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ON POSITIVE INTEGERS OF MINIMAL TYPE CONCERNED WITH THE CONTINUED FRACTION EXPANSION

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1. Introduction

In [3], Kawamoto and Tomita introduced the notion of the "minimal type" concerned with the continued fraction expansion for approaching Gauss' Conjecture. Let us explain it as follows:

Let α be a quadratic irrational whose continued fraction expansion is of the form

$$\alpha = [a_0, \overline{a_1, a_2, \ldots, a_\ell}] \text{ (the periodic part begins with } a_1\text{)},$$
$$a_i = a_{\ell-i} \ (1 \leq i \leq \ell - 1) \text{ (the symmetric property holds).}$$

(These properties hold if, for example, a quadratic irrational α is an algebraic integer.) Then we call the string $a_1, a_2, \ldots, a_{\ell-1}$ the *symmetric part* of the continued fraction expansion of α. For such α, we define nonnegative integers p_i, q_i, r_i by using the partial quotients a_i $(0 \leq i \leq \ell)$:

$$\text{(1.1)} \qquad \begin{cases} p_0 = 1, & p_1 = a_0, \quad p_i = a_{i-1}p_{i-1} + p_{i-2} \ (2 \leq i \leq \ell + 1), \\ q_0 = 0, & q_1 = 1, \quad q_i = a_{i-1}q_{i-1} + q_{i-2} \ (2 \leq i \leq \ell + 1), \\ r_0 = 1, & r_1 = 0, \quad r_i = a_{i-1}r_{i-1} + r_{i-2} \ (2 \leq i \leq \ell + 1). \end{cases}$$

For brevity, we put

$$A := q_\ell, \ B := q_{\ell-1}, \ C := r_{\ell-1},$$

and define linear polynomials $g(x), h(x)$ and a quadratic polynomial $f(x)$ by

$$g(x) = Ax - (-1)^\ell BC, \ h(x) = Bx - (-1)^\ell C^2, \ f(x) = g(x)^2 + 4h(x).$$

Moreover, let s_0 be the least integer x for which $g(x) > 0$. We remark that $g(x), h(x), f(x)$ and s_0 are determined only by the symmetric part because A, B and C do not depend on a_0, a_ℓ.

Definition 1 ([3, Definition 3.1]). Let d be a non-square positive integer. By results of Friesen [1] and Halter-Koch [2], d is uniquely of the form $d = f(s)/4$ with some integer $s \geq s_0$, where $f(x)$ and s_0 are obtained as

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above from the symmetric part $a_1, a_2, \ldots, a_{\ell-1}$ of the continued fraction expansion of $\sqrt{d}$ and ℓ is the minimal period (cf. [3, Theorem 3.1]). If $s = s_0$, that is, $d = f(s_0)/4$ holds, then we say that d is a *positive integer with period ℓ of minimal type for $\sqrt{d}$*. When $d \equiv 1 \pmod 4$ in addition, d is uniquely of the form $d = f(s)$ with some integer $s \geq s_0$, where $f(x)$ and s_0 are obtained as above from the symmetric part $a_1, a_2, \ldots, a_{\ell-1}$ of the continued fraction expansion of $(1+\sqrt{d})/2$ and ℓ is the minimal period. If $s = s_0$, that is, $d = f(s_0)$ holds, then we say that d is a *positive integer with period ℓ of minimal type for $(1+\sqrt{d})/2$*.

Furthermore, for a square-free positive integer d, we say that $\mathbb{Q}(\sqrt{d})$ is a *real quadratic field with period ℓ of minimal type*, if d is a positive integer with period ℓ of minimal type for $\sqrt{d}$ when $d \equiv 2, 3 \pmod 4$, and if d is a positive integer with period ℓ of minimal type for $(1+\sqrt{d})/2$ when $d \equiv 1 \pmod 4$.

Also, they proved in [3] the following:

Theorem ([3, Proposition 4.4]). *There exist exactly* 51 *real quadratic fields of class number* 1 *that are not of minimal type, with one more possible exception.*

For any positive integers ℓ and h, on the other hand, Sasaki [6] and Lachaud [5] showed that there exist at most finitely many real quadratic fields with period ℓ of class number h. Hence we have to examine a construction of real quadratic fields of minimal type in order to find many real quadratic fields of class number 1. Thus, the following problem arises.

Problem. For each positive integer ℓ, do there exist (infinitely many) real quadratic fields with period ℓ of minimal type?

For this problem, the following are known.

