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A CHARACTERIZATION OF THE GLAUBERMAN-WATANABE CORRESPONDING BLOCKS AS BIMODULES

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Abstract. We give a characterization of the Glauberman-Watanabe corresponding blocks viewed as bimodules as a direct summand of a restricted or an induced module from the block in terms of a vertex and a multiplicity.

1. Introduction

Let p be a prime. Let $\mathcal{O}$ be a complete discrete valuation ring having an algebraically closed residue field k of characteristic p and having a quotient field of characteristic zero which will be assumed be large enough. Let b be a (p-)block (idempotent) of a finite group G over $\mathcal{O}$.

Below, for groups H_1 and H_2, an $(\mathcal{O}H_1, \mathcal{O}H_2)$-bimodule M will be identified with an $\mathcal{O}[H_1 \times H_2]$-module in the usual way: $(h_1, h_2) \cdot m = h_1 \cdot m \cdot {h_2}^{-1}$ where $(h_1, h_2) \in H_1 \times H_2$ and $m \in M$.

By the action $g_1 \cdot x \cdot g_2 = g_1 x g_2$ where $g_1, g_2 \in G$ and $x \in \mathcal{O}G$, block (algebra) $\mathcal{O}Gb$ is an indecomposable $(\mathcal{O}G, \mathcal{O}G)$-bimodule, and b has a defect group D if and only if $\mathcal{O}Gb$ has a vertex $\Delta D = \{(d, d) \in D \times D \mid d \in D\}$, see [3, Theorem 1].

For a block c which is a Brauer correspondent of b where H is a subgroup of G containing $N_G(D)$, $\mathcal{O}[G \times G]$-module $\mathcal{O}Gb$ and $\mathcal{O}[H \times H]$-module $\mathcal{O}Hc$ are the Green corresponding modules with respect to $(G \times G, \Delta D, H \times H)$, see [3, Lemma 4.2c] and [4, Theorem 2]. That is, as a bimodule $\mathcal{O}Hc$ can be characterized as a unique indecomposable direct summand of $\mathcal{O}Gb{\downarrow}^{G \times G}_{H \times H}$ with a vertex ΔD. Also as a bimodule $\mathcal{O}Gb$ is characterized as a unique indecomposable direct summand of $\mathcal{O}Hc{\uparrow}^{G \times G}_{H \times H}$ with a vertex ΔD.

Let q be a prime such that $q \nmid |G|$. Let S be a cyclic group of order q acting on G. Then with this action, we can consider the semi-direct product $G \rtimes S$, denoted by E. Denote by G' the centralizer $C_G(S)$ of S in G. For an S-invariant irreducible character χ of G, there is a unique irreducible character of G', called Glauberman correspondent of χ, such that its multiplicity (in fact, it is ± 1 modulo q) in $\chi{\downarrow}^G_{G'}$ is not divisible by q, and χ is a unique S-invariant irreducible character of G such that its multiplicity (in fact, it is ± 1 modulo q) in $\chi'{\uparrow}^G_{G'}$ is not divisible by q,

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see [2]. Assume that b has a defect group D centralized by S, and denote by b' the Glauberman-Watanabe corresponding block of b. That is, b' is a block of G' such that $\mathrm{Irr}(b') = \{\chi' \mid \chi{\in}\mathrm{Irr}(b){=}\mathrm{Irr}(b)^S\}$, see [9]. Let $\ddot{S}$ be a subgroup of $S{\times}S(\subset E{\times}E)$ such that the canonical projections $S{\times}S \to S{\times}1$ and $S{\times}S \to 1{\times}S$ induce isomorphisms from $\ddot{S}$ to $S{\times}1$ and $1{\times}S$. Assume that $\mathcal{O}$ contains $q|G|^2$-th root of unity, see [7].

As in the case of the Brauer corresponding blocks, we can characterize the Glauberman-Watanabe corresponding blocks viewed as bimodules in terms of a vertex and a multiplicity as a direct summand of a restricted or an induced module from the block.

Theorem 1.1.

(1) $\mathcal{O}G'b'$ is a unique indecomposable direct summand of $\mathcal{O}Gb{\downarrow}^{G\times G}_{G'\times G'}$ with a vertex ΔD and with a multiplicity not divisible by q. In fact, its multiplicity is 1 modulo q.

(2) $\mathcal{O}Gb$ is a unique $\ddot{S}$-invariant indecomposable direct summand of $\mathcal{O}G'b'{\uparrow}^{G\times G}_{G'\times G'}$ with a vertex ΔD and with a multiplicity not divisible by q. In fact, its multiplicity is 1 modulo q.

We use Puig's theory as described in [8]. A multiplicity module of an H-algebra A for a group H with respect to a local pointed group D_γ is denoted by $V_A(D_\gamma)$.

