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NOTE ON THE COHOMOLOGICAL INVARIANT OF PFISTER FORMS

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Abstract. The cohomological invariant ring of the n-Pfister forms is isomorphic to the invariant ring under a $GL_n(\mathbb{Z}/2)$-action in that of an elementary abelian 2-group of rank n.

1. Introduction

Let G be an algebraic group over k with $ch(k) \neq 2$. The cohomological invariant ring $Inv^*(G;\mathbb{Z}/2) = \oplus_i Inv^i(G;\mathbb{Z}/2)$ of G is the ring generated by natural functors $H^1(K;G) \to H^i(K;\mathbb{Z}/2)$ for the category of finitely generated fields K over k. (For details, see the excellent book [Ga-Me-Se]). Moreover, we can define the cohomological invariant ring $Inv^*(Pfister_n;\mathbb{Z}/2)$ of n-Pfister forms, although the corresponding group G does not exist for $n \geq 4$. This ring has been computed by Serre using elementary but very elegant arguments in Theorem 18.1 in [Ga-Me-Se].

In this note, we show that this ring for n-Pfister forms can be identified with the invariant ring under a $GL_n(\mathbb{Z}/2)$-action in that of an elementary abelian 2-group of rank n, namely,

$$Inv^*(Pfister_n;\mathbb{Z}/2) \cong Inv^*((\mathbb{Z}/2)^n;\mathbb{Z}/2)^{GL_n(\mathbb{Z}/2)}.$$

To show this, we use some machinery from motivic cohomology and Dickson algebras, which can be applied for other groups G.

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2. Motivic cohomology and cohomological invariant

We recall the motivic cohomology $H^{*,*'}(X;\mathbb{Z}/2) = \oplus_{p,q} H^{p,q}(X;\mathbb{Z}/2)$ for a smooth scheme X over k with $ch(k) \neq 2$. By the Milnor conjecture (which has been solved by Voevodsky), we know $H^{p,q}(X;\mathbb{Z}/2) \cong H^p_{et}(X;\mathbb{Z}/2)$ for $p \leq q$. Consider $\tau \in H^{0,1}(Spec(k);\mathbb{Z}/2) \cong \mathbb{Z}/2$ as a nonzero element. It is known that $H^{p,q}(X;\mathbb{Z}/2) = 0$ for $(p-q) > dim(X)$. Hence we obtain

$$H^{*,*'}(Spec(k);\mathbb{Z}/2) \cong H^*_{et}(Spec(k);\mathbb{Z}/2)[\tau].$$

Let us write $H^{*,*'} = H^{*,*'}(Spec(k);\mathbb{Z}/2)$ and $H^* = H^*_{et}(Spec(k);\mathbb{Z}/2)$ so that $H^{*,*'} \cong H^*[\tau]$.

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Let BG be the classifying space of G ([To], [Vo2]). Let $H^p(X;H^q_{\mathbb{Z}/2})$ be the sheaf cohomology where $H^q_{\mathbb{Z}/2}$ is the Zariski sheaf induced from the presheaf $H^q_{et}(V;\mathbb{Z}/2)$ for each open subset V of X. Then Totaro proved that

$$Inv^*(BG;\mathbb{Z}/2) \cong H^0(BG;H^*_{\mathbb{Z}/2})$$

in a letter to Serre [Ga-Me-Se]. The Milnor conjecture implies the Beilinson-Lichtenbaum conjecture (see [Vo2,3]). This fact implies the following long exact sequence of motivic and sheaf cohomology theories (Lemma 3.1 in [Or-Vi-Vo], [Vo3])

$$\to H^{p,q-1}(X;\mathbb{Z}/2) \stackrel{\times\tau}{\to} H^{p,q}(X;\mathbb{Z}/2)$$

$$\to H^{p-q}(X;H^q_{\mathbb{Z}/2}) \to H^{p+1,q-1}(X;\mathbb{Z}/2) \stackrel{\times\tau}{\to} ...$$