Theorem ([3, Example 3.4, Example 3.5], [4, Theorem 1.1]). (1) *Only* $\mathbb{Q}(\sqrt{5})$ *is a real quadratic field with period* 1 *of minimal type.*
(2) *There does not exist a real quadratic field with period* 2, 3 *of minimal type.*
(3) *Let $\ell \geq 4$ be an even integer with $8 \nmid \ell$. Then there exist infinitely many real quadratic field with period ℓ of minimal type.*

In this article, we study quadratic irrationals $\sqrt{d}$ (resp. $(1+\sqrt{d})/2$) whose continued fraction expansion has the symmetric part $b, t, t, \ldots, t, b$ and give a necessary and sufficient condition for such d to be a positive integer with period ℓ of minimal type for $\sqrt{d}$ (resp. $(1+\sqrt{d})/2$). As a consequence, we can show the following result:

Main Theorem (Theorem 3). *Let $\ell \geq 4$ be an integer. Then there exist infinitely many positive integers d with period ℓ of minimal type for each $\sqrt{d}$ or $(1+\sqrt{d})/2$.*

2. Preliminary

Let ℓ be a positive integer and $a_0, \ldots, a_\ell$ be positive integers which satisfy the symmetric property $a_i = a_{\ell-i}\,(1 \leq i \leq \ell-1)$. Define nonnegative integers p_i, q_i, r_i by (1.1). Then it is well-known that

$$p_i = a_0 q_i + r_i \ (0 \leq i \leq \ell+1), \tag{2.1}$$

$$p_i q_{i-1} - p_{i-1} q_i = (-1)^i \ (1 \leq i \leq \ell), \tag{2.2}$$

$$q_\ell r_{\ell-1} - q_{\ell-1}^2 = (-1)^{\ell-1}. \tag{2.3}$$

(See, for example [3,(2.4)], [3,(2.3)], [3,(2.6)], respectively.) Moreover, for a variable λ, we have

$$[a_0, \ldots, a_i, \lambda] = \frac{\lambda p_{i+1} + p_i}{\lambda q_{i+1} + q_i} \ (0 \leq i \leq \ell). \tag{2.4}$$

(See [3,(2.2)].)

Theorem 1. *Under the above notation, put $k := a_0$, $s := (2k+(-1)^\ell BC)/A$ (resp. $s := (2k-1+(-1)^\ell BC)/A$) and $d := f(s)/4$ (resp. $d := f(s)$). Then we have*

$$d = k^2 + \frac{2kB+C}{A} \ (\textit{resp. } d = (2k-1)^2 + 4\frac{(2k-1)B+C}{A}) \tag{2.5}$$

and d is a positive rational number with $d \notin \mathbb{Q}^{\times 2}$. Moreover, the continued fraction expansion of $\sqrt{d}$ (resp. $(1+\sqrt{d})/2$) is

$$\sqrt{d} = [k, \overline{a_1, \ldots, a_{\ell-1}, 2k}] \ (\textit{resp. } \frac{1+\sqrt{d}}{2} = [k, \overline{a_1, \ldots, a_{\ell-1}, 2k-1}]). \tag{2.6}$$

Proof. Like the proof of [3, Theorem 3.1], we put

$$\alpha = k \text{ (resp. } \alpha = k-1), \ a_\ell = 2k \text{ (resp. } a_\ell = 2k-1).$$

Then by the definition of s, we have

$$g(s) = As - (-1)^\ell BC = a_\ell.$$

By using (2.3), we have

$$\begin{aligned} h(s) = Bs - (-1)^\ell C^2 &= \frac{a_\ell B + (-1)^\ell B^2 C}{A} - (-1)^\ell C^2 \\ &= \frac{a_\ell B + C(-1)^\ell (B^2 - AC)}{A} = \frac{a_\ell B + C}{A}. \end{aligned}$$

Hence we see from the relation $f(s) = g(s)^2 + 4h(s)$ that (2.5) holds and d is a positive rational number.

Next we consider an irrational number

$$\omega := [k, \overline{a_1, \ldots, a_{\ell-1}, a_\ell}]$$

to prove (2.6). By using (2.1), (2.2) and (2.3), we have

$$\begin{aligned} p_l &= kA + B, \\ p_{\ell-1} &= (p_\ell q_{\ell-1} - (-1)^\ell)/q_\ell = \{(kq_\ell + q_{\ell-1})q_{\ell-1} - (-1)^\ell\}/q_\ell \\ &= kq_{\ell-1} + (q_{\ell-1}^2 - (-1)^\ell)/q_\ell = kB + C. \end{aligned}$$

Since

$$\alpha + \omega = [a_\ell, \overline{a_1, \ldots, a_{\ell-1}, a_\ell}] = [\overline{a_\ell, a_1, \ldots, a_{\ell-1}}]$$

by the definition of α, we see from the case $i = \ell - 1$, $\lambda = \alpha + \omega$ in (2.4) that

$$\begin{aligned} \omega &= [k, a_1, \ldots, a_{\ell-1}, \overline{a_\ell, a_1, \ldots, a_{\ell-1}}] \\ &= [k, a_1, \ldots, a_{\ell-1}, \alpha + \omega] = \frac{(\alpha+\omega)p_\ell + p_{\ell-1}}{(\alpha+\omega)A + B}. \end{aligned}$$

