $\Pi_0 \cap \operatorname{supp} v_1$ never arrives at B_{+d}. The rigorous proof is done by taking Theorems A.3 and A.4 and by making repeated use of integration by parts. We consider the kernel of the first operator. We note that $\chi_{-\rho}$ is a function of the x_1 variable only and that the mapping j_ρ defined by (4.1) acts as the identity over $\operatorname{supp} v_0 \subset \left\{|x_2| < \rho^{1-\mu}\right\}$. We compute

$$\begin{aligned} v_0\Gamma_0 &= v_0[Q_{0\rho},\chi_{-\rho}] = v_0 \exp(ig_{0\rho})[K_0,\chi_{-\rho}]\exp(-ig_{0\rho}) \\ &= \exp(ig_{0\rho})v_0\left\{-2\,(\partial_1\chi_{-\rho})\,\partial_1 - \chi''_{-\rho}\right\}\exp(-ig_{0\rho}) \end{aligned}$$

$$= \quad v_0\Big\{-2\,(\partial_1\chi_{-\rho})\,\partial_1 + O(d^{-2})\Big\}.$$

It follows from (4.7) that

$$\begin{aligned} j_{+d}R(\zeta;Q_{+\rho})v_0 &= j_{+d}\exp(ig_{-\rho})R(\zeta;P_{+\rho})\exp(-ig_{-\rho})v_0, \\ R(\zeta;Q_{-\rho})j_{-d} &= \exp(ig_{+\rho})R(\zeta;P_{-\rho})\exp(-ig_{+\rho})j_{-d}. \end{aligned}$$

By (4.5), we also have

$$g_{-\rho}(x) = \alpha_-\gamma(x-\rho_-;-\omega_1) = \alpha_-\pi + O(d^{-\mu})$$

on B_{+d} or on $\Pi_0 \cap \operatorname{supp} v_0$, and

$$g_{+\rho}(x) = \alpha_+\gamma(x-\rho_+;\omega_1) = \alpha_+\pi + O(d^{-\mu})$$

on B_{-d} or on $\Pi_0 \cap \operatorname{supp} v_0$. Thus the kernel under consideration takes the following integral form:

$$2\int R(\zeta;P_{+\rho})(x,z)\partial_1\chi_{-\rho}(z_1)v_0(z_2)\partial_1 R(\zeta;P_{-\rho})(z,y)\,dz +$$
$$\int R(\zeta;P_{+\rho})(x,z)\Big(O(d^{-\mu})\partial_1\chi_{-\rho}(z_1) + O(d^{-2})\Big)v_0(z_2)\partial_1 R(\zeta;P_{-\rho})(z,y)\,dz$$

for $(x,y) \in B_{+d}\times B_{-d}$. We now apply [8, Proposition 6.1] to the first integral and [8, Proposition 6.2] to the second one. Then the first integral behaves like

$$e^{ik|x_1-y_1|}|x_1-y_1|^{-1/2}\left(c_0(\zeta)\pi_0 + O(d^{-(1/2-\mu)})\right),$$

and the second integral obeys the bound $e^{ik|x_1-y_1|}|x_1-y_1|^{-1/2}O(d^{-\mu})$. Since $\left|e^{ik|x_1-y_1|}/d^{1/2}\right| = O\left(d^{3\delta_0/2}\right)$ by (1.15) and since $\mu < 1/2$ strictly, we can take $\delta_0 > 0$ and $0 < \mu - 1/2 \ll 1$ so small that the desired leading term $R_0(\zeta;d)$ is obtained from $j_{+d}\Lambda_0(\zeta;\rho)j_{-d}$. □

Proof of Proposition 2.2. By (4.16) and (4.11), we have

$$\Gamma_0 = [Q_{0\rho},\chi_{-\rho}] = [\chi_{+\rho},Q_{+\rho}].$$

Hence

$$j_{+d}\Lambda_0\tilde{j}_{+d} = j_{+d}R(\zeta;Q_{+\rho})\Gamma_0 R(\zeta;Q_{+\rho})\tilde{j}_{+d} = 0,$$

and also we have by Lemmas 5.4 $\sim$ 5.7 that the leading term $\tilde{R}_0(\zeta;d)$ comes from

$$j_{+d}\Lambda\tilde{j}_{+d} = j_{+d}R(\zeta;Q_{+\rho})\tilde{j}_{+d} = j_{+d}\exp(ig_{-\rho})R(\zeta;P_{+\rho})\exp(-ig_{-\rho})\tilde{j}_{+d}.$$

