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ON MEANS OF BANACH-SPACE-VALUED FUNCTIONS

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Abstract. We continue to study relations among exponential and polynomial growth orders of the γ-th order Cesàro means ($\gamma \geq 0$) and of the Abel mean for a Banach-space-valued function u on the interval $[0, \infty)$. We have already studied the problem for a continuous function u. Now we assume that u is a locally integrable function in a Banach space or an improperly locally integrable positive function in a Banach lattice.

1. Introduction and preliminaries

Let X be a Banach space. We define

$$L_1^{loc}([0,\infty),X) := \left\{ u : [0,\infty) \to X \;\middle|\; \begin{array}{l} u \text{ is Bochner integrable on} \\ [0,\, b] \text{ for all } 0 < b < \infty \end{array} \right\}.$$

When X is a Banach lattice, we also define

$$L_1^{improp.\,loc}([0,\infty),X^+)$$

$$:= \left\{ u : [0,\infty) \to X^+ \;\middle|\; \begin{array}{l} u \text{ is Bochner integrable on } [a,\, b] \text{ for all} \\ 0 < a < b < \infty \text{ and } \int_0^1 u(s)\, ds := \\ \lim_{a \downarrow 0} \int_a^1 u(s)\, ds \text{ exists (in } X\text{-norm)} \end{array} \right\},$$

where X^+ denotes the positive cone of X. We note that $L_1^{improp.\,loc}([0,\infty), X^+)$ is not a subset of $L_1^{loc}([0,\infty),X)$ in general.

Unless the contrary is explicitely specified, we assume below that
(A) u is a function in $L_1^{loc}([0,\infty),X)$ with X a Banach space, or
(B) u is a function in $L_1^{improp.\,loc}([0,\infty),X^+)$ with X a Banach lattice.

Assuming that u is continuous on $[0,\infty)$, we have studied relations among exponetial and polynomial growth orders of the γ-th order Cesàro mean ($\gamma \geq 0$) and of the Abel mean for u (cf. Chen-Sato-Shaw [3]). In this paper we continue to study the problem under the assumption that u satisfies (A) or (B); the aim is to generalize the results of [3] to such a function u. (See also Chen-Sato [2], Li-Sato-Shaw [5]–[7], and Sato [9]–[12].)

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Let $\gamma \geq 0$ be a real number. Then the *γ-th oder Cesàro mean* $\mathfrak{c}_t^\gamma$ of u over $[0,t]$ is defined as $\mathfrak{c}_0^\gamma := u(0)$ and, for $t > 0$,

$$(1)\quad \mathfrak{c}_t^\gamma = \mathfrak{c}_t^\gamma(u) \ := \ \begin{cases} u(t) & \text{if } \gamma = 0, \\ \gamma t^{-\gamma} \int_0^t (t-s)^{\gamma-1} u(s)\, ds & \text{if } \gamma > 0 \text{ and the integral exists} \end{cases}$$
$$= \ (k_{\gamma+1}(t))^{-1}(k_\gamma * u)(t),$$

where $k_0 := \delta_0$, the Dirac measure at 0, and $k_\gamma(t) := t^{\gamma-1}/\Gamma(\gamma)$ for $t \geq 0$ if $\gamma > 0$. In particular, it follows that $\mathfrak{c}_t^1 = t^{-1}\int_0^t u(s)\, ds$ for all $t > 0$.

We note that under the assumption (A) [resp. (B)] the following hold. (a) If $0 < \gamma < 1$, then the Bochner integral $\int_0^t (t-s)^{\gamma-1} u(s)\, ds$ [resp. $\int_0^t (t-s)^{\gamma-1} u(s)\, ds = \lim_{a\downarrow 0} \int_a^t (t-s)^{\gamma-1} u(s)\, ds$] does not necessarily exist for all $t > 0$, but exists for dt-almost all $t > 0$. (b) If $\gamma \geq 1$, then the Bochner integral $\int_0^t (t-s)^{\gamma-1} u(s)\, ds$ [resp. $\int_0^t (t-s)^{\gamma-1} u(s)\, ds = \lim_{a\downarrow 0} \int_a^t (t-s)^{\gamma-1} u(s)\, ds$] exists for all $t > 0$, and the function $t \mapsto \int_0^t (t-s)^{\gamma-1} u(s)\, ds$ becomes continuous on $(0,\infty)$ and satisfies

$$(2)\qquad \lim_{t\downarrow 0} \left\| \int_0^t (t-s)^{\gamma-1} u(s)\, ds \right\| = 0.$$

Further we note that if u satisfies the additional hypothesis $\|\chi_{[a,\,b]} u\|_\infty < \infty$ for all $0 < a < b < \infty$, then $\int_0^t (t-s)^{\gamma-1} u(s)\, ds$ exists for all $\gamma > 0$ and $t > 0$, and the function $t \mapsto \int_0^t (t-s)^{\gamma-1} u(s)\, ds$ becomes continuous on $(0,\infty)$; but in general (2) cannot be expected for $0 < \gamma < 1$. (For example, let $u(s) := s^{\beta-1}$ for $s \geq 0$, where $\beta > 0$. Then $u \in L_1^{loc}([0,\infty), \mathbb{R}^+)$, and $\int_0^t (t-s)^{\gamma-1} u(s)\, ds = \int_0^t (t-s)^{\gamma-1} s^{\beta-1}\, ds = t^{\gamma+\beta-1}\int_0^1 (1-s)^{\gamma-1} s^{\beta-1}\, ds = t^{\gamma+\beta-1} B(\gamma,\beta)$. It follows that $\lim_{t\downarrow 0} \int_0^t (t-s)^{\gamma-1} u(s)\, ds = \infty$ whenever $0 < \gamma < 1-\beta$.)

For $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$ the *Abel mean* $\mathfrak{a}_\lambda$ of u is defined as

$$(3)\qquad \mathfrak{a}_\lambda = \mathfrak{a}_\lambda(u) := \lambda \int_0^\infty e^{-\lambda s} u(s)\, ds = \lambda \lim_{t\to\infty} \int_0^t e^{-\lambda s} u(s)\, ds$$

if the limit exists. The *abscissa of convergence* $\sigma(u)$ of the Laplace integral $\widehat{u}(\lambda) = \int_0^\infty e^{-\lambda s} u(s)\, ds := \lim_{t\to\infty} \int_0^t e^{-\lambda s} u(s)\, ds$ is defined as

$$\sigma(u) := \inf \left\{ \mathrm{Re}\lambda \ : \ \lim_{t\to\infty} \int_0^t e^{-\lambda s} u(s)\, ds \text{ exists} \right\}.$$

u is said to be *Laplace transformable* if $\sigma(u) < \infty$. It is known that the Laplace integral $\widehat{u}(\lambda)$ exists for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > \sigma(u)$, and the vector-valued function $\widehat{u} : \lambda \to \widehat{u}(\lambda)$ is analytic on the domain $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > \sigma(u)\}$ (see e.g. Theorems 1.4.1 and 1.5.1 of [1]). If the function $s \mapsto e^{-\lambda s} u(s)$ is Bochner integrable on $[0,\infty)$ for some $\lambda = \lambda_0$, then, for all $\lambda \in \mathbb{C}$ with

$\mathrm{Re}\lambda \geq \lambda_0$, the Bochner integral $\int_0^\infty e^{-\lambda s}u(s)\,ds$ exists and it agrees with $\widehat{u}(\lambda)$ by the dominated convergence theorem. In particular, we have $\lambda_0 \geq \sigma(u)$.

The function u is said to be *dt-exponentially bounded* if $\|u(t)\| \leq Me^{wt}$ for some $M > 0$, $w \in \mathbb{R}$ and dt-almost all $t \geq 0$. If there exist $M > 0$, $K > 0$ and $w \in \mathbb{R}$ such that $\|u(t)\| \leq Me^{wt}$ for dt-almost all $t \geq K$, then u is said to be *dt-exponentailly bounded* (at ∞), and we write $\|u(t)\| = O(e^{wt})$ (mod dt) as $t \to \infty$. We then define the *dt-exponential growth order* $w_0(u)$ of u (at ∞) as

$$w_0(u) := \inf\{w \in \mathbb{R} \,:\, \|u(t)\| = O(e^{wt}) \text{ (mod } dt) \text{ as } t \to \infty\}.$$

It is clear that $\int_0^\infty e^{-\lambda s}u(s)\,ds = \lim_{t\to\infty}\int_0^t e^{-\lambda s}u(s)\,ds$ exists for all λ with $\mathrm{Re}\lambda > w_0(u)$. It follows that $\sigma(u) \leq w_0(u)$. When $w_0(u) \leq 0$, u is said to be *sub-exponential*. When $\sigma(u) \leq 0$, one can define the *growth order* $\alpha_0(\mathfrak{a}.)$ of $\mathfrak{a}.$ (at 0) as

$$\alpha_0(\mathfrak{a}.) := \inf\{\alpha \in \mathbb{R} \,:\, \|\mathfrak{a}_\lambda\| = O(\lambda^{-\alpha}) \text{ as } \lambda \downarrow 0\}. \tag{4}$$

Similarly, one can define the *dt-polynomial growth order* $\alpha_0(u)$ of u (at ∞) as

$$\alpha_0(u) := \inf\{\alpha \in \mathbb{R} \,:\, \|u(t)\| = O(t^\alpha) \text{ (mod } dt) \text{ as } t \to \infty\}, \tag{5}$$

where $\|u(t)\| = O(t^\alpha)$ (mod dt) as $t \to \infty$ [resp. $t \to +0$] means that there exist $M > 0$ and $K > 0$ such that $\|u(t)\| \leq Mt^\alpha$ for dt-almost all $t \geq K$ [resp. $0 < t \leq K$]. If $\alpha_0(u) < \infty$, then u is said to be *dt-polynomially bounded* (at ∞).

Finally, $\|u(t)\| = o(t^\alpha)$ (mod dt) as $t \to \infty$ [resp. $t \to +0$] means that for any $\epsilon > 0$ there exists $K > 0$ such that $\|u(t)\| < \epsilon\, t^\alpha$ for dt-almost all $t \geq K$ [resp. $0 < t \leq K$]. If there exists $x \in X$ such that $\|u(t) - x\| = o(1)$ (mod dt) as $t \to \infty$ [resp. $t \to +0$], we write $x = \lim_{t\to\infty} u(t)$ (mod dt) [resp. $x = \lim_{t\to+0} u(t)$ (mod dt)].

2. Estimates of growth orders

The next lemma is formulated and proved in [3] (see Lemma 2.1 therein).

Lemma 2.1. *Let $\gamma \geq 0$. Then the Laplace transform $\widehat{k_\gamma}$ of k_γ is given as $\widehat{k_\gamma}(\lambda) = \lambda^{-\gamma}$ for all $\lambda > 0$. Therefore, for all $r, s \geq 0$, $\widehat{k_{r+s}} = \widehat{k_r}\widehat{k_s}$ so that $k_r * k_s = k_{r+s}$, where $k_r * k_s$ denotes the convolution of k_r and k_s.*

Lemma 2.2. *Suppose assumption (A) holds. Let γ, $\beta > 0$. Then $(k_\gamma * u)(t)$ exists for dt-almost all $t > 0$, and the function $t \mapsto (k_\gamma * u)(t)$ belongs to $L_1^{loc}([0,\infty), X)$. Further*

$$(k_\beta * (k_\gamma * u))(t) = (k_{\gamma+\beta} * u)(t) \tag{6}$$

for dt-almost all $t > 0$.

Proof. Define $u_D(s) := \chi_{[0,D)}(s)u(s)$ for $D > 0$. Then $u_D \in L_1([0,\infty), X)$ and $(k_\gamma * u)(t) = (k_\gamma * u_D)(t)$ for all $0 < t < D$. Thus by Fubini's theorem

$$\int_0^D \|(k_\gamma * u)(t)\|\, dt \le \int_0^D \int_0^t k_\gamma(t-s)\|u(s)\|\, ds\, dt$$
$$= \int_0^D \Big(\int_s^D k_\gamma(t-s)\, dt\Big)\|u(s)\|\, ds \le \Big(\int_0^D k_\gamma(t)\, dt\Big)\|u_D\|_1 < \infty.$$

It follows that $(k_\gamma * u)(t)$ exists for dt-almost all $t > 0$, and the function $k_\gamma * u$ belongs to $L_1^{loc}([0,\infty), X)$. Similarly

$$(k_\beta * (k_\gamma * u))(t) = (k_\beta * (k_\gamma * u_D))(t) = ((k_\beta * k_\gamma) * u_D)(t)$$
$$= (k_{\gamma+\beta} * u_D)(t) = (k_{\gamma+\beta} * u)(t)$$

for dt-almost all $0 < t < D$, where the third equality comes from Lemma 2.1. This completes the proof. □

Lemma 2.3. *Suppose asumption (B) holds. Let γ, $\beta > 0$. Then $(k_\gamma * u)(t)$ exists for dt-almost all $t > 0$, and the function $t \mapsto (k_\gamma * u)(t)$ belongs to $L_1^{improp.\,loc}([0,\infty), X^+)$. Further (6) holds whenever either side of (6) exists.*

Proof. The function $u_D(s) = \chi_{[0,\,D)}(s)u(s)$ satisfies $u_D \in L_1^{improp.\,loc}([0,\infty), X^+)$ and $u - u_D \in L_1^{loc}([0,\infty), X^+)$ for all $D > 0$. Since $\int_0^t k_\gamma(t-s)u_D(s)\, ds$ exists for all $t > D$, we apply Lemma 2.2 to obtain that

$$\int_0^t k_\gamma(t-s)u(s)\, ds = \int_0^t k_\gamma(t-s)u_D(s)\, ds + \int_0^t k_\gamma(t-s)(u-u_D)(s)\, ds$$

exists for dt-almost all $t > D$. Consequently $(k_\gamma * u)(t)$ exists for dt-almost all $t > 0$, and the function $t \mapsto (k_\gamma * u)(t)$ becomes a positive X-valued strongly measurable function on $(0,\infty)$.