Hence we get

$$A\omega^2 + (\alpha A + B - p_\ell)\omega = \alpha p_\ell + p_{\ell-1}$$

and by the above,

$$A\omega^2 + (\alpha - k)A\omega = \alpha k A + a_\ell B + C.$$

Since $\omega > 0$ and $\omega^2 = k^2 + (a_\ell B + C)/A$ (resp. $\omega^2 - \omega = k(k-1) + (a_\ell B + C)/A$), we see from (2.5) that

$$\omega = \sqrt{k^2 + \frac{a_\ell B + C}{A}} = \sqrt{d}$$

$$\left(\text{resp. } \omega = \frac{1 + \sqrt{1 + 4k(k-1) + 4\frac{a_\ell B + C}{A}}}{2} = \frac{1+\sqrt{d}}{2}\right).$$

Hence we obtain $d \notin \mathbb{Q}^{\times 2}$ and the desired continued fraction expansion. Thus the theorem is now proved. □

Remark 1. Since $As \in \mathbb{Z}$, $B(As) - (-1)^\ell AC^2 = a_\ell B + C$ as we have seen in the above proof and A is co-prime to B by (2.2), we have

$$s \in \mathbb{Z} \iff A \mid 2kB + C \text{ (resp. } A \mid (2k-1)B + C),$$

k being a positive integer. By (2.5), the last condition is equivalent to $d \in \mathbb{Z}$ (resp. $d \in \mathbb{Z}$ and $d \equiv 1 \pmod 4$).

3. Quadratic irrationals with special type of continued fraction expansion

In this section, we study quadratic irrationals $\alpha^{(j)}$ $(j = 1, 2)$ whose continued fraction expansions are of the form

$$\alpha^{(j)} = [k, \overline{\underbrace{b, t, t, \ldots, t, b}_{n}, k^{(j)}}], \quad \begin{cases} k^{(1)} = 2k, \\ k^{(2)} = 2k-1 \end{cases} \tag{3.1}$$

with (not necessary minimal) period $n + 1 \geq 4$.

For positive integers b, k, t, define infinite sequence of integers $\{S_i\}$ by

$$S_0 = 1,\ S_1 = 0,\ S_i = tS_{i-1} + S_{i-2}\ (i \geq 2)$$

and two finite sequences of integers $\{L_i\}$ and $\{H_i\}$ by

$$L_1 = 1,\ L_2 = b, \qquad L_i = tL_{i-1} + L_{i-2}\ (3 \leq i \leq n),\ \ L_{n+1} = bL_n + L_{n-1},$$
$$H_1 = k,\ H_2 = bk + 1,\ H_i = tH_{i-1} + H_{i-2}\ (3 \leq i \leq n),\ H_{n+1} = bH_n + H_{n-1}.$$

Then we have the following:

Proposition 1. *Let the notation be as above. Then we have*

$$\sqrt{k^2 + \frac{2kL_n + S_n}{L_{n+1}}} = [k, \overline{\underbrace{b, t, t, \ldots, t, b}_{n}, 2k}],$$

$$\frac{1 + \sqrt{(2k-1)^2 + 4\frac{(2k-1)L_n + S_n}{L_{n+1}}}}{2} = [k, \overline{\underbrace{b, t, t, \ldots, t, b}_{n}, 2k-1}].$$

Proof. From the definition, p_i, q_i, r_i which are obtained from the continued fraction expansion of quadratic irrational $\alpha^{(j)}$ with (3.1) can be expressed by $\{S_i\}, \{L_i\}, \{H_i\}$ as

$$\begin{aligned} p_i &= H_i\ (1 \leq i \leq n+1), \\ q_i &= L_i\ (1 \leq i \leq n+1), \\ r_i &= S_i\ (0 \leq i \leq n), \\ p_{n+2} &= k^{(j)} H_{n+1} + H_n, \\ q_{n+2} &= k^{(j)} L_{n+1} + L_n. \end{aligned}$$

Then the proposition is obtained from Theorem 1 immediately. $\square$

Next we will give a necessary and sufficient condition for d to be a positive integers d with period $n+1$ of minimal type for $\sqrt{d}$ (resp. $(1+\sqrt{d})/2$), where $\alpha^{(1)} = \sqrt{d}$ (resp. $\alpha^{(2)} = (1 + \sqrt{d})/2$) with (3.1) and n is odd.

Theorem 2. *Let $n \geq 3$ be an odd integer.*
(1) *Let d be a rational number with*

$$\sqrt{d} = [k, \overline{\underbrace{b, t, t, \ldots, t, b}_{n}, 2k}]$$

and suppose that $d = f(s_0)/4$. Then d is a positive integer with period $n+1$ of minimal type for $\sqrt{d}$ if and only if one of the following conditions holds:
(a) *t is even, $n = 3$ and $b \nmid t$;*
(b) *t is even, $n > 3$ and $b \neq t$;*
(c) *t is odd, b is even, $n \not\equiv 0 \pmod 3$ and $s_0 \equiv 0 \pmod 2$;*
(d) *t is odd, b is odd, $n \not\equiv 2 \pmod 3$ and $s_0 \equiv 0 \pmod 2$.*
(2) *Let d be a rational number with*

$$\frac{1+\sqrt{d}}{2} = [k, \overline{\underbrace{b, t, t, \ldots, t, b}_{n}, 2k-1}]$$

and suppose that $d = f(s_0)$ holds. Then d is a positive integer with period $n+1$ of minimal type for $(1+\sqrt{d})/2$ if and only if the following three conditions hold:
(a) *t is odd*;
(b) *$b \nmid t$ if $n = 3$ and $b \neq t$ if $n > 3$;*
(c) *either $n \equiv 0 \pmod 3$ or $s_0 \equiv 1 \pmod 2$ if b is even, and either $n \equiv 2 \pmod 3$ or $s_0 \equiv 1 \pmod 2$ if b is odd.*