We apply Theorem A.3 in Appendix ([8, Theorem 1.3]) to $R(\zeta;P_{+\rho})$. If we note that

$$g_{-\rho}(x) - g_{-\rho}(y) = \alpha_-(\gamma(\hat{x}_{-\rho};-\hat{y}_{-\rho}) - \pi)$$

for $(x,y) \in B_{+d} \times \tilde{B}_{+d}$, then the kernel of this operator behaves like

$$(i/4)H_0(k|x-y|)a_0(x,y;\rho) + e^{ik(|x_{+\rho}|+|y_{+\rho}|)}O(d^{-1}).$$

This yields the desired leading term $\tilde{R}_0(\zeta;d)$. □

5.3. We prove Lemmas 5.4 ∼ 5.7 which have remained unproved. The proof uses the following two auxiliary lemmas.

Lemma 5.8. *The following operators are all bounded with bound $O(d^{-N})$ uniformly in $\zeta \in D_d$ for any $N \gg 1$:*

$$\begin{aligned}
&j_{+d}R(\zeta;Q_{+\rho})p_0T_{10} : L^2(\Pi_0) \to L^2(\Omega_\rho),\\
&T_{01}v_1\Gamma_0R(\zeta;Q_{-\rho})j_{-d} : L^2(\Omega_\rho) \to L^2(\Pi_0),\\
&T_{01}v_1\Gamma_0R(\zeta;Q_{+\rho})\tilde{j}_{+d} : L^2(\Omega_\rho) \to L^2(\Pi_0),\\
&j_{+d}R(\zeta;Q_{+\rho})p_0T_{11}v_1\Gamma_0R(\zeta;Q_{-\rho})j_{-d} : L^2(\Omega_\rho) \to L^2(\Omega_\rho),\\
&j_{+d}R(\zeta;Q_{+\rho})p_0T_{11}v_1\Gamma_0R(\zeta;Q_{+\rho})\tilde{j}_{+d} : L^2(\Omega_\rho) \to L^2(\Omega_\rho).
\end{aligned}$$

Proof. The proof uses Theorem A.4 in Appendix (see [3, Propositions 6.3 and 6.4] also). In principle, it is based on the same idea as the proof of Lemma 5.1 ([3, Lemma 7.1]). For example, the bound on the first operator follows from the fact that outgoing particles starting from $\Pi_0 \cap \operatorname{supp} \tilde{v}_0$ and passing over $\Pi_0 \cap \operatorname{supp} v_1$ after scattered by the obstacle $\mathcal{O}_{-\rho}$ or $\mathcal{O}_{+\rho}$ never reach B_{+d}. This is made rigorous by repeated use of integration by parts. We do not go into details. □

Lemma 5.9. *The following statements hold true uniformly in $\zeta \in D_d$.*

(1) *The operator*

$$j_{+d}R(\zeta;Q_{+\rho})p_0\tilde{v}_0 : L^2(\Pi_0) \to L^2(\Omega_\rho)$$

is bounded with the bound $O(d^{1/2-\mu/2})O\left(d^{(\kappa_+ + \kappa_0/8)(1/2+3\delta_0/2)}\right)$.

(2) *The operator*

$$\tilde{v}_0\Gamma_0R(\zeta;Q_{-\rho})j_{-d} : L^2(\Omega_\rho) \to L^2(\Pi_0)$$

is bounded with the bound $O(d^{-1/2-\mu/2})O\left(d^{(\kappa_- + \kappa_0/8)(1/2+3\delta_0/2)}\right)$.

(3) *The operator*

$$\tilde{v}_0\Gamma_0R(\zeta;Q_{+\rho})\tilde{j}_{+d} : L^2(\Omega_\rho) \to L^2(\Pi_0)$$

is bounded with the bound $O(d^{-1/2-\mu/2})O(d^{\delta})O\left(d^{(\kappa_+ + \kappa_0/8)(1/2+3\delta_0/2)}\right)$.

Proof. The proof uses Theorems A.1 and A.2 in Appendix ([8, Theorems 1.1 and 1.2]). If $x \in B_{+d}$ and $y \in \Pi_0 \cap \operatorname{supp} \tilde{v}_0$, then

$$|x-y| = |x_1 - y_1| \left(1 + O(d^{-2\mu})\right)$$

and $|x_1 - y_1| < \rho/16 + \kappa_+ d = (\kappa_0/8 + \kappa_+)d$. This implies

$$\left| e^{ik|x-y|} |x-y|^{-1/2} \right| = O(d^{-1/2}) O\left(d^{(\kappa_+ + \kappa_0/8)(1/2 + 3\delta_0/2)} \right).$$

Statement (1) is verified by evaluating the Hilbert–Schmidt norm of the operator. Similar arguments apply to the other statements. □