We next prove that the function $k_\gamma * u$ belongs to $L_1^{improp.\,loc}([0,\infty), X^+)$. For this purpose, let $0 < \delta < \epsilon$. If $0 < \eta < \delta$ and $\delta \le t \le \epsilon$, then

$$\Big\|(k_\gamma * u)(t) - \int_\eta^t k_\gamma(t-s)u(s)\, ds\Big\| = \Big\|\int_0^\eta k_\gamma(t-s)u(s)\, ds\Big\|$$
$$\le \max\{k_\gamma(\delta-\eta),\, k_\gamma(\epsilon)\}\,\Big\|\int_0^\eta u(s)\, ds\Big\| \to 0 \quad \text{as } \eta \downarrow 0.$$

Since the function $t \mapsto \int_\eta^t k_\gamma(t-s)u(s)\, ds$ is Bochner integrable on $[\delta,\epsilon]$, it follows that the function $k_\gamma * u$ is Bochner integrable on $[\delta,\epsilon]$. Further we have

$$\int_\delta^\epsilon (k_\gamma * u)(t)dt = \int_\delta^\epsilon \left\{\int_0^t k_\gamma(t-s)u(s)ds\right\} dt = \lim_{\eta\downarrow 0}\int_\delta^\epsilon \left\{\int_\eta^t k_\gamma(t-s)u(s)ds\right\} dt$$

(by Lebesgue's convergence theorem)

$$= \lim_{\eta\downarrow 0}\int_\eta^\delta \left\{\int_\delta^\epsilon k_\gamma(t-s)dt\right\} u(s)\, ds + \int_\delta^\epsilon \left\{\int_s^\epsilon k_\gamma(t-s)dt\right\} u(s)\, ds$$

(by Fubini's theorem)

$$= \int_0^\delta \left\{\int_\delta^\epsilon k_\gamma(t-s)dt\right\} u(s)\, ds + \int_\delta^\epsilon \left\{\int_s^\epsilon k_\gamma(t-s)dt\right\} u(s)\, ds \in X^+.$$

Given $\eta > 0$, we can choose $\epsilon^\sim > 0$ so that

$$\int_0^{\epsilon^\sim} k_\gamma(s)ds < \eta \quad \text{and} \quad \left\|\int_0^{\epsilon^\sim} u(s)ds\right\| < \eta.$$

Then $0 < \delta < \epsilon < \epsilon^\sim$ implies

$$\left\|\int_\delta^\epsilon (k_\gamma * u)(t)dt\right\| \le \left\|\int_0^\delta \left\{\int_\delta^\epsilon k_\gamma(t-s)dt\right\} u(s)ds\right\| + \left\|\int_\delta^\epsilon \left\{\int_s^\epsilon k_\gamma(t-s)dt\right\} u(s)ds\right\|$$

$$< \eta\left\|\int_0^\delta u(s)ds\right\| + \eta\left\|\int_\delta^\epsilon u(s)ds\right\| < 2\eta^2.$$

Hence $\left\|\int_\delta^\epsilon (k_\gamma * u)(t)dt\right\| \to 0$ as $\epsilon \downarrow 0$ with $0 < \delta < \epsilon$. It follows that $\int_0^1 (k_\gamma * u)(t)\, dt = \lim_{a\downarrow 0}\int_a^1 (k_\gamma * u)(t)\, dt$ exists, and thus $k_\gamma * u \in L_1^{improp.\, loc}([0,\infty), X^+)$.

Let $t > 0$ be such that $(k_\beta * (k_\gamma * u))(t)$ exists. Then

$$(k_\beta * (k_\gamma * u))(t) = \int_0^t k_\beta(t-s)(k_\gamma * u)(s)ds = \lim_{\epsilon\downarrow 0}\int_\epsilon^{t-\epsilon} k_\beta(t-s)(k_\gamma * u)(s)ds.$$

Writing $P(\epsilon) := \int_\epsilon^{t-\epsilon} k_\beta(t-s)(k_\gamma * u)(s)ds$, where $0 < 2\epsilon < t$, we see that

$$P(\epsilon) = \lim_{\eta\downarrow 0}\int_\epsilon^{t-\epsilon} k_\beta(t-s)\left\{\int_\eta^s k_\gamma(s-r)u(r)dr\right\} ds$$

(by Lebesgue's convergence theorem)

$$= \lim_{\eta\downarrow 0}\int_\eta^\epsilon \left\{\int_\epsilon^{t-\epsilon} k_\beta(t-s)k_\gamma(s-r)ds\right\} u(r)dr$$

$$+ \int_\epsilon^{t-\epsilon} \left\{\int_r^{t-\epsilon} k_\beta(t-s)k_\gamma(s-r)ds\right\} u(r)dr$$

(by Fubini's theorem).

Since

$$\lim_{\eta\downarrow 0}\int_\eta^\epsilon\left\{\int_\epsilon^{t-\epsilon}k_\beta(t-s)k_\gamma(s-r)ds\right\}u(r)dr \le \int_0^\epsilon k_{\gamma+\beta}(t-r)u(r)dr$$

$$\le \max\{k_{\gamma+\beta}(t),k_{\gamma+\beta}(t-\epsilon)\}\int_0^\epsilon u(r)dr \to 0$$

as $\epsilon\downarrow 0$, and since

$$\lim_{\epsilon\downarrow 0}\int_\epsilon^{t-\epsilon}\Big\{\int_r^{t-\epsilon}k_\beta(t-s)k_\gamma(s-r)\,ds\Big\}u(r)\,dr = \lim_{\epsilon\downarrow 0}\int_\epsilon^{t-\epsilon}k_{\gamma+\beta}(t-r)u(r)\,dr$$

(by Lemma 2.1)

whenever either side of this equation exists, it follows that

$$(k_\beta*(k_\gamma*u))(t)=\lim_{\epsilon\downarrow 0}P(\epsilon)=\lim_{\epsilon\downarrow 0}\int_\epsilon^{t-\epsilon}k_{\gamma+\beta}(t-r)u(r)\,dr.$$

Hence $(k_{\gamma+\beta}*u)(t)$ exists, and $(k_{\gamma+\beta}*u)(t)=(k_\beta*(k_\gamma*u))(t)$.

Similarly, if $(k_{\gamma+\beta}*u)(t)$ exists for some $t>0$, then the existence of $(k_\beta*(k_\gamma*u))(t)$ can be proved. We may omit the details. □

Theorem 2.4. (Cf. Theorem 2.2 of [3].) *Suppose assumption (A) or (B) holds. Let $\gamma\ge 0$, and $\beta>0$. Then the following hold.*

(i) *If $t>0$ and $\|\chi_{[0,t]}(\cdot)\mathfrak{c}_\cdot^\gamma\|_\infty<\infty$, then $\|\mathfrak{c}_t^{\gamma+\beta}\|\le\|\chi_{[0,t]}(\cdot)\mathfrak{c}_\cdot^\gamma\|_\infty$.*

(ii) *If $\|\mathfrak{c}_t^\gamma\|\le Me^{wt}$ for some $M>0$ and $w\ge 0$ and dt-almost all $t>0$, then $\|\mathfrak{c}_t^{\gamma+\beta}\|\le Me^{wt}$ for dt-almost all $t>0$.*

Furthermore, the function $F_0(\gamma):=\max\{w_0(\mathfrak{c}_\cdot^\gamma),0\}$ is decreasing on $[0,\infty)$.

Proof. (i) Using Lemmas 2.2 and 2.3, we have for dt-almost all $t>0$

$$\begin{aligned}
\|\mathfrak{c}_t^{\gamma+\beta}\| &= (k_{\gamma+\beta+1}(t))^{-1}\|(k_{\gamma+\beta}*u)(t)\|\\
&= (k_{\gamma+\beta+1}(t))^{-1}\|(k_\beta*(k_\gamma*u))(t)\|\\
&= (k_{\gamma+\beta+1}(t))^{-1}\|(k_\beta*(k_{\gamma+1}\mathfrak{c}_\cdot^\gamma))(t)\|\\
&\le (k_{\gamma+\beta+1}(t))^{-1}(k_\beta*k_{\gamma+1})(t)\|\chi_{[0,t]}(\cdot)\mathfrak{c}_\cdot^\gamma\|_\infty\\
&= \|\chi_{[0,t]}(\cdot)\mathfrak{c}_\cdot^\gamma\|_\infty<\infty.
\end{aligned}$$

(ii) Since the hypothesis implies $\|\chi_{[0,t]}(\cdot)\mathfrak{c}_\cdot^\gamma\|_\infty\le Me^{wt}$ for all $t>0$, it follows from (i) that $\|\mathfrak{c}_t^{\gamma+\beta}\|\le Me^{wt}$ for dt-almost all $t>0$.

To prove $F_0(\gamma)\ge F_0(\gamma+\beta)$, suppose $w>F_0(\gamma)$. Since $w>w_0(\mathfrak{c}_\cdot^\gamma)$, there exist $M>0$ and $K>0$ such that $\|\mathfrak{c}_t^\gamma\|\le Me^{wt}$ for dt-almost all $t\ge K$.

Then we write

$$\mathfrak{c}_t^{\gamma+\beta} = (k_{\gamma+\beta+1}(t))^{-1}\left(\int_0^K + \int_K^t\right) k_{\gamma+\beta}(t-s)u(s)\,ds =: I + II.$$

Here if assumption (B) holds, then for all $t > 2K$

$$\begin{aligned}\|I\| &= (k_{\gamma+\beta+1}(t))^{-1} M_{ab}\left\|\int_0^K (t-s)^{\gamma+\beta-1}u(s)\,ds\right\| \\ &= \frac{M_{ab}}{t^{\gamma+\beta}}\, t^{\gamma+\beta-1}\left\|\int_0^K \left(1-\frac{s}{t}\right)^{\gamma+\beta-1} u(s)\,ds\right\| \\ &\leq \frac{M_{ab}}{t}\, \max\{(1/2)^{\gamma+\beta-1}, 1\}\,\left\|\int_0^K u(s)\,ds\right\|,\end{aligned}$$

where M_{ab} denotes an absolute constant not necessarily the same at each occurence. Similarly if assumption (A) holds, then for all $t > 2K$

$$\|I\| \leq \frac{M_{ab}}{t}\, \max\{(1/2)^{\gamma+\beta-1}, 1\}\int_0^K \|u(s)\|\,ds.$$

Thus, in either case,

$$\|I\| = O(t^{-1}) \quad (t \to \infty).$$

Next, since $u_K(s) = \chi_{[0,\,K)}(s)u(s)$, we have

$$\begin{aligned}\|II\| &= (k_{\gamma+\beta+1}(t))^{-1}\left\|\int_0^t k_{\gamma+\beta}(t-s)(u-u_K)(s)\,ds\right\| \\ &= (k_{\gamma+\beta+1}(t))^{-1}\|(k_\beta * (k_\gamma * (u-u_K))(t)\|\end{aligned}$$

for dt-almost all $t > K$, where $(k_\gamma * (u-u_K))(s) = 0$ for all $0 < s \leq K$.

Suppose assumption (B) holds. Then, since

$$\|(k_\gamma * (u-u_K))(s)\| \leq \|(k_\gamma * u)(s)\| = k_{\gamma+1}(s)\|\mathfrak{c}_s^\gamma\| \leq k_{\gamma+1}(s)Me^{ws}$$

for ds-almost all $s \geq K$, it follows that

$$\begin{aligned}\|II\| &= (k_{\gamma+\beta+1}(t))^{-1}\left\|\int_0^t k_\beta(t-s)(k_\gamma * (u-u_K))(s)\,ds\right\| \\ &\leq (k_{\gamma+\beta+1}(t))^{-1}\int_K^t k_\beta(t-s)k_{\gamma+1}(s)Me^{ws}ds \\ &\leq (k_{\gamma+\beta+1}(t))^{-1}Me^{wt}\int_0^t k_\beta(t-s)k_{\gamma+1}(s)ds \\ &= (k_{\gamma+\beta+1}(t))^{-1}Me^{wt}(k_\beta * k_{\gamma+1})(t) = Me^{wt}\end{aligned}$$

for dt-almost all $t > K$. Consequently for dt-almost all $t > 2K$

$$\|\mathfrak{c}_t^{\gamma+\beta}\| \leq \|I\| + \|II\| \leq O(t^{-1}) + Me^{wt},$$

and hence $\|\mathfrak{c}_t^{\gamma+\beta}\| = O(e^{wt})$ (mod dt) as $t \to \infty$ by the fact that $w > 0$. Therefore $w \geq F_0(\gamma+\beta)$, and $F_0(\gamma) \geq F_0(\gamma+\beta)$.

