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AN EXPLICIT EFFECT OF NON-SYMMETRY OF RANDOM WALKS ON THE TRIANGULAR LATTICE

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Abstract. In the present paper, we study an explicit effect of non-symmetry on asymptotics of the n-step transition probability as $n \to \infty$ for a class of non-symmetric random walks on the triangular lattice. Realizing the triangular lattice into $\mathbb{R}^2$ appropriately, we observe that the Euclidean distance in $\mathbb{R}^2$ naturally appears in the asymptotics. We characterize this realization from a geometric view point of Kotani-Sunada's standard realization of crystal lattices. As a corollary of the main theorem, we obtain that the transition semigroup generated by the non-symmetric random walk approximates the heat semigroup generated by the usual Brownian motion on $\mathbb{R}^2$.

1. Introduction

Let $G = (V, E)$ be a locally finite, connected, oriented graph. Here V is the set of vertices and E is the set of oriented edges. For an oriented edge $e \in E$, the *origin* and the *terminus* of e are denoted by $o(e)$ and $t(e)$, respectively. The *inverse edge* of e is denoted by $\overline{e}$. A *random walk* on G is given by a non-negative valued function p on E satisfying

$$\sum_{e \in E_x} p(e) = 1 \quad \text{for all } x \in V,$$

where $E_x = \{e \in E |\ o(e) = x\}$. Here $p(e)$ is the probability that a particle at $o(e)$ moves to $t(e)$ along the edge e in one unit time. Then the transition probability $p(n, x, y)$ that a particle starting at $x \in V$ reaches $y \in V$ at time n is given by

$$p(n, x, y) = \sum_{c=(e_1,e_2,\ldots,e_n)} p(e_1)p(e_2)\cdots p(e_n),$$

where the sum is taken over all paths $c = (e_1, e_2, \ldots, e_n)$ with $t(e_i) = o(e_{i+1})$, $i = 1, \ldots, n-1$ and $o(e_1) = x$, $t(e_n) = y$. If there exists a positive valued function m_V on V such that

$$p(e)m_V(o(e)) = p(\overline{e})m_V(t(e)), \quad e \in E, \tag{1.1}$$

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the random walk is said to be (m_V-)symmetric.

A principal theme for random walks is to investigate the properties of $p(n,x,y)$ as $n\to\infty$. One of the most classical problems is the recurrence-transience problem which is related to the divergence-convergence of the Green function $G(x):=\sum_{n=1}^{\infty}p(n,x,x)$, and the local central limit theorem gives us a useful criterion for the divergence-convergence of the series. For this reason, this theme has been discussed intensively in various settings by many authors. See Spitzer [10], Lawler and Limic [8] and Woess [15] for an overview of random walks.

In particular, Kotani, Shirai and Sunada investigated long time asymptotics of $p(n,x,y)$ of (m_V-)symmetric random walks on a *crystal lattice*, a covering graph of a finite graph whose covering transformation group is abelian. In [4], as the precision of the local central limit theorem (cf. [7]), they established the asymptotic expansion

$$\begin{aligned} &p(n,x,y)m_V(y)^{-1} \\ &\sim a_0 n^{-r/2}\exp\Big(-\frac{d(x,y)^2}{4n}\Big)\cdot\Big(1+a_1(x,y)n^{-1}+a_2(x,y)n^{-2}+\cdots\Big) \end{aligned} \tag{1.2}$$

as $n\to\infty$, where $d(x,y)$ is a Euclidean distance appeared through the *standard realization* of the graph into $\mathbb{R}^r$. In their proof, spectral theoretic arguments due to the periodicity of the graph and the symmetry of the random walk play crucial roles.

Later in [2, 14], Uchiyama and his coauthor also established the formula (1.2) for non-symmetric random walks on periodic graphs (i.e., crystal lattices) in the Euclidean space by a probabilistic approach. Their result implies that the effect of the non-symmetry on the coefficient $a_1(x,y)$ highly depends not only on the underlying periodic graph but also on the choice of the 1-step transition probability p even if the zero mean condition (see condition **(P1)** in Section 2) is imposed.

In view of these results, it is a meaningful problem to determine an explicit effect of the non-symmetry on the coefficient $a_1(x,y)$ on a specific graph. In the present paper, we give an affirmative answer to this problem in the case of the triangular lattice. As we will mention later, there exist non-symmetric random walks on the triangular lattice satisfying the zero mean condition **(P1)**. On the other hand, for example, on the square lattice and the hexagonal lattice, we note that the zero mean condition is equivalent to the symmetry of the random walk since these graphs are maximal abelian coverings (see Kotani and Sunada [6, page 842]). In the proof of the main theorem (Theorem 2.2), we make use of the probabilistic approach as in [2, 14] with the idea of the standard realization by Kotani and Sunada [4, 5].

As a corollary of Theorem 2.2, we establish the functional analytic central limit theorem (Corollary 2.4) that the transition semigroup generated by the non-symmetric random walk on the triangular lattice approximates the heat semigroup generated by the usual Brownian motion on $\mathbb{R}^2$. It says that the effect of the non-symmetry of the random walk does not appear in the appropriate space-time scaling limit.

Throughout the present paper, $O(\cdot)$ stands for the Landau symbol. When the dependence of the $O(\cdot)$ term is significant, we specify as $O_N(\cdot)$, etc.

2. Framework and Results

First of all, we prepare some notations and formulate our problem. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be linearly independent vectors in $\mathbb{R}^2$ and we set $\mathbf{e}_3 := \mathbf{e}_2 - \mathbf{e}_1$ and $K := \|\mathbf{e}_1\|_{\mathbb{R}^2} + \|\mathbf{e}_2\|_{\mathbb{R}^2}$. In particular, we denote two unit vectors by $\widehat{\mathbf{e}}_1 := {}^t(1,0)$ and $\widehat{\mathbf{e}}_2 := {}^t(0,1)$. We define the triangular lattice $G = (V, E)$ by

$$V = \{x = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 |\ (x_1, x_2) \in \mathbb{Z}^2\},$$
$$E = \{(x, y) \in V \times V |\ x - y \in \{\pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3\}\}$$

(see Figure 1). We remark that G is isomorphic to the Cayley graph of the

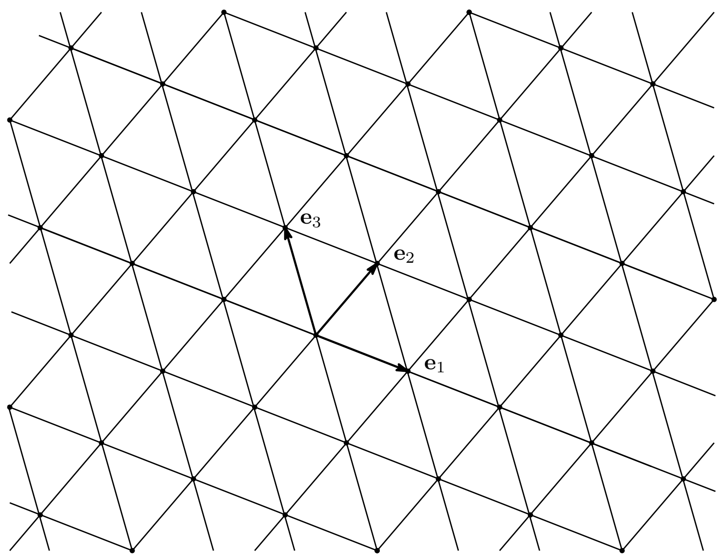

Figure 1. Triangular lattice

abelian group $\mathbb{Z}^2$ with a set of generators $S = \{\pm(1,0), \pm(0,1), \pm(-1,1)\}$ identified by $V \simeq \mathbb{Z}^2$ and $\mathbf{e}_1 \simeq (1,0)$, $\mathbf{e}_2 \simeq (0,1)$, $\mathbf{e}_3 \simeq (-1,1)$.

We consider a $\mathbb{Z}^2$-invariant random walk on the triangular lattice G satisfying

$$p\big((x, x + \mathbf{e}_1)\big) = \alpha, \quad p\big((x, x - \mathbf{e}_1)\big) = \alpha',$$

$$p\big((x,x+\mathbf{e}_2)\big)=\beta', \quad p\big(x,x-\mathbf{e}_2)\big)=\beta,$$
$$p\big((x,x+\mathbf{e}_3)\big)=\gamma, \quad p\big((x,x+\mathbf{e}_3)\big)=\gamma',$$

for every $x \in V$, where $\alpha,\alpha',\beta,\beta',\gamma,\gamma'$ are nonnegative constants with

$$\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1.$$

Throughout the present paper, we impose the following conditions on the 1-step transition probability p:

(P1): Zero mean condition:

$$\sum_{e\in E_0} p(e)e=0.$$

(P2):

$$\Gamma(p):=\widehat{\alpha}\widehat{\beta}+\widehat{\beta}\widehat{\gamma}+\widehat{\gamma}\widehat{\alpha}>0,$$

where $\widehat{\alpha}:=\alpha+\alpha', \widehat{\beta}:=\beta+\beta', \widehat{\gamma}:=\gamma+\gamma'$.