Proof of Lemma 5.4. The lemma follows immediately from Lemmas 5.8 and 5.9. □

Proof of Lemma 5.5. By Lemma 5.2, $\| Z_{\mathrm{rem}} \left(Id + Z_{\mathrm{rem}} \right)^{-1} \| = O(d^{-3/8})$ as an operator on $L^2(\Pi_0)$. This, together with Lemmas 5.8 and 5.9, implies that

$$\| j_{+d} \Lambda_2(\zeta; \rho) j_{-d} \| = O(d^{-3/8-\mu}) O\left(d^{(\kappa_+ + \kappa_- + \kappa_0/4)(1/2 + 3\delta_0/2)} \right)$$

as an operator on $L^2(\Omega_\rho)$. Recall from (2.2) that $\mu > 2/5$. Since $\kappa_- + \kappa_+ = 1$ and $\kappa_0 < 1/2$ strictly, we can take $\delta_0 > 0$ so small that

$$-3/8 - \mu + (1 + \kappa_0/4)(1/2 + 3\delta_0/2) < -\mu + 3/16 < 0.$$

Next we prove the lemma for $j_{+d} \Lambda_2(\zeta; \rho) \tilde{j}_{+d}$. Recall the representation for $Z_{\mathrm{rem}}(\zeta; \rho)$ from (5.5). By Lemma 5.2, we have the expansion

$$\begin{aligned} Z_{\mathrm{rem}} \left(Id + Z_{\mathrm{rem}} \right)^{-1} = {} & Z_1 - Z_1^2 + \cdots + (-1)^n Z_1^{n+1} \\ & + \ (-1)^{n+1} Z_1^{n+2} \left(Id + Z_{\mathrm{rem}} \right)^{-1} + \mathrm{Op}\left(d^{-N} \right) \end{aligned}$$

by the Neumann series. We have shown in the course of the proof of [3, Lemma 7.2] that the kernel $Z_1(\zeta; \rho)(x, y)$ of $Z_1(\zeta; \rho)$ takes the form

$$\begin{aligned} & Z_1(\zeta; \rho)(x, y) = \\ & v_0(x_2) p_0(x_1) e^{ik|x_{-\rho}|} |x_{-\rho}|^{-1/2} z_-(x, y; \zeta, \rho) |y_{-\rho}|^{-1/2} e^{ik|y_{-\rho}|} p_0(y_1) \tilde{v}_0(y_2) \\ & + \ v_0(x_2) p_0(x_1) e^{ik|x_{+\rho}|} |x_{+\rho}|^{-1/2} z_+(x, y; \zeta, \rho) |y_{+\rho}|^{-1/2} e^{ik|y_{+\rho}|} p_0(y_1) \tilde{v}_0(y_2), \end{aligned}$$

where $z_\pm(x, y; \zeta, \rho)$ satisfies

$$\left| \partial_x^l \partial_y^m z_\pm(x, y; \zeta, \rho) \right| = O\left(d^{-1-\mu-(|l|+|m|)(1-\mu)} \right)$$

uniformly in $\zeta \in D_d$ and in $x \in \Pi_0 \cap \operatorname{supp} v_0$ and $y \in \Pi_0 \cap \operatorname{supp} \tilde{v}_0$. We observe that

$$\left| (\partial/\partial z_1)(|x - z| + |z_{+\rho}|) \right| = \left| (\partial/\partial z_1)(|x - z| + |\rho_+ - z|) \right| > c > 0$$

for $x \in B_{+d}$ and $z \in \Pi_0 \cap \operatorname{supp} v_0$. If we take account of Theorems A.1 and A.2 in Appendix, then we have

$$\big(\Gamma_0 R(\zeta; Q_{+\rho})\big)(z,y) = O(d^{-3/2}) \left| e^{ik|z-y|} \right|$$

for $z \in \Pi_0 \cap \operatorname{supp} v_0$ and $y \in \tilde{B}_{+d}$, and

$$R(\zeta; Q_{+\rho})(x,z) = O(d^{-1/2}) \left| e^{ik|x-z|} \right|$$

for $x \in B_{+d}$ and $z \in \Pi_0 \cap \operatorname{supp} v_0$. Since $|z_2| = O\left(d^{1-\mu}\right)$ on $\operatorname{supp} v_0$ or $\operatorname{supp} \tilde{v}_0$, we see by repeated use of partial integration that

$$\big(R(\zeta, Q_{+\rho}) p_0 Z_1 \Gamma_0 R(\zeta; Q_{+\rho})\big)(x,y) = O(d^{-3\mu}) \left| e^{ik|x-\rho|} \right| \left| e^{ik|y-\rho|} \right|$$

uniformly in $x \in B_{+d}$ and $y \in \tilde{B}_{+d}$, so that

$$\| j_{+d} R(\zeta, Q_{+\rho}) p_0 Z_1 \Gamma_0 R(\zeta; Q_{+\rho}) \tilde{j}_{+d} \| = O(d^{-3\mu+\delta}) O\left(d^{2(\kappa_0+\kappa_+)(1/2+3\delta_0/2)} \right)$$

as a bounded operator on $L^2(\Omega_\rho)$. If we note that $\kappa_0 < \kappa_-$, then