Next suppose assumption (A) holds. Then

$$(k_\gamma * u_K)(s) = (k_\gamma * u)(s) - (k_\gamma * (u - u_K))(s)$$

for ds-almost all $s > 0$. Since $\|u_K(t)\| = 0$ on $[K,\infty)$, and the function $t \mapsto \|u_K(t)\|$ belongs to $L_1^{loc}([0,\infty),\mathbb{R}^+)$, it follows easily that $0 \leq \mathfrak{c}_t^{\gamma+\beta}(\|u_K(\cdot)\|) = o(e^{wt})$ as $t \to \infty$. Thus for dt-almost all $t > K$

$$\begin{aligned}
\|II\| &= (k_{\gamma+\beta+1}(t))^{-1} \left\| \int_0^t k_\beta(t-s)(k_\gamma * (u-u_K))(s)\, ds \right\| \\
&\leq (k_{\gamma+\beta+1}(t))^{-1} \left(\left\| \int_K^t k_\beta(t-s)(k_\gamma * u)(s)\, ds \right\| \right. \\
&\qquad\qquad \left. + \int_K^t k_\beta(t-s)(k_\gamma * \|u_K(\cdot)\|)(s)\, ds \right) \\
&\leq (k_{\gamma+\beta+1}(t))^{-1} \int_K^t k_\beta(t-s) k_{\gamma+1}(s) M e^{ws}\, ds + \mathfrak{c}_t^{\gamma+\beta}(\|u_K(\cdot)\|) \\
&\leq M e^{wt} + o(e^{wt}) = O(e^{wt}).
\end{aligned}$$

Hence $\|\mathfrak{c}_t^{\gamma+\beta}\| \leq \|I\| + \|II\| \leq O(t^{-1}) + O(e^{wt}) = O(e^{wt})$ (mod dt) as $t \to \infty$.

This completes the proof. □

Remarks. **(a)** It is clear that $L_1^{improp.\,loc}([0,\infty),\mathbb{R}^+)$ is a subset of $L_1^{loc}([0,\infty),\mathbb{R})$. But if X is a Banach lattice, then $L_1^{improp.\,loc}([0,\infty),X^+)$ is not necessarily a subset of $L_1^{loc}([0,\infty),X)$. To see this we give the following example.

Example 1. Let $X := \ell_2 = \{(a_n)_{n=1}^\infty : a_n \in \mathbb{R},\, \sum_{n=1}^\infty a_n^2 < \infty\}$, with $\|(a_n)_{n=1}^\infty\| := (\sum_{n=1}^\infty a_n^2)^{1/2}$. For each $n \geq 1$, let u_n be the continuous nonnegative function on $(0,\infty)$ defined by $u_n = n$ on $[(n+1)^{-1},\, n^{-1}]$, $u_n = 0$ on $[0,\,(n+2)^{-1}] \cup [n^{-1} + (n(n+1))^{-1},\,\infty)$, and u_n is linear on $[(n+2)^{-1},\,(n+1)^{-1}]$ and $[n^{-1},\, n^{-1} + (n(n+1))^{-1}]$. Define a function $u : [0,\infty) \to X^+$ by $u(s) := (u_n(s))_{n=1}^\infty$. It is clear that u is continuous on $(0,\infty)$. If $0 < s \leq 1$, then there exists a unique $n(s) \in \mathbb{N}$ such that

$$\frac{1}{n(s)+1} < s \leq \frac{1}{n(s)}.$$

By the definition of $u_{n(s)}$ we have $u_{n(s)}(s) = n(s)$, and thus $\|u(s)\| \geq u_{n(s)}(s) = n(s) \sim s^{-1}$. Here $a(s) \sim b(s)$ means that both the ratios $a(s)/b(s)$ and $b(s)/a(s)$ are bounded on the domain considered. Since $\int_0^1 s^{-1} ds = \infty$, it follows that $\int_0^1 \|u(s)\| ds = \infty$. Next we show that $\int_0^1 u(s) ds =$

$\lim_{a\downarrow 0}\int_a^1 u(s)\,ds$ exists. For this purpose, note that by the definition of u_n

$$a_n := \int_0^1 u_n(s)ds = \frac{n}{2(n+1)(n+2)} + \frac{n}{n(n+1)} + \frac{n}{2n(n+1)} \le \frac{2}{n+1} \qquad (n \ge 1).$$

It follows that $(a_n)_{n=1}^{\infty} \in X$, and

$$\int_0^1 u(s)ds = \lim_{a\downarrow 0}\int_a^1 u(s)ds = (a_n)_{n=1}^{\infty} \qquad \text{(in } X\text{-norm)}.$$

(b) The following example shows that there exists a nonnegative real-valued function u on the interval $[0,\infty)$ such that u is continuous on $(0,\infty)$ and $\int_0^\infty u(s)\,ds$
$< \infty$, but $\int_0^1 \mathfrak{c}_s^\gamma\,ds = \infty$ (i.e., $\mathfrak{c}_\cdot^\gamma \notin L_1^{improp.\,loc}([0,\infty),\mathbb{R}^+)$) for all $\gamma > 0$. This implies that $\sigma(\mathfrak{c}_\cdot^\gamma)$, where $\gamma > 0$, cannot be defined in general, although $\sigma(k_\gamma * u)$ can be done because $k_\gamma * u \in L_1^{loc}([0,\infty),\mathbb{R}^+)$ by Lemma 2.2. We note that if the vector-valued function $u : t \to X$ is continuous on $[0,\infty)$, then the function $t \to \mathfrak{c}_t^\gamma$ is also continuous on $[0,\infty)$ and satisfies $\sigma(\mathfrak{c}_\cdot^\gamma) = \sigma(k_\gamma * u)$ for all $\gamma > 0$ (see Theorem 2.3(i) of [3]).

Example 2. Let u be a nonnegative real-valued function on $[0,\infty)$ such that u is continuous on $(0,\infty)$, $\int_0^1 u(s)\,ds < \infty$, $\int_0^1 u(s)|\log s|\,ds = \infty$, and $u = 0$ on $[1,\infty)$. Then we have $\int_0^1 \mathfrak{c}_s^\gamma\,ds = \infty$ for all $\gamma > 0$.

To see this, let $0 < \gamma < k$, where k is a positive integer. Then, by Fubini's theorem,

$$\begin{aligned}\int_0^1 \mathfrak{c}_t^\gamma\,dt &= \int_0^1 \left(\frac{\gamma}{t^\gamma}\int_0^t (t-s)^{\gamma-1}u(s)\,ds\right)dt = \int_0^1\int_0^t \frac{\gamma}{t^\gamma}(t-s)^{\gamma-1}u(s)\,dsdt \\ &= \int_0^1 \left(\int_s^1 \frac{\gamma}{t^\gamma}(t-s)^{\gamma-1}\,dt\right)u(s)\,ds,\end{aligned}$$

and, by the fact $\gamma - 1 < k$,

$$\int_s^1 \frac{\gamma}{t^\gamma}(t-s)^{\gamma-1}\,dt = \gamma\int_s^1 \frac{1}{t}\Big(1-\frac{s}{t}\Big)^{\gamma-1}dt \ge \gamma\int_s^1 \frac{1}{t}\Big(1-\frac{s}{t}\Big)^{k}dt$$

for all $0 < s < 1$. Here

$$\begin{aligned}\int_s^1 \frac{1}{t}\Big(1-\frac{s}{t}\Big)^k dt &= \int_s^1 \frac{1}{t}\,dt + \sum_{l=1}^{k}\binom{k}{l}(-1)^l \int_s^1 \frac{s^l}{t^{l+1}}\,dt \\ &= -\log s + \sum_{l=1}^{k}\binom{k}{l}(-1)^l s^l \int_s^1 t^{-(l+1)}\,dt \\ &= -\log s + \sum_{l=1}^{k}\binom{k}{l}(-1)^l \cdot \frac{1-s^l}{l}\end{aligned}$$

for all $0 < s < 1$, whence

$$\int_0^1 \mathfrak{c}_t^\gamma\,dt \geq \gamma \int_0^1 u(s)|\log s|\,ds + \gamma \sum_{l=1}^{k}\binom{k}{l}(-1)^l \int_0^1 \frac{(s^l-1)}{l} u(s)\,ds = \infty.$$

Theorem 2.5. (Cf. Theorem 2.3 of [3].) *Suppose assumption (A) or (B) holds. Then the following hold.*

(i) *For all* $\gamma \geq 0$,

$$\max\{\sigma(k_\gamma * u), 0\} = \max\{w_0(k_{\gamma+1} * u), 0\} = \max\{w_0(\mathfrak{c}_\cdot^{\gamma+1}), 0\}. \tag{7}$$

Consequently, the function $F_1(\gamma) := \max\{\sigma(k_\gamma * u), 0\}$ *is decreasing on* $[0,\infty)$.

(ii) *For all* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}\lambda > \max\{\sigma(u), 0\}$ and $\gamma \geq 0$,

$$\mathfrak{a}_\lambda = \lambda^{\gamma+1} \int_0^\infty e^{-\lambda t}(k_\gamma * u)(t)\,dt = \lambda^{\gamma+1} \int_0^\infty e^{-\lambda t} k_{\gamma+1}(t)\mathfrak{c}_t^\gamma\,dt. \tag{8}$$

Proof. (i) It is known (cf. [1, p. 31]) that $\sigma(u) < \infty$ if and only if $w_0(1 * u) < \infty$, and $\max\{\sigma(u), 0\} = \max\{w_0(1 * u), 0\}$. Applying this to the function $k_\gamma * u \in L_1^{loc}([0,\infty), X) \cup L_1^{improp.\,loc}([0,\infty), X^+)$, and using the equations $1*(k_\gamma * u) = k_{\gamma+1} * u$ (by Lemmas 2.2 and 2.3) and $(k_{\gamma+1} * u)(t) = k_{\gamma+2}(t)\mathfrak{c}_t^{\gamma+1} = (t^{\gamma+1}/\Gamma(\gamma+2))\mathfrak{c}_t^{\gamma+1}$ for dt-almost all $t > 0$, we deduce (7) at once. Since $F_1(\gamma) = F_0(\gamma+1)$ for $\gamma \geq 0$, F_1 is decreasing by Theorem 2.4.

(ii) The case $\gamma = 0$ is trivial, and so we consider the case $\gamma > 0$. Let λ be such that $\mathrm{Re}\lambda > \max\{\sigma(u), 0\}$. Since $\max\{\sigma(u), 0\} \geq \max\{w_0(1 * u), \sigma(k_\gamma * u), 0\} \geq w_0(k_{\gamma+1} * u)$, integration by parts gives

$$\begin{aligned}\int_0^\infty e^{-\lambda t} u(t)\,dt &= \lambda \int_0^\infty e^{-\lambda t}(1 * u)(t)\,dt, \\ \int_0^\infty e^{-\lambda t}(k_\gamma * u)(t)\,dt &= \lambda \int_0^\infty e^{-\lambda t}(1 * (k_\gamma * u))(t)\,dt \\ &= \lambda \int_0^\infty e^{-\lambda t}(k_{\gamma+1} * u)(t)\,dt \qquad \text{(by (6))},\end{aligned}$$

and Fubini's theorem yields

$$\begin{aligned}
\lambda \int_0^\infty e^{-\lambda t}(k_{\gamma+1} * u)(t)\,dt \; &= \; \lambda \lim_{\eta\downarrow 0} \int_\eta^\infty e^{-\lambda t}(k_\gamma * (1 * u))(t)\,dt \qquad \text{(by (6))} \\
&= \lambda \lim_{\eta\downarrow 0} \int_\eta^\infty \int_0^t e^{-\lambda(t-s)} k_\gamma(t-s) e^{-\lambda s}(1 * u)(s)\,dsdt \\
&= \lambda \lim_{\eta\downarrow 0} \Big(\int_0^\infty e^{-\lambda t} k_\gamma(t)\,dt \cdot \int_\eta^\infty e^{-\lambda s}(1 * u)(s)\,ds \\
&\qquad\qquad + \int_0^\eta \Big\{ \int_{\eta-s}^\infty e^{-\lambda t} k_\gamma(t)\,dt \Big\} e^{-\lambda s}(1 * u)(s)\,ds \Big) \\
&= \lambda \lim_{\eta\downarrow 0} \Big(\lambda^{-\gamma} \int_\eta^\infty e^{-\lambda s}(1 * u)(s)\,ds \\
&\qquad\qquad + \int_0^\eta \Big\{ \int_{\eta-s}^\infty e^{-\lambda t} k_\gamma(t)\,dt \Big\} e^{-\lambda s}(1 * u)(s)\,ds \Big) \\
&= \lambda^{1-\gamma} \int_0^\infty e^{-\lambda s}(1 * u)(s)\,ds \\
&= \lambda^{-\gamma} \int_0^\infty e^{-\lambda s} u(s)\,ds = \lambda^{-(\gamma+1)} \mathfrak{a}_\lambda,
\end{aligned}$$

which completes the proof. □

Remarks. **(a)** Let $u \neq 0$ be a function in $L_1^{loc}([0,\infty), X)$. If $\sigma(k_\gamma * u) \geq 0$ for some $\gamma \geq 0$, then, by Theorem 2.5(i), $\sigma(k_\gamma * u) \geq \sigma(k_\beta * u)$ for all $\beta > \gamma$. Thus the function $\gamma \mapsto \sigma(k_\gamma * u)$ is nonnegative and decreasing on $[0, D(u))$, where

$$D(u) := \inf\{\gamma > 0 \;:\; \sigma(k_\gamma * u) < 0\};$$

and we have $\sigma(k_\gamma * u) \leq 0$ for all $\gamma \in (D(u), \infty)$. Here we would like to note that *if* $D(u) \neq \infty$*, then:*

(i) $\sigma(k_{D(u)} * u) < 0$*;*

(ii) $\{\gamma \geq D(u) \;:\; \sigma(k_\gamma * u) \neq 0\}$ *is a finite set.*

To see this we make the following preparations: Suppose $\sigma(u) < 0$, with $u \neq 0$, and define

$$U(\lambda) := \int_0^\infty e^{-\lambda t} u(t)\,dt \qquad (\,\mathrm{Re}\lambda > \sigma(u)\,). \tag{9}$$

It is known (cf. [1, Theorem 1.5.1]) that the vector-valued function U is ananlytic on $\{\lambda \;:\; \mathrm{Re}\lambda > \sigma(u)\}$. Further we note the following:

(I) If $U(0) \neq 0$, then $\sigma(k_\gamma * u) = 0$ for all $\gamma > 0$.

