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MODULAR DIFFERENTIAL EQUATIONS WITH REGULAR SINGULARITIES AT ELLIPTIC POINTS FOR THE HECKE CONGRUENCE SUBGROUPS OF LOW-LEVELS

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Abstract. In this paper, we give explicit expressions of modular differential equations with regular singularities at elliptic points for the Hecke subgroups of level 2, 3, and 4, and their solutions expressed in terms of the Gauss hypergeometric series. We also give quasimodular-form solutions for some modular differential equations.

1. Introduction

In general, a definition of modular differential equations of second-order on the upper half-plane is as follows. For a Fuchsian group of finite covolume $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ and a rational number k, we consider a second-order linear differential equation with regular singularities:

$$f''(\tau) + A(\tau)f'(\tau) + B(\tau)f(\tau) = 0, \tag{1}$$

where τ is a variable in the upper half-plane, the symbol $'$ stands for a differential operator with respect to $2\pi i\tau$, and $A(\tau)$ and $B(\tau)$ are meromorphic functions on the upper half-plane, which are at most of polynomial growth in $\mathrm{Im}(\tau)^{-1}$ in a neighrborhood of every cusp of Γ. Then, we call Eq. (1) a modular differential equation of weight k for Γ if its solution space is invariant under the weight k action of Γ, namely, if $f(\tau)$ is a solution of Eq. (1), then $(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right)$ is also a solution for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Note that this condition does not depend on a choice of branches of $(c\tau+d)^{-k}$ because the differential equation is linear.

Historically, Kaneko and Koike in [4] constructed various modular-form solutions of a certain modular differential equation of second-order, whose coefficient functions $A(\tau)$ and $B(\tau)$ are holomorphic on the upper half-plane. It was originally studied in [7] in connection to supersingular j-polynomials. This differential equation has a property that the space of solutions is invariant under the action of the modular group $\mathrm{SL}_2(\mathbb{Z})$, and modular solutions in [4] are all expressed in terms of the Gauss hypergeometric polynomials. Later, Tsutsumi in [12] studied a larger class of second-order differential

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equations, and called it "modular differential equations" of second-order, which allows regular singularities at elliptic points of $\mathrm{SL}_2(\mathbb{Z})$, and described modular solutions also in terms of hypergeometric series.

In this paper, because we are particularly interested in the case of the Hecke congruence subgroups, we consider modular differential equations of second-order with regular singularities at elliptic points for the Hecke subgroup $\Gamma_0(N)$ $(N = 2, 3, \text{and } 4)$. The result in [12] essentially uses the property that $\mathrm{SL}_2(\mathbb{Z})$ is a non-compact arithmetic triangular group. From Takeuchi's result in [8], all the Hecke congruence subgroups having a similar property, namely, being non-compact arithmetic triangular groups are only $\Gamma_0(N)$ $(N = 2, 3, \text{and } 4)$, except for $\mathrm{SL}_2(\mathbb{Z})$. Therefore, for these groups, it is natural to seek modular differential equations of second order with regular singularities at elliptic points because we expect to get a result similar to the case of $\mathrm{SL}_2(\mathbb{Z})$. Also, we give modular solutions explicitly in some cases, and quasimodular solutions for some of these modular differential equations. Because the proof of our results in this paper is similar to [5, 6, 12], we give only its sketch.

2. Normal forms

We define functions which will be needed in the sequel. For this purpose and for readers' convenience, we give the following: the expression of necessary forms (*forms*), all of their zero points (up to equivalence) (*zeros*), Hauptmodul of the field of the modular functions (*Hauptmodul*), and structure of the space (we denote $\mathcal{M}_k$) of modular forms of weight k (*structure*). We also give analogs of the discriminant functions, which are

$$\Delta_1(\tau) = \eta(\tau)^{24}, \quad \Delta_2(\tau) = \frac{\eta(2\tau)^{16}}{\eta(\tau)^8}, \quad \Delta_3(\tau) = \frac{\eta(3\tau)^9}{\eta(\tau)^3}, \quad \Delta_4(\tau) = \frac{\eta(4\tau)^8}{\eta(2\tau)^4},$$

where $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta function, and $q = e^{2\pi i\tau}$. Here, the "analog" has the following properties:

(i) a logarithmic derivative of $\Delta_N(\tau)$ $(N = 1, 2, 3, \text{and } 4)$ with respect to $2\pi i\tau$ is equal to $E_2^{(N)}(\tau)$, the Eisenstein series of weight 2 at $i\infty$ for the Hecke congruence subgroup of level N, i.e., $E_2^{(N)}(\tau) = (\log \Delta_N(\tau))'$,

(ii) it has zero points only on the cusp $i\infty$.