$$-3\mu + \delta + (\kappa_0 + \kappa_+)(1 + 3\delta_0) < -\mu$$

for $\delta_0 > 0$ and $\delta > 0$ small enough, and hence

$$\| j_{+d} R(\zeta, Q_{+\rho}) p_0 Z_1 \Gamma_0 R(\zeta; Q_{+\rho}) \tilde{j}_{+d} \| = O(d^{-\mu})$$

uniformly in $\zeta \in D_d$. The better bound is expected for the higher power Z_1^n with $n \ge 2$. It is easy to see that the contribution from the even power Z_1^{2m} is negligible. We consider the odd power Z_1^{2m+1}. We have chosen μ, $0 < \mu < 1/2$, so close to $1/2$ so that

$$O(d^{-\mu}) \left| e^{ik|z-\rho|} \right| \left| e^{ik|z+\rho|} \right| = O(d^{-\mu}) \left| e^{ik\rho} \right| = O(1)$$

for $z \in \Pi_0 \cap \operatorname{supp} v_0$. This yields

$$\big(R(\zeta, Q_{+\rho}) p_0 Z_1^{2m+1} \Gamma_0 R(\zeta; Q_{+\rho})\big)(x,y) = O(d^{-(2m+3)\mu}) \left| e^{ik|x-\rho|} \right| \left| e^{ik|y-\rho|} \right|$$

uniformly in $x \in B_{+d}$ and $y \in \tilde{B}_{+d}$, and hence it follows that

$$\| j_{+d} R(\zeta, Q_{+\rho}) p_0 Z_1^{2m+1} \Gamma_0 R(\zeta; Q_{+\rho}) \tilde{j}_{+d} \| = O\left(d^{-(2m+1)\mu} \right).$$

By Lemma 5.2, the reminder operator

$$Z_1^{n+2} \left(Id + Z_{\mathrm{rem}} \right)^{-1} : L^2(\Pi_0) \to L^2(\Pi_0)$$

obeys the bound $O\left(d^{-3(n+2)/8}\right)$. Thus we can take $n \gg 1$ so large that the operator $j_{+d} \Lambda_2(\zeta; \rho) \tilde{j}_{+d}$ under consideration obeys the bound $O(d^{-\nu})$ for some $\nu > 0$. This proves the lemma. □

Proof of Lemma 5.6. First we consider the operator $j_{+d} \Lambda_3 j_{-d}$. Recall the representation for S_3 from (5.13). Then S_3 is decomposed into the sum of four operators. Among these operators, it suffices to prove the bound only for the operator

$$j_{+d}R(\zeta;Q_{+\rho})p_0 Z_3 v_0 \Gamma_0 R(\zeta;Q_{-\rho})j_{-d} : L^2(\Omega_\rho) \to L^2(\Omega_\rho).$$

By Lemma 5.8, the other three operators are shown to be negligible. Recall from (5.6) and (5.11) that $\tilde{Z}_{\pm\pm}$ and $\tilde{Z}_{\pm\mp}$ are defined by $\tilde{Z}_{\pm\pm} = \tilde{u}_\pm \otimes \tilde{w}_\pm$ and $\tilde{Z}_{\pm\mp} = \tilde{u}_\pm \otimes \tilde{w}_\mp$, where

$$\tilde{u}_\pm = (Id + Z_{\mathrm{rem}}(\zeta;\rho))^{-1}u_\pm = u_\pm - Z_{\mathrm{rem}}(Id + Z_{\mathrm{rem}})^{-1}u_\pm,$$
$$\tilde{w}_\pm = (Id + Z_{\mathrm{rem}}(\zeta;\rho)^*)^{-1}w_\pm = w_\pm - Z_{\mathrm{rem}}^*(Id + Z_{\mathrm{rem}}^*)^{-1}w_\pm,$$

and $u_\pm$ and $w_\pm$ are defined by (5.1). We now define the operators

$$\Lambda_{\pm\pm}(\zeta;\rho) = j_{+d}R(\zeta;Q_{+\rho})p_0\tilde{Z}_{\pm\pm}v_0\Gamma_0 R(\zeta;Q_{-\rho})j_{-d},$$
$$\Lambda_{\pm\mp}(\zeta;\rho) = j_{+d}R(\zeta;Q_{+\rho})p_0\tilde{Z}_{\pm\mp}v_0\Gamma_0 R(\zeta;Q_{-\rho})j_{-d}$$

and we assert that

$$\|\Lambda_{\pm\pm}(\zeta;\rho)\| = O\left(d^{-\nu}\right), \qquad \|\Lambda_{\pm\mp}(\zeta;\rho)\| = O\left(d^{-\nu}\right) \tag{5.15}$$

uniformly in $\zeta \in D_d$ for some $\nu > 0$. Then the desired bound on the operator $j_{+d}\Lambda_3 j_{-d}$ in question is obtained.