Before the proof of Theorem 2, we will state properties of S_i and L_i.

Lemma 1. (1) *For the parity of S_i, the following holds*:
(i) *If t is even, then*

$$S_i \equiv 0 \pmod 2 \iff i \equiv 1 \pmod 2.$$

(ii) *If t is odd, then*

$$S_i \equiv 0 \pmod 2 \iff i \equiv 1 \pmod 3.$$

(2) *For the parity of L_i, the following holds*:
(i) *If b and t are both even, then*

$$L_i \equiv 0 \pmod 2 \iff i \equiv 0 \pmod 2 \ (1 \leq i \leq n),$$

$$L_{n+1} \equiv \begin{cases} 1 \pmod 2 & \text{if } n \equiv 0 \pmod 2, \\ 0 \pmod 2 & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

(ii) *If b is even and t is odd, then*

$$L_i \equiv 0 \pmod 2 \iff i \equiv 2 \pmod 3 \ (1 \leq i \leq n),$$

$$L_{n+1} \equiv \begin{cases} 0 \pmod 2 & \text{if } n \equiv 0 \pmod 3, \\ 1 \pmod 2 & \text{if } n \equiv 1, 2 \pmod 3. \end{cases}$$

(iii) *If b is odd and t is even, then*

$$L_i \equiv 1 \pmod 2 \ (1 \le i \le n),$$
$$L_{n+1} \equiv 0 \pmod 2.$$

(iv) *If b and t are both odd, then*

$$L_i \equiv 0 \pmod 2 \iff i \equiv 0 \pmod 3 \ (1 \le i \le n),$$

$$L_{n+1} \equiv \begin{cases} 0 \pmod 2 & \text{if } n \equiv 2 \pmod 3, \\ 1 \pmod 2 & \text{if } n \equiv 0, 1 \pmod 3. \end{cases}$$

Proof. We can easily prove by mathematical induction. □

Lemma 2. *For $3 \le i \le n$, we have*

$$L_{i-1}^2 - L_i L_{i-2} = (-1)^{i-1}(b^2 - tb - 1). \tag{3.2}$$

Proof. This is also proved by mathematical induction.

For $i = 3$, we see that

$$L_2^2 - L_3 L_1 = b^2 - (tb + 1) = b^2 - tb - 1.$$

Assume that (3.2) holds for $i = j$ $(3 \le j \le n-1)$. Then we have

$$L_{j-1}^2 - L_j L_{j-2} = (-1)^{j-1}(b^2 - tb - 1).$$

From the definition of $\{L_i\}$, we have

$$\begin{aligned} L_j^2 - L_{j+1}L_{j-1} &= L_j^2 - (tL_j + L_{j-1})L_{j-1} \\ &= L_j(L_j - tL_{j-1}) - L_{j-1}^2 \\ &= L_j L_{j-2} - L_{j-1}^2 \\ &= -(L_{j-1}^2 - L_j L_{j-2}) \\ &= (-1)^j(b^2 - tb - 1), \end{aligned}$$

and hence (3.2) holds for $i = j + 1$. □

For the case $b = t$, the following holds:

Proposition 2. *For a quadratic irrational $\alpha^{(1)} = \sqrt{d}$ (resp. $\alpha^{(2)} = (1 + \sqrt{d})/2$) with a non-square positive integer d and (3.1), we assume $b = t$. Then the followings hold.*

(1) *We have $s_0 = (-1)^{n+1}L_{n-2}$.*

(2) *d is not a positive integer with period $n + 1$ of minimal type for $\sqrt{d}$ (resp. $(1 + \sqrt{d})/2$).*

Proof. When $b = t$, we have $S_n = L_{n-1}$, and hence

$$g(x) = L_{n+1}x - (-1)^{n+1}L_nL_{n-1},$$
$$h(x) = L_nx - (-1)^{n+1}L_{n-1}^2.$$