For finite groups H, H' and an $(\mathcal{O}H, \mathcal{O}H')$-bimodule M, $\mathrm{End}_{\mathcal{O}}(M)$ has an interior $H{\times}H'$-algebra structure, see [8, Example 10.6], and its subalgebra $\mathrm{End}_{\mathcal{O}}(M)^{1\times H'}$ consisting of $1{\times}H'$-invariant elements has an interior H-algebra structure determined by the left $\mathcal{O}H$-module structure of M. The proof depends in particular on Puig correspondence, see [5, 2.10.3] [8, Section 19], and the following characterization of the bimodule inducing a Morita equivalence, see [6, Proposition 6.5]:

$(\mathcal{O}H, \mathcal{O}H')$-bimodule M induces a Morita equivalence between block algebras $\mathcal{O}Hc$ and $\mathcal{O}H'c'$ of H and H' if and only if the structural map $\mathcal{O}H \to \mathrm{End}_{\mathcal{O}}(M)^{1\times H'}$ induces an interior H-algebra isomorphism $\mathcal{O}Hc \simeq \mathrm{End}_{\mathcal{O}}(M)^{1\times H'}$.

For a direct summand X of a module Y, the multiplicity of X in Y is denoted by $m(X, Y)$.

2. Reduction to the case $D \lhd G$

Lemma 2.1. *The statement of Theorem 1.1 holds if the statement of Theorem 1.1 holds in the case $D \lhd G$.*

Proof Let c be a block of $N = N_G(D)$ which is a Brauer correspondent of b, and let c' be a block of $N' = N_{G'}(D)$ which is a Brauer correspondent of b'. Note that c' is a Glauberman-Watanabe correspondent of c, see [9, Proposition 4(i)].

(1) Considering Green correspondence with respect to $(G{\times}G, \Delta D, N{\times}N)$, we have $\mathcal{O}Gb{\downarrow}^{G\times G}_{N\times N} \simeq \mathcal{O}Nc{\oplus}X$ for an $\mathcal{O}[N{\times}N]$-module X any of whose indecomposable direct summands has a vertex not $N{\times}N$-conjugate to ΔD.

Then, since the statement of Theorem 1.1(1) holds for $\mathcal{O}Nc$ by the assumption and none of the indecomposable direct summands of $X{\downarrow}^{N\times N}_{N'\times N'}$ has a vertex ΔD, we have $\mathcal{O}Gb{\downarrow}^{G\times G}_{N'\times N'} \simeq \mathcal{O}N'c'{\oplus}Y$ for an $\mathcal{O}[N'{\times}N']$-module Y any of whose indecomposable direct summands with a vertex ΔD has a multiplicity divisible by q.

On the other hand, Green correspondence with respect to $(G'{\times}G', \Delta D, N'{\times}N')$ induces a multiplicity preserving bijection between the indecomposable direct summands of $\mathcal{O}Gb{\downarrow}^{G\times G}_{G'\times G'}$ with a vertex ΔD and the indecomposable direct summands of $\mathcal{O}Gb{\downarrow}^{G\times G}_{N'\times N'}$ with a vertex ΔD.

Hence, noting that $\mathcal{O}G'b'$ and $\mathcal{O}N'c'$ are the Green corresponding modules, we have the statement.

(2) Firstly, we note that for a group H, a subgroup K of H and an indecomposable $\mathcal{O}K$-module Z with a vertex R, if an indecomposable direct summand of $Z{\uparrow}^H_K$ does not have a vertex R, then its vertex has an order strictly smaller than $|R|$.

Considering Green correspondence with respect to $(G'{\times}G', \Delta D, N'{\times}N')$, we have $\mathcal{O}N'c'{\uparrow}^{G'\times G'}_{N'\times N'} \simeq \mathcal{O}G'b'{\oplus}X$ for an $\mathcal{O}[G'{\times}G']$-module X none of whose indecomposable direct summands has a vertex ΔD. Hence, the indecomposable direct summands of $\mathcal{O}G'b'{\uparrow}^{G\times G}_{G'\times G'}$ with a vertex ΔD are the indecomposable direct summands of $\mathcal{O}N'c'{\uparrow}^{G\times G}_{N'\times N'}$ with a vertex ΔD, and so we consider $\mathcal{O}N'c'{\uparrow}^{G\times G}_{N'\times N'}$.

By the assumption, we have $\mathcal{O}N'c'{\uparrow}^{N\times N}_{N'\times N'} \simeq \mathcal{O}Nc{\oplus}Y$ for an $\mathcal{O}[N{\times}N]$-module Y any of whose $\ddot{S}$-invariant indecomposable direct summands with a vertex ΔD has a multiplicity divisible by q. Note that Y is $\ddot{S}$-invariant since $\mathcal{O}N'c'{\uparrow}^{N\times N}_{N'\times N'}$ and $\mathcal{O}Nc$ are $\ddot{S}$-invariant. We have $\mathcal{O}N'c'{\uparrow}^{G\times G}_{N'\times N'} \simeq \mathcal{O}Nc{\uparrow}^{G\times G}_{N\times N}{\oplus}Y{\uparrow}^{G\times G}_{N\times N}$.

$\mathcal{O}Gb$ is a unique indecomposable direct summand of $\mathcal{O}Nc{\uparrow}^{G\times G}_{N\times N}$ with a vertex ΔD, considering Green correspondence with respect to $(G{\times}G, \Delta D, N{\times}N)$.