To see this, let $\gamma > 0$ and $\lambda > 0$. We have by Theorem 2.5(ii) that

$$\int_0^\infty e^{-\lambda t}(k_\gamma * u)(t)\,dt = \lambda^{-\gamma} \int_0^\infty e^{-\lambda t} u(t)\,dt = \lambda^{-\gamma} U(\lambda),$$

so that
$$\lim_{\lambda\downarrow 0}\left\|\int_0^\infty e^{-\lambda t}(k_\gamma * u)(t)\,dt\right\| = \lim_{\lambda\downarrow 0}\lambda^{-\gamma}\|U(0)\| = \infty,$$
which implies $\sigma(k_\gamma * u) = 0$.

(II) If $U(0) = 0$, then (since $u \neq 0$ implies $U \neq 0$) there exists $n_0 \in \mathbb{N}$ and an analytic function W on $\{\lambda : \mathrm{Re}\lambda > \sigma(u)\}$, with $W(0) \neq 0$, such that

$$U(\lambda) = \lambda^{n_0} W(\lambda) \qquad (\mathrm{Re}\lambda > \sigma(u)). \tag{10}$$

Then we have
$$\begin{cases} 0 > \sigma(u) \geq \sigma(k_1 * u) \geq \ldots \geq \sigma(k_{n_0} * u), \text{ and} \\ \sigma(k_\gamma * u) = 0 \quad \text{for all } \ 0 < \gamma \notin \{1, 2, \ldots, n_0\}. \end{cases}$$

To see this we use the fact that $\int_0^\infty u(t)\,dt = U(0) = 0$. Then by Theorem 1.4.3 of [1], $\sigma(u) = w_0(k_1 * u)$. Since $w_0(k_1 * u) \geq \sigma(k_1 * u)$, it then follows that $0 > \sigma(u) \geq \sigma(k_1 * u)$. Applying this together with (8)–(10) we see inductively that
$$\int_0^\infty u(t)\,dt = \int_0^\infty (k_1 * u)(t)\,dt = \ \ldots \ = \int_0^\infty (k_{n_0-1} * u)(t)\,dt = 0,$$
and
$$0 > \sigma(u) \geq \sigma(k_1 * u) \geq \ \ldots \ \geq \sigma(k_{n_0} * u).$$
Next, let $0 < \gamma \notin \{1, 2, \ldots, n_0\}$. To prove $\sigma(k_\gamma * u) = 0$, we assume the contrary: $\sigma(k_\gamma * u) < 0$. Since $W(0) \neq 0$, there exists $x^* \in X^*$ such that $\langle W(0), x^*\rangle \neq 0$, where X^* is the dual space of X. Then the complex-valued function
$$\lambda \mapsto \frac{\langle \int_0^\infty e^{-\lambda t}(k_\gamma * u)(t)\,dt,\, x^*\rangle}{\langle W(\lambda),\, x^*\rangle}$$
is analytic on $\{\lambda : |\lambda| < \epsilon\}$ for some $\epsilon > 0$. But by Theorem 2.5(ii)
$$\lambda^{n_0-\gamma} = \frac{\lambda^{-\gamma}\langle U(\lambda),\, x^*\rangle}{\langle W(\lambda),\, x^*\rangle} = \frac{\langle \int_0^\infty e^{-\lambda t}(k_\gamma * u)(t)\,dt,\, x^*\rangle}{\langle W(\lambda),\, x^*\rangle}$$
for all $\lambda > 0$. This is a contradiction, because $n_0 - \gamma \notin \mathbb{N}_0 := \{0\} \cup \mathbb{N}$ implies that the function $\lambda \mapsto \lambda^{n_0-\gamma}$ $(\lambda > 0)$ cannot be extended analytically to the domain $\{\lambda : |\lambda| < \epsilon\}$. Hence we must have $0 \leq \sigma(k_\gamma * u) \leq \max\{\sigma(u), 0\} = 0$ by Theorem 2.5(i).

Proofs of (i) and (ii). If $\sigma(k_{D(u)} * u) \geq 0$, then there must exist γ_1, γ_2 such that $D(u) < \gamma_1 < \gamma_2 < D(u) + 1$ and $\sigma(k_{\gamma_i} * u) < 0$ for $i = 1, 2$. But this contradicts (II) because $0 < \gamma_2 - \gamma_1 < 1$ (replace u with $k_{\gamma_1} * u$ in (II)). Hence we must have $\sigma(k_{D(u)} * u) < 0$, and (ii) is direct from (I) and (II). □

(b) To explain the behaviour of the function $\gamma \mapsto \sigma(k_\gamma * u)$ on $[0, \infty)$ we give the following examples.

Example 3. Let $\lambda_0 > 0$, and define $u(t) := e^{\lambda_0 t}$ for $t \geq 0$. Then

$$(k_\gamma * u)(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} e^{\lambda_0 s}\, ds = \frac{e^{\lambda_0 t}}{\Gamma(\gamma)} \lambda_0^{-\gamma} \int_0^{\lambda_0 t} s^{\gamma-1} e^{-s}\, ds \tag{11}$$

for all $\gamma > 0$ and $t > 0$. Since $\lim_{t\to\infty} \int_0^{\lambda_0 t} s^{\gamma-1} e^{-s}\, ds = \Gamma(\gamma) > 0$, it follows that $\sigma(k_\gamma * u) = \sigma(u) = \lambda_0 > 0$ for all $\gamma > 0$. (See also Theorem 2.7 below.)

Example 4. Let $u(t) := \sin t$ for $t \geq 0$. Then $\sigma(k_\gamma * u) = \sigma(u) = 0$ for all $\gamma > 0$. This will be proved in Example 7 below.

Example 5. Let $f \in C^\infty([0,\infty), \mathbb{R})$ be such that $f(t) = 0$ on $[0, \delta]$, where $0 < \delta < \log \frac{\pi}{2}$, $-\cos e^t \leq f(t) \leq 0$ on $[\delta, \log \frac{\pi}{2}]$, and $f(t) = -\cos e^t$ on $[\log \frac{\pi}{2}, \infty)$. Define $u(t) := f''(t)$ for $t \geq 0$. Then the following holds.

$$\sigma(k_\gamma * u) = \begin{cases} 1-\gamma & (0 \leq \gamma < 1), \\ 0 & (1 \leq \gamma < 2), \\ -1 & (\gamma = 2), \\ 0 & (\gamma > 2). \end{cases} \tag{12}$$

To see this we first note that $u(t) = f''(t)$, $(k_1 * u)(t) = f'(t)$ and $(k_2 * u)(t) = f(t)$ for all $t \in [0, \infty)$, and

$$\begin{cases} f''(t) = e^t \sin e^t + e^{2t} \cos e^t & \text{on } [\log \frac{\pi}{2}, \infty), \\ f'(t) = e^t \sin e^t & \text{on } [\log \frac{\pi}{2}, \infty), \\ f(t) = -\cos e^t & \text{on } [\log \frac{\pi}{2}, \infty). \end{cases} \tag{13}$$

(I) We prove $\sigma(k_2 * u) = -1$. To do this, let $0 < \lambda < 1$. Since integration by parts gives

$$\int_0^b e^{-\lambda t} \sin e^t\, dt = \left[\frac{1}{-\lambda} e^{-\lambda t} \sin e^t \right]_{t=0}^b + \frac{1}{\lambda} \int_0^b e^{(1-\lambda)t} \cos e^t\, dt,$$

it follows that

$$\int_0^\infty e^{-\lambda t} \sin e^t\, dt = \frac{1}{\lambda} \sin 1 + \frac{1}{\lambda} \int_0^\infty e^{(1-\lambda)t} \cos e^t\, dt.$$

This shows that $\sigma(\cos e^t) \leq -1$. Since $(k_2 * u)(t) + \cos e^t = 0$ on $[\log \frac{\pi}{2}, \infty)$, it follows that $\sigma(k_2 * u) = \sigma(-\cos e^t) = \sigma(\cos e^t) \leq -1$.

Next suppose $\lambda > 1$. Putting, for $n \geq 1$,

$$s(n,1) := \log\left(2n\pi - \frac{\pi}{4}\right) \quad \text{and} \quad s(n,2) := \log\left(2n\pi + \frac{\pi}{4}\right),$$

we have

$$\int_{s(n,1)}^{s(n,2)} e^{\lambda t} \cos e^t\, dt \geq \frac{1}{\sqrt{2}} \int_{s(n,1)}^{s(n,2)} e^{\lambda t}\, dt = \frac{1}{\sqrt{2}\,\lambda} \left(e^{\lambda s(n,2)} - e^{\lambda s(n,1)} \right)$$

$$= \frac{1}{\sqrt{2}\,\lambda}\left(\left(2n\pi + \frac{\pi}{4}\right)^{\lambda} - \left(2n\pi - \frac{\pi}{4}\right)^{\lambda}\right) \geq \frac{\pi}{2\sqrt{2}}\left(2n\pi - \frac{\pi}{4}\right)^{\lambda-1},$$

and, by the assumption $\lambda > 1$,

$$\lim_{n\to\infty}\left(2n\pi - \frac{\pi}{4}\right)^{\lambda-1} = \infty.$$

It follows that $\lim_{b\to\infty}\int_0^b e^{\lambda t}\cos e^t\,dt$ does not exist, and thus $\sigma(\cos e^t) \geq -1$. Consequently $\sigma(k_2 * u) = \sigma(\cos e^t) = -1$.

(II) To see that $\sigma(k_\gamma * u) = 0$ for all $\gamma > 2$, it is sufficient to check that $\int_0^\infty (k_2 * u)(t)\,dt \neq 0$. (This is due to the fact that if $\gamma > 2$, then $\sigma(k_\gamma * u) \leq 0$ since $\sigma(k_2 * u) = -1$ (cf. Theorem 2.5(i)), and $\lim_{\lambda\downarrow 0}\left\|\int_0^\infty e^{-\lambda t}(k_\gamma * u)(t)\,dt\right\| = \lim_{\lambda\downarrow 0}\lambda^{-(\gamma-2)}\left\|\int_0^\infty e^{-\lambda t}(k_2 * u)(t)\,dt\right\| = \lim_{\lambda\downarrow 0}\lambda^{-(\gamma-2)}\left\|\int_0^\infty (k_2 * u)(t)\,dt\right\| = \infty$ when $\int_0^\infty (k_2 * u)(t)\,dt \neq 0$.) Since $(k_2 * u)(t) = f(t)$ on $[0, \infty)$, and $f(t) \geq -\cos e^t$ on $[0, \infty)$ by the definition of f, it is also sufficient to check that $\int_0^\infty \cos e^t\,dt < 0$. To do this, we put $t = \log s$. Then $\int_0^\infty \cos e^t\,dt = \int_1^\infty s^{-1}\cos s\ ds$, and the relations

$$\left|\frac{\cos s}{s}\right| \geq \left|\frac{\cos(s+\pi)}{s+\pi}\right| \quad \text{and} \quad \cos s = -\cos(s+\pi)$$

yield

$$\int_1^\infty \frac{\cos s}{s}\,ds < \int_1^{\pi/2}\frac{\cos s}{s}\,ds + \int_{\pi/2}^{5\pi/2}\frac{\cos s}{s}\,ds =: A + B.$$

Since

$$A = \int_1^{\pi/2}\frac{\cos s}{s}\,ds < \int_1^{\pi/2}\cos s\,ds = 1 - \sin 1,$$

and

$$B = \int_{\pi/2}^{3\pi/2}\frac{\pi\cos s}{(s+\pi)s}\,ds < -\frac{4}{15\pi}\int_{-\pi/2}^{\pi/2}\cos s\,ds = -\frac{8}{15\pi},$$

it follows that

$$\int_0^\infty \cos e^t\,dt = \int_1^\infty \frac{\cos s}{s}\,ds < A + B < 1 - \sin 1 - \frac{8}{15\pi} < 0,$$

which is the desired result.

(III) To see that $\sigma(k_\gamma * u) = 1 - \gamma$ for all $\gamma \in [0, 1)$, it is sufficient to check that $w_0(k_{\gamma+1} * u) = 1 - \gamma$ for $\gamma \in [0, 1)$ (cf. Theorem 2.5(i)). To do this, let $0 \leq \gamma < 1$. Since $(k_1 * u)(t) = e^t \sin e^t$ on $[\log\frac{\pi}{2}, \infty)$, it follows easily (cf. Theorem 2.4) that $w_0(k_{\gamma+1} * u) = 1 - \gamma$ if and only if $w_0((k_\gamma * e^s \sin e^s)(\cdot)) = 1 - \gamma$. Since $w_0((k_0 * e^s \sin e^s)(\cdot)) = 1$, we will prove the latter equality for $0 < \gamma < 1$. Define a real-valued function A on $[0, \infty)$ by

$$A(t) := \Gamma(\gamma)(k_\gamma * e^s \sin e^s)(t) = \int_0^t (t-s)^{\gamma-1} e^s \sin e^s \, ds.$$

Putting $s = \log r$, we have

$$A(t) = \int_1^{e^t} (t - \log r)^{\gamma-1} \sin r \, dr.$$

Since the function $r \mapsto (t - \log r)^{\gamma-1}$ is increasing on the interval $[1, e^t]$, and $\sin r$ is a periodic function of r with period 2π, if we let $n(t) := \max\{k \in \mathbb{N}_0 : k\pi < e^t\}$ for $t \geq 0$, then $|A(t)| \leq M(t)$, where $M(t)$ denotes the maximum of the numbers

$$\Big|\int_1^\pi (t - \log r)^{\gamma-1} \sin r \, dr\Big|, \qquad \Big|\int_{n(t)\pi}^{e^t} (t - \log r)^{\gamma-1} \sin r \, dr\Big|$$

and

$$\Big|\int_{(l-1)\pi}^{l\pi} (t - \log r)^{\gamma-1} \sin r \, dr\Big| \qquad (2 \leq l \leq n(t)).$$

Hence

$$|A(t)| \leq M(t) \leq \int_{e^t-\pi}^{e^t} (t - \log r)^{\gamma-1} \, dr < \int_{e^t-\pi}^{e^t} \left(\frac{e^t - r}{e^t}\right)^{\gamma-1} dr,$$

where the last inequality comes from the left-hand inequality of the relations

$$\frac{1}{e^t} = (\log)'(e^t) < \frac{\log(e^t) - \log r}{e^t - r} = \frac{t - \log r}{e^t - r} < (\log)'(r) = \frac{1}{r} \tag{14}$$

for all $1 < r < e^t$. Thus

$$|A(t)| < e^{(1-\gamma)t} \int_{e^t-\pi}^{e^t} (e^t - r)^{\gamma-1} \, dr = e^{(1-\gamma)t} \gamma^{-1} \pi^\gamma,$$

and hence $w_0((k_\gamma * e^s \sin e^s)(\cdot)) = w_0(A) \leq 1 - \gamma$.