(1) By using Lemma 2, we have

$$\begin{aligned} g((-1)^{n+1}L_{n-2}) &= (-1)^{n+1}L_{n+1}L_{n-2} - (-1)^{n+1}L_nL_{n-1} \\ &= (-1)^{n+1}\{(tL_n + L_{n-1})L_{n-2} - L_nL_{n-1}\} \\ &= (-1)^{n+1}\{tL_nL_{n-2} + L_{n-1}(L_{n-2} - L_n)\} \\ &= (-1)^{n+1}t(L_nL_{n-2} - L_{n-1}^2) \\ &= (-1)^{n+1}t(-1)^n(t^2 - t^2 - 1) = t > 0, \\ g((-1)^{n+1}L_{n-2} - 1) &= t - L_{n+1} = L_1 - L_{n+1} < 0. \end{aligned}$$

Thus we get

$$s_0 = (-1)^{n+1}L_{n-2}.$$

(2) By also using Lemma 2, we have

$$h((-1)^{n+1}L_{n-2}) = (-1)^{n+1}L_nL_{n-2} - (-1)^{n+1}L_{n-1}^2 = 1,$$

and hence

$$f((-1)^{n+1}L_{n-2}) = t^2 + 4.$$

First, assume on the contrary that d is a positive integer with period $n+1$ of minimal type for $\sqrt{d}$. Then we have

$$d = \frac{f((-1)^{n+1}L_{n-2})}{4} = \left(\frac{t}{2}\right)^2 + 1.$$

Hence the integer part k of $\sqrt{d}$ is $k = t/2$, and so $t = 2k$. Then we have

$$\sqrt{d} = [k, \overline{2k, 2k, \ldots, 2k}] = [k, \overline{2k}].$$

This contradicts that the minimal period is $n + 1$.

Next we assume that d is a positive integer with period $n + 1$ of minimal type for $(1 + \sqrt{d})/2$. Then we have

$$d = f((-1)^{n+1}L_{n-2}) = t^2 + 4.$$

It follows from $d \equiv 1 \pmod 4$ that t is odd. Since

$$t^2 < (t+1)^2 < t^2 + 4 < (t+2)^2 \ \text{ if } t = 1,$$
$$t^2 < t^2 + 4 < (t+1)^2 < (t+2)^2 \ \text{ if } t \geq 3,$$

the integer part k of $(1+\sqrt{d})/2$ is $k=(t+1)/2$, and hence $t=2k-1$. Therefore, we have

$$\frac{1+\sqrt{d}}{2}=[k,\overline{2k-1,2k-1,\ldots,2k-1}]=[k,\overline{2k-1}].$$

This contradicts that the minimal period is $n+1$. The proof is now completed. $\square$

Proposition 3. *Let $n \geq 3$ be an integer.*
(1) *Let d be a non-square positive integer with*

$$\sqrt{d}=[k,\overline{\underbrace{b,t,t,\ldots,t,b}_{n},2k}].$$

Assume that $d=f(s_0)/4$. Then the minimal period is $n+1$ if and only if $b \nmid t$ when $n=3$ and $b \neq t$ when $n>3$.
(2) *Let $d \equiv 1 \pmod 4$ be a non-square positive integer with*

$$\frac{1+\sqrt{d}}{2}=[k,\overline{\underbrace{b,t,t,\ldots,t,b}_{n},2k-1}].$$

Assume that $d=f(s_0)$. Then the minimal period is $n+1$ if and only if $b \nmid t$ when $n=3$ and $b \neq t$ when $n>3$.

Proof. (1) First suppose that $n=3$. Then the minimal period is 4 if and only if $t \neq 2k$. Hence we have only to show that

$$b \mid t \iff t=2k.$$

Suppose that $b \mid t$. It is obtained from the symmetric part b,t,b that

$$g(x)=(tb^2+2b)x-(tb+1)t,$$
$$h(x)=(tb+1)x-t^2.$$

From the definition of s_0, it must hold that

$$g(s_0)>0,\ g(s_0-1)<0.$$

Then we have inequalities

$$\frac{t}{b}-\frac{t}{b(tb+2)}<s_0<\frac{t}{b}+1-\frac{t}{b(tb+2)}.$$

By the assumption $b \mid t$, therefore, we have $s_0=t/b$, and hence

$$d=\frac{f(s_0)}{4}=\frac{g(s_0)^2}{4}+h(s_0)=\left(\frac{t}{2}\right)^2+\frac{t}{b}.$$

It follows from $b \mid t$ and $d \in \mathbb{Z}$ that t is even. Since

$$\left(\frac{t}{2}\right)^2 < \left(\frac{t}{2}\right)^2 + \frac{t}{b} < \left(\frac{t}{2}+1\right)^2,$$

the integer part k of $\sqrt{d}$ is $k = t/2$. Then we have $t = 2k$.

Conversely, suppose that $t = 2k$, that is,

$$\sqrt{d} = [k, \overline{b, 2k, b, 2k}].$$

Then by Proposition 1, we have

$$d = k^2 + \frac{tL_3 + S_3}{L_4} = k^2 + \frac{t(tb+1)+t}{tb^2+2b} = k^2 + \frac{t(tb+2)}{b(tb+2)} = k^2 + \frac{t}{b}.$$

Since $d \in \mathbb{Z}$, we get $b \mid t$.