Remark 2.1. Condition **(P1)** is equivalent to the following condition:

- There exists a constant $0 \le \kappa \le \frac{1}{3}$ such that $\alpha-\alpha'=\beta-\beta'=\gamma-\gamma'=\kappa$.

In the case $\kappa=0$, condition **(P1)** implies that $p(e)=p(\overline{e})$ holds for every $e \in E$. Namely, our random walk is symmetric with $m_V \equiv 1$. On the other hand, by a simple calculation, we see that there does not exist the function m_V satisfying (1.1) unless $\kappa=0$. Hence our random walk is non-symmetric in the case $0<\kappa\le\frac{1}{3}$. We can regard the constant κ as intensity of the non-symmetry.

We set

$$M_q(\theta):=\sum_{e\in E_0} p(e)\langle e,\theta\rangle^q, \quad \theta=(\theta_1,\theta_2)\in\mathbb{R}^2, q\in\mathbb{N},$$

where $\langle\cdot,\cdot\rangle$ stands for the scalar product on $\mathbb{R}^2$. Condition **(P1)** implies $M_1(\theta)\equiv 0$. Besides, the explicit form of $M_q(\theta)$ is easily calculated as

$$M_q(\theta)=\begin{cases}\kappa\big(\langle\mathbf{e}_1,\theta\rangle^q-\langle\mathbf{e}_2,\theta\rangle^q+\langle\mathbf{e}_3,\theta\rangle^q\big) & \text{(if } q \text{ is odd)}\ ,\\ \widehat{\alpha}\langle\mathbf{e}_1,\theta\rangle^q+\widehat{\beta}\langle\mathbf{e}_2,\theta\rangle^q+\widehat{\gamma}\langle\mathbf{e}_3,\theta\rangle^q & \text{(if } q \text{ is even).}\end{cases}$$

Note that $M_q(\theta)\equiv 0$ for every odd number q if the random walk is symmetric.

We define the covariance matrix Q by

$$\langle Q\theta,\theta\rangle=M_2(\theta), \quad \theta\in\mathbb{R}^2.$$

In the case of $\mathbf{e}_1 = \widehat{\mathbf{e}}_1$ and $\mathbf{e}_2 = \widehat{\mathbf{e}}_2$, the corresponding covariance matrix is easily calculated as

$$\widehat{Q} := \begin{pmatrix} \widehat{\alpha} + \widehat{\gamma} & -\widehat{\gamma} \\ -\widehat{\gamma} & \widehat{\beta} + \widehat{\gamma} \end{pmatrix},$$

and hence we obtain $\det\widehat{Q} = \Gamma(p)$. For a general pair of two vectors $\mathbf{e}_1$, $\mathbf{e}_2$, we can decompose the covariance matrix Q as

$$Q = T\widehat{Q}\,{}^tT, \tag{2.1}$$

where $T = [\mathbf{e}_1, \mathbf{e}_2]$ stands for the matrix formed by column vectors $\mathbf{e}_1, \mathbf{e}_2$. It follows from condition **(P2)** and linear independence of $\mathbf{e}_1$, $\mathbf{e}_2$ that the covariance matrix Q is positive definite, i.e., $\langle \theta, Q\theta \rangle \geq \lambda \|\theta\|^2_{\mathbb{R}^2}, \theta \in \mathbb{R}^2$ for some $\lambda > 0$. By (2.1), we also observe that

$$\langle Q^{-1}x, y \rangle = \langle \widehat{Q}^{-1}(x_1\widehat{\mathbf{e}}_1 + x_2\widehat{\mathbf{e}}_2), y_1\widehat{\mathbf{e}}_1 + y_2\widehat{\mathbf{e}}_2 \rangle \tag{2.2}$$

holds for all $x = x_1\mathbf{e}_1 + x_2\mathbf{e}_2, y = y_1\mathbf{e}_1 + y_2\mathbf{e}_2 \in V$. This means that the left-hand side of (2.2) is independent of the realization of the triangular lattice G.

Let

$$A(G) := \frac{1}{3\Gamma(p)^{1/2}}, \quad l := \left(\frac{\widehat{\beta} + \widehat{\gamma}}{3\Gamma(p)} \right)^{1/2} \tag{2.3}$$

and we introduce three vectors by

$$\mathbf{h}_1 := {}^t(l, 0), \quad \mathbf{h}_2 := {}^t\Big(\frac{\widehat{\gamma}}{\widehat{\beta} + \widehat{\gamma}} l, \frac{\Gamma(p)^{1/2}}{\widehat{\beta} + \widehat{\gamma}} l \Big), \quad \mathbf{h}_3 := \mathbf{h}_2 - \mathbf{h}_1$$

(see Figure 2). Note that $A(G)$ is equal to the area of the parallelogram

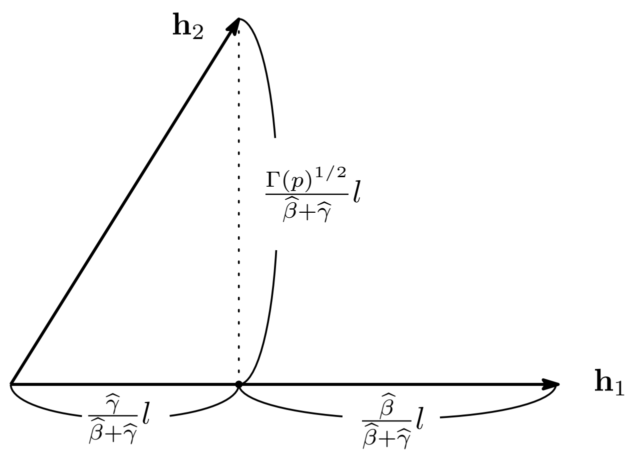

FIGURE 2. $\mathbf{h}_1$ and $\mathbf{h}_2$

spanned by $\mathbf{h}_1$ and $\mathbf{h}_2$. In the case of $\mathbf{e}_1 = \mathbf{h}_1$ and $\mathbf{e}_2 = \mathbf{h}_2$, the corresponding covariance matrix is

$$Q = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix}. \tag{2.4}$$

Recalling (2.2) and (2.4), we easily see $\langle Q^{-1}\mathbf{e}_i, \mathbf{e}_j\rangle = 3\langle \mathbf{h}_i, \mathbf{h}_j\rangle, i = 1,2,3$. Then a direct calculation of the inner product $\langle \mathbf{h}_i, \mathbf{h}_j\rangle, i, j = 1,2,3$ implies

$$\begin{aligned} &\langle Q^{-1}\mathbf{e}_1, \mathbf{e}_1\rangle = \tfrac{\widehat{\beta}+\widehat{\gamma}}{\Gamma(p)}, \quad \langle Q^{-1}\mathbf{e}_2, \mathbf{e}_2\rangle = \tfrac{\widehat{\gamma}+\widehat{\alpha}}{\Gamma(p)}, \quad \langle Q^{-1}\mathbf{e}_3, \mathbf{e}_3\rangle = \tfrac{\widehat{\alpha}+\widehat{\beta}}{\Gamma(p)}, \\ &\langle Q^{-1}\mathbf{e}_1, \mathbf{e}_2\rangle = \tfrac{\widehat{\gamma}}{\Gamma(p)}, \quad \langle Q^{-1}\mathbf{e}_2, \mathbf{e}_3\rangle = \tfrac{\widehat{\alpha}}{\Gamma(p)}, \quad \langle Q^{-1}\mathbf{e}_3, \mathbf{e}_1\rangle = \tfrac{-\widehat{\beta}}{\Gamma(p)}. \end{aligned} \tag{2.5}$$

Now, we are in a position to state the main result of the present paper.