To see the reverse inequality, suppose $t = \log(2n\pi + \pi)$. Then as above

$$\begin{aligned} A(t) &= \left(\int_1^\pi + \int_\pi^{3\pi} + \ldots + \int_{2n\pi-\pi}^{2n\pi+\pi}\right) (t - \log r)^{\gamma-1} \sin r \, dr \\ &> \int_{2n\pi-\pi}^{2n\pi+\pi} (t - \log r)^{\gamma-1} \sin r \, dr \\ &= \int_{2n\pi}^{2n\pi+\pi} \left\{(t - \log r)^{\gamma-1} - (t - \log(r-\pi))^{\gamma-1}\right\} \sin r \, dr \\ &> \frac{1}{\sqrt{2}} \int_{2n\pi+\frac{1}{4}\pi}^{2n\pi+\frac{3\pi}{4}} \left\{(t - \log r)^{\gamma-1} - (t - \log(r-\pi))^{\gamma-1}\right\} dr. \end{aligned}$$

Here we note that if $1 < e^t - \pi < r < e^t$, then, by the right-hand inequality of (14),

$$0 < t - \log r < \frac{1}{r}(e^t - r) < \frac{e^t - r}{e^t - \pi} = (1 + o(1))e^{-t}(e^t - r)$$

as $t = \log(2n\pi + \pi) \to \infty$, and also by the left-hand inequality of (14),

$$t - \log(r - \pi) > e^{-t}(e^t - (r - \pi)).$$

Hence, by the inequality $1 - \gamma > 0$,

$$(t - \log r)^{\gamma-1} > \left(\frac{e^t - r}{e^t - \pi}\right)^{\gamma-1} = (1 + o(1))\, e^{(1-\gamma)t}\, (e^t - r)^{\gamma-1}$$

as $t = \log(2n\pi + \pi) \to \infty$, and

$$(t - \log(r - \pi))^{\gamma-1} < e^{(1-\gamma)t}\, (e^t - (r - \pi))^{\gamma-1}.$$

Using these inequalities we obtain that

$$\begin{aligned} A(t) &> \frac{e^{(1-\gamma)t}}{\sqrt{2}} \int_{2n\pi+\frac{1}{4}\pi}^{2n\pi+\frac{3}{4}\pi} \left\{(1 + o(1))(e^t - r)^{\gamma-1} - (e^t - (r - \pi))^{\gamma-1}\right\} dr \\ &= \frac{e^{(1-\gamma)t}}{\sqrt{2}} \int_{2n\pi+\frac{1}{4}\pi}^{2n\pi+\frac{3}{4}\pi} \left\{(1 + o(1))(2n\pi + \pi - r)^{\gamma-1} - (2n\pi + 2\pi - r))^{\gamma-1}\right\} dr \\ &= \frac{e^{(1-\gamma)t}}{\sqrt{2}} \left\{(1 + o(1)) \int_{\pi/4}^{3\pi/4} r^{\gamma-1}\, dr - \int_{5\pi/4}^{7\pi/4} r^{\gamma-1}\, dr\right\}, \end{aligned}$$

whence the reverse inequality $w_0((k_\gamma * e^s \sin e^s)(\cdot)) = w_0(A) \geq 1 - \gamma$ follows.

(IV) Finally we prove that $\sigma(k_\gamma * u) = 0$ for all $\gamma \in [1, 2)$. Since $(k_1 * (k_1 * u))(t) = f(t) = -\cos e^t$ on the interval $[\log \frac{\pi}{2}, \infty)$, it follows that $w_0(k_1 * (k_1 * u)) = 0$, and $\lim_{t\to\infty}(k_1 * (k_1 * u))(t)$ does not exists. Thus, by Theorem 1.4.3 of [1], $\sigma(k_1 * u) = w_0(k_1 * (k_1 * u)) = 0$. If $\gamma \in (1, 2)$, then, since $\sigma(k_1 * u) = 0$, it follows that $\sigma(k_\gamma * u) \leq 0$. Here, if we assume that $\sigma(k_\gamma * u) < 0$, then, since $0 < 2 - \gamma < 1$, the argument in the above Remark (a) yields that $\sigma(k_2 * u) = 0$, which contradicts $\sigma(k_2 * u) = -1$. Thus we must have $\sigma(k_\gamma * u) = 0$.

The next lemma is formulated and proved in [3] (see Lemma 2.5 therein).

Lemma 2.6. *Suppose assumption (B) holds. Let $\lambda > 0$, $\gamma > 0$ and $x \in X$. Then*

$$\int_0^\infty e^{-\lambda t} u(t)\, dt \left(:= \lim_{b\to\infty} \int_0^b e^{-\lambda t} u(t)\, dt\right) = x$$

if and only if

$$\lambda^\gamma \int_0^\infty e^{-\lambda t}(k_\gamma * u)(t)\, dt = x.$$

Theorem 2.7. (Cf. Theorem 2.6 of [3].) *Suppose assumption (B) holds. Then $\sigma(k_\gamma * u) = \sigma(u)$ for all $\gamma > 0$ if $\sigma(u) > 0$, and $\sigma(k_\gamma * u) = 0$ for all $\gamma > 0$ if $\sigma(u) \le 0$ and $u \ne 0$.*

Proof. Suppose $\sigma(u) > 0$. Then it follows from Lemma 2.6 that $\sigma(k_\gamma * u) = \sigma(u) > 0$ for all $\gamma > 0$. Next, suppose $\sigma(u) \le 0$ with $u \ne 0$, and let $\gamma > 0$. Then $\sigma(k_\gamma * u) \le 0$ by Lemma 2.6 (or Theorem 2.5(i)). Since $\frac{d}{dt}\int_0^t u(s)\, ds = u(t) \in X^+$ for almost all $t > 0$, it follows that the X^+-valued function $t \mapsto \int_0^t u(s)\, ds$ is non-zero, increasing, and continuous on $[0, \infty)$. Thus $\int_0^\infty e^{-\lambda t}u(t)\, dt > 0$ for all $\lambda > 0$. Hence $\int_0^\infty e^{-\lambda t}u(t)\, dt \ge \int_0^\infty e^{-\beta t}u(t)\, dt > 0$ if $\beta > \lambda > 0$. Therefore, by Lemma 2.6

$$\lim_{\lambda\downarrow 0} \left\| \int_0^\infty e^{-\lambda t}(k_\gamma * u)(t)\, dt \right\| = \lim_{\lambda\downarrow 0} \lambda^{-\gamma} \left\| \int_0^\infty e^{-\lambda t}u(t)\, dt \right\| = \infty,$$

which implies $\sigma(k_\gamma * u) \ge 0$. Consequently $\sigma(k_\gamma * u) = 0$. This completes the proof. ☐

The following proposition may be regarded as a continuous version of the classical theorem of Abel for power series (cf. [15, §1.22], [16, Chapters 2 and 5]).

Proposition 2.8. *Suppose assumption (A) or (B) holds. Assume that $\int_0^\infty u(s)\, ds := \lim_{t\to\infty} \int_0^t u(s)\, ds$ exists. Then, for any $0 < \delta < \pi/2$, $\int_0^\infty e^{-\lambda s}u(s)\, ds := \lim_{t\to\infty} \int_0^t e^{-\lambda s}u(s)\, ds$ exists uniformly for all λ in $D(0;\delta)$, where $D(0;\delta) := \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0, |\arg\lambda| < \delta\}$. Consequently, the function $\lambda \mapsto \int_0^\infty e^{-\lambda s}u(s)\, ds$ is continuous on $\{0\} \cup D(0;\delta)$.*

Proof. By hypothesis, given an $\epsilon > 0$, there exists $K > 0$ such that $\| \int_a^b u(s)\, ds\| < \epsilon$ for all $K < a < b < \infty$. Let $a > K$. Since $\sigma(u) = \sigma(u - u_a) \le 0$, where $u_a(s) := \chi_{[0,\, a)}(s)u(s)$ as before, we have

$$\int_a^\infty e^{-\lambda s}u(s)\, ds = \int_0^\infty e^{-\lambda s}(u - u_a)(s)\, ds = \lambda \int_0^\infty e^{-\lambda s}(k_1 * (u - u_a))(s)\, ds$$

for all $\lambda \in D(0;\delta)$. Since

$$\|(k_1 * (u - u_a))(s)\| = \left\| \int_0^s \chi_{[a,\infty)}(r)u(r)\, dr \right\| < \epsilon \qquad (s > 0),$$

it follows that

$$\begin{aligned}\Big\|\int_0^\infty e^{-\lambda s}u(s)\,ds-\int_0^a e^{-\lambda s}u(s)\,ds\Big\| &= \Big\|\int_a^\infty e^{-\lambda s}u(s)\,ds\Big\| \\ &\le |\lambda|\int_0^\infty e^{-(\mathrm{Re}\lambda)s}\|(k_1*(u-u_a))(s)\|\,ds \\ &< \frac{|\lambda|}{\mathrm{Re}\lambda}\,\epsilon \;<\; \frac{1}{\cos\delta}\,\epsilon \qquad (a>K),\end{aligned}$$

completing the proof. □

Remarks. (a) The following example shows that Proposition 2.8 does not hold when the condition $0<\delta<\pi/2$ is replaced with $\delta=\pi/2$.

Example 6. Let $X := \{(a_n)_{n=1}^\infty : a_n \in \mathbb{C},\ \lim_{n\to\infty} a_n = 0\}$. Then X becomes a Banach space with the norm $\|(a_n)\| := \sup_{n\ge 1}|a_n|$. Choose a strictly positive continuous function f on $[0,\infty)$ such that f is decreasing on $[0,\infty)$, $f(0)=1$, $\lim_{t\to\infty} f(t)=0$, and $\int_0^\infty f(t)\,dt=\infty$. For $n\ge 1$, define a function u_n on $[0,\infty)$ by $u_n(s) := e^{is/n}\,n^{-2}f(s)$, and put

$$u(s) := \big(u_n(s)\big)_{n=1}^\infty = \big(e^{is/n}\,n^{-2}f(s)\big)_{n=1}^\infty \ (\in X).$$

It is clear that $u:[0,\infty)\to X$ is continuous. Let $B>0$. Then we have

$$\int_0^B u(s)\,ds = \Big(\int_0^B u_n(s)\,ds\Big)_{n=1}^\infty,$$

and

$$\begin{aligned}\int_0^B u_n(s)\,ds &= \frac{1}{n^2}\int_0^B e^{is/n}f(s)\,ds = \frac{1}{n}\int_0^{B/n} e^{is}f(ns)\,ds \\ &= \frac{1}{n}\Big(\int_0^{B/n} f(ns)\cos s\,ds + i\int_0^{B/n} f(ns)\sin s\,ds\Big) \\ &=: \frac{1}{n}\,I_n(B) + i\frac{1}{n}\,II_n(B),\end{aligned}$$

where, as is easily seen, $\lim_{B\to\infty} I_n(B)$ and $\lim_{B\to\infty} II_n(B)$ exist. Further

$$-\frac{\pi}{2} \le I_n(B) = \int_0^{B/n} f(ns)\cos s\,ds \le \int_0^{\pi/2} f(ns)\cos s\,ds \le \int_0^{\pi/2} f(s)\,ds \le \frac{\pi}{2},$$

and

$$0 \le II_n(B) = \int_0^{B/n} f(ns)\sin s\,ds \le \int_0^{\pi} f(ns)\sin s\,ds \le \pi.$$

Therefore

$$\Big|\int_0^B u_n(s)\,ds\Big| \le \frac{1}{n}\,\big|I_n(B)\big| + \frac{1}{n}\,\big|II_n(B)\big| \le \frac{1}{n}\cdot\frac{\pi}{2} + \frac{1}{n}\cdot\pi = \frac{3\pi}{2n} \qquad (B>0),$$

and hence $\left|\int_0^\infty u_n(s)\,ds\right| = \lim_{B\to\infty}\left|\int_0^B u_n(s)\,ds\right| \le (2n)^{-1}3\pi$ for each $n \ge 1$. It follows that

$$\int_0^\infty u(s)\,ds = \lim_{B\to\infty}\int_0^B u(s)\,ds = \lim_{B\to\infty}\left(\int_0^B u_n(s)\,ds\right)_{n=1}^\infty = \left(\int_0^\infty u_n(s)\,ds\right)_{n=1}^\infty$$

exists in X. Next, let $\lambda_k := \delta_k + ik^{-1}$ for $k \ge 1$, where $\delta_k > 0$ will be determined later. Then

$$\int_0^\infty e^{-\lambda_k s}\,u(s)\,ds = \left(\int_0^\infty e^{-(\delta_k+ik^{-1})s}\,u_n(s)\,ds\right)_{n=1}^\infty \in X,$$

where, in particular,

$$\int_0^\infty e^{-(\delta_k+ik^{-1})s}\,u_n(s)\,ds = \int_0^\infty e^{-\delta_k s}\,k^{-2}f(s)\,ds \quad \text{when } n = k.$$