Next suppose that $n > 3$. If $b \neq t$, it is obviously that the minimal period is $n+1$. If $b = t$, we have seen in Proposition 2 that the minimal period is not $n+1$.

(2) First suppose that $n = 3$. Then the minimal period is 4 if and only if $t \neq 2k-1$. Hence we have only to show that

$$b \mid t \iff t = 2k-1.$$

Suppose that $b \mid t$. It is obtained from the symmetric part b, t, b that $s_0 = t/b$ as we have seen in the proof of (1). Then we have

$$d = f(s_0) = t^2 + \frac{4t}{b}.$$

It follows from $b \mid t$ and $d \equiv 1 \pmod 4$ that t is odd. Since

$$t^2 < (t+1)^2 < t^2 + \frac{4t}{b} < (t+2)^2 \quad \text{if } b = 1,$$

$$t^2 < t^2 + \frac{4t}{b} < (t+1)^2 < (t+2)^2 \quad \text{if } b \geq 2,$$

the integer part k of $(1+\sqrt{d})/2$ is $k = (t+1)/2$. Hence we get $t = 2k-1$.

Conversely, suppose that $t = 2k-1$, that is

$$\frac{1+\sqrt{d}}{2} = [k, \overline{b, 2k-1, b, 2k-1}].$$

Then by Proposition 1, we have

$$d = (2k-1)^2 + 4\frac{tL_3 + S_3}{L_4} = (2k-1)^2 + \frac{4t}{b}.$$

Since $d \equiv 1 \pmod 4$, we obtain $b \mid t$.

Next suppose that $n > 3$. If $b \neq t$, it is obviously that the minimal period is $n+1$. If $b = t$, we have seen in Proposition 2 that the minimal period is not $n+1$. $\square$

Proof of Theorem 2. Noting that n is odd, we see from Lemma 1 that

$$g(x) = L_{n+1}x - L_n S_n \equiv 0 \pmod 2 \text{ for any integer } x, \tag{3.3}$$

if t is even, and

$$g(s_0) \equiv 0 \pmod 2 \iff \begin{cases} n \not\equiv 0 \pmod 3 \text{ and } s_0 \equiv 0 \pmod 2 & \text{if } b \text{ is even,} \\ n \not\equiv 2 \pmod 3 \text{ and } s_0 \equiv 0 \pmod 2 & \text{if } b \text{ is odd,} \end{cases} \tag{3.4}$$

if t is odd.

(1) From the definition, d is a positive integer with period $n+1$ of minimal type for $\sqrt{d}$ if and only if $d \in \mathbb{Z}$ and the minimal period is $n+1$.

When t is even, it follows from (3.3) that

$$d = \frac{f(s_0)}{4} = \left(\frac{g(s_0)}{2}\right)^2 + h(s_0) \in \mathbb{Z}.$$

Moreover, by Proposition 3 we see that

$$\text{the minimal period is } n+1 \iff \begin{cases} b \nmid t & \text{if } n = 3, \\ b \neq t & \text{if } n > 3. \end{cases}$$

When t is odd, it holds that $t \neq 2k$. Then we see from Proposition 3 that the minimal period is $n+1$. Since

$$d = \frac{f(s_0)}{4} = \left(\frac{g(s_0)}{2}\right)^2 + h(s_0),$$

we see from (3.4) that

$$\begin{aligned} d \in \mathbb{Z} &\iff g(s_0) \equiv 0 \pmod 2 \\ &\iff \begin{cases} n \not\equiv 0 \pmod 3 \text{ and } s_0 \equiv 0 \pmod 2 & \text{if } b \text{ is even,} \\ n \not\equiv 2 \pmod 3 \text{ and } s_0 \equiv 0 \pmod 2 & \text{if } b \text{ is odd.} \end{cases} \end{aligned}$$

(2) From the definition, d is a positive integer with period $n+1$ of minimal type for $(1+\sqrt{d})/2$ if and only if $d \equiv 1 \pmod 4$ and the minimal period is $n+1$.

If t is even, then by $g(s_0) \equiv 0 \pmod 2$, we have

$$d = f(s_0) = g(s_0)^2 + 4h(s_0) \equiv 0 \not\equiv 1 \pmod 4.$$

Hence d is not a positive integer with period $n+1$ of minimal type for $(1+\sqrt{d})/2$.

Suppose that t is odd. Then by Proposition 3 we see that

$$\text{the minimal period is } n+1 \iff \begin{cases} b \nmid t & \text{if } n = 3, \\ b \neq t & \text{if } n > 3. \end{cases}$$

Since $d = f(s_0) \equiv g(s_0)^2 \pmod 4$, we see from (3.4) that

$$
\begin{aligned}
d \equiv 1 \pmod 4 \iff & g(s_0) \equiv 1 \pmod 2 \\
\iff & \begin{cases} n \equiv 0 \pmod 3 \text{ or } s_0 \equiv 1 \pmod 2 & \text{if } b \text{ is even,} \\ n \equiv 2 \pmod 3 \text{ or } s_0 \equiv 1 \pmod 2 & \text{if } b \text{ is odd.} \end{cases}
\end{aligned}
$$

Theorem 2 is completely proved. □

4. Main Theorem

The following is the key proposition for the proof of our main theorem (Theorem 3).