We analyze only $\Lambda_{-+}(\zeta;\rho)$ in some details. By Theorems A.1 and A.2, the function $\left(R(\zeta;Q_{+\rho})p_0u_-\right)(x)$ satisfies

$$|(R(\zeta;Q_{+\rho})p_0u_-)(x)| = O(d^{-\mu})\left|e^{ik|x-\rho|}\right|$$

uniformly in $x \in B_{+d}$. Similarly we have

$$\left|\left((\Gamma_0 R(\zeta;Q_{-\rho})j_{-d})^* w_+\right)(y)\right| = \left|\left(R(\zeta;Q_{-\rho})^*\Gamma_0^* w_+\right)(y)\right| = O(d^{-\mu})\left|e^{ik|y+\rho|}\right|$$

uniformly in $y \in B_{-d}$. Hence the operator

$$I_0 = \left(j_{+d}R(\zeta;Q_{+\rho})p_0u_-\right) \otimes \left((\Gamma_0 R(\zeta;Q_{-\rho})j_{-d})^* w_+\right) : L^2(\Omega_\rho) \to L^2(\Omega_\rho)$$

is bounded by $O(d^{-2\mu})O\left(d^{(2\kappa_0+\kappa_++\kappa_-)(1/2+3\delta_0/2)}\right)$. Since $\kappa_+ + \kappa_- = 1$ and since $\kappa_0 < 1/2$ strictly, we can take $\delta_0 > 0$ so small that

$$-2\mu + (2\kappa_0 + \kappa_+ + \kappa_-)(1/2 + 3\delta_0/2) < 0.$$

Thus we have $\|I_0\| = O(d^{-\nu})$ for some $\nu > 0$. Next we evaluate the norm of remainder operators such as

$$I_1 = \left(j_{+d}R(\zeta;Q_{+\rho})(\tilde{u}_- - u_-)\right) \otimes \left((\Gamma_0 R(\zeta;Q_{-\rho})j_{-d})^* w_+\right),$$
$$\left(j_{+d}R(\zeta;Q_{+\rho})u_-\right) \otimes \left((\Gamma_0 R(\zeta;Q_{-\rho})j_{-d})^*(\tilde{w}_+ - w_+)\right),$$
$$\left(j_{+d}R(\zeta;Q_{+\rho})(\tilde{u}_- - u_-)\right) \otimes \left((\Gamma_0 R(\zeta;Q_{-\rho})j_{-d})^*(\tilde{w}_+ - w_+)\right).$$

By Lemma 5.9 (1), we see that the operator

$$j_{+d}R(\zeta;Q_{+\rho})p_0\tilde{v}_0 : L^2(\Pi_0) \to L^2(\Omega_\rho)$$

obeys the bound $O(d^{1/2-\mu/2})O\left(d^{(\kappa_0/8+\kappa_+)(1/2+3\delta_0/2)}\right)$. We also have

$$\|\tilde{u}_- - u_-\|_0 = O(d^{-3/8})O(d^{-7/32-\mu/2}) = O(d^{-19/32-\mu/2})$$

by Lemma 5.2 and (5.12). Thus we are able to evaluate the operator norm of the first operator I_1 as follows:

$$\|I_1\| = O(d^{-3/32-\mu})O(d^{-\mu})O\left(d^{(9\kappa_0/8+\kappa_++\kappa_-)(1/2+3\delta_0/2)}\right).$$

We can take $\delta_0 > 0$ so small that

$$-3/32 - 2\mu + (1 + 9\kappa_0/8)(1/2 + 3\delta_0/2) < 0$$

strictly. Hence we have $\|I_1\| = O\left(d^{-\nu}\right)$ for some $\nu > 0$. The other two remainder operators are dealt with in a similar way, and we obtain (5.15) for $\Lambda_{-+}(\zeta;\rho)$.

To prove (5.15) for $\Lambda_{\pm\pm}(\zeta;\rho)$ and $\Lambda_{+-}(\zeta;\rho)$, we observe that the function $\left(R(\zeta;Q_{+\rho})p_0u_+\right)(x)$ behaves like

$$\left(R(\zeta;Q_{+\rho})p_0u_+\right)(x) = O(d^{-N})$$

over B_{+d}, which follows from Theorems A.1 and A.2. Intuitively, this follows from the fact that particles outgoing from ρ_+ to B_{+d} never pass over $\Pi_0 \cap \operatorname{supp} v_0$. For a similar reason, we also have

$$\left(R(\zeta;Q_{-\rho})^*\Gamma_0^*w_-\right)(y) = O(d^{-N})$$

over B_{-d}. In fact, particles incoming to ρ_- from B_{-d} never pass over $\Pi_0 \cap \operatorname{supp} \tilde{v}_0$. If we take these facts into account, then we can show (5.15) for the other three operators.