Since $\lim_{\delta_k\downarrow 0}\int_0^\infty e^{-\delta_k s}f(s)\,ds = \int_0^\infty f(s)\,ds = \infty$, we can choose δ_k so that $0 < \delta_k < k^{-1}$ and

$$\int_0^\infty e^{-\delta_k s}\,k^{-2}f(s)\,ds \ge k.$$

Then the sequence $\{\delta_k + ik^{-1}\}_{k=1}^\infty$ satisfies that $\delta_k + ik^{-1} \in D(0;\pi/2)$ for all $k \ge 1$, $\lim_{k\to\infty}(\delta_k + ik^{-1}) = 0$, and

$$\lim_{k\to\infty}\int_0^\infty e^{-(\delta_k+ik^{-1})s}u(s)\,ds = \lim_{k\to\infty}\left(\int_0^\infty e^{-(\delta_k+ik^{-1})s}u_n(s)\,ds\right)_{n=1}^\infty$$

does not exist in X, because

$$\begin{aligned}\lim_{k\to\infty}\left\|\left(\int_0^\infty e^{-(\delta_k+ik^{-1})s}u_n(s)\,ds\right)_{n=1}^\infty\right\| &\ge \lim_{k\to\infty}\int_0^\infty e^{-\delta_k s}\,k^{-2}f(s)\,ds \\ &\ge \lim_{k\to\infty} k = \infty.\end{aligned}$$

This proves that the function $\lambda \mapsto \int_0^\infty e^{-\lambda s}u(s)\,ds$ from $\{0\}\cup D(0;\pi/2)$ to X is not continuous at 0. Thus the uniform convergence of $\int_0^\infty e^{-\lambda s}u(s)\,ds = \lim_{t\to\infty}\int_0^t e^{-\lambda s}u(s)\,ds$ does not hold on the domain $D(0;\pi/2)$.

(b) The existence of the limit

$$\lim_{D(0;\pi/2)\ni\lambda\to 0}\int_0^\infty e^{-\lambda s}u(s)\,ds$$

does not imply the existence of $\lim_{t\to\infty}\int_0^t u(s)\,ds$. For example, let $u(t) := \sin t$ for $t \ge 0$. Then

$$\int_0^\infty e^{-\lambda s}u(s)\,ds = \int_0^\infty e^{-\lambda s}\sin s\,ds = \frac{1}{1+\lambda^2}$$

for all λ with $\mathrm{Re}\lambda > 0$. Thus $\lim_{\mathrm{Re}\lambda>0,\,\lambda\to 0} \int_0^\infty e^{-\lambda s}u(s)\,ds = 1$. On the other hand, $\lim_{t\to\infty} \int_0^t \sin s\,ds = \lim_{t\to\infty}(1-\cos t)$ does not exist.

Theorem 2.9. (Cf. Theorem 2.4 of [3].) *Suppose assumption (A) or (B) holds. Then the following hold.*

(i) *If* $0 \le \sigma(u) < \infty$, *then* $\sup_{w<\lambda<K} \|\mathfrak{a}_\lambda\| < \infty$ *for all* $\sigma(u) < w < K < \infty$.

(ii) *If* $\sigma(u) < 0$, *then* $\sup_{\lambda>0} \|\lambda^{-1}\mathfrak{a}_\lambda\| < \infty$ *and* $\sup_{0<\lambda<K} \|\mathfrak{a}_\lambda\| < \infty$ *for all* $0 < K < \infty$.

Proof. (i) By Proposition 2.8 (or [1, Theorem 1.5.1]), the function $\lambda \mapsto \int_0^\infty e^{-\lambda t}u(t)\,dt$ is continous on the interval $(\sigma(u),\infty)$. Thus, if $\sigma(u) \ge 0$, then the function $\lambda \mapsto \mathfrak{a}_\lambda = \lambda \int_0^\infty e^{-\lambda t}u(t)\,dt$ is continuous on $(\sigma(u),\infty)$. Hence (i) follows.

(ii) By Theorem 2.5

$$\mathfrak{a}_\lambda = \lambda \int_0^\infty e^{-\lambda t}u(t)\,dt = \lambda^2 \int_0^\infty e^{-\lambda t}(1*u)(t)\,dt$$

for all $\lambda > 0$. Since $\lim_{t\to\infty}(1*u)(t) = \int_0^\infty u(s)\,ds$ exists, we have $M := \sup_{t>0} \|(1*u)(t)\| < \infty$ and $\|\mathfrak{a}_\lambda\| \le \lambda^2 \int_0^\infty e^{-\lambda t}M\,dt = M\lambda$ for all $\lambda > 0$. Hence $\sup_{\lambda>0} \|\lambda^{-1}\mathfrak{a}_\lambda\| \le M$. By this and (i), $\sup_{0<\lambda<K} \|\mathfrak{a}_\lambda\| < \infty$ for all $0 < K < \infty$. The proof is complete. $\square$

Remarks. (a) The hypothesis $0 \le \sigma(u) < w < K < \infty$ cannot be sharpened as $0 \le \sigma(u) = w < K < \infty$, or $0 \le \sigma(u) < w < K = \infty$ in Theorem 2.9(i). To see this, let $\lambda_0 \ge 0$ and $\delta > 0$. Define $u(s) := e^{\lambda_0 s}s^{\delta-1}$ for $s \ge 0$. Then $\sigma(u) = \lambda_0$ and, for all $\lambda > \lambda_0$,

$$\begin{aligned}\mathfrak{a}_\lambda &= \lambda \int_0^\infty e^{-\lambda s}u(s)\,ds = \lambda \int_0^\infty e^{-(\lambda-\lambda_0)s}s^{\delta-1}\,ds \\ &= \frac{\lambda}{(\lambda-\lambda_0)^\delta} \int_0^\infty e^{-s}s^{\delta-1}\,ds = \frac{\lambda}{(\lambda-\lambda_0)^\delta}\Gamma(\delta).\end{aligned}$$

Thus if $\delta > 1$, then $\lim_{\lambda\downarrow\lambda_0} \|\mathfrak{a}_\lambda\| = \lim_{\lambda\downarrow\lambda_0} \Gamma(\delta)\lambda/(\lambda-\lambda_0)^\delta = \infty$. Similarly, if $1 > \delta > 0$, then $\lim_{\lambda\to\infty} \|\mathfrak{a}_\lambda\| = \infty$.

(b) From the proof of Theorem 2.9(ii) we see that if the limit $\int_0^\infty u(s)\,ds = \lim_{t\to\infty} \int_0^t u(s)\,ds$ exists, then $\alpha_0(\mathfrak{a}.) \le -1$, and in particular if $\int_0^\infty u(s)\,ds \ne 0$, then $\alpha_0(\mathfrak{a}.) = -1$. Further, the condition $0 < K < \infty$ cannot be sharpened as $K = \infty$ in Theorem 2.9(ii). To see this, let $v(s) := e^{-\lambda_0 s}s^{\delta-1}$ for $s \ge 0$, where $\lambda_0 > 0$ and $1 > \delta > 0$. Then $\sigma(v) = -\lambda_0 < 0$, and for all $\lambda > 0$

$$\mathfrak{a}_\lambda = \lambda \int_0^\infty e^{-(\lambda+\lambda_0)s}s^{\delta-1}\,ds = \frac{\lambda}{(\lambda+\lambda_0)^\delta}\Gamma(\delta).$$

Thus $\lim_{\lambda\uparrow\infty}\|\mathfrak{a}_\lambda\| = \infty$.

Theorem 2.10. (Cf. Theorem 2.7 of [3].) *Suppose assumption (A) or (B) holds. Let $\gamma \geq 0$, $\alpha > -1-\gamma$, and $M > 0$. Assume that $\|\mathfrak{c}_t^\gamma\| \leq Mt^\alpha$ for dt-almost all $t > 0$. Then the following hold.*

(i) *If $\beta > 0$, then for dt-almost all $t > 0$*

$$\|\mathfrak{c}_t^{\gamma+\beta}\| \leq M\frac{\Gamma(\gamma+\alpha+1)}{\Gamma(\gamma+1)}\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma+\alpha+\beta+1)}t^\alpha, \tag{15}$$

where the right-hand side of (15) can be replaced with Mt^α when $\alpha \geq 0$.

(ii) *If $\sigma(u) \leq 0$, then for all $\lambda > 0$*

$$\|\mathfrak{a}_\lambda\| \leq M\frac{\Gamma(\gamma+\alpha+1)}{\Gamma(\gamma+1)}\lambda^{-\alpha}. \tag{16}$$

Proof. (i) Since

$$k_{\gamma+1}(t)\|\mathfrak{c}_t^\gamma\| \leq \frac{t^\gamma}{\Gamma(\gamma+1)}Mt^\alpha = M\frac{\Gamma(\gamma+\alpha+1)}{\Gamma(\gamma+1)}k_{\gamma+\alpha+1}(t) \tag{17}$$

for dt-almost all $t > 0$, it follows that

$$\begin{aligned}
\|\mathfrak{c}_t^{\gamma+\beta}\| &= \|(k_{\gamma+\beta+1}(t))^{-1}(k_\beta * (k_{\gamma+1}\mathfrak{c}_\cdot^\gamma))(t)\| \\
&\leq M\frac{\Gamma(\gamma+\alpha+1)}{\Gamma(\gamma+1)}\frac{\Gamma(\gamma+\beta+1)}{t^{\gamma+\beta}}(k_\beta * k_{\gamma+\alpha+1})(t) \\
&= M\frac{\Gamma(\gamma+\alpha+1)}{\Gamma(\gamma+1)}\frac{\Gamma(\gamma+\beta+1)}{t^{\gamma+\beta}}k_{\gamma+\alpha+\beta+1}(t) \qquad \text{(by Lemma 2.1)} \\
&= M\frac{\Gamma(\gamma+\alpha+1)}{\Gamma(\gamma+1)}\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma+\alpha+\beta+1)}t^\alpha
\end{aligned}$$

for dt-almost all $t > 0$. In particular if $\alpha \geq 0$, then $\|\mathfrak{c}_s^\gamma\| \leq Ms^\alpha \leq Mt^\alpha$ for ds-almost all $0 < s < t$, so that

$$\begin{aligned}
\|\mathfrak{c}_t^{\gamma+\beta}\| &= \|(k_{\gamma+\beta+1}(t))^{-1}(k_\beta * (k_{\gamma+1}\,\mathfrak{c}_\cdot^\gamma)(t)\| \\
&\leq (k_{\gamma+\beta+1}(t))^{-1}(k_\beta * k_{\gamma+1})(t)\,Mt^\alpha = Mt^\alpha
\end{aligned}$$

for dt-almost all $t > 0$. (It is possible to prove directly that $\Gamma(\gamma+\alpha+1)\Gamma(\gamma+\beta+1)\{\Gamma(\gamma+1)\Gamma(\gamma+\alpha+\beta+1)\}^{-1} \leq 1$ if $\alpha \geq 0$ (cf. the proof of Theorem 2.7(i) in [3]).)

(ii) Since $\sigma(u) \le 0$, we can apply Theorem 2.5(ii), together with (17) and Lemma 2.1, to obtain the following estimation for all $\lambda > 0$:

$$\begin{aligned}
\|\mathfrak{a}_\lambda\| &= \lambda^{\gamma+1}\left\|\int_0^\infty e^{-\lambda t}k_{\gamma+1}(t)\mathfrak{c}_t^\gamma\,dt\right\| \le \lambda^{\gamma+1}\int_0^\infty e^{-\lambda t}k_{\gamma+1}(t)\|\mathfrak{c}_t^\gamma\|\,dt \\
&\le \lambda^{\gamma+1}M\frac{\Gamma(\gamma+\alpha+1)}{\Gamma(\gamma+1)}\int_0^\infty e^{-\lambda t}k_{\gamma+\alpha+1}(t)\,dt \\
&= \lambda^{\gamma+1}M\frac{\Gamma(\gamma+\alpha+1)}{\Gamma(\gamma+1)}\lambda^{-(\gamma+\alpha+1)} = M\frac{\Gamma(\gamma+\alpha+1)}{\Gamma(\gamma+1)}\lambda^{-\alpha}.
\end{aligned}$$

This completes the proof. □

Remarks. **(a)** The assumption $\|\mathfrak{c}_t^\gamma\| \le Mt^\alpha$ for dt-almost all $t > 0$ does not imply $\sigma(u) \le 0$ in the case where assumption (A) holds. For example, the function u in Example 5 satisfies $\sigma(u) = 1$ by (12) and $\mathfrak{c}_t^2 = (k_3(t))^{-1}(k_2 * u)(t) = \Gamma(3)t^{-2}f(t) = -2t^{-2}\cos e^t$ on $[\log\frac{\pi}{2},\infty)$ by (13). Hence $\sup_{t>0}|t^{-\alpha}\mathfrak{c}_t^2| < \infty$ whenever $\alpha \ge -2$, since $\mathfrak{c}_t^2 = f(t) = 0$ on $[0, \delta]$. On the other hand, if $0 \le \gamma \le 1$ and $\|\mathfrak{c}_t^\gamma\| = O(t^\alpha)$ (mod dt) as $t \to \infty$, then $\sigma(u) \le 0$. This can be proved by using Theorems 2.5(i) and 2.4 as follows.

$$\sigma(u) \le \max\{\sigma(u), 0\} = \max\{w_0(\mathfrak{c}_\cdot^1), 0\} \le \max\{w_0(\mathfrak{c}_\cdot^\gamma), 0\} \le 0.$$

Further we note that if assumption (B) holds, then, for any $\gamma \ge 0$, the condition $\|\mathfrak{c}_t^\gamma\| = O(t^\alpha)$ (mod dt) as $t \to \infty$ implies $\sigma(u) \le 0$, and thus Theorem 2.10(ii) holds without the assumption $\sigma(u) \le 0$. This can be proved by using Theorems 2.7, 2.5(i), and 2.4 as follows.

$$\begin{aligned}
\sigma(u) \le \max\{\sigma(u),0\} &= \max\{\sigma(k_\gamma * u),0\} = \max\{w_0(\mathfrak{c}_\cdot^{\gamma+1}),0\} \\
&\le \max\{w_0(\mathfrak{c}_\cdot^\gamma),0\} = 0.
\end{aligned}$$

(b) Suppose assumption (A) or (B) holds. Then $\sigma(u) = \sigma(w)$, where $w(t) := t^\beta u(t)$ for $t \ge 0$ with $\beta > 0$. This follows from an easy modification of the argument in [1, Proposition 1.4.1]. (For the details, see the proof of Theorem 2.3(i) in [3].) This will be used implicitely in Theorem 3.3 below.