From these properties, we also have the following transformation formula:

$$E_2^{(N)}\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^2 E_2^{(N)}(\tau) + \frac{6c(c\tau+d)}{\pi i[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \tag{2}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Hereinafter, N denotes the levels 2, 3, or 4.

Case of $\mathrm{SL}_2(\mathbb{Z})$. ($k$: positive even integer)

forms and zeros: Eisenstein series of weight 4 and 6:

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}(\sum_{d|n} d^3)q^n, \qquad \text{a simple zero at } \rho_1^{(1)} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,$$

$$E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}(\sum_{d|n} d^5)q^n, \qquad \text{a simple zero at } \rho_1^{(2)} = i.$$

Hauptmodul and structure:

$$j_1(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)}, \qquad \mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z})) = \bigoplus_{\substack{4v+6w=k \\ v,w\geq 0}} \mathbb{C}E_4^v E_6^w.$$

Case of $N = 2$. (k: positive even integer)

forms and zeros:

$$H_2(\tau) = 1 + 24\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{d|n \\ d:\text{odd}}} d\Bigg)q^n, \qquad \text{a simple zero at } \rho_2 = -\frac{1}{2} + \frac{i}{2}.$$

Hauptmodul and structure:

$$j_2(\tau) = \frac{H_2(\tau)^2}{\Delta_2(\tau)}, \qquad \mathcal{M}_k(\Gamma_0(2)) = \bigoplus_{\substack{2v+4w=k \\ v,w\geq 0}} \mathbb{C}H_2^v \Delta_2^w.$$

Case of $N = 3$. (k: positive integer)

forms and zeros:

$$I_3(\tau) = 1 + 6\sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{d}{3}\right)\right)q^n, \qquad \text{a simple zero at } \rho_3 = -\frac{1}{2} + \frac{i}{2\sqrt{3}}.$$

Hauptmodul and structure:

$$j_3(\tau) = \frac{I_3(\tau)^3}{\Delta_3(\tau)}, \qquad \mathcal{M}_k(\Gamma_0(3), \left(\frac{\cdot}{3}\right)^k) = \bigoplus_{\substack{v+3w=k \\ v,w\geq 0}} \mathbb{C}I_3^v \Delta_3^w.$$

Remark 1. We use $\left(\frac{\cdot}{\cdot}\right)$ to denote the Legendre symbol.

Case of $N=4$. (k: positive half integer)

forms and zeros:

$$\theta(\tau)=\sum_{n\in\mathbb{Z}} q^{n^2}, \qquad \theta(\tau)^4 \text{ has a simple zero at } -\frac{1}{2}.$$

Hauptmodul and structure:

$$j_4(\tau)=\frac{\theta(\tau)^4}{\Delta_4(\tau)}, \qquad \mathcal{M}_k(\Gamma_0(4),\chi_k)=\bigoplus_{\substack{\frac{v}{2}+2w=k\\ v,w\geq 0}} \mathbb{C}\theta^v\Delta_4^w,$$

where $\chi_k=\left(\frac{-1}{\cdot}\right)^k$ if k is an odd integer, otherwise χ_k is trivial.

Furthermore, the table of all elliptic points and cusps (up to equivalence) is a following:

	$\mathrm{SL}_2(\mathbb{Z})$	$\Gamma_0(2)$	$\Gamma_0(3)$	$\Gamma_0(4)$
elliptic points	$\rho_1^{(1)},\ \rho_1^{(2)}$	ρ_2	ρ_3	None
cusps	∞	$0,\ \infty$	$0,\ \infty$	$0,\ -\frac{1}{2},\ \infty$

Then, we get the following from the above information.