Theorem 2.2. (1) *Let us consider the case $0 \le \kappa < 1/3$. Then we have*

$$\begin{aligned} 2\pi n \cdot p(n,x,y) &= 3A(G)\exp\Big(-\frac{1}{2n}\langle Q^{-1}(y-x), y-x\rangle\Big) \\ &\quad \times \Big(1 + \sum_{j=1}^{N} n^{-j/2} P_j\big(\frac{y-x}{\sqrt{n}}\big)\Big) + O_N\big(n^{-\frac{N+1}{2}}\big), \quad N \in \mathbb{N}\cup\{0\} \end{aligned} \tag{2.6}$$

as $n \to \infty$ uniformly for all $x, y \in V$, where $P_j = P_j(y)$, $j \in \mathbb{N}$ is a polynomial of degree at most $3j$ in the variables y_1, y_2 and an odd or even function depending on whether j is odd or even. Furthermore, let $a_1(y-x;\kappa)n^{-1}$ denote the leading term of $\sum_{j=1}^{N} n^{-j/2}P_j\big(\frac{y-x}{\sqrt{n}}\big), N \ge 2$, that is,

$$\lim_{n\to\infty} n\Big(\sum_{j=1}^{N} n^{-j/2}P_j\Big(\frac{y-x}{\sqrt{n}}\Big) - a_1(y-x;\kappa)n^{-1}\Big) = 0, \quad x, y \in V.$$

Then the coefficient $a_1(y;\kappa)$ is explicitly obtained by

$$a_1(y;\kappa) = a_1^{(0)} + \kappa a_1^{(1)}(y) + \kappa^2 a_1^{(2)}, \tag{2.7}$$

where

$$\begin{aligned} a_1^{(0)} &= -1 + \frac{1}{8\Gamma(p)^2}\Big\{\widehat{\alpha}(\widehat{\beta}+\widehat{\gamma})^2 + \widehat{\beta}(\widehat{\gamma}+\widehat{\alpha})^2 + \widehat{\gamma}(\widehat{\alpha}+\widehat{\beta})^2\Big\}, \\ a_1^{(1)}(y) &= \frac{1}{\Gamma(p)^2}\Big\{(\widehat{\alpha}\widehat{\beta} - 2\widehat{\beta}\widehat{\gamma} + \widehat{\gamma}\widehat{\alpha})y_1 + (-\widehat{\alpha}\widehat{\beta} - \widehat{\beta}\widehat{\gamma} + 2\widehat{\gamma}\widehat{\alpha})y_2\Big\}, \\ &\qquad\qquad \textit{for } y = y_1\mathbf{e}_1 + y_2\mathbf{e}_2, \\ a_1^{(2)} &= \frac{3}{8\Gamma(p)^2}\Big(-1 + \frac{5\widehat{\alpha}\widehat{\beta}\widehat{\gamma}}{\Gamma(p)}\Big). \end{aligned}$$

If, in addition, we realize the triangular lattice G along $\mathbf{h}_1$ and $\mathbf{h}_2$, then we have

$$2\pi n \cdot p(n,x,y) = 3A(G)\exp\Big(-\frac{3}{2n}\|y-x\|^2_{\mathbb{R}^2}\Big) \times \Big(1+\sum_{j=1}^{N} n^{-j/2}P_j\Big(\frac{y-x}{\sqrt{n}}\Big)\Big) + O_N\big(n^{-\frac{N+1}{2}}\big), \quad N \in \mathbb{N}\cup\{0\},$$

as $n \to \infty$ uniformly for all $x, y \in V$.
(2) *Let us consider the case $\kappa = 1/3$, i.e., $\alpha=\beta=\gamma=1/3$, $\alpha'=\beta'=\gamma'=0$. We set*

$$V_k := \big\{x = x_1\widetilde{\mathbf{e}}_1 + x_2\widetilde{\mathbf{e}}_2 \mid (x_1,x_2)\in\mathbb{Z}^2\big\} + k\mathbf{e}_1, \quad k=0,1,2,$$

where $\widetilde{\mathbf{e}}_1 = 2\mathbf{e}_1 - \mathbf{e}_2$, $\widetilde{\mathbf{e}}_2 = \mathbf{e}_1+\mathbf{e}_2$. For $k,l = 0,1,2$, we take $x \in V_k$ and $y \in V_l$. Then we have

$$2\pi n \cdot p(n,x,y) = 9A(G)\exp\Big(-\frac{1}{2n}\langle Q^{-1}(y-x), y-x\rangle\Big) \times \Big(1+\sum_{j=1}^{N} n^{-j/2}P_j\Big(\frac{y-x}{\sqrt{n}}\Big)\Big) + O_N\big(n^{-\frac{N+1}{2}}\big), \quad N \in \mathbb{N}\cup\{0\} \tag{2.8}$$

as $n = 3m + (l-k) \to \infty$ uniformly for all x and y. In this case, $A(G) = 1/\sqrt{3}$ and the coefficient of the leading term is

$$a_1(y-x;1/3) = -\frac{2}{3}.$$

If, in addition, we realize the triangular lattice G along $\mathbf{h}_1 = {}^t(\sqrt{2/3}, 0)$ and $\mathbf{h}_2 = {}^t(1/\sqrt{6}, 1/\sqrt{2})$ (see Figure 3), then we have

$$2\pi n \cdot p(n,x,y) = 9A(G)\exp\Big(-\frac{3}{2n}\|y-x\|^2_{\mathbb{R}^2}\Big) \times \Big(1+\sum_{j=1}^{N} n^{-j/2}P_j\Big(\frac{y-x}{\sqrt{n}}\Big)\Big) + O_N\big(n^{-\frac{N+1}{2}}\big), \quad N \in \mathbb{N}\cup\{0\}$$

as $n = 3m + (l-k) \to \infty$ uniformly for all x and y.

Now, we characterize the pair $\mathbf{h}_1$, $\mathbf{h}_2$ through a variational problem on the crystal lattices based on the idea of Kotani and Sunada [4, 5].

We set $\mathbf{e}_1 := {}^t(u,0), \mathbf{e}_2 := {}^t(v_1,v_2)$. Without loss of generality, we may assume $u, v_2 > 0$. Then the energy of the (quotient graph of the) triangular lattice G is given by

$$\mathcal{E}(G) := \frac{1}{2}\sum_{e\in E_0} p(e)\|t(e)-o(e)\|^2_{\mathbb{R}^2}$$

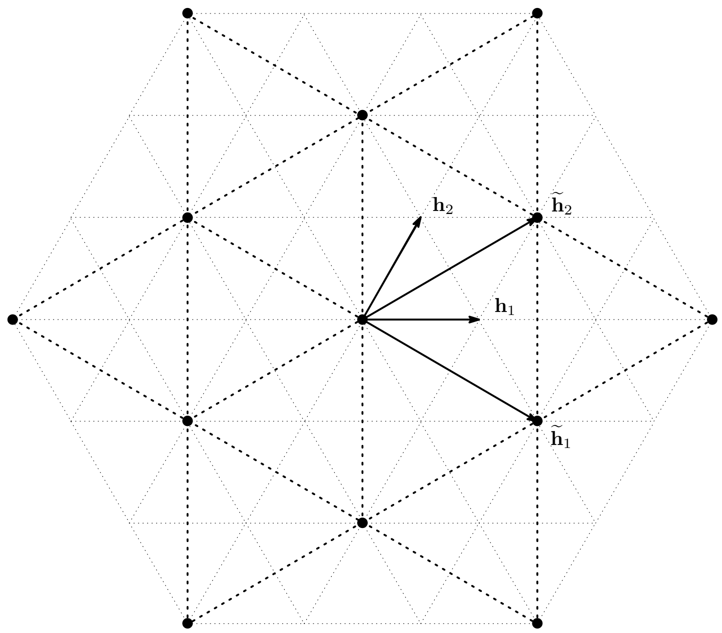

FIGURE 3. V_0 in the case of $\mathbf{e}_1 = \mathbf{h}_1$, $\mathbf{e}_2 = \mathbf{h}_2$ and $\kappa = 1/3$

$$= \frac{1}{2}\Big[\widehat{\alpha}u^2 + \widehat{\beta}(v_1^2 + v_2^2) + \widehat{\gamma}\big\{(v_1 - u)^2 + v_2^2\big\}\Big].$$

Minimizing $\mathcal{E}(G)$ with respect to (u, v_1, v_2) under the condition

$$\det(\langle \mathbf{e}_i, \mathbf{e}_j\rangle)_{i,j=1}^2 = A(G)^2,$$

we obtain

$$u = l, \quad v_1 = \frac{\widehat{\gamma}}{\widehat{\beta} + \widehat{\gamma}}l, \quad v_2 = \frac{\Gamma(p)^{1/2}}{\widehat{\beta} + \widehat{\gamma}}l.$$

Hence we have derived $\mathbf{e}_1 = \mathbf{h}_1$, $\mathbf{e}_2 = \mathbf{h}_2$.

Remark 2.3. In the case $\alpha = \alpha' = \beta = \beta' = \gamma = \gamma' = 1/6$, the random walk is said to be simple. In this case, we have

$$A(G) = \frac{1}{\sqrt{3}}, \quad l = \sqrt{\frac{2}{3}}, \quad \mathbf{h}_1 = {}^t\left(\sqrt{\frac{2}{3}}, 0\right), \quad \mathbf{h}_2 = {}^t\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right).$$

We note the volume of the Albanese torus is equal to $1/\sqrt{3}$ (see [4, page 640], [7, Section 10]). The standard realization of the triangular lattice is the equilateral triangular lattice in $\mathbb{R}^2$ each of whose edge has length $\sqrt{2/3}$. The quantity $A(G)$ introduced in (2.3) can be regarded as a generalization of the notion of the volume of the Albanese torus to some non-symmetric cases. Henceforth, we call the realization of the triangular lattice G along $\mathbf{h}_1$ and $\mathbf{h}_2$ the standard realization even if the random walk is non-symmetric.