Next we show the lemma for the operator $j_{+d}\Lambda_3\tilde{j}_{+d}$. We can obtain similar bounds on the operators

$$\tilde{\Lambda}_{\pm\pm}(\zeta;\rho) = j_{+d}R(\zeta;Q_{+\rho})p_0\tilde{Z}_{\pm\pm}v_0\Gamma_0R(\zeta;Q_{+\rho})\tilde{j}_{+d},$$
$$\tilde{\Lambda}_{\pm\mp}(\zeta;\rho) = j_{+d}R(\zeta;Q_{+\rho})p_0\tilde{Z}_{\pm\mp}v_0\Gamma_0R(\zeta;Q_{+\rho})\tilde{j}_{+d}.$$

For example, we consider $\tilde{\Lambda}_{--}(\zeta;\rho)$. We repeat the same argument as above to get

$$\begin{aligned}&\left\|\left(j_{+d}R(\zeta;Q_{+\rho})u_-\right)\otimes\left((\Gamma_0R(\zeta;Q_{+\rho})\tilde{j}_{+d})^*w_-\right)\right\|\\ &\quad = O(d^{-2\mu})O(d^{\delta})O\left(d^{2(\kappa_0+\kappa_+)(1/2+3\delta_0/2)}\right)\end{aligned}$$

as a bounded operator on $L^2(\Omega_\rho)$. Since $\kappa_0 < \kappa_-$ strictly, we can take $\delta_0 > 0$ and $\delta > 0$ so small that

$$-2\mu + 2(\kappa_0 + \kappa_+)(1/2 + 3\delta_0/2) + \delta < 0.$$

The norm of such a remainder operator as

$$\Big(j_{+d} R(\zeta; Q_{+\rho})(\tilde{u}_- - u_-)\Big) \otimes \Big((\Gamma_0 R(\zeta; Q_{+\rho})\tilde{j}_{+d})^* w_-\Big)$$

is bounded by

$$O(d^{-3/32-\mu}) O\left(d^{(\kappa_0/8+\kappa_+)(1/2+3\delta_0/2)}\right) O(d^{-\mu}) O(d^{\delta}) O\left(d^{(\kappa_0+\kappa_+)(1/2+3\delta_0/2)}\right).$$

Since $\kappa_0 < \kappa_-$ (and hence $9\kappa_0/8 - 2\kappa_- < 0$) and $\kappa_+ + \kappa_- = 1$, we have

$$-3/32 - 2\mu + (9\kappa_0/8 + 2\kappa_+)(1/2 + 3\delta_0/2) + \delta < 0$$

for $\delta_0 > 0$ and $\delta > 0$ small enough. The other remainder operators are dealt with in a similar way. Hence we can show that $\|\tilde{\Lambda}_{--}(\zeta;\rho)\| = O(d^{-\nu})$ uniformly in $\zeta \in D_d$ for some $\nu > 0$. A similar argument applies to $\tilde{\Lambda}_{++}(\zeta;\rho)$ and $\tilde{\Lambda}_{\pm\mp}(\zeta;\rho)$. We skip the details. Thus the proof of the lemma is now complete. □

Proof of Lemma 5.7. The proof uses Theorems A.1 ∼ A.4. According to these results,

$$p_0 R(\zeta; Q_{-\rho}) j_{-d}, \quad p_0 R(\zeta; Q_{+\rho}) \tilde{j}_{+d} : L^2(\Omega_\rho) \to L^2(\Pi_0),$$

and $j_{+d} R(\zeta; Q_{+\rho}) p_0 : L^2(\Pi_0) \to L^2(\Omega_\rho)$ are bounded with bound $O(d^\nu)$ for some $\nu > 0$. In particular, Theorem A.4 is used to evaluate the bounds when $|x_2| \gg 1$. We note that $H_0\big(k r_\rho(x,y)\big)$ rapidly falls off even for $\mathrm{Im}\, k < 0$ as $|x_2| \to \infty$. Hence the lemma follows at once. □

Appendix

In the previous work [8], we have studied the asymptotic properties, particularly along forward directions of resolvent kernels (the Green functions) with spectral parameters in the lower half plane of the complex plane (unphysical sheet) for magnetic Schrödinger operators in two dimensions. Here we refer to these results as the three theorems below in the form adapted to the application to the present problem.

Let $b \in C_0^\infty(\boldsymbol{R}^2)$ be a given magnetic field such that b has α as a magnetic flux and $\mathrm{supp}\, b \subset \mathcal{O} \subset \{|x| < 1\}$ for some simply connected bounded domain $\mathcal{O}$ with the smooth boundary. We take

$$A(x) = \alpha\, \Phi(x), \qquad x \in \Omega = \boldsymbol{R}^2 \setminus \overline{\mathcal{O}}$$

as the vector potential corresponding to b, where Φ is the Aharonov–Bohm potential defined by (1.3). We consider the self–adjoint operator

$$P = H(A,0) = (-i\nabla - A)^2, \qquad \mathcal{D}(P) = H^2(\Omega) \cap H_0^1(\Omega),$$

in $L^2(\Omega)$. The notation d still denotes the large parameter, and we use D_d and μ with the meanings ascribed by (1.9) and (2.2), respectively. We also define