Let M be an $\ddot{S}$-invariant indecomposable direct summand of $Y{\uparrow}^{G\times G}_{N\times N}$ with a vertex ΔD, and let L be an indecomposable direct summand of Y such that $M \mid L{\uparrow}^{G\times G}_{N\times N}$. Then L has a vertex ΔD. If L is $\ddot{S}$-invariant, then

$q \mid m(L,Y)$ and so $q \mid m(M, Y{\uparrow}_{N\times N}^{G\times G})$. If L is not $\ddot{S}$-invariant, then we have $M \simeq M^{t_i} \mid (L{\uparrow}_{N\times N}^{G\times G})^{t_i} \simeq L^{t_i}{\uparrow}_{N\times N}^{G\times G}$ and $L^{t_i}|Y$ for any $t_i \in \ddot{S}$ and we have $L^{t_i} \not\simeq L^{t_j}$ if $i \neq j$. Hence, any $\ddot{S}$-invariant indecomposable direct summand of $Y{\uparrow}_{N\times N}^{G\times G}$ with a vertex ΔD has a multiplicity divisible by q.

Hence, we have the statement. □

3. The case $D \lhd G$

In this section, we assume

$$D \lhd G$$

and show Theorem under $D \lhd G$.

We cite the facts used later from [7]. (Note that $q \not| n(V, V')$ in the statement of [7, Lemma 3.2(ii)] is a misprint of $q \not| n(U'', U)$.)

Lemma 3.1. *(*[7, Lemma 3.2]*) Let H be a group acted by S with $q \not| |H|$. Let $\Gamma = H \rtimes S$ and $H' = C_H(S)$. Let $\tilde{A}$ be a simple k-algebra having a Γ-algebra structure. Let $\tilde{Z}$ be a unique simple $\tilde{A}$-module, which has the $k_*\hat{\Gamma}$-module structure associated with the Γ-algebra structure of $\tilde{A}$, see* [8, Example 10.8]*. If a direct summand U of $\tilde{Z}{\downarrow}_{H'}^{\Gamma}$ is simple projective and there exists an S-invariant simple projective direct summand V of $U{\uparrow}_{H'}^{H}$ such that $q \not| m(V, U{\uparrow}_{H'}^{H})$, then any S-invariant indecomposable direct summand of $U{\uparrow}_{H'}^{H}$ not isomorphic to V has a multiplicity divisible by q.*

Lemma 3.2. *(*[7, Corollary 4.5, Theorem 4.9]*) Any indecomposable direct summand of $\mathcal{O}Gb{\downarrow}_{G\times G'}^{G\times G}$ has a vertex ΔD, and there is a primitive idempotent $f \in (\mathcal{O}Gb)^{G'}$ such that $\mathcal{O}Gf$ is a unique indecomposable direct summand of $\mathcal{O}Gb{\downarrow}_{G\times G'}^{G\times G}$ with a multiplicity m not divisible by q. In fact, $m \equiv \pm 1 \pmod q$. Similar for $\mathcal{O}Gb{\downarrow}_{G'\times G}^{G\times G}$ and $f\mathcal{O}G$. Moreover, $\mathcal{O}Gf$ and $f\mathcal{O}G$ induce a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}G'b'$. In particular, $fb' = f$, $f\mathcal{O}G \otimes_{\mathcal{O}G} \mathcal{O}Gf \simeq \mathcal{O}G'b'$ as $(\mathcal{O}G', \mathcal{O}G')$-bimodules and $\mathcal{O}Gf \otimes_{\mathcal{O}G'} f\mathcal{O}G \simeq \mathcal{O}Gb$ as $(\mathcal{O}G, \mathcal{O}G)$-bimodules.*

Proof of Theorem 1.1(1) Since we have $\mathcal{O}Gb{\downarrow}_{G'\times G'}^{G\times G} \simeq \mathcal{O}Gb{\downarrow}_{G'\times G}^{G\times G} \otimes_{\mathcal{O}G} \mathcal{O}Gb{\downarrow}_{G\times G'}^{G\times G}$, the statement follows from Lemma 3.2. □

Next, we show Theorem 1.1(2). Below, let an idempotent f and an integer m be as in Lemma 3.2.

Note that $\mathcal{O}Gb$ is $\ddot{S}$-invariant, since there is some unit element u_s of $\mathcal{O}Gb$ such that $s^{-1}u_s$ is in the center of $\mathcal{O}Gb$ and so we have $\mathcal{O}Gb^{(s_1,s_2)} \simeq$

$s_1^{-1}\mathcal{O}Gbs_2 \simeq s_1^{-1}u_{s_1}\mathcal{O}Gbu_{s_2}^{-1}s_2 \simeq \mathcal{O}Gb$, see [9, p.551 and Proposition 1]. Similarly, $\mathcal{O}Gf$ is also $\ddot{S}$-invariant.

Note also that $\mathcal{O}Gb$ are covered by q isomorphic blocks and the restriction $\downarrow^E_G$ gives an isomorphism between the category of the $\mathcal{O}E$-modules in a block covering b and the category of the $\mathcal{O}Gb$-modules which preserves a vertex of the modules, see [9, p.553]. Similar for b' and the blocks of E' covering b'.