Theorem 1. *Any modular differential equation of weight k for $\Gamma_0(N)$ which has regular singularities only at elliptic points for $\Gamma_0(N)$ is given by*

$$f''(\tau)+A^{(N)}(\tau)f'(\tau)+B^{(N)}(\tau)f(\tau)=0, \tag{3}$$

where

$$\begin{aligned} A^{(2)}(\tau)&=-\frac{k+1}{2}E_2^{(2)}(\tau)+\frac{\alpha_2H_2(\tau)^2+\beta_2\Delta_2(\tau)}{H_2(\tau)},\\ A^{(3)}(\tau)&=-\frac{2(k+1)}{3}E_2^{(3)}(\tau)+\frac{\alpha_3I_3(\tau)^3+\beta_3\Delta_3(\tau)}{I_3(\tau)},\\ A^{(4)}(\tau)&=-(k+1)E_2^{(4)}(\tau)+\alpha_4\theta(\tau)^4+\beta_4\Delta_4(\tau) \end{aligned} \tag{4}$$

and

$$\begin{aligned} B^{(2)}(\tau)=\frac{k(k+1)}{16}E_2^{(2)}(\tau)^2-\frac{k}{4}\frac{\alpha_2H_2(\tau)^2+\beta_2\Delta_2(\tau)}{H_2(\tau)}E_2^{(2)}(\tau)\\ +\frac{\gamma_2H_2(\tau)^4+\widetilde{\delta}_2H_2(\tau)^2\Delta_2(\tau)+\varepsilon_2\Delta_2(\tau)^2}{H_2(\tau)^2}, \end{aligned} \tag{5}$$

$$B^{(3)}(\tau) = \frac{k(k+1)}{9} E_2^{(3)}(\tau)^2 - \frac{k}{3} \frac{\alpha_3 I_3(\tau)^3 + \beta_3 \Delta_3(\tau)}{I_3(\tau)} E_2^{(3)}(\tau) + \frac{\gamma_3 I_3(\tau)^6 + \widetilde{\delta_3} I_3(\tau)^3 \Delta_3(\tau) + \varepsilon_3 \Delta_3(\tau)^2}{I_3(\tau)^2},$$

$$B^{(4)}(\tau) = \frac{k(k+1)}{4} E_2^{(4)}(\tau)^2 - \frac{k}{2} (\alpha_4 \theta(\tau)^4 + \beta_4 \Delta_4(\tau)) E_2^{(4)}(\tau) + \gamma_4 \theta(\tau)^8 + \widetilde{\delta_4} \theta(\tau)^4 \Delta_4(\tau) + \varepsilon_4 \Delta_4(\tau)^2$$

with some constants $\alpha_N, \beta_N, \gamma_N, \widetilde{\delta_N}, \varepsilon_N \in \mathbb{C}$.

Proof. Because this proof is completely parallel to Theorem C in [12], we give its sketch only for the case of $\Gamma_0(2)$, other cases being similarly proved.

Because any elliptic point for $\Gamma_0(2)$ is equivalent to ρ_2, by the definition of regular singular points, the functions $H_2(\tau) \cdot A^{(2)}(\tau)$ and $H_2(\tau)^2 \cdot B^{(2)}(\tau)$ are holomorphic on the upper half plane. The modularity of the space of solutions for Eq. (3) gives

$$A^{(2)}\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^2 A^{(2)}(\tau) - \frac{k+1}{\pi i} c(c\tau+d) \tag{6}$$

and

$$B^{(2)}\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^4 B^{(2)}(\tau) - \frac{k}{2\pi i} c(c\tau+d)^3 A^{(2)}(\tau) + \frac{k(k+1)}{(2\pi i)^2} c^2 (c\tau+d)^2.$$

From the quasimodular property of $E_2^{(2)}(\tau)$ as well as the holomorphy of $H_2(\tau) \cdot A^{(2)}(\tau)$, we conclude from Eq. (6) that the function

$$H_2(\tau)\Big(A^{(2)}(\tau) + \frac{k+1}{2} E_2^{(2)}(\tau)\Big)$$

is a holomorphic modular form of weight 4 for $\Gamma_0(2)$. Because the space of modular forms of weight 4 for $\Gamma_0(2)$ is spanned by $H_2(\tau)^2$ and $\Delta_2(\tau)$, we have Eq. (4) for some $\alpha_2, \beta_2 \in \mathbb{C}$. Similarly, the function

$$H_2(\tau)^2\Big(B^{(2)}(\tau) - \frac{k(k+1)}{16} E_2^{(2)}(\tau)^2\Big) + \frac{k}{4} E_2^{(2)}(\tau) H_2(\tau)(\alpha_2 H_2(\tau)^2 + \beta_2 \Delta_2(\tau))$$

is a holomorphic modular form of weight 8 for $\Gamma_0(2)$, thus contained in the space spanned by $H_2(\tau)^4$, $H_2(\tau)^2 \Delta_2(\tau)$ and $\Delta_2(\tau)^2$. Therefore we have Eq. (5) for some $\gamma_2, \widetilde{\delta_2}, \varepsilon_2 \in \mathbb{C}$. □