$$\sigma(x;y) = \gamma(\hat{x};\hat{y}) - \pi = \gamma(x;y) - \pi, \quad \hat{x} = x/|x|,$$

where $\gamma(\theta;\omega)$ again denotes the azimuth angle from $\omega \in S^1$ to θ. With the notation above, we are now in a position to state the asymptotic properties of the resolvent kernel $R(\zeta;P)(x,y)$ with $\zeta \in D_d$ when $|x-y| \gg 1$ with $|\sigma(x,y)| \ll 1$. The following three theorems (Theorems A.1, A.2 and A.3) are obtained as particular cases of Theorems 1.1, 1.2 and 1.3 in [8], respectively. In fact, these theorems remain true for ζ such that $\left|\operatorname{Re}\zeta - E_0\right| < E_0/2$ and $\left|\operatorname{Im}\zeta\right| < c\left((\log|d|)/|d|\right)$ for some $c > 0$.

Theorem A.1. *Assume that $\zeta \in D_d$ and that x and y fulfill*

$$d/c \le |x|,\ |y| \le c\,d, \qquad |\sigma(x,y)| \le c\,d^{-(1-\mu)}$$

for some $c > 1$. Then the kernel $R(\zeta;P)(x,y)$ takes the asymptotic form

$$\begin{aligned} R(\zeta;P)(x,y) &= (i/4)\cos(\alpha\pi)e^{i\alpha(\gamma(\hat{x};\hat{y})-\pi)}H_0(k|x-y|) \\ &\quad + e^{ik(|x|+|y|)}\left(|x|+|y|\right)^{-1/2} r_1(x,y;\zeta,d), \end{aligned}$$

where the remainder term r_1 is analytic in $\zeta \in D_d$ and obeys

$$\left|\partial_x^n \partial_y^m r_1\right| = O\left(d^{\mu-1/2-(|n|+|m|)/2}\right)$$

uniformly in x, y and ζ.

Theorem A.2. *Assume that $\zeta \in D_d$ and that x and y fulfill*

$$d/c \le |x|,\ |y| \le c\,d, \qquad d^{-(1-\mu)}/c \le |\sigma(x,y)| \le c\,d^{-\mu}$$

for some $c > 1$. Let $c_0(\zeta)$ be defined by (2.4) and let $z_0 = z_0(x,y;\zeta)$ be defined by

$$z_0 = \Big(|x||y|/(|x|+|y|)\Big)^{1/2}|\sigma(x,y)|\zeta^{1/4}.$$

Then $R(\zeta;P)(x,y)$ behaves like

$$\begin{aligned} &R(\zeta;P)(x,y) = (i/4)e^{i\alpha(\gamma(\hat{x};-\hat{y})-\pi)}H_0(k|x-y|) \\ &\pm\ c_0(\zeta)\frac{i\sin(\alpha\pi)}{\pi}\left(\frac{e^{ik|x-y|}}{|x-y|^{1/2}}\right)\left(\pi - (2\pi)^{1/2}e^{-i\pi/4}\int_0^{z_0} e^{is^2/2}\,ds\right) \\ &+\ e^{ik|x-y|}|x-y|^{-1/2}r_{\pm 2}(x,y;\zeta,d) \end{aligned}$$

according as $\pm\sigma(x,y) > 0$, *where* $r_{\pm 2}$ *is analytic in* $\zeta \in D_d$ *and obeys*

$$\left|\partial_x^n \partial_y^m r_{\pm 2}\right| = O\left(d^{-\mu-(|n|+|m|)\mu}\right)$$

uniformly in x, y *and* ζ.

Theorem A.3. *Assume that* $\zeta \in D_d$ *and that* x *and* y *fulfill*

$$d/c \le |x|,\ |y| \le c\,d, \qquad |\sigma(x,y)| > d^{-\mu}/c$$

for some $c > 1$. *Then*

$$\begin{aligned} R(\zeta;P)(x,y) &= (i/4)e^{i\alpha(\gamma(\hat{x};-\hat{y})-\pi)}H_0(k|x-y|) \\ &\quad + e^{ik(|x|+|y|)}\left(|x|+|y|\right)^{-1/2} r_3(x,y;\zeta,d), \end{aligned}$$

where r_3 *is analytic in* $\zeta \in D_d$ *and obeys*

$$\left|\partial_x^n \partial_y^m r_3\right| = |\sigma(x,y)|^{-1-(|n|+|m|)}\, O\left(d^{-1/2-(|n|+|m|)}\right)$$

uniformly in x, y *and* ζ.