Lemma 3.3. *$\mathcal{O}Gb$ is an $\ddot{S}$-invariant indecomposable direct summand of $f\mathcal{O}G{\uparrow}^{G\times G}_{G'\times G}$ with a vertex ΔD and with a multiplicity not divisible by q.*

Proof Since $b\big(f\mathcal{O}G{\uparrow}^{G\times G}_{G'\times G}\big) \simeq \mathcal{O}Gb{\downarrow}^{G\times G}_{G\times G'} \otimes_{\mathcal{O}G'} f\mathcal{O}G$, the statement follows from Lemma 3.2. □

Let $E' = G'S$ and let $\hat{b}'$ be a block of E' covering b'.

Noting that b' and b are covered by isomorphic blocks of E' and E, we have $f\mathcal{O}G \simeq \hat{f}\mathcal{O}E{\downarrow}^{E'\times E}_{G'\times G}$ for some primitive idempotent $\hat{f} \in (\mathcal{O}Eb)^{E'}$ such that $\hat{b}'\hat{f} = \hat{f}$ and let $\hat{b}$ be a block of E such that $\hat{b}\hat{f} = \hat{f}$, see [7, Lemma 5.3]. Then $\hat{f}\mathcal{O}E \otimes_{\mathcal{O}E\hat{b}} - \simeq \big(\mathcal{O}E'\hat{b}' \otimes_{\mathcal{O}G'b'} f\mathcal{O}G \otimes_{\mathcal{O}Gb} \mathcal{O}E\hat{b}\big) \otimes_{\mathcal{O}E\hat{b}} -$ induces a Morita equivalence between $\mathcal{O}E'\hat{b}'$ and $\mathcal{O}E\hat{b}$, and so by [6, Proposition 6.5] we have $\mathrm{End}_{\mathcal{O}}(\hat{f}\mathcal{O}E)^{1\times E} \simeq \mathcal{O}E'\hat{b}'$ as interior E'-algebras, which is used in the proof of Lemma 3.4(viii)⇔(ix).

Let $D_{\gamma'}$ be a defect pointed group of $\mathcal{O}E'\hat{b}'$. We also denote by $D_{\gamma'}$ the corresponding local pointed group of $\mathrm{Ind}^E_{E'}(\mathcal{O}E'\hat{b}')$ through the canonical embedding from $\mathcal{O}E'\hat{b}'$ to $\mathrm{Res}^E_{E'}\mathrm{Ind}^E_{E'}(\mathcal{O}E'\hat{b}')$, see [8, Proposition 15.1].

In Lemma 3.4(x), $S(\subset E)$ and its image in $N_E(D_{\gamma'})/D$ is identified.

Lemma 3.4. *There is a multiplicity preserving bijection between the following sets:*

(i) The isomorphism classes of the $\ddot{S}$-invariant indecomposable direct summands of $\mathcal{O}G'b'{\uparrow}^{G\times G'}_{G'\times G'}$ with a vertex ΔD.

(ii) The isomorphism classes of the $\ddot{S}$-invariant indecomposable direct summands of $\mathcal{O}E'\hat{b}'{\uparrow}^{E\times E'}_{E'\times E'}{\downarrow}^{E\times E'}_{G\times E'}$ with a vertex ΔD.

(iii) The $\ddot{S}$-invariant points of $G\times E'$ of the interior $E\times E'$-algebra $\mathrm{End}_{\mathcal{O}}\big(\mathcal{O}E'\hat{b}'{\uparrow}^{E\times E'}_{E'\times E'}\big)$ with a defect group ΔD.

(iv) The S-invariant points of G of the interior E-algebra $\mathrm{End}_{\mathcal{O}}\big(\mathcal{O}E'\hat{b}'{\uparrow}^{E\times E'}_{E'\times E'}\big)^{1\times E'}$ with a defect group D.

(v) The isomorphism classes of the $\ddot{S}$-invariant indecomposable direct summands of $f\mathcal{O}G{\uparrow}^{G\times G}_{G'\times G}$ with a vertex ΔD.

(vi) The isomorphism classes of the $\ddot{S}$-invariant indecomposable direct summands of $\hat{f}\mathcal{O}E{\uparrow}^{E\times E}_{E'\times E}{\downarrow}^{E\times E}_{G\times E}$ with a vertex ΔD.

(vii) *The $\ddot{S}$-invariant points of $G\times E$ of the interior $E\times E$-algebra* $\mathrm{End}_{\mathcal{O}}\big(\hat{f}\mathcal{O}E{\uparrow}_{E'\times E}^{E\times E}\big)$ *with a defect group* ΔD.

(viii) *The S-invariant points of G of the interior E-algebra* $\mathrm{End}_{\mathcal{O}}\big(\hat{f}\mathcal{O}E{\uparrow}_{E'\times E}^{E\times E}\big)^{1\times E}$ *with a defect group D.*

(ix) *The S-invariant points of G of the interior E-algebra* $\mathrm{Ind}_{E'}^{E}\big(\mathcal{O}E'\hat{b}'\big)$ *with a defect group D.*

(x) *The isomorphism classes of the S-invariant projective indecomposable direct summands of* $V_{\mathrm{Ind}_{E'}^{E}(\mathcal{O}E'\hat{b}')}(D_{\gamma'}){\downarrow}_{N_G(D_{\gamma'})/D}^{N_E(D_{\gamma'})/D}$.