Now, we take the standard realization of the triangular lattice G as above. As an application of Theorem 2.2, we obtain the following corollary which says that the transition semigroup generated by the random walk approximates the heat semigroup generated by the usual Brownian motion on $\mathbb{R}^2$. See e.g., [4, Theorem 8.5] for the detailed proof.

Corollary 2.4. *Let $t > 0$ and $\{\delta_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers satisfying $\lim_{n\to\infty} n\delta_n^2 = 3t$. Then for every continuous function f on $\mathbb{R}^2$ with compact support and for a sequence $\{x_n\}_{n=1}^{\infty}$ in V with $\lim_{n\to\infty} \delta_n x_n = x \in \mathbb{R}^2$, we have*

$$\lim_{n\to\infty} \sum_{y\in V} p(n, x_n, y) f(\delta_n y) = \frac{1}{2\pi t} \int_{\mathbb{R}^2} \exp\left(-\frac{\|x-z\|_{\mathbb{R}^2}^2}{2t}\right) f(z)dz.$$

Remark 2.5. Corollary 2.4 can be also obtained as an immediate consequence of the approximation theory due to Trotter [13] (see also Kotani [3]).

3. Preliminaries

In this section, we collect several basic facts which are useful in the proof of Theorem 2.2. In what follows, we denote $x_3 := x_2 - x_1$ and $(\frac{\partial}{\partial x_3}) := (\frac{\partial}{\partial x_2}) - (\frac{\partial}{\partial x_1})$ for convenience.

First, we recall an elementary fact about the Fourier transform

$$\int_{\mathbb{R}^2} e^{-\frac{1}{2}\langle Q\theta,\theta\rangle} e^{-\sqrt{-1}\langle x,\theta\rangle} d\theta = 2\pi(\det Q)^{-1/2} e^{-\frac{1}{2}\langle Q^{-1}x,x\rangle}, \quad x \in \mathbb{R}^2. \tag{3.1}$$

Differentiating both sides of (3.1) with respect to x_i, $i = 1, 2, 3$, we have

$$\begin{aligned}&\int_{\mathbb{R}^2} \langle \mathbf{e}_i, \theta\rangle e^{-\frac{1}{2}\langle Q\theta,\theta\rangle} e^{-\sqrt{-1}\langle x,\theta\rangle} d\theta \\ &= 2\pi\sqrt{-1}\cdot(\det Q)^{-1/2}\big(-\langle Q^{-1}\mathbf{e}_i, x\rangle\big) e^{-\frac{1}{2}\langle Q^{-1}x,x\rangle}, \quad x \in \mathbb{R}^2, i = 1, 2, 3.\end{aligned}$$

Repeating this argument several times, we obtain the following proposition:

Proposition 3.1. *For $N \in \mathbb{N}$ and $i_1, \dots, i_N = 1, 2, 3$, we define a function $F(i_1, \dots, i_N)$ by*

$$F(i_1, \dots, i_N)(x) := \int_{\mathbb{R}^2} \big(\prod_{k=1}^{N} \langle \mathbf{e}_{i_k}, \theta\rangle\big) e^{-\frac{1}{2}\langle Q\theta,\theta\rangle} e^{-\sqrt{-1}\langle x,\theta\rangle} d\theta, \quad x \in \mathbb{R}^2.$$

Then we have

$$F(i_1, \dots, i_N)(x) = 2\pi(\sqrt{-1})^N (\det Q)^{-1/2} e^{-\frac{1}{2}\langle Q^{-1}x,x\rangle} G(i_1, \dots, i_N)(x),$$

where $\{G(i_1, \dots, i_k)(x) : x \in \mathbb{R}^2, k = 1, \dots, N\}$ is determined as the solution of the recursive system of the following equations starting from $k = 1$ to

$k = N$:

$$
\begin{cases} G(i_1,\dots,i_k)(x) = -\langle Q^{-1}\mathbf{e}_{i_k}, x\rangle G(i_1,\dots,i_{k-1})(x) \\ \qquad\qquad\qquad\qquad\qquad\qquad + \dfrac{\partial}{\partial x_{i_k}} G(i_1,\dots,i_{k-1})(x), \\ G(i_1)(x) = -\langle Q^{-1}\mathbf{e}_{i_1}, x\rangle. \end{cases} \tag{3.2}
$$

Remark 3.2. $G(i_1,\dots,i_N)(x)$ is decomposed by

$$
G(i_1,\dots,i_N)(x) = \begin{cases} \displaystyle\sum_{l=0}^{\frac{N-1}{2}} G(i_1,\dots,i_N)_{(2l+1)}(x) & \text{(if } N \text{ is odd)}, \\ \displaystyle\sum_{l=0}^{\frac{N}{2}} G(i_1,\dots,i_N)_{(2l)}(x) & \text{(if } N \text{ is even)}\ , \end{cases}
$$

where $G(i_1,\dots,i_N)_{(k)}(x)$, $k = 0, 1, \dots, N$ is a homogeneous polynomial of degree k in the variables $\langle Q^{-1}\mathbf{e}_1, x\rangle$, $\langle Q^{-1}\mathbf{e}_2, x\rangle$ and $\langle Q^{-1}\mathbf{e}_3, x\rangle$. In particular, $G(i_1, i_2)(x)$ and $G(i_1, i_2, i_3)(x)$, $i_1, i_2, i_3 = 1, 2, 3$ are obtained by

$$
\begin{aligned} G(i_1,i_2)(x) &= \langle Q^{-1}\mathbf{e}_{i_1}, x\rangle\langle Q^{-1}\mathbf{e}_{i_2}, x\rangle - \langle Q^{-1}\mathbf{e}_{i_1}, \mathbf{e}_{i_2}\rangle, \\ G(i_1,i_2,i_3)(x) &= -\langle Q^{-1}\mathbf{e}_{i_1}, x\rangle\langle Q^{-1}\mathbf{e}_{i_2}, x\rangle\langle Q^{-1}\mathbf{e}_{i_3}, x\rangle \\ &\quad + \langle Q^{-1}\mathbf{e}_{i_1}, \mathbf{e}_{i_2}\rangle\langle Q^{-1}\mathbf{e}_{i_3}, x\rangle + \langle Q^{-1}\mathbf{e}_{i_2}, \mathbf{e}_{i_3}\rangle\langle Q^{-1}\mathbf{e}_{i_1}, x\rangle \\ &\quad + \langle Q^{-1}\mathbf{e}_{i_3}, \mathbf{e}_{i_1}\rangle\langle Q^{-1}\mathbf{e}_{i_2}, x\rangle. \end{aligned}
$$

Next, we define the characteristic function of the 1-step transition probability p by

$$
\varphi(\theta) := \sum_{e\in E_0} p(e)\exp\{\sqrt{-1}\langle e, \theta\rangle\}, \quad \theta = (\theta_1, \theta_2) \in \mathbb{R}^2. \tag{3.3}
$$

We denote φ especially by ϕ in the case of $\mathbf{e}_1 = \widehat{\mathbf{e}}_1$ and $\mathbf{e}_2 = \widehat{\mathbf{e}}_2$. Noting that the characteristic function of the n-step transition probability of the random walk starting at the origin is equal to φ^n, we easily have

$$
\begin{aligned} p(n,x,y) = \frac{|\mathrm{det}T|}{(2\pi)^2}\int_{({}^tT)^{-1}(D)} \varphi(\theta)^n \exp\{-\sqrt{-1}\langle y - x, \theta\rangle\} d\theta, \\ x, y \in V,\ n \in \mathbb{N}, \end{aligned} \tag{3.4}
$$

where $D := [-\pi, \pi]^2$. (See e.g., Lawler and Limic [8, Section 2.2.2] for details.)

For later use, we give

Lemma 3.3. *Let* $\varphi : ({}^tT)^{-1}(D) \to \mathbb{C}$ *be the characteristic function defined in* (3.3). *Then we have the following:*

(1) *In the case* $0 \le \kappa < 1/3$, $|\varphi(\theta)| = 1$ *holds only when* $\theta = 0$.
(2) *In the case* $\kappa = 1/3$, $|\varphi(\theta)| = 1$ *holds only when*

$$\theta = (0,0),\ ({}^tT)^{-1}\big((\frac{2\pi}{3}, -\frac{2\pi}{3})\big), ({}^tT)^{-1}\big((-\frac{2\pi}{3}, \frac{2\pi}{3})\big).$$

Proof. Since (2) is obvious, we only prove (1). By virtue of

$$\varphi(\theta) = \phi({}^tT\theta), \quad \theta \in \mathbb{R}^2,$$

it is sufficient to show that $|\phi(\theta)| = 1$ on D implies $\theta = 0$.