Thus we have

Theorem 2.1. *There is an additive isomorphism*

$$Inv^p(G;\mathbb{Z}/2) \cong H^{p,p}(BG;\mathbb{Z}/2)/(\tau) \oplus Ker(\tau)|H^{p+1,p-1}(BG;\mathbb{Z}/2).$$

As an application, we first consider the case where $G=\mathbb{Z}/2$. The $mod(2)$ motivic cohomology is computed in [Vo1,2].

$$H^{*,*'}(B\mathbb{Z}/2;\mathbb{Z}/2) \cong H^{*,*'}[y]\otimes\Delta(x) = H^{*,*'}[y]\{1,x\}$$

with $\beta(x)=y$, hence $deg(y)=(2,1)$ and $deg(x)=(1,1)$. Here Voevodsky shows ([Vo1,2]) as follows:

$$x^2=\tau y+\rho x \quad with\ \rho=(-1)\in H^1=k^*/(k^*)^2.$$

Next, we consider their product $G=(\mathbb{Z}/2)^n$. The cohomology $H^{*,*'}(B\mathbb{Z}/2;\mathbb{Z}/2)$ has the Kunneth formula (by Voevodsky [Vo2]). Hence the motivic cohomology is given by

$$H^{*,*'}(BG;\mathbb{Z}/2) \cong H^{*,*'}[y_1,...,y_n]\otimes\Delta(x_1,..,x_n)$$

where $\beta(x_i)=y_i$ and $x_i^2=\tau y_i+\rho x_i$. Hence from Theorem 2.1, we obtain (as stated 16.4 in [Ga-Me-Se])

Lemma 2.2. *Let G be an elementary abelian 2-group of rank n. Then $Inv^*(G;\mathbb{Z}/2)\cong H^*\otimes\Delta(x_1,...,x_n)$ with $x_i^2=\rho x_i$.*

3. Dickson invariants

Recall that the mod 2 (topological) cohomology

$$H^*(B(\mathbb{Z}/2)^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, ..., x_n] \quad |x_i| = 1.$$

It is well known that the invariant ring under the $GL_n(\mathbb{Z}/2)$-action is the Dickson algebra

$$H^*(B(\mathbb{Z}/2)^n;\mathbb{Z}/2)^{GL_n(\mathbb{Z}/2)} \cong \mathbb{Z}/2[d_{n,0}, ..., d_{n,n-1}]$$

where each generator (Dickson element) $d_{n,i}$ is given by

$$\begin{aligned} w_t(x) &= \Pi_{\epsilon_i=0 \ or \ 1}(t + \epsilon_1 x_1 + ... + \epsilon_n x_n) \\ &= t^{2^n} + d_{n,n-1}t^{2^{n-1}} + d_{n,n-2}t^{2^{n-2}} + ... + d_{n,0}t. \end{aligned}$$

Examples. Let w_i be the i-th elementary symmetric function for x_j in $H^*(B(\mathbb{Z}/2)^n;\mathbb{Z}/2)$. Then,

$$\begin{cases} d_{2,1} = x_1^2 + x_1x_2 + x_2^2 = w_1^2 + w_2 \\ d_{2,0} = x_1^2x_2 + x_1x_2^2 = w_1w_2. \end{cases}$$

To recognize the Dickson elements in the cohomological invariant ring, let us consider it in the following ring

$$U = \mathbb{Z}/2[\rho] \otimes \Delta(x_1, ..., x_n), \quad x_i^2 = \rho x_i.$$

For example, in U, we see $d_{2,0} = \rho x_1x_2 + x_1\rho x_2 = 0$, and

$$d_{2,1} = \rho x_1 + x_1x_2 + \rho x_2 = \rho w_1 + w_2.$$

Lemma 3.1. *In U, we have $d_{n,i} = 0$ for $i < n-1$ and*

$$d_{n,n-1} = \sum_{i\geq 1}^{n} w_i\rho^{2^{n-1}-i} = (\rho + x_1)...(\rho + x_n)\rho^{2^{n-1}-n} + \rho^{2^{n-1}}.$$