Proof (i)⇔(ii): We have an isomorphism

$$\mathcal{O}G'b'{\uparrow}_{G'\times G'}^{G\times G'} \simeq \mathcal{O}E'\hat{b}'{\uparrow}_{E'\times E'}^{E\times E'}{\downarrow}_{G\times E'}^{E\times E'}{\downarrow}_{G\times G'}^{G\times E'},$$

and the restriction ${\downarrow}_{G\times G'}^{G\times E'}$, which is compatible with the action of $\ddot{S}$, gives a vertex preserving bijection between the isomorphism classes of $(\mathcal{O}G,\mathcal{O}E'\hat{b}')$-bimodules and the isomorphism classes of $(\mathcal{O}G,\mathcal{O}G'b')$-bimodules.

(ii)⇔(iii): See [8, Example 13.4 and Proposition 18.11].

(iii)⇔(iv): Since the idempotents of the both sides of

$$\mathrm{End}_{\mathcal{O}}\big(\mathcal{O}E'\hat{b}'{\uparrow}_{E'\times E'}^{E\times E'}\big)^{G\times E'} = \Big(\mathrm{End}_{\mathcal{O}}\big(\mathcal{O}E'\hat{b}'{\uparrow}_{E'\times E'}^{E\times E'}\big)^{1\times E'}\Big)^{G}$$

are the same and $1\times S$ acts trivially on them, there is a multiplicity preserving bijection between the $\ddot{S}$-invariant points of $G\times E'$ of $\mathrm{End}_{\mathcal{O}}\big(\mathcal{O}E'\hat{b}'{\uparrow}_{E'\times E'}^{E\times E'}\big)$ and the S-invariant points of G of $\mathrm{End}_{\mathcal{O}}\big(\mathcal{O}E'\hat{b}'{\uparrow}_{E'\times E'}^{E\times E'}\big)^{1\times E'}$.

Moreover, the points of the former with a defect group ΔD correspond bijectively to the points of the latter with a defect group D, since we have

$$\begin{aligned}
& \mathrm{Tr}_{\Delta P}^{G\times E'}\Big(\mathrm{End}_{\mathcal{O}}\big(\mathcal{O}E'\hat{b}'{\uparrow}_{E'\times E'}^{E\times E'}\big)^{\Delta P}\Big) \\
= \ & \mathrm{Tr}_{P\times E'}^{G\times E'}\Big(\mathrm{Tr}_{\Delta P}^{P\times E'}\Big(\mathrm{End}_{\mathcal{O}}\big(\mathcal{O}E'\hat{b}'{\uparrow}_{E'\times E'}^{E\times E'}\big)^{\Delta P}\Big)\Big) \\
= \ & \mathrm{Tr}_{P\times E'}^{G\times E'}\Big(\mathrm{End}_{\mathcal{O}}\big(\mathcal{O}E'\hat{b}'{\uparrow}_{E'\times E'}^{E\times E'}\big)^{P\times E'}\Big) \\
= \ & \mathrm{Tr}_{P}^{G}\Big(\Big(\mathrm{End}_{\mathcal{O}}\big(\mathcal{O}E'\hat{b}'{\uparrow}_{E'\times E'}^{E\times E'}\big)^{1\times E'}\Big)^{P}\Big),
\end{aligned}$$

for a subgroup P of D. Here, the second equality holds, since we have

$$\begin{aligned}
\mathcal{O}E'\hat{b}'{\uparrow}_{E'\times E'}^{E\times E'}{\downarrow}_{P\times E'}^{E\times E'} \ \Big| \ & \mathcal{O}{\uparrow}_{\Delta D}^{E'\times E'}{\uparrow}_{E'\times E'}^{E\times E'}{\downarrow}_{P\times E'}^{E\times E'} \\
& \simeq \mathcal{O}{\uparrow}_{\Delta D}^{E\times E'}{\downarrow}_{P\times E'}^{E\times E'} \\
& \simeq \textstyle\bigoplus_{(c_1,c_2)\in[\Delta D\backslash E\times E'/P\times E']}\mathcal{O}{\uparrow}_{\Delta D^{(c_1,c_2)}\cap(P\times E')}^{P\times E'} \\
& \simeq \textstyle\bigoplus_{(c_1,c_2)\in[\Delta D\backslash E\times E'/P\times E']}\mathcal{O}{\uparrow}_{(\Delta D^{(c_1,c_2)}\cap(P\times E'))^{(1,c_2^{-1}c_1)}}^{P\times E'}
\end{aligned}$$

(here, c_1 is chosen so that $c_1 \in E'$ since $E = C_E(D)E'$ when $D \lhd E$, see the proof of [9, Corollary 2]) and

$$\Big(\Delta D^{(c_1,c_2)} \cap (P\times E')\Big)^{(1,c_2^{-1}c_1)} = \Delta D^{(c_1,c_1)} \cap (P\times E') = \Delta D \cap (P\times E') = \Delta P$$

and so $\mathrm{Tr}_{\Delta P}^{P\times E'}$ is surjective by [8, Corollary 17.3(a)(c)].