Proposition 4. *Let $n \geq 3$ be an odd (resp. an even) integer and let s_0 be an integer which is obtained from the symmetric part $a_1, a_2, \ldots, a_n$ as in §1. Moreover, we put $m := \max\{a_2, a_3, \ldots, a_{n-1}\}$ and define nonnegative integers u_i by*

$$u_0 = 1, \ u_1 = 0, \ u_i = m u_{i-1} + u_{i-2} \ (i \geq 2).$$

If $a_1 \geq u_n$, then we have $s_0 = 1$ (resp. $s_0 = 0$).

Proof. Recall that

$$g(x) = q_{n+1} x - (-1)^{n+1} q_n r_n.$$

Now we suppose that $a_1 \geq u_n$. Then by the definition of u_i, we have $u_n \geq r_n$, and hence $a_1 \geq r_n$. This gives that

$$q_{n+1} - q_n r_n = a_n q_n + q_{n-1} - q_n r_n = (a_n - r_n) q_n + q_{n-1} = (a_1 - r_n) q_n + q_{n-1} > 0.$$

If n is odd, then we have

$$
\begin{aligned}
g(0) &= -q_n r_n < 0, \\
g(1) &= q_{n+1} - q_n r_n > 0,
\end{aligned}
$$

and so $s_0 = 1$. If n is even, then we have

$$
\begin{aligned}
g(-1) &= -q_{n+1} + q_n r_n < 0, \\
g(0) &= q_n r_n > 0,
\end{aligned}
$$

and so $s_0 = 0$. □

For the case where the symmetric part is the string $b, t, t, \ldots, t, b$, the converse of Proposition 4 is true. Namely,

Proposition 5. *Let $n \geq 3$ be an odd (resp. an even) integer and let s_0 be an integer which is obtained from the symmetric part $b, t, t, \ldots, t, b$. Then we have*

$$b \geq S_n \iff s_0 = 1 \ (\textit{resp. } s_0 = 0).$$

Proof. The "⇒" part is easily proved using Proposition 4. Indeed, we have $m = t$ in this case. Hence it holds that $u_i = S_i$ $(i \geq 0)$ and $a_1 = b$.

Let us prove the "⇐" part. First, we consider the case where n is odd and $s_0 = 1$. Suppose, on the contrary, that $b < S_n$. Then we have $b - S_n \leq -1$, and hence

$$\begin{aligned} g(s_0) &= L_{n+1} - L_n S_n = bL_n + L_{n-1} - L_n S_n = (b - S_n)L_n + L_{n-1} \\ &\leq -L_n + L_{n-1} = -(t-1)L_{n-1} - L_{n-2} < 0. \end{aligned}$$

This contradicts $g(s_0) > 0$. Therefore we get $b \geq S_n$.

Next, we consider the case where n is even and $s_0 = 0$. Suppose, on the contrary, that $b < S_n$. Then by $-(b - S_n) \geq 1$, we have

$$\begin{aligned} g(-1) &= -L_{n+1} + L_n S_n = -(b - S_n)L_n - L_{n-1} \\ &\geq L_n - L_{n-1} = (t-1)L_{n-1} + L_{n-2} > 0. \end{aligned}$$

This contradicts $s_0 = 0$. Hence we have $b \geq S_n$. □

Theorem 3. *Let $\ell \geq 4$ be an integer. Then there exist infinitely many non-square positive integers d with period ℓ of minimal type for each $\sqrt{d}$ or $(1+\sqrt{d})/2$ whose continued fraction expansion has the symmetric part $b, t, t, \ldots, t, b$.*

Proof. Let $\ell \geq 4$ be an integer and put $n := \ell - 1$. Recall that

$$\begin{aligned} g(x) &= L_{n+1}x - (-1)^{n+1} L_n S_n, \\ h(x) &= L_n x - (-1)^{n+1} S_n^2. \end{aligned}$$

First we consider the case where n is odd. Suppose that t is even (resp. odd) and b is a positive integer with

$$b \geq S_n, \quad \begin{cases} b \nmid t & \text{if } n = 3, \\ b \neq t & \text{if } n > 3. \end{cases} \tag{4.1}$$

By Proposition 5, it follows that $s_0 = 1$, and hence

$$f(s_0) = g(s_0)^2 + 4h(s_0) = L_{n+1}^2 - 2L_{n+1}L_n S_n + 4L_n + S_n^2(L_n^2 - 4).$$