If $f(\tau)$ is a solution of a modular differential equation in Theorem 1, then we can see that the product of $f(\tau)$ and a suitable power of $\Delta_N(\tau)$ is a solution for the modular differential equation $\mathcal{D}_k^{(N)}(\alpha_N, \beta_N, \delta_N, \varepsilon_N)$ (its definition is below). In other words, we can shift it to the space of solutions for $\mathcal{D}_k^{(N)}$ by a power of $\Delta_N(\tau)$. Therefore, without loss of generality, we also assume that the modular differential equation has a power series solution of the form $1 + h_1 q + h_2 q^2 + \cdots$ ($h_n \in \mathbb{C}$). Under this assumption, we reduce the number of parameters by one and obtain the following normalized form of the modular differential equation.

Theorem 2. *If a modular differential equation* (3) *of weight k for $\Gamma_0(N)$ has regular singularities only at elliptic points for $\Gamma_0(N)$ and has a power series solution of the form $1 + h_1 q + h_2 q^2 + \cdots$, then the equation is given as*

$$\mathcal{D}_k^{(N)}(\alpha_N, \beta_N, \delta_N, \varepsilon_N) \ : \ f''(\tau) + A^{(N)}(\tau) f'(\tau) + B^{(N)}(\tau) f(\tau) = 0,$$

where $A^{(N)}(\tau)$ is the same in Theorem 1, and

$$\begin{aligned}
B^{(2)}(\tau) &= \frac{k(k+1)}{4} E_2^{(2)}(\tau)' - \frac{k}{4} \frac{(\alpha_2 H_2(\tau)^2 + \beta_2 \Delta_2(\tau))'}{H_2(\tau)} \\
&\qquad + \frac{\Delta_2(\tau)(\delta_2 H_2(\tau)^2 + \varepsilon_2 \Delta_2(\tau))}{H_2(\tau)^2}, \\
B^{(3)}(\tau) &= \frac{k(k+1)}{3} E_2^{(3)}(\tau)' - \frac{k}{3} \frac{(\alpha_3 I_3(\tau)^3 + \beta_3 \Delta_3(\tau))'}{I_3(\tau)} \\
&\qquad + \frac{\Delta_3(\tau)(\delta_3 I_3(\tau)^3 + \varepsilon_3 \Delta_3(\tau))}{I_3(\tau)^2}, \\
B^{(4)}(\tau) &= \frac{k(k+1)}{2} E_2^{(4)}(\tau)' - \frac{k}{2} (\alpha_4 \theta(\tau)^4 + \beta_4 \Delta_4(\tau))' \\
&\qquad + \Delta_4(\tau)(\delta_4 \theta(\tau)^4 + \varepsilon_4 \Delta_4(\tau))
\end{aligned}$$

with some constants $\alpha_N, \beta_N, \delta_N, \varepsilon_N \in \mathbb{C}$, $\delta_N = \widetilde{\delta}_N - \xi_N^2 \cdot k(k+1-\alpha_N)$, $\xi_N = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$.

Proof. Using the relations

$$\begin{aligned}
4E_2^{(2)}(\tau)' &= E_2^{(2)}(\tau)^2 - H_2(\tau)^2 + 64\Delta_2(\tau), \\
2H_2(\tau)' &= H_2(\tau) E_2^{(2)}(\tau) - H_2(\tau)^2 + 64\Delta_2(\tau), \\
3E_2^{(3)}(\tau)' &= E_2^{(3)}(\tau)^2 - I_3(\tau)(I_3(\tau)^3 - 27\Delta_3(\tau)), \\
3I_3(\tau)' &= I_3(\tau) E_2^{(3)}(\tau) - I_3(\tau)^3 + 27\Delta_3(\tau), \\
2E_2^{(4)}(\tau)' &= E_2^{(4)}(\tau)^2 - \theta(\tau)^4(\theta(\tau)^4 - 16\Delta_4(\tau)), \\
4\theta(\tau)' &= \theta(\tau)(E_2^{(4)}(\tau) - \theta(\tau)^4 + 16\Delta_4(\tau)),
\end{aligned}$$

and the characteristic polynomial of Eq. (3) at $q = 0$, we can check it. $\square$