We calculate the characteristic function $\phi(\theta)$ as

$$\begin{aligned} \phi(\theta) &= \big(\widehat{\alpha}\cos(\theta_1) + \sqrt{-1}\kappa\sin(\theta_1)\big) + \big(\widehat{\beta}\cos(-\theta_2) + \sqrt{-1}\kappa\sin(-\theta_2)\big) \\ &\qquad + \big(\widehat{\gamma}\cos(\theta_2 - \theta_1) + \sqrt{-1}\kappa\sin(\theta_2 - \theta_1)\big) \\ &=: \phi_1(\theta) + \phi_2(\theta) + \phi_3(\theta), \quad \theta = (\theta_1, \theta_2) \in D. \end{aligned} \tag{3.5}$$

Note that the assumption $0 \le \kappa < \frac{1}{3}$ implies $\min\{\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}\} > \kappa \ge 0$. Then we have

$$|\phi_1(\theta)| \le \widehat{\alpha}, \quad |\phi_2(\theta)| \le \widehat{\beta}, \quad |\phi_3(\theta)| \le \widehat{\gamma}, \quad \theta = (\theta_1, \theta_2) \in D. \tag{3.6}$$

We also observe that $|\phi_1(\theta)| = \widehat{\alpha}$, $|\phi_2(\theta)| = \widehat{\beta}$ and $|\phi_3(\theta)| = \widehat{\gamma}$ imply $\phi_1(\theta) = \pm\widehat{\alpha}$, $\phi_2(\theta) = \pm\widehat{\beta}$ and $\phi_3(\theta) = \pm\widehat{\gamma}$, respectively.

Now, we suppose $|\phi(\theta)| = 1$ on D. By combining (3.5) and (3.6) with $\widehat{\alpha} + \widehat{\beta} + \widehat{\gamma} = 1$, we deduce

$$(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) = (\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}), (-\widehat{\alpha}, -\widehat{\beta}, -\widehat{\gamma}), \quad \theta \in D.$$

We easily see that the first equation $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) = (\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})$ has the solution $(\theta_1, \theta_2) = (0,0)$. On the other hand, there exists no solution of the second equation $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) = (-\widehat{\alpha}, -\widehat{\beta}, -\widehat{\gamma})$. Hence we conclude $\theta = 0$. □

Remark 3.4. In the case $0 \le \kappa < 1/3$, by (1) of Lemma 3.3, the random walk is aperiodic. Namely, the period of the random walk $d(p) := \gcd\{n \in \mathbb{N} : p(n, 0, 0) > 0\}$ is equal to 1. See e.g., Spitzer [10, P8 in Section 7] for the proof. On the other hand, (2) of Lemma 3.3 means that the random walk is periodic with $d(p) = 3$ in the case $\kappa = 1/3$.

Before closing this section, we present an asymptotic expansion formula of the characteristic function φ which plays a crucial role in the next section. We set

$$\chi_q(\theta) := (\sqrt{-1})^{-q}\big(\frac{d}{dt}\big)^q\Big|_{t=0} \log\varphi(t\theta), \quad \theta \in \mathbb{R}^2, q \in \mathbb{N}. \tag{3.7}$$

(See Bhattacharya and Ranga Rao [1, page 47].) We note that $\chi_q(\theta)$ is a polynomial in the variables $M_2(\theta), \dots, M_q(\theta)$. Noting condition **(P1)**, we have

$$\chi_1(\theta) \equiv 0, \quad \chi_2(\theta) = M_2(\theta), \quad \chi_3(\theta) = M_3(\theta), \quad \chi_4(\theta) = M_4(\theta) - 3M_2(\theta)^2.$$

The following proposition is taken from [1, Lemma 7.1 and Theorem 9.11].

Proposition 3.5. *Let $n \in \mathbb{N}$ and $N \in \mathbb{N} \cup \{0\}$. Then there exist positive constants $C_1(N), C_2(N)$ such that for all $\theta \in \mathbb{R}^2$ with*

$$\langle Q\theta, \theta\rangle^{1/2} \le C_1(N)\Big(\sum_{e \in E_0} p(e)\langle Q^{-1}e, e\rangle^{N/2}\Big)^{-\frac{1}{N+3}} \cdot n^{\frac{N+1}{2(N+3)}},$$

we have

$$\Big|\varphi\Big(\frac{\theta}{\sqrt{n}}\Big)^n - \exp\Big(-\frac{1}{2}\langle Q\theta, \theta\rangle\Big) \cdot \Big(b_0(\theta) + b_1(\theta)n^{-1/2} + \cdots + b_N(\theta)n^{-N/2}\Big)\Big|$$
$$\le C_2(N)\exp\Big(-\frac{1}{4}\langle Q\theta, \theta\rangle\Big)\Big(\langle Q\theta, \theta\rangle^{N+3} + \langle Q\theta, \theta\rangle^{3(N+1)}\Big) \cdot n^{-\frac{N+1}{2}}.$$

Here $b_0(\theta) \equiv 1$ and $b_j(\theta)$, $j = 1, \dots, N$ is written as

$$b_j(\theta) = (\sqrt{-1})^{3j} b_j^{(3j)}(\theta) + (\sqrt{-1})^{3j-2} b_j^{(3j-2)}(\theta) + \cdots + (\sqrt{-1})^{j+2} b_j^{(j+2)}(\theta), \tag{3.8}$$

where $b_j^{(k)}(\theta)$, $k = j+2, j+4, \dots, 3j$ is a polynomial in the variables $\chi_3(\theta), \dots, \chi_k(\theta)$ and it can be regarded as a homogeneous polynomial of degree k in the variables $\langle \mathbf{e}_1, \theta\rangle, \langle \mathbf{e}_2, \theta\rangle$ and $\langle \mathbf{e}_3, \theta\rangle$. In particular,

$$b_1(\theta) = (\sqrt{-1})^3\Big(\frac{\chi_3(\theta)}{6}\Big), \quad b_2(\theta) = (\sqrt{-1})^6\Big(\frac{\chi_3(\theta)^2}{72}\Big) + (\sqrt{-1})^4\Big(\frac{\chi_4(\theta)}{24}\Big).$$

4. Proof of Theorem 2.2

In this section, we prove Theorem 2.2 based on the standard Laplace method argument as in [8, 10]. For the reader's convenience, we nevertheless give a detailed proof. Without loss of generality, we may suppose $x = 0$ throughout the proof.

First, we recall that the covariance matrix Q is positive definite and Lemma 3.3. Then we can choose a positive constant R sufficiently small such that both

$$\eta := \sup\{|\varphi(\theta)| : \theta \in ({}^tT)^{-1}(D), \|\theta\|_{\mathbb{R}^2} \ge R\} < 1, \tag{4.1}$$

and

$$|\varphi(\theta)| \le \exp\Big(-\frac{1}{4}\langle Q\theta, \theta\rangle\Big), \quad \|\theta\|_{\mathbb{R}^2} < R \tag{4.2}$$

hold. See e.g., [10, P7 in Section 7] and also Shiga [9, Lemma 6.15] for the proof of (4.2).

Recalling (2.1), (2.3), (3.4) and performing the change of variables $\sqrt{n}\theta' = \theta$, we have

$$
\begin{aligned}
&2\pi n \cdot p(n,0,y) \\
&= \frac{|\mathrm{det}T|n}{2\pi}\int_{({}^tT)^{-1}(D)} \varphi(\theta')^n e^{-\sqrt{-1}\langle y,\theta'\rangle} d\theta' \\
&= \frac{3A(G)}{2\pi}(\mathrm{det}Q)^{1/2}\int_{\sqrt{n}({}^tT)^{-1}(D)} \varphi\Big(\frac{\theta}{\sqrt{n}}\Big)^n e^{-\sqrt{-1}\langle y,\frac{\theta}{\sqrt{n}}\rangle} d\theta.
\end{aligned} \tag{4.3}
$$