If we put

$$k := \frac{g(s_0)}{2} = \frac{L_{n+1} - L_n S_n}{2} \ \left(\text{resp. } k := \frac{g(s_0)+1}{2} = \frac{L_{n+1} - L_n S_n + 1}{2}\right),$$

then $k > 0$ by $g(s_0) > 0$. Noting the parity of n and t, it follows from Lemma 1 that

$$L_{n+1} \equiv S_n \equiv 0 \pmod 2 \ (\text{resp. } L_{n+1} - L_n S_n \equiv 1 \pmod 2),$$

and hence k is a positive integer. Since

$$s_0 = \frac{2k + L_n S_n}{L_{n+1}} \text{ (resp. } s_0 = \frac{2k - 1 + L_n S_n}{L_{n+1}}\text{)}$$

from $s_0 = 1$, if we put

$$d_1 := \frac{f(s_0)}{4} \text{ (resp. } d_2 := f(s_0)\text{)},$$

then we see from Theorem 1 and Remark 1 that $d_1 \in \mathbb{Z}$, $d_1 \notin \mathbb{Q}^{\times 2}$ (resp. $d_2 \in \mathbb{Z}$, $d_2 \notin \mathbb{Q}^{\times 2}$, $d_2 \equiv 1 \pmod 4$) and

$$d_1 = k^2 + \frac{2kL_n + S_n}{L_{n+1}} \text{ (resp. } d_2 = (2k-1)^2 + 4\frac{(2k-1)L_n + S_n}{L_{n+1}}\text{)},$$

$$\sqrt{d_1} = [k, \overline{\underbrace{b, t, t, \dots, t, b}_{n}, 2k}] \text{ (resp. } \frac{1+\sqrt{d_2}}{2} = [k, \overline{\underbrace{b, t, t, \dots, t, b}_{n}, 2k-1}]\text{)}.$$

Then by Theorem 2, d_1 (resp. d_2) is a positive integer with period $n+1$ of minimal type for $\sqrt{d_1}$ (resp. $(1+\sqrt{d_2})/2$).

There are infinitely many positive integers b which satisfies (4.1) for each t because S_n does not depend on b. From this, the infiniteness is obtained.

Next, we consider the case where n is even. Let t be an even positive integer and b an even (resp. an odd) positive integer with

$$b \geq S_n, \ b \neq t. \tag{4.2}$$

Then it follows from Proposition 5 that $s_0 = 0$, and hence

$$f(s_0) = S_n^2(L_n^2 + 4).$$

If we put

$$k := \frac{g(s_0)}{2} = \frac{L_n S_n}{2} \text{ (resp. } k := \frac{g(s_0)+1}{2} = \frac{L_n S_n + 1}{2}\text{)},$$

then $k > 0$. Noting the parity of n, b and t, it follows from Lemma 1 that

$$L_n \equiv 0 \pmod 2 \text{ (resp. } L_n \equiv S_n \equiv 1 \pmod 2\text{)},$$

and hence k is a positive integer. Since

$$s_0 = \frac{2k - L_n S_n}{L_{n+1}} \text{ (resp. } s_0 = \frac{2k - 1 - L_n S_n}{L_{n+1}}\text{)}$$

from $s_0 = 0$, if we put

$$d_3 := \frac{f(s_0)}{4} \text{ (resp. } d_4 := f(s_0)\text{)},$$

then we see from Theorem 1 and Remark 1 that $d_3 \in \mathbb{Z}$, $d_3 \notin \mathbb{Q}^{\times 2}$ (resp. $d_4 \in \mathbb{Z}$, $d_4 \notin \mathbb{Q}^{\times 2}$, $d_4 \equiv 1 \pmod 4$) and

$$d_3 = k^2 + \frac{2kL_n + S_n}{L_{n+1}} \text{ (resp. } d_4 = (2k-1)^2 + 4\frac{(2k-1)L_n + S_n}{L_{n+1}}\text{)},$$

$$\sqrt{d_3} = [k, \overline{\underbrace{b, t, t, \ldots, t, b}_{n}, 2k}] \text{ (resp. } \frac{1+\sqrt{d_4}}{2} = [k, \overline{\underbrace{b, t, t, \ldots, t, b}_{n}, 2k-1}]\text{)}.$$

By Proposition 3, we see from $b \neq t$ that the minimal period is $n+1$. Hence d_3 (resp. d_4) is a positive integer with period $n+1$ of minimal type for $\sqrt{d_3}$ (resp. $(1+\sqrt{d_4})/2$).

The infiniteness follows from the fact that there are infinitely many even (resp. odd) positive integers b with (4.2) for each t. $\square$

Remark 2. In [3], they classified three cases by the parity of A and C:
(I) $A \equiv 1 \pmod 2$;
(II) $(A, C) \equiv (0, 0) \pmod 2$;
(III) $(A, C) \equiv (0, 1) \pmod 2$.
It is easily seen that d_1, d_3 and d_4 are in Case (II), (I) and (III), respectively. Furthermore, d_2 is in Case (I) if either b is even, $n \not\equiv 0 \pmod 3$ or b is odd, $n \not\equiv 2 \pmod 3$, and d_2 is in Case (III) if either b is even, $n \equiv 0 \pmod 3$ or b is odd, $n \equiv 2 \pmod 3$.

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