(v)$\Leftrightarrow$(vi)$\Leftrightarrow$(vii)$\Leftrightarrow$(viii): Similar to (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv).

(iv)$\Leftrightarrow$(ix): There are isomorphisms of interior E-algebras

$$\begin{aligned}
&\mathrm{End}_{\mathcal{O}}\big(\mathcal{O}E'\hat{b}'{\uparrow}_{E'\times E'}^{E\times E'}\big)^{1\times E'}\\
\simeq&\mathrm{Ind}_{E'\times E'}^{E\times E'}\Big(\mathrm{End}_{\mathcal{O}}\big(\mathcal{O}E'\hat{b}'\big)\Big)^{1\times E'}\\
\simeq&\mathrm{Ind}_{E'}^{E}\Big(\mathrm{End}_{\mathcal{O}}\big(\mathcal{O}E'\hat{b}'\big)^{1\times E'}\Big)\\
\simeq&\mathrm{Ind}_{E'}^{E}\big(\mathcal{O}E'\hat{b}'\big).
\end{aligned}$$

Here, for the first isomorphism, see [8, Example 16.4]. For the second isomorphism, note that $\sum_{i,j}(g_i,1)\otimes_{\mathcal{O}[E'\times E']} x_{i,j} \otimes_{\mathcal{O}[E'\times E']} (g_j^{-1},1)$ belongs to $\mathrm{Ind}_{E'\times E'}^{E\times E'}\Big(\mathrm{End}_{\mathcal{O}}\big(\mathcal{O}E'\hat{b}'\big)\Big)$, where $g_i, g_j \in [E/E']$ and $x_{i,j} \in \mathrm{End}_{\mathcal{O}}\big(\mathcal{O}E'\hat{b}'\big)$, is $1\times E'$-invariant if and only if $x_{i,j}$ is $1\times E'$-invariant for any i, j.

(viii)$\Leftrightarrow$(ix): There are isomorphisms of interior E-algebras

$$\begin{aligned}
&\mathrm{End}_{\mathcal{O}}\big(\hat{f}\mathcal{O}E{\uparrow}_{E'\times E}^{E\times E}\big)^{1\times E}\\
\simeq&\mathrm{Ind}_{E'\times E}^{E\times E}\Big(\mathrm{End}_{\mathcal{O}}\big(\hat{f}\mathcal{O}E\big)\Big)^{1\times E}\\
\simeq&\mathrm{Ind}_{E'}^{E}\Big(\mathrm{End}_{\mathcal{O}}\big(\hat{f}\mathcal{O}E\big)^{1\times E}\Big)\\
\simeq&\mathrm{Ind}_{E'}^{E}\big(\mathcal{O}E'\hat{b}'\big).
\end{aligned}$$

(ix)$\Leftrightarrow$(x): Let D_δ be a defect pointed group of a point of G of $\mathrm{Ind}_{E'}^{E}(\mathcal{O}E'\hat{b}')$. Then D_δ and $D_{\gamma'}$ are conjugate by an element of E, see [8, Proposition 16.7 and Corollary 18.4]. In fact, they are conjugate by an element of G, since a pointed group $D_{\gamma'}$ of $\mathcal{O}E'\hat{b}'$ is S-invariant. Hence, any point of G of $\mathrm{Ind}_{E'}^{E}(\mathcal{O}E'\hat{b}')$ with a defect group D has a defect pointed group $D_{\gamma'}$.

Then, the statement follows from Puig correspondence, see [8, Theorem 19.1], since the action of S is compatible with the correspondence, see for example the first paragraph in p.341 of [7]. $\square$

Lemma 3.5. *$V_{\mathrm{Ind}_{E'}^{E}(\mathcal{O}E'\hat{b}')}(D_{\gamma'}){\downarrow}_{N_G(D_{\gamma'})/D}^{N_E(D_{\gamma'})/D}$ is projective and has a unique S-invariant indecomposable direct summand with a multiplicity not divisible by q.*

Proof $V_{\mathcal{O}E'\hat{b}'}(D_{\gamma'})\downarrow^{N_{E'}(D_{\gamma'})/D}_{N_{G'}(D_{\gamma'})/D}$ is simple projective, since it is isomorphic to a defect multiplicity module of $\mathcal{O}G'b'$, see [8, Corollary 37.6(a)]. Through the canonical embedding, $V_{\mathcal{O}E'\hat{b}'}(D_{\gamma'})\downarrow^{N_{E'}(D_{\gamma'})/D}_{N_{G'}(D_{\gamma'})/D}$ can be viewed as a direct summand of $V_{\mathrm{Ind}^E_{E'}(\mathcal{O}E'\hat{b}')}(D_{\gamma'})\downarrow^{N_E(D_{\gamma'})/D}_{N_{G'}(D_{\gamma'})/D}$, see [8, Corollary 15.5], and we have