Proof. Decompose that

$$w_t(x) = \Pi(t + \epsilon_1x_1 + ...\epsilon_{n-1}x_{n-1}) \times \Pi(t + x_n + \epsilon_1x_1 + ...\epsilon_{n-1}x_{n-1}).$$

By induction on n, we assume that this element is

$$(t^{2^{n-1}} + d_{n-1,n-2}t^{2^{n-2}})((t + x_n)^{2^{n-1}} + d_{n-1,n-2}(t + x_n)^{2^{n-2}}).$$

By setting $d_{n-1,n-2} = d$, $t^{2^{n-2}} = T$ and $x_n^{2^{n-2}} = X$, the above formula is written as

$$\begin{aligned} &(T^2 + dT)(T^2 + X^2 + dT + dX) \\ &= T^4 + (d^2 + dX + X^2)T^2 + (d^2X + dX^2)T. \end{aligned}$$

Here note $X^2 = \rho^m X = \rho^\ell x_n$, $d^2 = \rho^m d$ (since $(\rho + x)^2 = \rho(\rho + x)$). So we obtain $(d^2X + dX^2) = 0$.

Let $a_i = (\rho + x_1)...(\rho + x_i)$ so that $a_{n-1} = d + \rho^k$. Then we have

$$\begin{aligned} d_{n,n-1} &= d^2 + dX + X^2 = \rho^m d + \rho^{m-1} dx_n + \rho^\ell x_n \\ &= \rho^s(a_{n-1} + \rho^k) + \rho^{s-1}(a_{n-1} + \rho^k)x_n + \rho^\ell x_n \\ &= \rho^{s-1} a_{n-1}(\rho + x_n) + \rho^r = \rho^{s-1} a_n + \rho^r \end{aligned}$$

for some $m, \ell, k, s, r \geq 1$, as desired. □

Corollary 3.2. *Let us write*

$$e_n = \rho^{-2^{n-1}+n} d_{n,n-1} = \sum_{i \geq 1}^{n} w_i \rho^{n-i} = (\rho + x_1)...(\rho + x_n) + \rho^n.$$

Then we have the ring isomorphism

$$Inv^*((\mathbb{Z}/2)^n; \mathbb{Z}/2)^{GL_n(\mathbb{Z}/2)} \cong H^*\{1, e_n\}, \quad e_n^2 = \rho^n e_n.$$

Proof. By $Ideal(\rho)$, we consider the associated graded algebra

$$gr(H^* \otimes \Delta(x_1, ..., x_n)) \cong gr(H^*) \otimes \Lambda(x_1, ..., x_n), \quad (x_i^2 = 0).$$

Note $e_n = w_n$ in the above graded algebra. We can see

$$\Lambda(x_1, ..., x_n)^{GL_n(\mathbb{Z}/2)} \cong \mathbb{Z}/2\{1, w_n\}$$

from the following arguments.

The invariant ring of $\mathbb{Z}/2[x_1, ..., x_n]$ under the n-th Symmetric group is multiplicatively genarated by w_s for $s \leq n$. Here $x_i^2 = 0$ in $gr(U)$, and hence its invariant ring is additively generated by w_s for $s \leq n$.

Suppose $s < n$ and write

$$w_s = x_1(\sum_{1 \neq i_k} x_{i_1}...x_{i_{s-1}}) + (\sum_{1 \neq i_k} x_{i_1}...x_{i_s}).$$

Consider the action $x_{12} : x_1 \mapsto x_1 + x_2$ but $x_{12} : x_i \mapsto x_i$ for $i > 1$. Then

$$(x_{12} - 1)w_s = x_2(\sum_{1 \neq i_k} x_{i_1}...x_{i_{s-1}}) = x_2 x_3 ... x_{s+1} + ... \neq 0 \quad \text{in } \Lambda(x_1, ..., x_n).$$