Let J_ρ be the mapping defined by (4.2). We define the complex scaled operator

$$\tilde{Q}_\rho = J_\rho P J_\rho^{-1}, \qquad \mathcal{D}(\tilde{Q}_\rho) = H^2(\Omega) \cap H_0^1(\Omega),$$

for the operator P and

$$\tilde{Q}_{0\rho} = J_\rho K_0 J_\rho^{-1}, \qquad \mathcal{D}(\tilde{Q}_{0\rho}) = H^2(\boldsymbol{R}^2),$$

for the free Hamiltonian $K_0 = -\Delta$. Then the resolvent kernel $R(\zeta;\tilde{Q}_\rho)(x,y)$ is given by

$$R(\zeta;\tilde{Q}_\rho)(x,y) = \left[\det\left(\partial j_\rho/\partial x\right)\right]^{1/2} R(\zeta;P)(j_\rho(x),j_\rho(y)) \left[\det\left(\partial j_\rho/\partial y\right)\right]^{1/2},$$

where $\det\left(\partial j_\rho/\partial x\right)$ denotes the Jacobian of the mapping j_ρ defined by (4.1). By definition, the mapping j_ρ acts as the identity over the strip

$$W_\rho = \left\{x = (x_1,x_2) : |x_2| < \rho\right\}, \quad \rho = 2\kappa_0 d,$$

and hence we have the relation

$$R(\zeta;\tilde{Q}_\rho)(x,y) = R(\zeta;P)(j_\rho(x),j_\rho(y)) = R(\zeta;P)(x,y)$$

for $(x,y) \in W_\rho \times W_\rho$. If we take account of the relation $R(\zeta;K_0)(x,y) = (i/4)H_0(k|x-y|)$, then we have

$$R(\zeta;\tilde{Q}_{0\rho})(x,y) = \frac{i}{4}\left[\det\left(\partial j_\rho/\partial x\right)\right]^{1/2} H_0(kr_\rho(x,y)) \left[\det\left(\partial j_\rho/\partial y\right)\right]^{1/2},$$

where

$$r_\rho(x,y) = \left((x_1-y_1)^2 + \left((x_2+i\eta_\rho(x_2)x_2) - (y_2+i\eta_\rho(y_2)y_2)\right)^2\right)^{1/2}$$

and $\eta_\rho(t)$ is defined in (4.1). We set

$$U_{\pm\rho} = \Big\{x = (x_1, x_2) : |x_2| > \rho/c,\ \rho/c < \pm x_1 < c\rho\Big\}$$

for $c > 1$. Then the next theorem is verified in almost the same way as [3, Proposition 6.3].

Theorem A.4. *Assume that $x \in W_\rho$ fulfills $\rho/c < x_1 < c\rho$ and $y \in U_{-\rho}$ for some $c > 1$, and define*

$$\psi_{-\rho}(x, y) = \gamma(j_\rho(x); -\omega_1) - \gamma(j_\rho(y); -\omega_1), \quad \omega_1 = (1, 0),$$

for x and y as above. Then $R(\zeta; \tilde{Q}_\rho)(x, y)$ admits the decomposition

$$R(\zeta; \tilde{Q}_\rho)(x, y) = \exp(i\alpha\psi_{-\rho}(x, y))R(\zeta; \tilde{Q}_{0\rho})(x, y) + R_{\mathrm{sc}}(x, y; \zeta)$$

and the analytic function $R_{\mathrm{sc}}(x, y; \zeta)$ over D_d satisfies the following estimates uniformly in $\zeta \in D_d$.

(1) *If $|y_2| > L\rho$ for $L \gg 1$ fixed arbitrarily, then*

$$R_{\mathrm{sc}}(x, y; \zeta) = O\left((|x| + |y|)^{-\sigma L}\right)$$

for some $\sigma > 0$ independent of L together with the derivatives $\partial R_{\mathrm{sc}}/\partial x_1$ and $\partial R_{\mathrm{sc}}/\partial y_1$.

(2) *If $|y_2| < 2L\rho$ for $L \gg 1$ fixed, then R_{sc} takes the form*

$$R_{\mathrm{sc}}(x, y; \zeta) = \exp(ikr_\rho(x))q_0(x, y; \zeta)\exp(ikr_\rho(y))$$

and $q_0(x, y; \zeta)$ satisfies

$$\left|(\partial/\partial x_2)^j\,(\partial/\partial y_2)^l\,q_0\right| = O(\rho^{-1-j-l}).$$

Similar estimates hold true for $\partial q_0/\partial x_1$ and $\partial q_0/\partial y_1$.

A similar relation remains true when $x \in U_{-\rho}$ and $y \in W_\rho$ fulfills $\rho/c < y_1 < c\rho$ for some $c > 1$.

Remark. If $x \in W_\rho$ fulfills $-c\rho < x_1 < -\rho/c$ and $y \in U_{+\rho}$ for some $c > 1$ or if $x \in U_{+\rho}$ and $y \in W_\rho$ fulfills $-c\rho < y_1 < -\rho/c$, then the same results as above remain true for $\psi_{-\rho}$ replaced by

$$\psi_{+\rho}(x, y) = \gamma(j_\rho(x); \omega_1) - \gamma(j_\rho(y); \omega_1).$$

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