3. Hypergeometric solutions of modular differential equations

The Gauss hypergeometric differential equation is defined by

$$x(1-x)\frac{d^2y}{dx^2}+\big(\gamma-(\alpha+\beta+1)x\big)\frac{dy}{dx}-\alpha\beta y=0 \tag{7}$$

where $\alpha,\beta,\gamma\in\mathbb{C}$. If γ, $\alpha-\beta$ and $\gamma-\alpha-\beta$ are not integers, the two functions

$$F(\alpha,\beta,\gamma;x) \qquad \text{and} \qquad x^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;x) \tag{8}$$

give linearly independent solutions around $x=0$, where the Gauss hypergeometric series $F={}_2F_1$ is defined by

$$F(\alpha,\beta,\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{x^n}{n!},$$

where $(\alpha)_n=\alpha(\alpha+1)\cdots(\alpha+n-1)$. The series $F(\alpha,\beta,\gamma;x)$ is a solution of Eq. (7) when γ is not a non-positive integer. When α and γ (resp. β and γ) are negative integers with $\alpha>\gamma$ (resp. $\beta>\gamma$), $F(\alpha,\beta,\gamma;x)$ becomes a polynomial.

In a following theorem, we give the conditions that modular differential equations with regular singularities at elliptic points for $\Gamma_0(N)$ have hypergeometric solutions.

Theorem 3. *For given $\alpha_N,\beta_N,\delta_N,\varepsilon_N\in\mathbb{C}$, put*

$$\begin{aligned}
s_2&=\frac{1}{128}\Big(64\alpha_2+\beta_2-\sqrt{(64\alpha_2+\beta_2)^2-256\delta_2-4\varepsilon_2}\Big), & r_2&=\frac{k-4s_2}{2},\\
s_3&=\frac{1}{54}\Big(27\alpha_3+\beta_3-\sqrt{(27\alpha_3+\beta_3)^2-108\delta_3-4\varepsilon_3}\Big), & r_3&=k-3s_3,\\
s_4&=\frac{1}{32}\Big(16\alpha_4+\beta_4-\sqrt{(16\alpha_4+\beta_4)^2-64\delta_4-4\varepsilon_4}\Big), & r_4&=2k-4s_4,
\end{aligned}$$

$$c_2=\alpha_2-\frac{k-1}{2},\quad c_3=\alpha_3-\frac{2k-1}{3},\quad c_4=\alpha_4-k,$$

and let a_2 and b_2 be the solutions of the equation

$$X^2+\Big(r_2+\frac{\beta_2}{64}-\frac{1}{2}\Big)X+\frac{r_2}{4}\Big(r_2+\frac{\beta_2}{32}-1\Big)+\frac{\varepsilon_2}{4096}=0,$$

a_3 and b_3 the solutions of the equation

$$X^2+\Big(\frac{2r_3}{3}+\frac{\beta_3}{27}-\frac{1}{3}\Big)X+\frac{r_3}{9}\Big(r_3+\frac{\beta_3}{9}-1\Big)+\frac{\varepsilon_3}{729}=0,$$

and a_4 and b_4 the solutions of the equation

$$X^2+\Big(\frac{r_4}{2}+\frac{\beta_4}{16}\Big)X+\frac{r_4}{16}\Big(r_4+\frac{\beta_4}{4}\Big)+\frac{\varepsilon_4}{256}=0.$$

Suppose either (i) c_N *is not an integer, or* (ii) a_N *and* c_N *are negative integers with* $a_N > c_N$*, or* (iii) b_N *and* c_N *are negative integers with* $b_N > c_N$*. Then, the differential equation* $\mathcal{D}^{(2)}(\alpha_2, \beta_2, \delta_2, \varepsilon_2)$ *has two linearly independent solutions*

$$H_2^{r_2}(H_2^2 - 64\Delta_2)^{s_2} F\Big(a_2, b_2, c_2; \frac{64}{j_2}\Big)$$

and

$$H_2^{r_2}(H_2^2 - 64\Delta_2)^{s_2} F\Big(a_2 - c_2 + 1, b_2 - c_2 + 1, 2 - c_2; \frac{64}{j_2}\Big)\Big(\frac{64}{j_2}\Big)^{1-c_2}$$