All elements in H^* and e_n are invariants in $H^* \otimes \Delta(x_1, ..., x_n) = H^* \otimes_{\mathbb{Z}/2[\rho]} U$. Thus we have the corollary. □

Let $n = 2$ and $G = SO_3$, or $n = 3$ and $G = G_2$ the exceptional group. Then G has only one conjugacy class A_n of maximal elementary abelian 2-groups of rank n. The Weyl group $W_G(A_n)$ is isomorphic to $GL_n(\mathbb{Z}/2)$. Hence we have the restriction map

$$Inv^*(G; \mathbb{Z}/2) \to Inv^*(A_n; \mathbb{Z}/2)^{W_G(A_n)} \cong H^*\{1, e_n\}.$$

The result in [Ga-Me-Se] shows that this map is an isomorphism. This fact holds true for $n \geq 4$ as stated in the next section, although the corresponding group G does not exist.

4. Pfister forms

The most important quadratic forms are Pfister forms. Given $a = (a_1, ..., a_n) \in (k^*/(k^*)^2)^{\times n}$, the n-th Pfister form P_a is defined as

$$P_a = \langle\langle a_1, ..., a_n\rangle\rangle = \langle 1, -a_1\rangle \otimes ... \otimes \langle 1, -a_n\rangle$$
$$= \oplus_{1\le i_1<...<i_s\le n}\langle (-a_{i_1})...(-a_{i_s})\rangle.$$

Given a quadratic form $q_a = \langle a_1, ..., a_n\rangle$, the total Stiefel-Whitney class is given by

$$w_t(q_a) = \Pi(t + x_i) \in H^*[t] \quad with\ x_i = (a_i) \in H^1.$$

Hence we obtain

$$w_t(P_a) = \Pi_{\epsilon_i=0\ or\ 1}(t + \epsilon_1(\rho + x_1) + ... + \epsilon_n(\rho + x_n))$$

identifying $x_i = (a_i)$. Thus the following proposition follows the preceding lemma. (Substitute $x_i + \rho$ for x_i on the right-hand side of the equation in Lemma 3.1.)

Proposition 4.1. *Let $x_i = (a_i) \in k^*/(k^*)^2$ and $w_n = x_1...x_n$. Then*

$$w_t(P_a) = t^{2^n} + (w_n + \rho^n)\rho^{2^{n-1}-n}t^{2^{n-1}}.$$

Next, we consider the map from $(k^*/(k^*)^2)^{\times n}$ to the set $Pfist_n$ of n-th Pfister forms defined by

$$p : a = (a_1, ..., a_n) \mapsto P_{-a} = \langle\langle -a_1, ..., -a_n\rangle\rangle = \langle 1, a_1\rangle \otimes ... \otimes \langle 1, a_n\rangle.$$

This map induces the map of cohomological invariant rings

$$p^* : Inv^*(Pfister_n; \mathbb{Z}/2) \to Inv^*((k^*/(k^*)^2)^{\times n}; \mathbb{Z}/2).$$

Here the last invariant ring is isomorphic to

$$Inv^*((\mathbb{Z}/2)^n; \mathbb{Z}/2) \cong H^* \otimes \Delta(x_1, ..., x_n).$$

On $H^* \otimes \Delta(x_1, ..., x_n)$, we can define the usual $GL_n(\mathbb{Z}/2)$-action. This action is also written as follows. Consider the Bruhat decomposition

$$GL_n(\mathbb{Z}/2) = \coprod_{w\in S_n} BwB$$

where B is the Borel group generated by upper triangular matrices, and S_n is the n-th symmetric group generated by transition matrices. The group B is generated by $x_{ij} = 1 + e_{ij}$; the elementary matrix with (i, j) entries 1 with the following relations

$$x_{ij}^2 = 1, \quad [x_{ij}, x_{kl}] = \begin{cases} x_{il} & \text{if } j = k\,; \\ 0 & otherwise. \end{cases}$$

Define $w(x_i) = x_{w(i)}$ for $w \in S_n$ and

$$x_{ij}(x_i) = x_i + x_j, \quad x_{ij}(x_k) = x_k \;\; for\ i \neq k.$$

Then the $GL_n(\mathbb{Z}/2)$-action is decided on $Inv^*((\mathbb{Z}/2)^n;\mathbb{Z}/2)$.