We take a positive constant r sufficiently small such that

$$
r < \min\{C_1(N)K^{-\frac{2N+3}{N+3}}\lambda^{\frac{N}{2(N+3)}}, R\}, \tag{4.4}
$$

and divide the range of the above integration $\sqrt{n}({}^tT)^{-1}(D)$ into three parts according as $\|\theta\|_{\mathbb{R}^2} \le rn^{1/6}$; $rn^{1/6} < \|\theta\|_{\mathbb{R}^2} \le R\sqrt{n}$; $\|\theta\|_{\mathbb{R}^2} > R\sqrt{n}$. Then we can write as

$$
\begin{aligned}
&\int_{\sqrt{n}({}^tT)^{-1}(D)} \varphi\Big(\frac{\theta}{\sqrt{n}}\Big)^n e^{-\sqrt{-1}\langle y,\frac{\theta}{\sqrt{n}}\rangle} d\theta \\
&\quad = I(n)(y) + J_1(n)(y) + J_2(n)(y) + J_3(n)(y) + J_4(n)(y),
\end{aligned}
$$

where

$$
\begin{aligned}
I(n)(y) &= \sum_{j=0}^{N} I_j(n)(y) \\
&:= \sum_{j=0}^{N}\Big\{n^{-j/2}\int_{\mathbb{R}^2} b_j(\theta)e^{-\frac{1}{2}\langle Q\theta,\theta\rangle}e^{-\sqrt{-1}\langle y,\frac{\theta}{\sqrt{n}}\rangle}d\theta\Big\}, \\
J_1(n)(y) &:= \int_{\|\theta\|_{\mathbb{R}^2}\le rn^{1/6}} \Big\{\varphi\Big(\frac{\theta}{\sqrt{n}}\Big)^n - e^{-\frac{1}{2}\langle Q\theta,\theta\rangle}\Big(\sum_{j=0}^{N} b_j(\theta)n^{-j/2}\Big)\Big\} \\
&\qquad \times e^{-\sqrt{-1}\langle y,\frac{\theta}{\sqrt{n}}\rangle}d\theta, \\
J_2(n)(y) &:= -\int_{\|\theta\|_{\mathbb{R}^2}>rn^{1/6}} e^{-\frac{1}{2}\langle Q\theta,\theta\rangle}\Big(\sum_{j=0}^{N} b_j(\theta)n^{-j/2}\Big)e^{-\sqrt{-1}\langle y,\frac{\theta}{\sqrt{n}}\rangle}d\theta, \\
J_3(n)(y) &:= \int_{rn^{1/6}<\|\theta\|_{\mathbb{R}^2}\le R\sqrt{n}} \varphi\Big(\frac{\theta}{\sqrt{n}}\Big)^n e^{-\sqrt{-1}\langle y,\frac{\theta}{\sqrt{n}}\rangle}d\theta, \\
J_4(n)(y) &:= \int_{\|\theta\|_{\mathbb{R}^2}>R\sqrt{n},\ \theta\in\sqrt{n}({}^tT)^{-1}(D)} \varphi\Big(\frac{\theta}{\sqrt{n}}\Big)^n e^{-\sqrt{-1}\langle y,\frac{\theta}{\sqrt{n}}\rangle}d\theta.
\end{aligned}
$$

Our first task is to estimate the error terms $J_1(n)(y), J_2(n)(y), J_3(n)(y)$ and $J_4(n)(y)$. By using (4.1) and (4.2), we have

$$\sup_{y\in V}|J_4(n)(y)| \leq \int_{\sqrt{n}({}^tT)^{-1}(D)} \eta^n d\theta = (2\pi)^2|\mathrm{det}T|\cdot n\eta^n,$$

and

$$\sup_{y\in V}|J_3(n)(y)| \leq \int_{\|\theta\|_{\mathbb{R}^2}>rn^{1/6}} e^{-\frac{1}{4}\langle\theta,Q\theta\rangle}d\theta \leq \int_{\|\theta\|_{\mathbb{R}^2}>rn^{1/6}} e^{-\frac{\lambda}{4}\|\theta\|^2_{\mathbb{R}^2}}d\theta$$

Thus both $J_3(n)(y)$ and $J_4(n)(y)$ converge to zero as $n\to\infty$ exponentially fast uniformly for all $y\in V$. Similarly, we have

$$\sup_{y\in V}|J_2(n)(y)| \leq e^{-\frac{\lambda}{4}n^{1/4}}\sum_{j=0}^{N} n^{-j/2}\Big(\int_{\mathbb{R}^2} e^{-\frac{\lambda}{4}\|\theta\|^2_{\mathbb{R}^2}}|b_j(\theta)|d\theta\Big),$$

and since each $b_j(\theta)$ has polynomial growth, we also see that $J_2(n)(y)$ converges to zero as $n\to\infty$ exponentially fast uniformly for all $y\in V$.

By virtue of $|M_q(\theta)|\leq K^q\|\theta\|^q_{\mathbb{R}^2}, q\in\mathbb{N}$, we have

$$\sum_{e\in E_0} p(e)\langle Q^{-1}e,e\rangle^{N/2} \leq K^N\lambda^{-N/2}, \quad N\in\mathbb{N}\cup\{0\}.$$

Furthermore recalling (4.4), we observe that $\|\theta\|_{\mathbb{R}^2}\leq rn^{1/6}$ implies

$$\begin{aligned}\langle Q\theta,\theta\rangle^{1/2} &\leq K\|\theta\|_{\mathbb{R}^2}\\ &\leq C_1(N)K^{-\frac{N}{N+3}}\lambda^{\frac{N}{2(N+3)}}n^{\frac{1}{6}}\\ &\leq C_1(N)\Big(\sum_{e\in E_0}p(e)\langle Q^{-1}e,e\rangle^{N/2}\Big)^{-\frac{1}{N+3}}n^{\frac{N+1}{2(N+3)}}.\end{aligned}$$

Thus we may apply Proposition 3.5, and we obtain

$$\begin{aligned}&\sup_{y\in V}|J_1(n)(y)|\\ &\leq C_2(N)n^{-\frac{N+1}{2}}\int_{\|\theta\|_{\mathbb{R}^2}\leq rn^{1/6}} e^{-\frac{1}{4}\langle Q\theta,\theta\rangle}\big(\langle Q\theta,\theta\rangle^{N+3}+\langle Q\theta,\theta\rangle^{3(N+1)}\big)d\theta\\ &\leq O_N(n^{-\frac{N+1}{2}}).\end{aligned}$$

Now, we calculate the principal terms

$$I_j(n)(y) := n^{-j/2}\int_{\mathbb{R}^2} b_j(\theta)e^{-\frac{1}{2}\langle Q\theta,\theta\rangle}e^{-\sqrt{-1}\langle y,\frac{\theta}{\sqrt{n}}\rangle}d\theta, \quad y\in V,\ j=0,1,\dots,N. \tag{4.5}$$

It follows directly from (3.1) that

$$I_0(n)(y) = 2\pi(\mathrm{det}Q)^{-1/2}\exp\big(-\frac{1}{2n}\langle Q^{-1}y,y\rangle\big).$$

Applying Proposition 3.1, we obtain

$$\begin{aligned} I_1(n)(y) &= (\sqrt{-1})^3 n^{-1/2}\int_{\mathbb{R}^2}\Big(\frac{M_3(\theta)}{6}\Big)e^{-\frac{1}{2}\langle Q\theta,\theta\rangle}e^{-\sqrt{-1}\langle y,\frac{\theta}{\sqrt{n}}\rangle}d\theta \\ &= 2\pi n^{-1/2}(\mathrm{det}Q)^{-1/2}\exp\big(-\frac{1}{2n}\langle Q^{-1}y,y\rangle\big)P_1\big(\frac{y}{\sqrt{n}}\big), \end{aligned}$$

where

$$P_1(y) := \frac{\kappa(\sqrt{-1})^6}{6}\big(G(1,1,1)(y) - G(2,2,2)(y) + G(3,3,3)(y)\big).$$

Besides, it follows from Remark 3.2 that

$$\begin{aligned} G(i,i,i)\big(\frac{y}{\sqrt{n}}\big) &= -n^{-3/2}\langle Q^{-1}\mathbf{e}_i,y\rangle^3 + 3n^{-1/2}\langle Q^{-1}\mathbf{e}_i,\mathbf{e}_i\rangle\langle Q^{-1}\mathbf{e}_i,y\rangle, \\ & \qquad i=1,2,3. \end{aligned}$$

Thus the explicit form of $P_1\big(\frac{y}{\sqrt{n}}\big)$ is given by

$$\begin{aligned} P_1\big(\frac{y}{\sqrt{n}}\big) &= \frac{\kappa}{6}n^{-3/2}\Big\{\langle Q^{-1}\mathbf{e}_1,y\rangle^3 - \langle Q^{-1}\mathbf{e}_2,y\rangle^3 + \langle Q^{-1}\mathbf{e}_3,y\rangle^3\Big\} \\ &\quad + \frac{\kappa}{2}n^{-1/2}\Big\{\langle Q^{-1}\mathbf{e}_1,\mathbf{e}_2\rangle\langle Q^{-1}\mathbf{e}_3,y\rangle + \langle Q^{-1}\mathbf{e}_2,\mathbf{e}_3\rangle\langle Q^{-1}\mathbf{e}_1,y\rangle \\ &\quad + \langle Q^{-1}\mathbf{e}_3,\mathbf{e}_1\rangle\langle Q^{-1}\mathbf{e}_2,y\rangle\Big\}. \end{aligned} \tag{4.6}$$

It follows from Proposition 3.5 that

$$\begin{aligned} I_2(n)(y) &= n^{-2/2}\int_{\mathbb{R}^2}(\sqrt{-1})^4\Big(\frac{M_4(\theta)}{24} - \frac{M_2(\theta)^2}{8}\Big)e^{-\frac{1}{2}\langle Q\theta,\theta\rangle}e^{-\sqrt{-1}\langle y,\frac{\theta}{\sqrt{n}}\rangle}d\theta \\ &\quad + n^{-2/2}\int_{\mathbb{R}^2}(\sqrt{-1})^6\Big(\frac{M_3(\theta)^2}{72}\Big)e^{-\frac{1}{2}\langle Q\theta,\theta\rangle}e^{-\sqrt{-1}\langle y,\frac{\theta}{\sqrt{n}}\rangle}d\theta. \end{aligned}$$