Theorem 2.11. (Cf. Corollary 2.8 of [3].) *Suppose assumption (A) or (B) holds. Let $\gamma \ge 0$, and $\alpha > -1-\gamma$. Assume that $\|\mathfrak{c}_t^\gamma\| = O(t^\alpha)$ (mod dt) as $t \to \infty$. Then the following hold.*

(i) *For all $\beta > 0$, $\|\mathfrak{c}_t^{\gamma+\beta}\| = O(t^\alpha)$ (mod dt) as $t \to \infty$.*

(ii) *If $\sigma(u) \le 0$, then $\|\mathfrak{a}_\lambda\| = O(\lambda^{-\alpha})$ as $\lambda \downarrow 0$.*

Proof. By the assumption there exist $M > 0$ and $K > 0$ such that $\|\mathfrak{c}_t^\gamma\| \le Mt^\alpha$ for dt-almost all $t \ge K$.

(i) For dt-almost all $t > 0$ we have

$$\begin{aligned}\mathfrak{c}_t^{\gamma+\beta} &= (k_{\gamma+\beta+1}(t))^{-1}(k_\beta * (k_\gamma * u))(t) \\ &= (k_{\gamma+\beta+1}(t))^{-1}\left(\int_0^K + \int_K^t\right) k_\beta(t-s)(k_\gamma * u)(s)ds =: I + II.\end{aligned}$$

To estimate $\|I\|$, first suppose assumption (B) holds. Then, for all $t > 2K$,

$$\begin{aligned}\|I\| &= \frac{M_{ab}}{t^{\gamma+\beta}}\left\|\int_0^K (t-s)^{\beta-1}(k_\gamma * u)(s)\,ds\right\| \\ &= \frac{M_{ab}}{t^{\gamma+\beta}}\, t^{\beta-1}\left\|\int_0^K \left(1-\frac{s}{t}\right)^{\beta-1}(k_\gamma * u)(s)\,ds\right\| \\ &\leq \frac{M_{ab}}{t^{1+\gamma}}\, \max\{(1/2)^{\beta-1}, 1\}\,\left\|\int_0^K (k_\gamma * u)(s)\,ds\right\|.\end{aligned}$$

Next suppose assumption (A) holds. Then

$$\|I\| \leq \frac{M_{ab}}{t^{1+\gamma}}\,\max\{(1/2)^{\beta-1}, 1\}\,\int_0^K \|(k_\gamma * u)(s)\|\,ds \qquad (t > 2K).$$

Thus, in either case,

$$\|I\| = O(t^{-1-\gamma}) \qquad (t > 2K).$$

The estimation of $\|II\|$ is done as follows (see (17)).

$$\begin{aligned}\|II\| &= (k_{\gamma+\beta+1}(t))^{-1}\left\|\int_K^t k_\beta(t-s)(k_{\gamma+1}(s)\mathfrak{c}_s^\gamma)\,ds\right\| \\ &\leq (k_{\gamma+\beta+1}(t))^{-1}\int_0^t k_\beta(t-s)k_{\gamma+1}(s)(Ms^\alpha)\,ds \\ &= M\frac{\Gamma(\gamma+\alpha+1)}{\Gamma(\gamma+1)}\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma+\alpha+\beta+1)}t^\alpha\end{aligned}$$

for all $t > K$. Thus $\alpha > -1-\gamma$ implies

$$\|\mathfrak{c}_t^{\gamma+\beta}\| \leq \|I\| + \|II\| = O(t^{-1-\gamma}) + O(t^\alpha) = O(t^\alpha) \;\; (\mathrm{mod}\ dt) \;\text{ as }\; t \to \infty.$$

(ii) Since $\sigma(u) \leq 0$, we have

$$\mathfrak{a}_\lambda = \lambda^{\gamma+1}\left(\int_0^K + \int_K^\infty\right) e^{-\lambda t}(k_\gamma * u)(t)\,dt =: III + IV$$

for all $\lambda > 0$. Here if assumption (B) holds, then

$$\|III\| \leq \lambda^{\gamma+1}\left\|\int_0^K (k_\gamma * u)(s)\,ds\right\| = O(\lambda^{\gamma+1}) \qquad (\lambda > 0);$$

and if assumption (A) holds, then

$$\|III\| \le \lambda^{\gamma+1} \int_0^K \|(k_\gamma * u)(s)\|\, ds = O(\lambda^{\gamma+1}) \qquad (\lambda > 0).$$

The estimation of $\|IV\|$ is done as follows.

$$\begin{aligned} \|IV\| &\le \lambda^{\gamma+1} \int_K^\infty e^{-\lambda t} k_{\gamma+1}(t) \|\mathfrak{c}_t^\gamma\|\, dt \\ &\le \lambda^{\gamma+1} M \frac{\Gamma(\gamma+\alpha+1)}{\Gamma(\gamma+1)} \int_K^\infty e^{-\lambda t} k_{\gamma+\alpha+1}(t)\, dt \\ &\le M \frac{\Gamma(\gamma+\alpha+1)}{\Gamma(\gamma+1)} \lambda^{-\alpha}. \end{aligned}$$

Thus $\alpha > -1-\gamma$ implies

$$\|\mathfrak{a}_\lambda\| \le \|III\| + \|IV\| = O(\lambda^{\gamma+1}) + O(\lambda^{-\alpha}) = O(\lambda^{-\alpha}) \quad \text{as } \lambda \downarrow 0.$$

This completes the proof. □

Remark. If the assumption "$\|\mathfrak{c}_t^\gamma\| = O(t^\alpha)$ (mod dt) as $t \to \infty$ in Theorem 2.11 is replaced with the local assumption "$\|\mathfrak{c}_t^\gamma\| = O(t^\alpha)$ (mod dt) as $t \downarrow 0$", then the following hold.

(i) For all $\beta > 0$, $\|\mathfrak{c}_t^{\gamma+\beta}\| = O(t^\alpha)$ as $t \to +0$.

(ii) If $\sigma(u) < \infty$, then $\|\mathfrak{a}_\lambda\| = O(\lambda^{-\alpha})$ as $\lambda \uparrow \infty$.

For, by the assumption, there exist $M > 0$ and $K > 0$ such that $\|\mathfrak{c}_t^\gamma\| \le Mt^\alpha$ for dt-almost all $0 < t \le K$. Then, as in the proof of Theorem 2.11(i),

$$\begin{aligned} \|\mathfrak{c}_t^{\gamma+\beta}\| &\le \big(k_{\gamma+\beta+1}(t)\big)^{-1} \int_0^t k_\beta(t-s) k_{\gamma+1}(s) \big(Ms^\alpha\big)\, ds \\ &= M\, \frac{\Gamma(\gamma+\alpha+1)}{\Gamma(\gamma+1)}\, \frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma+\alpha+\beta+1)}\, t^\alpha \end{aligned}$$

for dt-almost all $0 < t \le K$, so that (i) follows. To prove (ii), note that $\|\mathfrak{c}_t^{\gamma+1}\| = O(t^\alpha)$ (mod dt) as $t \to +0$ by (i), whence there exist $M_1 > 0$ and $K > 0$ such that $\|\mathfrak{c}_t^{\gamma+1}\| \le M_1 t^\alpha$ for dt-almost all $0 < t \le K$. Then for all $\lambda > \max\{\sigma(u), 0\}$ we have

$$\mathfrak{a}_\lambda = \lambda^{\gamma+2} \Big(\int_0^K + \int_K^\infty \Big) e^{-\lambda t} k_{\gamma+2}(t) \mathfrak{c}_t^{\gamma+1}\, dt =: I + II,$$

where

$$\begin{aligned}\|I\| &\le \lambda^{\gamma+2}\int_0^K e^{-\lambda t}k_{\gamma+2}(t)\big(M_1t^\alpha\big)\,dt\\ &\le \lambda^{\gamma+2}\,M_1\,\frac{\Gamma(\gamma+\alpha+2)}{\Gamma(\gamma+2)}\int_0^\infty e^{-\lambda t}k_{\gamma+\alpha+2}(t)\,dt\\ &= M_1\,\frac{\Gamma(\gamma+\alpha+2)}{\Gamma(\gamma+2)}\,\lambda^{-\alpha}=O(\lambda^{-\alpha}).\end{aligned}$$

As for $\|II\|$, since $w_0(k_{\gamma+1}*u)\le\max\{\sigma(u),0\}$ by Theorem 2.5(i), we see that

$$\begin{aligned}\|II\| &\le \lambda^{\gamma+2}\int_K^\infty e^{-\lambda t}\|(k_{\gamma+1}*u)(t)\|\,dt\\ &\le \lambda^{\gamma+2}\,e^{-(\lambda/2)K}\int_K^\infty e^{-(\lambda/2)t}\|(k_{\gamma+1}*u)(t)\|\,dt\\ &= \lambda^{\gamma+2}\,e^{-(\lambda/2)K}\,o(1)=o(\lambda^{-\alpha})\quad\text{ as }\lambda\uparrow\infty.\end{aligned}$$

Consequently $\|\mathfrak{a}_\lambda\|\le\|I\|+\|II\|=O(\lambda^{-\alpha})+o(\lambda^{-\alpha})=O(\lambda^{-\alpha})$ as $\lambda\uparrow\infty$, which is the desired result.

As an immediate consequence of Theorem 2.11 we have the next corollary.

Corollary 2.12. *Suppose assumption (A) or (B) holds. Then the following hold.*

(i) *If* $\gamma'>\gamma\ge0$, *then* $\alpha_0(\mathfrak{c}_\cdot^{\gamma'})\le\max\{\alpha_0(\mathfrak{c}_\cdot^\gamma),-1-\gamma\}$.

(ii) *If* $\sigma(u)\le0$, *then* $\alpha_0(\mathfrak{a}_\cdot)\le\max\{\alpha_0(\mathfrak{c}_\cdot^\gamma),-1-\gamma\}$ *for all* $\gamma\ge0$.

Remark. Let $\lambda_0>0$. Then the function $u(t):=t^{\lambda_0-1}$ for $t\ge0$ satisfies

$$\mathfrak{c}_t^\gamma=\frac{\gamma}{t^\gamma}\int_0^t(t-s)^{\gamma-1}s^{\lambda_0-1}\,ds=\gamma\,t^{\lambda_0-1}\int_0^1(1-s)^{\gamma-1}s^{\lambda_0-1}\,ds=\gamma\,t^{\lambda_0-1}B(\gamma,\lambda_0)$$

for all $\gamma>0$ and $t>0$, and

$$\mathfrak{a}_\lambda=\lambda\int_0^\infty e^{-\lambda t}t^{\lambda_0-1}\,dt=\lambda^{-(\lambda_0-1)}\int_0^\infty e^{-t}t^{\lambda_0-1}\,dt=\lambda^{-(\lambda_0-1)}\Gamma(\lambda_0)$$

for all $\lambda>0$. Thus we have $\alpha_0(\mathfrak{c}_\cdot^\gamma)=\alpha_0(u)=\lambda_0-1=\alpha_0(\mathfrak{a}_\cdot)$ for all $\gamma\ge0$. Of course this is a special case. In general the function $\gamma\mapsto\alpha_0(\mathfrak{c}_\cdot^\gamma)$ is not constant on $[0,\infty)$. To see this the following examples would be interesting.