$$\begin{aligned}&V_{\mathrm{Ind}^E_{E'}(\mathcal{O}E'\hat{b}')}(D_{\gamma'})\downarrow^{N_E(D_{\gamma'})/D}_{N_G(D_{\gamma'})/D}\\ \simeq&V_{\mathcal{O}E'\hat{b}'}(D_{\gamma'})\uparrow^{N_E(D_{\gamma'})/D}_{N_{E'}(D_{\gamma'})/D}\downarrow^{N_E(D_{\gamma'})/D}_{N_G(D_{\gamma'})/D}\\ \simeq&V_{\mathcal{O}E'\hat{b}'}(D_{\gamma'})\downarrow^{N_{E'}(D_{\gamma'})/D}_{N_{G'}(D_{\gamma'})/D}\uparrow^{N_G(D_{\gamma'})/D}_{N_{G'}(D_{\gamma'})/D},\end{aligned}$$

see the proof of [1, Proposition 7.4] or the second paragraph of the proof of [7, Corollary 3.6(ii)].

Hence, $V_{\mathrm{Ind}^E_{E'}(\mathcal{O}E'\hat{b}')}(D_{\gamma'})\downarrow^{N_E(D_{\gamma'})/D}_{N_G(D_{\gamma'})/D}$ is projective since $V_{\mathcal{O}E'\hat{b}'}(D_{\gamma'})\downarrow^{N_{E'}(D_{\gamma'})/D}_{N_{G'}(D_{\gamma'})/D}$ is projective, and for the remaining assertion, by Lemma 3.1, it suffices to show that $V_{\mathrm{Ind}^E_{E'}(\mathcal{O}E'\hat{b}')}(D_{\gamma'})\downarrow^{N_E(D_{\gamma'})/D}_{N_G(D_{\gamma'})/D}$ has an S-invariant simple direct summand with a multiplicity not divisible by q.

$\mathcal{O}Gb$ is an $\ddot{S}$-invariant indecomposable direct summand of $f\mathcal{O}G\uparrow^{G\times G}_{G'\times G}$ with a vertex ΔD and with a multiplicity not divisible by q, see Lemma 3.3. Noting an $(\mathcal{O}E,\mathcal{O}E)$-bimodule isomorphism $\mathcal{O}E\hat{b}\simeq\mathcal{O}E\hat{f}\otimes_{\mathcal{O}E'}\hat{f}\mathcal{O}E$, $\mathcal{O}E\hat{b}\downarrow^{E\times E}_{G\times E}$ is an indecomposable direct summand of $\hat{f}\mathcal{O}E\uparrow^{E\times E}_{E'\times E}\downarrow^{E\times E}_{G\times E}$ such that $\mathcal{O}Gb\simeq\mathcal{O}E\hat{b}\downarrow^{E\times E}_{G\times E}\downarrow^{G\times E}_{G\times G}$ (Lemma 3.4(v)(vi)). Let $\ddot{\sigma}$ be a point of $G\times E$ of $\mathrm{End}_{\mathcal{O}}\big(\hat{f}\mathcal{O}E\uparrow^{E\times E}_{E'\times E}\big)$ corresponding to $\mathcal{O}E\hat{b}\downarrow^{E\times E}_{G\times E}$ (Lemma 3.4(vi)(vii)). Then the localized algebra $\mathrm{End}_{\mathcal{O}}\big(\hat{f}\mathcal{O}E\uparrow^{E\times E}_{E'\times E}\big)_{\ddot{\sigma}}$ is isomorphic to an interior $G\times E$-algebra $\mathrm{End}_{\mathcal{O}}\big(\mathcal{O}E\hat{b}\downarrow^{E\times E}_{G\times E}\big)$, see [8, Lemma 12.4]. The set $\ddot{\sigma}$ can be viewed as a point $\dot{\sigma}$ of G of $\mathrm{End}_{\mathcal{O}}\big(\hat{f}\mathcal{O}E\uparrow^{E\times E}_{E'\times E}\big)^{1\times E}$ (Lemma 3.4(vii)(viii)). Note that $j\mathrm{End}_{\mathcal{O}}\big(\hat{f}\mathcal{O}E\uparrow^{E\times E}_{E'\times E}\big)^{1\times E}j=j\mathrm{End}_{\mathcal{O}}\big(\hat{f}\mathcal{O}E\uparrow^{E\times E}_{E'\times E}\big)j^{1\times E}$ for $j\in\dot{\sigma}=\ddot{\sigma}$. In fact, we have $j\mathrm{End}_{\mathcal{O}}\big(\hat{f}\mathcal{O}E\uparrow^{E\times E}_{E'\times E}\big)^{1\times E}j\subseteq j\mathrm{End}_{\mathcal{O}}\big(\hat{f}\mathcal{O}E\uparrow^{E\times E}_{E'\times E}\big)j^{1\times E}$ since j is $1\times E$-invariant, and we have $j\mathrm{End}_{\mathcal{O}}\big(\hat{f}\mathcal{O}E\uparrow^{E\times E}_{E'\times E}\big)^{1\times E}j\supseteq j\mathrm{End}_{\mathcal{O}}\big(\hat{f}\mathcal{O}E\uparrow^{E\times E}_{E'\times E}\big)j^{1\times E}$ since an $\mathcal{O}[1\times E]$-endomorphism of $j\big(\hat{f}\mathcal{O}E\uparrow^{E\times E}_{E'\times E}\big)$ can be extended to an $\mathcal{O}[1\times E]$-endomorphism of $\hat{f}\mathcal{O}E\uparrow^{E\times E}_{E'\times E}$. Let σ be a point of G of $\mathrm{Ind}^E_{E'}\big(\mathcal{O}E'\hat{b}'\big)$ corresponding to $\dot{\sigma}$ through the isomorphism $\mathrm{End}_{\mathcal{O}}\big(\hat{f}\mathcal{O}E\uparrow^{E\times E}_{E'\times E}\big)^{1\times E}\simeq\mathrm{Ind}^E_{E'}\big(\mathcal{O}E'\hat{b}'\big)$ (Lemma 3.4 (viii)(ix)), and let V