Then by repeating the same argument as above, we have

$$\begin{aligned} I_2(n)(y) &= 2\pi n^{-2/2}(\mathrm{det}Q)^{-1/2}\exp\big(-\frac{1}{2n}\langle Q^{-1}y,y\rangle\big) \\ &\quad \times\Big(P_2^{(1)}(n)\big(\frac{y}{\sqrt{n}}\big) + P_2^{(2)}(n)\big(\frac{y}{\sqrt{n}}\big)\Big), \end{aligned}$$

where

$$\begin{aligned} P_2^{(1)}(n)(y) &= \frac{(\sqrt{-1})^8}{24}\big\{\widehat{\alpha}G(1,1,1,1)(y) + \widehat{\beta}G(2,2,2,2)(y) \\ &\quad + \widehat{\gamma}G(3,3,3,3)(y)\big\} \end{aligned}$$

$$
\begin{aligned}
& -\frac{(\sqrt{-1})^8}{8}\big\{\widehat{\alpha}^2 G(1,1,1,1)(y) + \widehat{\beta}^2 G(2,2,2,2)(y) \\
& +\widehat{\gamma}^2 G(3,3,3,3)(y) + 2\widehat{\alpha}\widehat{\beta} G(1,1,2,2)(y) \\
& +2\widehat{\beta}\widehat{\gamma} G(2,2,3,3)(y) + 2\widehat{\gamma}\widehat{\alpha} G(1,1,3,3)(y)\big\},
\end{aligned}
$$

$$
\begin{aligned}
P_2^{(2)}(n)(y) &= \frac{\kappa^2(\sqrt{-1})^{12}}{72}\Big\{\sum_{i=1}^{3} G(i,i,i,i,i,i)(y) - 2G(1,1,1,2,2,2)(y) \\
&\quad +2G(1,1,1,3,3,3)(y) - 2G(2,2,2,3,3,3)(y)\Big\}.
\end{aligned}
$$

Besides, it follows from (3.2) that

$$
\begin{aligned}
G(i,i,i,j,j,j)_{(0)}(y) &= -9\langle Q^{-1}\mathbf{e}_i,\mathbf{e}_i\rangle\langle Q^{-1}\mathbf{e}_i,\mathbf{e}_j\rangle\langle Q^{-1}\mathbf{e}_j,\mathbf{e}_j\rangle \\
&\quad -6\langle Q^{-1}\mathbf{e}_i,\mathbf{e}_j\rangle^3, \\
G(i,i,j,j)_{(0)}(y) &= \langle Q^{-1}\mathbf{e}_i,\mathbf{e}_i\rangle\langle Q^{-1}\mathbf{e}_j,\mathbf{e}_j\rangle + 2\langle Q^{-1}\mathbf{e}_i,\mathbf{e}_j\rangle^2
\end{aligned}
$$

for $i,j = 1,2,3$.

Combining these identities with (2.5), we have

$$
\begin{aligned}
& 3\big(\widehat{\alpha}^2\langle Q^{-1}\mathbf{e}_1,\mathbf{e}_1\rangle^2 + \widehat{\beta}^2\langle Q^{-1}\mathbf{e}_2,\mathbf{e}_2\rangle^2 + \widehat{\gamma}^2\langle Q^{-1}\mathbf{e}_3,\mathbf{e}_3\rangle^2\big) \\
& \quad + 2\widehat{\alpha}\widehat{\beta}\big(\langle Q^{-1}\mathbf{e}_1,\mathbf{e}_1\rangle\langle Q^{-1}\mathbf{e}_2,\mathbf{e}_2\rangle + 2\langle Q^{-1}\mathbf{e}_1,\mathbf{e}_2\rangle^2\big) \\
& \quad + 2\widehat{\gamma}\widehat{\alpha}\big(\langle Q^{-1}\mathbf{e}_1,\mathbf{e}_1\rangle\langle Q^{-1}\mathbf{e}_3,\mathbf{e}_3\rangle + 2\langle Q^{-1}\mathbf{e}_1,\mathbf{e}_3\rangle^2\big) \\
& \quad + 2\widehat{\beta}\widehat{\gamma}\big(\langle Q^{-1}\mathbf{e}_2,\mathbf{e}_2\rangle\langle Q^{-1}\mathbf{e}_3,\mathbf{e}_3\rangle + 2\langle Q^{-1}\mathbf{e}_2,\mathbf{e}_3\rangle^2\big) = 8.
\end{aligned} \tag{4.7}
$$

Thus the constant term of $P_2^{(1)}$ and $P_2^{(2)}$ are obtained by

$$
-1 + \frac{1}{8}\Big(\widehat{\alpha}\langle Q^{-1}\mathbf{e}_1,\mathbf{e}_1\rangle^2 + \widehat{\beta}\langle Q^{-1}\mathbf{e}_2,\mathbf{e}_2\rangle^2 + \widehat{\gamma}\langle Q^{-1}\mathbf{e}_3,\mathbf{e}_3\rangle^2\Big) \tag{4.8}
$$

and

$$
\begin{aligned}
& -\frac{5}{24}\Big(\langle Q^{-1}\mathbf{e}_1,\mathbf{e}_1\rangle^3 + \langle Q^{-1}\mathbf{e}_2,\mathbf{e}_2\rangle^3 + \langle Q^{-1}\mathbf{e}_3,\mathbf{e}_3\rangle^3\Big) \\
& + \frac{1}{6}\Big(\langle Q^{-1}\mathbf{e}_1,\mathbf{e}_2\rangle^3 + \langle Q^{-1}\mathbf{e}_2,\mathbf{e}_3\rangle^3 - \langle Q^{-1}\mathbf{e}_3,\mathbf{e}_1\rangle^3\Big) \\
& + \frac{1}{4}\Big(\langle Q^{-1}\mathbf{e}_1,\mathbf{e}_1\rangle\langle Q^{-1}\mathbf{e}_1,\mathbf{e}_2\rangle\langle Q^{-1}\mathbf{e}_2,\mathbf{e}_2\rangle \\
& \qquad + \langle Q^{-1}\mathbf{e}_2,\mathbf{e}_2\rangle\langle Q^{-1}\mathbf{e}_2,\mathbf{e}_3\rangle\langle Q^{-1}\mathbf{e}_3,\mathbf{e}_3\rangle \\
& \qquad - \langle Q^{-1}\mathbf{e}_3,\mathbf{e}_3\rangle\langle Q^{-1}\mathbf{e}_3,\mathbf{e}_1\rangle\langle Q^{-1}\mathbf{e}_1,\mathbf{e}_1\rangle\Big),
\end{aligned} \tag{4.9}
$$

respectively. Summarizing (4.6), (4.7), (4.8), (4.9), and recalling (2.5) again, we conclude that the explicit form of the coefficient of the leading term $a_1(y;\kappa)n^{-1}$ is given by (2.7).

For general $j = 1, 2, \ldots, N$, by virtue of (3.8), we may write $I_j(n)(y)$ as

$$\begin{aligned} I_j(n)(y) &= \sum_{k=1}^{j} I_j^{(k)}(n)(y) \\ &:= \sum_{k=1}^{j} \Big\{ n^{-j/2}(\sqrt{-1})^{j+2k} \int_{\mathbb{R}^2} b_j^{(j+2k)}(\theta) e^{-\frac{1}{2}\langle Q\theta,\theta\rangle} e^{-\sqrt{-1}\langle y, \frac{\theta}{\sqrt{n}}\rangle} d\theta \Big\}. \end{aligned}$$

Applying Proposition 3.1 again, we obtain

$$\begin{aligned} I_j^{(k)}(n)(y) = 2\pi(\sqrt{-1})^{j+2k} n^{-j/2} (\det Q)^{-1/2} \exp\big(-\frac{1}{2n}\langle Q^{-1}y, y\rangle\big) \\ \times P_j^{(j+2k)}\big(\frac{y}{\sqrt{n}}\big), \end{aligned} \tag{4.10}$$

where

$$P_j^{(j+2k)}(y) := \begin{cases} \displaystyle\sum_{l=0}^{\frac{j-1}{2}+k} (\sqrt{-1})^{j+2k-2l} P_{j,j+2k-2l}^{(j+2k)}(y) & (\text{if } j \text{ is odd}), \\ \displaystyle\sum_{l=0}^{\frac{j}{2}+k} (\sqrt{-1})^{j+2k-2l} P_{j,j+2k-2l}^{(j+2k)}(y) & (\text{if } j \text{ is even}) \end{cases},$$

and each $P_{j,j+2k-2l}^{(j+2k)}(y)$ is a homogeneous polynomial of degree $(j+2k-2l)$ in the variables y_1, y_2. Noting $(j+2k-2l) \le 3j$ and $(\sqrt{-1})^{j+2k}(\sqrt{-1})^{j+2k-2l} \in \mathbb{R}$ for any j, k, l, we see that

$$P_j(y) := \sum_{k=1}^{j} (\sqrt{-1})^{j+2k} P_j^{(j+2k)}(y), \quad j = 1, 2, \ldots, N \tag{4.11}$$

is a real valued polynomial of degree at most $3j$ in the variables y_1, y_2 and it is an odd or even function depending on whether j is odd or even.