near $i\infty$*, and the differential equation* $\mathcal{D}^{(3)}(\alpha_3, \beta_3, \delta_3, \varepsilon_3)$ *has two linearly independent solutions*

$$I_3^{r_3}(I_3^3 - 27\Delta_3)^{s_3} F\Big(a_3, b_3, c_3; \frac{27}{j_3}\Big)$$

and

$$I_3^{r_3}(I_3^3 - 27\Delta_3)^{s_3} F\Big(a_3 - c_3 + 1, b_3 - c_3 + 1, 2 - c_3; \frac{27}{j_3}\Big)\Big(\frac{27}{j_3}\Big)^{1-c_3}$$

near $i\infty$*, and the differential equation* $\mathcal{D}^{(4)}(\alpha_4, \beta_4, \delta_4, \varepsilon_4)$ *has two linearly independent solutions*

$$\theta^{r_4}(\theta^4 - 16\Delta_4)^{s_4} F\Big(a_4, b_4, c_4; \frac{16}{j_4}\Big)$$

and

$$\theta^{r_4}(\theta^4 - 16\Delta_4)^{s_4} F\Big(a_4 - c_4 + 1, b_4 - c_4 + 1, 2 - c_4; \frac{16}{j_4}\Big)\Big(\frac{16}{j_4}\Big)^{1-c_4}$$

near $i\infty$*.*

Proof. We describe a sketch of the proof in the case for $\Gamma_0(2)$, other cases being similar and left to the reader.

We transform the Gauss hypergeometric equation (7) by a change of variable into the equation $\mathcal{D}^{(2)}(\alpha_2, \beta_2, \delta_2, \varepsilon_2)$. Putting $x = 64/j_2(\tau)$, Eq. (7) is transformed into the equation

$$\begin{aligned} g''(\tau) + \Big(\frac{(2c_2 - 1)H_2(\tau)^2 + 4^3(1 - 2a_2 - 2b_2)\Delta_2(\tau)}{2H_2(\tau)} - \frac{1}{2}E_2^{(2)}(\tau)\Big)g'(\tau) \\ - 4^3 a_2 b_2 \frac{\Delta_2(\tau)}{H_2(\tau)^2}(H_2(\tau)^2 - 64\Delta_2(\tau))g(\tau) = 0. \end{aligned}$$

Secondly, by changing the unknown

$$g(\tau) = H_2(\tau)^{-r_2}(H_2(\tau)^2 - 64\Delta_2(\tau))^{-s_2} f(\tau),$$

the above differential equation transforms into $f(\tau)$:

$$f''(\tau)+\widetilde{A^{(2)}}(\tau)f'(\tau)+\widetilde{B^{(2)}}(\tau)f(\tau)=0, \tag{9}$$

where $l=2r_2+4s_2$, and

$$\begin{aligned}
\widetilde{A^{(2)}}(\tau)&=\Big(\frac{l-1}{2}+c_2\Big)H_2(\tau)-\frac{l+1}{2}E_2^{(2)}(\tau)\\
&\quad+32(1-2a_2-2b_2-2r_2)\frac{\Delta_2(\tau)}{H_2(\tau)},\\
\widetilde{B^{(2)}}(\tau)&=\frac{l(l+1)}{4}E^{(2)}(\tau)'-\frac{l}{4}\Big(\frac{l-1}{2}+c_2\Big)\frac{(H_2(\tau)^2)'}{H_2(\tau)}\\
&\quad-\frac{l}{4}\cdot 32(1-2a_2-2b_2-2r_2)\frac{\Delta_2(\tau)'}{H_2(\tau)}\\
&\quad+(64s_2^2-64(a_2+b_2-c_2)s_2-16(2a_2+r_2)(2b_2+r_2))\Delta_2(\tau)\\
&\quad+1024(2a_2+r_2)(2b_2+r_2)\frac{\Delta_2(\tau)^2}{H_2(\tau)^2}.
\end{aligned}$$