be an indecomposable direct summand of $V_{\mathrm{Ind}_{E'}^{E}(\mathcal{O}E'\hat{b}')}(D_{\gamma'})\downarrow^{N_E(D_{\gamma'})/D}_{N_G(D_{\gamma'})/D}$ corresponding to G_σ by Puig correspondence (Lemma 3.4(ix)(x)).

Then V is S-invariant and has a multiplicity not divisible by q.

Moreover, V is simple, since V can be identified with a defect multiplicity module of the localized algebra $\mathrm{End}_{\mathcal{O}}\big(\hat{f}\mathcal{O}E\uparrow^{E\times E}_{E'\times E}\big)^{1\times E}{}_{\dot{\sigma}}$, we have isomorphisms of interior G-algebras

$$\begin{aligned}
&\mathrm{End}_{\mathcal{O}}\big(\hat{f}\mathcal{O}E\uparrow^{E\times E}_{E'\times E}\big)^{1\times E}{}_{\dot{\sigma}}\\
\simeq&\mathrm{End}_{\mathcal{O}}\big(\hat{f}\mathcal{O}E\uparrow^{E\times E}_{E'\times E}\big)_{\ddot{\sigma}}{}^{1\times E}\\
\simeq&\mathrm{End}_{\mathcal{O}}\big(\mathcal{O}E\hat{b}\downarrow^{E\times E}_{G\times E}\big)^{1\times E}\\
\simeq&\mathrm{Res}^{E}_{G}\big(\mathcal{O}E\hat{b}\big)\\
\simeq&\mathcal{O}Gb,
\end{aligned}$$

and $\mathcal{O}Gb$ has a simple defect multiplicity module.

Hence, the assertion holds. □

Proposition 3.6.

(1) $\mathcal{O}Gf$ is a unique $\ddot{S}$-invariant indecomposable direct summand of $\mathcal{O}G'b'\uparrow^{G\times G'}_{G'\times G'}$ with a vertex ΔD and with a multiplicity not divisible by q. In fact, its multiplicity is $m\equiv\pm1$ (mod q).

(2) $f\mathcal{O}G$ is a unique $\ddot{S}$-invariant indecomposable direct summand of $\mathcal{O}G'b'\uparrow^{G'\times G}_{G'\times G'}$ with a vertex ΔD and with a multiplicity not divisible by q. In fact, its multiplicity is $m\equiv\pm1$ (mod q).

(3) $\mathcal{O}Gb$ is a unique $\ddot{S}$-invariant indecomposable direct summand of $f\mathcal{O}G\uparrow^{G\times G}_{G'\times G}$ with a vertex ΔD and with a multiplicity not divisible by q. In fact, its multiplicity is $m\equiv\pm1$ (mod q).

(4) $\mathcal{O}Gb$ is a unique $\ddot{S}$-invariant indecomposable direct summand of $\mathcal{O}Gf\uparrow^{G\times G}_{G\times G'}$ with a vertex ΔD and with a multiplicity not divisible by q. In fact, its multiplicity is $m\equiv\pm1$ (mod q).

Proof (1) Since $b\big(\mathcal{O}G'b'\uparrow^{G\times G'}_{G'\times G'}\big)\simeq\mathcal{O}Gb\downarrow^{G\times G}_{G\times G'}b'$, $\mathcal{O}Gf$ is an $\ddot{S}$-invariant indecomposable direct summand of $\mathcal{O}G'b'\uparrow^{G\times G'}_{G'\times G'}$ with a vertex ΔD and with a multiplicity $m\equiv\pm1$ (mod q) by Lemma 3.2. Uniqueness part follows from Lemma 3.4(i)(x) and Lemma 3.5.

(2) By symmetry (2) is equivalent to (1).

(3) It follows from Lemma 3.3, Lemma 3.4(i)(v) and (1).

(4) By symmetry (4) is equivalent to (3). □

Proof of Theorem 1.1(2) The statement follows from Proposition 3.6(1)(4) (or (2)(3)) and the similar argument in the fifth paragraph of the proof of Lemma 2.1(2). □

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