Theorem 4.2. *The above map p^* induces the isomorphism of rings*

$$Inv^*(Pfister_n;\mathbb{Z}/2) \cong H^* \otimes \Delta(x_1, ..., x_n)^{GL_n(\mathbb{Z}/2)} \cong H^*\{1, e_n\}.$$

Proof. On $(k^*/(k^*)^2)^{\times n}$, we can define the $GL_n(\mathbb{Z}/2)$-action by

$$x_{ij}(a_1, ...a_n) = (a_1, ..., a_{i-1}, a_i a_j, a_{i+1}, ..., a_n),$$
$$w(a_1, ..., a_n) = (a_{w(1)}, ..., a_{w(n)}).$$

This induces the action on $Inv^*((\mathbb{Z}/2)^n;\mathbb{Z}/2)$ by $w(x_i) = x_{w(i)}$ and

$$x_{ij}(x_i) = x_i + x_j, \quad x_{ij}(x_k) = x_k \;\; for\ i \neq k.$$

Define a $GL_n(\mathbb{Z}/2)$ action on $Pfister_n$ by setting $x_{ij}p(a) = p(x_{ij}(a))$. Then this action is invariant, indeed,

$$\begin{aligned} x_{12}\langle\langle a_1, a_2\rangle\rangle &= px_{12}(-a_1, -a_2) = p(a_1a_2, -a_2) \\ &= \langle\langle -a_1a_2, a_2\rangle\rangle = \langle 1, a_1a_2, -a_2, -a_1a_2^2\rangle \\ &= \langle 1, a_1a_2, -a_2, -a_1\rangle = \langle\langle a_1, a_2\rangle\rangle. \end{aligned}$$

Hence we have the map

$$q^* : Inv^*(Pfister_n;\mathbb{Z}/2) \to H^* \otimes \Delta(x_1, ..., x_n)^{GL_n(\mathbb{Z}/2)}.$$

The following map

$$P_{-a} \mapsto x_1'...x_n' + \rho^n \in H^* \quad with\ x_i' = x_i + \rho = (-a_i)$$

represents a cohomological invariant for $Pfister_n$, which restricts to e on that for $(\mathbb{Z}/2)^n$. (In fact, the Pfister form P_a is defined from each symbol $x_1...x_n \in H^*$ which is also reconstructed from the Pfister form uniquely.) Hence the above map q^* is an epimorphism.

For each finitely generated field K over k, the map $p : (K^*/(K^*)^2)^n \to Pfister_n|_K$ is obviously an epimorphism. Hence the induced map $p^* : Inv^*(Pfister_n;\mathbb{Z}/2) \to Inv^*((k^*/(k^*)^2)^{\times n};\mathbb{Z}/2)$ is defined by

$$p^*(x) : (K^*/(K^*)^2)^n \to Pfister_n \xrightarrow{x} H^n(K;\mathbb{Z}/2),$$

which is always a monomorphism. □

If we consider the map

$$q : a = (a_1, ..., a_n) \mapsto P_a = \langle\langle a_1, ..., a_n\rangle\rangle,$$

then the map q^* also induces the isomorphism

$$Inv^*(Pfister_n;\mathbb{Z}/2) \cong H^* \otimes \Delta(x_1, ..., x_n)^{GL_n(\mathbb{Z}/2)_{II}} \cong H^*\{1, w_n\}$$

where $GL(\mathbb{Z}/2)_{II}$ is the unusual action defined by $x_{ij}(x_i) = \rho + x_i + x_j$.

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