Comparing the coefficients of $\mathcal{D}^{(2)}(\alpha_2,\beta_2,\delta_2,\varepsilon_2)$ with those of Eq. (9), we can get this theorem. Setting $x=27/j_3(\tau)$ and $g(\tau)=I_3(\tau)^{-r_3}(I_3(\tau)^3-27\Delta_3(\tau))^{-s_3}f(\tau)$ for the case of $N=3$, $x=16/j_4(\tau)$ and $g(\tau)=\theta(\tau)^{-r_4}(\theta(\tau)^4-16\Delta_4(\tau))^{-s_4}f(\tau)$ for the case of $N=4$, we can check it similarly. □

From this theorem, giving a suitable condition about α_N, β_N, δ_N, ε_N, and k, $\mathcal{D}_k^{(N)}(\alpha_N,\beta_N,\delta_N,\varepsilon_N)$ have modular-form solutions of weight k. The following is an example:

Example . Assume $\delta_2=\varepsilon_2=0$. Suppose k is a positive even integer.

(i) When $k\equiv 0 \pmod 4$ and if $\alpha_2-\frac{k-1}{2}$ is a negative integer with the additional condition $k>2(1+2\alpha_2)$, the equation $\mathcal{D}_k^{(2)}(\alpha_2,\beta_2,0,0)$ has the modular form

$$H_2(\tau)^{\frac{k}{2}}F\Big(-\frac{k}{4},-\frac{k-2}{4}-\frac{\beta_2}{64},\alpha_2-\frac{k-1}{2};\frac{64}{j_2(\tau)}\Big)$$

of weight k for $\Gamma_0(2)$ as a solution. In particular, the case of $\beta_2=0$, this is a modular form solution when $k\equiv 2 \pmod 4$.

To prove this, we only need to check that the hypergeometric series in each expression becomes a polynomial if the assumption is satisfied, and that the expression is indeed a holomorphic modular form,which is easily seen.

Remark 2. In [3], the special case ($\alpha_2=\beta_2=0$) of Example (i) is treated.

4. Quasimodular solution for $\mathcal{D}_k^{(N)}(\alpha_N, \beta_N, \delta_N, \varepsilon_N)$

In the previous section, we found that modular differential equations had modular-form solutions expressed in terms of hypergeometric series. When, however, the condition of Theorem 3 is not satisfied, e.g., when c_N is a non-positive integer we cannot consider $F(a_N, b_N, c_N; x)$ in general and we do not know whether the modular differential equation have modular solutions. But even in this case, with a suitable condition, we can have some "quasimodular" forms as solutions. (See [4, 5, 6, 10].) These solutions correspond to the condition that c_N is a non-positive integer. We will give some quasimodular forms as solutions of modular differential equations $\mathcal{D}_k^{(N)}(\alpha_N, \beta_N, \delta_N, \varepsilon_N)$.

First, we define the sequences of polynomials $P_n^{(N)}(X)$, $Q_n^{(N)}(X)$, $P_n^{(N*)}(X)$, and $Q_n^{(N*)}(X)$ by

$$\begin{aligned} P_0^{(N)}(X) &= P_0^{(N*)}(X) = 1, & P_1^{(N)}(X) &= P_1^{(N*)}(X) = X, \\ Q_0^{(N)}(X) &= Q_0^{(N*)}(X) = 0, & Q_1^{(N)}(X) &= Q_1^{(N*)}(X) = 1, \end{aligned}$$

$$\begin{aligned} P_{n+1}^{(N)}(X) &= XP_n^{(N)}(X) + \mu_n^{(N)} P_{n-1}^{(N)}(X) \quad (n \geq 1), \\ Q_{n+1}^{(N)}(X) &= XQ_n^{(N)}(X) + \mu_n^{(N)} Q_{n-1}^{(N)}(X) \quad (n \geq 1), \\ P_{n+1}^{(N*)}(X) &= XP_n^{(N*)}(X) + \mu_n^{(N*)} P_{n-1}^{(N*)}(X) \quad (n \geq 1), \\ Q_{n+1}^{(N*)}(X) &= XQ_n^{(N*)}(X) + \mu_n^{(N*)} Q_{n-1}^{(N*)}(X) \quad (n \geq 1), \end{aligned}$$