Then by (4.10) and (4.11), we obtain

$$I_j(n)(y) = 2\pi n^{-j/2} (\det Q)^{-1/2} \exp\big(-\frac{1}{2n}\langle Q^{-1}y, y\rangle\big) P_j\big(\frac{y}{\sqrt{n}}\big). \tag{4.12}$$

Here we mention that the term $n^{-j/2} P_j\big(\frac{y}{\sqrt{n}}\big)$ on the right-hand side of (4.12) is regarded as a polynomial of degree at most $2j$ in the variable n^{-1}.

Plugging the above all arguments into (4.3), and recalling (2.5), we finally obtain the desired asymptotic expansion formula (2.6). This completes the proof of (1).

Next, we prove (2). For simplicity, we only give a proof in the case $k = l = 0$. (In other cases, the proof goes through in a very similar way with a slight modification.) Let $p_{(3)} := p * p * p$ and we denote by $\varphi_{(3)}$

and $Q_{(3)}$ the characteristic function and the covariance matrix associated with the 1-step probability distribution $p_{(3)}$, respectively. We define $\chi_q^{(3)}(\theta)$, $q \in \mathbb{N}$ in the same way as in (3.7) with φ replaced by $\varphi_{(3)}$. Note $Q_{(3)} = 3Q$ and $\chi_q^{(3)}(\theta) = 3\chi_q(\theta)$, $q \in \mathbb{N}$.

Let $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ be an enlarged triangular lattice defined by $\widetilde{V} := V_0$ and $\widetilde{E} := \{(x, y) \in V_0 \times V_0 |\ x - y \in \{\pm\widetilde{\mathbf{e}}_1, \pm\widetilde{\mathbf{e}}_2, \pm(\widetilde{\mathbf{e}}_2 - \widetilde{\mathbf{e}}_1)\}\}$ (see Figure 3). We introduce a random walk on $\widetilde{G}$ whose 1-step transition probability distribution is given by $p_{(3)}$. We denote by $p_{(3)}(m, x, y)$ the m-step transition probability of the random walk. Namely, $p_{(3)}(m, x, y) = p(3m, x, y)$ for $x, y \in \widetilde{V}$.

Since $p_{(3)}(1, 0, 0) = 2/9 > 0$, the random walk on $\widetilde{G}$ is aperiodic. Thus we may follow the proof of (1) above, and we finally obtain the desired asymptotic expansion formula (2.8). This completes the proof of (2). □

5. Examples

Example 5.1 (Kotani-Sunada [4], Example 2)**.** We consider the simple random walk. Namely, we consider the case $\alpha = \alpha' = \beta = \beta' = \gamma = \gamma' = 1/6$. As we mentioned in Remark 2.3, the standard realization of the triangular lattice is the equilateral triangular lattice in $\mathbb{R}^2$ each of whose edge has length $\sqrt{2/3}$. Then we have

$$2\pi n \cdot p(n, x, y) = \sqrt{3} \exp\Big(- \frac{3}{2n} \|y - x\|^2_{\mathbb{R}^2}\Big) \times \Big(1 - \frac{1}{2} n^{-1} + \cdots + \frac{1}{n^{N/2}} P_N\Big(\frac{y - x}{\sqrt{n}}\Big)\Big) + O_N\big(n^{-\frac{N+1}{2}}\big), \quad N \geq 3$$

as $n \to \infty$ uniformly for all $x = x_1\mathbf{h}_1 + x_2\mathbf{h}_2, y = y_1\mathbf{h}_1 + y_2\mathbf{h}_2 \in V$.

Example 5.2 ([12])**.** We consider the case

$$\alpha = \beta = \gamma = (1/6) + \varepsilon,\ \alpha' = \beta' = \gamma' = (1/6) - \varepsilon \quad (0 < \varepsilon < 1/6).$$

In this case, the standard realization of the triangular lattice is same as Example 5.1. However, the corresponding random walk is non-symmetric with $\kappa = 2\varepsilon$. Then we have

$$\begin{aligned} 2\pi n \cdot p(n, x, y) \quad &= \quad \sqrt{3} \exp\Big(- \frac{3}{2n} \|y - x\|^2_{\mathbb{R}^2}\Big) \cdot \Big\{1 + \Big(- \frac{1}{2} - 6\varepsilon^2\Big) n^{-1} \\ &\quad + \cdots + \frac{1}{n^{N/2}} P_N\Big(\frac{y - x}{\sqrt{n}}\Big)\Big\} + O_N\big(n^{-\frac{N+1}{2}}\big), \quad N \geq 3 \end{aligned}$$

as $n \to \infty$ uniformly for all $x = x_1\mathbf{h}_1 + x_2\mathbf{h}_2, y = y_1\mathbf{h}_1 + y_2\mathbf{h}_2 \in V$.

Example 5.3. We consider the case

$$\alpha = 1/4, \alpha' = 1/12,\ \beta = 1/3, \beta' = 1/6,\ \gamma = 1/6, \gamma' = 0 \quad (\kappa = 1/6).$$

In this case,

$$A(G) = \frac{2}{\sqrt{11}},\ l = \frac{2\sqrt{2}}{\sqrt{11}},\ \mathbf{h}_1 = {}^t\left(\frac{2\sqrt{2}}{\sqrt{11}}, 0\right),\ \mathbf{h}_2 = {}^t\left(\frac{1}{\sqrt{22}}, \frac{1}{\sqrt{2}}\right)$$

(see Figure 4). Then we have

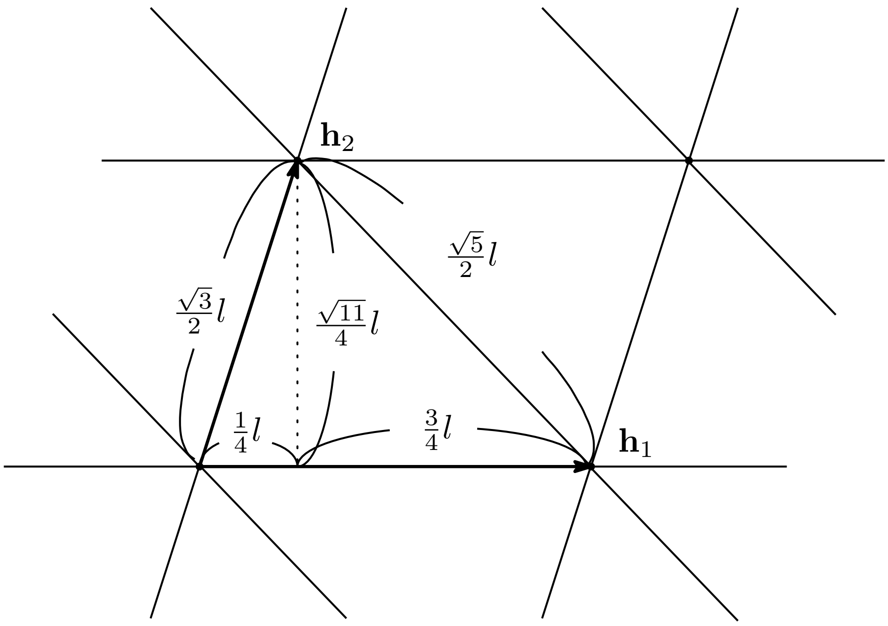

Figure 4. Example 5.3

$$\begin{aligned} 2\pi n \cdot p(n,x,y) \quad = \quad & \frac{6}{\sqrt{11}} \exp\Big(-\frac{3}{2n}\|y-x\|^2_{\mathbb{R}^2}\Big) \\ & \times\Big[1 + \Big\{-\frac{719}{1331} + \frac{6}{121}\Big(2(y_1-x_1) - 5(y_2-x_2)\Big)\Big\}n^{-1} \\ & \quad + \cdots + \frac{1}{n^{N/2}} P_N\Big(\frac{y-x}{\sqrt{n}}\Big)\Big] + O_N\big(n^{-\frac{N+1}{2}}\big), \quad N \geq 3 \end{aligned}$$

as $n \to \infty$ uniformly for all $x = x_1\mathbf{h}_1 + x_2\mathbf{h}_2, y = y_1\mathbf{h}_1 + y_2\mathbf{h}_2 \in V$.

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