where the constants $\mu_n^{(N)}$ and $\mu_n^{(N*)}$ are given by

$$\begin{aligned} \mu_n^{(2)} &= 4\Big(4 + \frac{1}{n}\Big)\Big(4 - \frac{1}{n+1}\Big), & \mu_n^{(2*)} &= 4\Big(4 - \frac{1}{n}\Big)\Big(4 + \frac{1}{n+1}\Big), \\ \mu_n^{(3)} &= 3\Big(3 + \frac{1}{n}\Big)\Big(3 - \frac{1}{n+1}\Big), & \mu_n^{(3*)} &= 3\Big(3 - \frac{1}{n}\Big)\Big(3 + \frac{1}{n+1}\Big), \\ \mu_n^{(4)} &= 4\Big(2 + \frac{1}{n}\Big)\Big(2 - \frac{1}{n+1}\Big), & \mu_n^{(4*)} &= 4\Big(2 - \frac{1}{n}\Big)\Big(2 + \frac{1}{n+1}\Big). \end{aligned}$$

Theorem 4.

(a) *Suppose that $k = \xi_N \cdot n - 1 (n = 1, 2, \ldots)$. Then the form*

$$K_n^{(N)} = \sqrt{\Delta_{NA}}^{\,n-1} P_{n-1}^{(N)}\Big(\frac{Z_N}{\sqrt{\Delta_{NA}}}\Big) V_N' - \sqrt{\Delta_{NA}}^{\,n} Q_{n-1}^{(N)}\Big(\frac{Z_N}{\sqrt{\Delta_{NA}}}\Big)$$

is a quasimodular form of weight $k+1$ and depth 1 for $\Gamma_0(N)$ whose order of zero at $i\infty$ is n, and is a solution of $\mathcal{D}_k^{(N)}((k+1)/\xi_N, 0, 0, 0)$, *where*

$$Z_N = \begin{cases} H_2^2 - 128\Delta_2 & \text{if } N = 2, \\ I_3^3 - 54\Delta_3 & \text{if } N = 3, \\ \theta^4 - 32\Delta_4 & \text{if } N = 4, \end{cases} \quad \Delta_{NA} = \begin{cases} \Delta_2(H_2^2 - 64\Delta_2) & \text{if } N = 2, \\ \Delta_3(I_3^3 - 27\Delta_3) & \text{if } N = 3, \\ \Delta_4(\theta^4 - 16\Delta_4) & \text{if } N = 4, \end{cases}$$

$$V_N = \begin{cases} H_2/24 & \text{if } N = 2, \\ I_3/6 & \text{if } N = 3, \\ (\log\theta)/2 & \text{if } N = 4. \end{cases}$$

(b) *Suppose that* $k = \xi_N \cdot n + 1 (n = 1, 2, \ldots)$. *Then the form*

$$K_n^{(N*)} = \sqrt{\Delta_{NA}}^{\,n-1} P_{n-1}^{(N*)}\Big(\frac{Z_N}{\sqrt{\Delta_{NA}}}\Big)(V_N^*)' - \sqrt{\Delta_{NA}}^{\,n} Q_{n-1}^{(N*)}\Big(\frac{Z_N}{\sqrt{\Delta_{NA}}}\Big) W_N^*$$

is a quasimodular form of weight $k+1$ *and depth* 1 *for* $\Gamma_0(N)$ *whose order of zero at* $i\infty$ *is* n, *and is a solution of* $\mathcal{D}_k^{(N)}((k+3)/\xi_N, l_N, 0, 0)$, *where*

$$V_N^* = \begin{cases} -(H_2^2 - 128\Delta_2)/80 & \text{if } N = 2, \\ -(I_3^3 - 54\Delta_3)/36 & \text{if } N = 3, \\ -(\theta_4 - 32\Delta_4)/24 & \text{if } N = 4, \end{cases} \quad W_N^* = \begin{cases} 2{E_2^{(2)}}^2 - H_2 & \text{if } N = 2, \\ 2{E_2^{(3)}}^2 - I_3^2 & \text{if } N = 3, \\ 2{E_2^{(4)}}^2 - \theta^4 & \text{if } N = 4, \end{cases}$$

$l_2 = -64$, $l_3 = -36$, *and* $l_4 = -32$.

Proof. We can establish that $K_n^{(N)}$ and $K_n^{(N*)}$ are solutions of each modular differential equation by induction on n. By looking at the exponent of each modular differential equation, we can easily find the order of zero at $i\infty$ of the solution is as stated. More detailed proof is given in [5], and the other cases can be shown similarly, hence we omit them. □

Remark 3. For the case of $\Gamma_0(2)$, (a) in Theorem 4 was proved by Kaneko-Koike in [5].

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