Example 7. Let $u(t):=\sin t$ for $t\ge0$. Then $\alpha_0(\mathfrak{a}_\cdot)=-1$, and

$$\alpha_0(\mathfrak{c}_\cdot^\gamma)=\begin{cases}-\gamma & (0\le\gamma<1),\\ -1 & (\gamma\ge1).\end{cases}$$

To see this we first note that

$$\mathfrak{a}_\lambda = \lambda \int_0^\infty e^{-\lambda t} \sin t \, dt = \frac{\lambda}{1+\lambda^2}$$

for all $\lambda > 0$, whence $\alpha_0(\mathfrak{a}_.) = -1$. On the other hand, since $\mathfrak{c}_t^1 = t^{-1}\int_0^t \sin s \, ds = t^{-1}(1-\cos t)$ for all $t > 0$, it follows that $\alpha_0(\mathfrak{c}_.^1) = -1$. Therefore, by Corollary 2.12, $\alpha_0(\mathfrak{c}_.^\gamma) = -1$ for all $\gamma \geq 1$. It is clear that $\alpha_0(\mathfrak{c}_.^0) = \alpha_0(u) = 0$. Next suppose $0 < \gamma < 1$. Then, for $t > 4\pi$, we have

$$\begin{aligned} \mathfrak{c}_t^\gamma &= \frac{\gamma}{t^\gamma}\int_0^t (t-s)^{\gamma-1}\sin s \, ds \\ &= \frac{\gamma}{t^\gamma}\left(\int_0^{t-K} + \int_{t-K}^t\right)(t-s)^{\gamma-1}\sin s \, ds \; =: \frac{\gamma}{t^\gamma}\big(I + II\big), \end{aligned}$$

where K is defined to be the constant such that $2\pi < K \leq 4\pi$ and $(t-K)(2\pi)^{-1}$ is a positive integer. Then

$$|II| \leq \int_{t-K}^t (t-s)^{\gamma-1}|\sin s| \, ds \leq \int_0^K s^{\gamma-1} \, ds = \frac{K^\gamma}{\gamma} \leq \frac{(4\pi)^\gamma}{\gamma}.$$

Since $t-K = 2\pi l$ for some integer $l \geq 1$, and since the function $s \mapsto (t-s)^{\gamma-1}$ is positive and strictly increasing on $[0,\, t]$, it follows that

$$\int_{2j\pi-2\pi}^{2j\pi}(t-s)^{\gamma-1}\sin s \, ds < 0 < \int_{2j\pi-\pi}^{2j\pi+\pi}(t-s)^{\gamma-1}\sin s \, ds$$

for all $1 \leq j \leq l-1$. Therefore, using $\int_0^\pi (t-s)^{\gamma-1}\sin s \, ds > 0$, we have

$$\begin{aligned} 0 &< -\int_0^{t-K}(t-s)^{\gamma-1}\sin s \, ds = -\int_0^{2l\pi}(t-s)^{\gamma-1}\sin s \, ds \\ &= -\left(\int_0^\pi + \sum_{j=1}^{l-1}\int_{2j\pi-\pi}^{2j\pi+\pi} + \int_{2l\pi-\pi}^{2l\pi}\right)(t-s)^{\gamma-1}\sin s \, ds \\ &< -\int_{2l\pi-\pi}^{2l\pi}(t-s)^{\gamma-1}\sin s \, ds < \int_{2l\pi-\pi}^{2l\pi}(t-s)^{\gamma-1} \, ds \\ &< \pi(t-2l\pi)^{\gamma-1} = \pi K^{\gamma-1} < \pi(2\pi)^{\gamma-1}, \end{aligned}$$

so that

$$|I| = \left|\int_0^{t-K}(t-s)^{\gamma-1}\sin s \, ds\right| < \pi(2\pi)^{\gamma-1}.$$

Hence

$$|\mathfrak{c}_t^\gamma| \leq \frac{\gamma}{t^\gamma}\big(|I| + |II|\big) \leq \frac{\gamma}{t^\gamma}\left(\pi(2\pi)^{\gamma-1} + \frac{(4\pi)^\gamma}{\gamma}\right),$$

and thus $\alpha_0(\mathfrak{c}^\gamma_\cdot) \le -\gamma$. By a similar argument,

$$\begin{aligned}\limsup_{t\to\infty}\int_0^t (t-s)^{\gamma-1}\sin s\,ds \;&\ge\; \limsup_{n\to\infty}\int_{2n\pi-\pi}^{2n\pi+\pi}(2n\pi+\pi-s)^{\gamma-1}\sin s\,ds\\ &\ge \int_\pi^{3\pi}(3\pi-s)^{\gamma-1}\sin s\,ds > 0,\end{aligned}$$

so that $\limsup_{t\to\infty}|t^\gamma \mathfrak{c}^\gamma_t| > 0$, and hence $\alpha_0(\mathfrak{c}^\gamma_\cdot) \ge -\gamma$. Consequently $\alpha_0(\mathfrak{c}^\gamma_\cdot) = -\gamma$.

Using Example 7 we prove here that the function $u(t) = \sin t$ satisfies $\sigma(k_\gamma * u) = \sigma(u) = 0$ for all $\gamma > 0$ (cf. Example 4). It is obvious that $\sigma(u) = 0$. Next suppose $\gamma > 0$. Since $\gamma + 1 > 1$, we have $\alpha_0(\mathfrak{c}^{\gamma+1}_\cdot) = -1$ by Example 7. Hence the equation

$$\Gamma(\gamma+1)\,(k_{\gamma+1} * u)(t) = \int_0^t (t-s)^\gamma \sin s\,ds = \frac{t^{\gamma+1}}{\gamma+1}\,\mathfrak{c}^{\gamma+1}_t \quad (t>0)$$

shows that $w_0(k_{\gamma+1} * u) = 0$ and that $\lim_{t\to\infty}(k_{\gamma+1} * u)(t)$ does not exist. Hence, by Theorem 1.4.3 of [1], $\sigma(k_\gamma * u) = w_0(1*(k_\gamma * u)) = w_0(k_{\gamma+1} * u) = 0$.

Example 8. For $n \ge 1$ let H_n be the linear subspace of $L_2([0,\,1],\mathbb{R})$ determined by the functions $v_l(t) := t^l$ on $[0,\,1]$ with $l = 0, 1, \ldots, n-1$. Denote by P_n the orthogonal projection operator from $L_2([0,\,1],\mathbb{R})$ to H_n and define

$$u := v_n - P_n v_n. \tag{18}$$

u can be regarded as a continuous function on $[0,\,1]$. Also we can regard u as a function on $[0,\infty)$ by setting $u(t) := 0$ for all $t > 1$. Then $u \in L_1([0,\infty),\mathbb{R})$ and the following hold.

$$\alpha_0(\mathfrak{c}^\gamma_\cdot) = \begin{cases} -\infty & \text{if } \gamma = 0, 1, \ldots, n,\\ -1-n & \text{if } \gamma \in [0,\infty)\setminus\{0,1,\ldots,n\},\end{cases} \tag{19}$$

and $\alpha_0(\mathfrak{a}_\cdot) = -1-n$. (We note that $\sigma(u) = -\infty$. See Remark (b) under Theorem 2.9.)

To see this we first note that $\sigma(u) = -\infty$ and $\alpha_0(\mathfrak{c}^0_\cdot) = \alpha_0(u) = -\infty$, since $u(t) = 0$ for all $t > 1$ by definition. If $\gamma = k$, where k is an integer satisfying $1 \le k \le n$, then, using the equation

$$\int_0^t s^l u(s)\,ds = \int_0^1 s^l u(s)\,ds = \int_0^1 s^l (v_n - P_n v_n)(s)\,ds = 0$$

for all $t > 1$ and $l = 0, 1, \ldots, k-1$, we obtain that

$$\mathfrak{c}^k_t \;=\; \frac{k}{t^k}\int_0^t (t-s)^{k-1}u(s)\,ds \;=\; \frac{k}{t^k}\sum_{l=0}^{k-1}\binom{k-1}{l} t^{k-1-l}(-1)^l \int_0^t s^l u(s)\,ds = 0$$

for all $t > 1$. It follows that $\alpha_0(\mathfrak{c}^k_\cdot) = -\infty$. On the other hand, since

$$\begin{aligned}\mathfrak{c}^{n+1}_t &= \frac{n+1}{t^{n+1}} \int_0^t (t-s)^n u(s)\, ds = \frac{n+1}{t^{n+1}} \int_0^1 (t-s)^n u(s)\, ds \\ &= \frac{n+1}{t^{n+1}} (-1)^n \int_0^1 s^n u(s)\, ds \qquad (t \geq 1),\end{aligned}$$

and

$$\int_0^1 s^n u(s)\, ds = \int_0^1 (v_n - P_n v_n)^2(s)\, ds > 0,$$

it follows that $\alpha_0(\mathfrak{c}^{n+1}_\cdot) = -1 - n$. Further,

$$(k_{n+1} * u)(t) = \frac{1}{n!} \int_0^t (t-s)^n u(s)\, ds = \frac{(-1)^n}{n!} \int_0^1 s^n u(s)\, ds \neq 0 \qquad (t \geq 1).$$

Thus, by Theorem 2.5(ii), for all $\lambda > 0$

$$\begin{aligned}\mathfrak{a}_\lambda &= \lambda^{n+2} \int_0^\infty e^{-\lambda t} (k_{n+1} * u)(t)\, dt \\ &= \lambda^{n+2} \int_0^1 e^{-\lambda t} \left((k_{n+1} * u)(t) - \frac{(-1)^n}{n!} \int_0^1 s^n u(s)\, ds \right) dt \\ &\quad + \lambda^{n+2} \int_0^\infty e^{-\lambda t} \left(\frac{(-1)^n}{n!} \int_0^1 s^n u(s)\, ds \right) dt \\ &= \lambda^{n+1} \left\{ \lambda \int_0^1 e^{-\lambda t} \left((k_{n+1} * u)(t) - \frac{(-1)^n}{n!} \int_0^1 s^n u(s)\, ds \right) dt \right. \\ &\qquad \left. + \frac{(-1)^n}{n!} \int_0^1 s^n u(s)\, ds \right\}.\end{aligned}$$

Since

$$\lim_{\lambda \downarrow 0} \lambda \int_0^1 e^{-\lambda t} \left((k_{n+1} * u)(t) - \frac{(-1)^n}{n!} \int_0^1 s^n u(s)\, ds \right) dt = 0,$$

it follows that

$$\lim_{\lambda \downarrow 0} \lambda^{-1-n} \mathfrak{a}_\lambda = \frac{(-1)^n}{n!} \int_0^1 s^n u(s)\, ds \neq 0.$$

Hence $\alpha_0(\mathfrak{a}_\cdot) = -1 - n$. Combining this with the result that $\alpha_0(\mathfrak{c}^{n+1}_\cdot) = -1-n$, and using Corollary 2.12, we have $\alpha_0(\mathfrak{c}^\gamma_\cdot) = -1-n$ for all $\gamma \geq n+1$.

Finally, suppose $k = 0, 1, \ldots, n$ and $0 < \beta < 1$. Then

$$\begin{aligned}\mathfrak{c}^{k+\beta}_t &= \frac{k+\beta}{t^{k+\beta}} \int_0^t (t-s)^k (t-s)^{\beta-1} u(s)\, ds \\ &= \frac{k+\beta}{t^{k+1}} \int_0^1 (t-s)^k \left(1 - \frac{s}{t}\right)^{\beta-1} u(s)\, ds\end{aligned}$$

for all $t > 1$. We write

$$\left(1-\frac{s}{t}\right)^{\beta-1} = \left(\sum_{l=0}^{n-k-1} + \sum_{l=n-k}^{\infty}\right) \frac{(1-\beta)(2-\beta)\dots(l-\beta)}{l!} \left(\frac{s}{t}\right)^{l} =: f(s)+g(s).$$

Since $f(s)$ is a polynomial function with degree $\max\{n-k-1, 0\}$, and

$$g(s) = \frac{(1-\beta)(2-\beta)\dots(n-k-\beta)}{(n-k)!} \left(\frac{s}{t}\right)^{n-k} (1+o(1)) \quad \text{as } t \to \infty,$$

it follows that

$$\int_0^1 (t-s)^k f(s)u(s)\,ds = 0,$$

and

$$\begin{aligned}
&\int_0^1 (t-s)^k g(s)u(s)\,ds \\
&\quad = \int_0^1 (t-s)^k \frac{(1-\beta)(2-\beta)\dots(n-k-\beta)}{(n-k)!} \left(\frac{s}{t}\right)^{n-k} (1+o(1))\,u(s)\,ds \\
&\quad = \frac{(1-\beta)(2-\beta)\dots(n-k-\beta)}{(n-k)!}(-1)^k t^{-(n-k)} \int_0^1 s^n (1+o(1))\,u(s)\,ds
\end{aligned}$$

as $t \to \infty$. Thus

$$\begin{aligned}
\lim_{t\to\infty} t^{n+1}\mathfrak{c}_t^{k+\beta} &= \frac{(k+\beta)(1-\beta)(2-\beta)\dots(n-k-\beta)}{(n-k)!}(-1)^k \int_0^1 s^n u(s)\,ds \\
&\neq 0,
\end{aligned}$$

which shows that $\alpha_0(\mathfrak{c}_\cdot^{k+\beta}) = -1-n$.

Theorem 2.13. (Cf. Theorem 2.9 of [3].) *Suppose assumption (B) holds. Let $\gamma \geq 1$, and $\alpha > -1-\gamma$. Then the following hold.*

(i) $\sup_{t>0} \|t^{-\alpha}\mathfrak{c}_t^\gamma\| < \infty$ *if and only if* $\sigma(u) \leq 0$ *and* $\sup_{\lambda>0}\|\lambda^\alpha \mathfrak{a}_\lambda\| < \infty$.

(ii) $\|\mathfrak{c}_t^\gamma\| = O(t^\alpha)$ *as* $t \to \infty$ *if and only if* $\sigma(u) \leq 0$ *and* $\|\mathfrak{a}_\lambda\| = O(\lambda^{-\alpha})$ *as* $\lambda \downarrow 0$.

Proof. First we note that, since $\gamma \geq 1$ by assumption, $t \mapsto \mathfrak{c}_t^\gamma$ becomes a continuous function on $(0,\infty)$, as was remarked in §1. Then each of the first conditions of (i) and (ii) implies $\sigma(u) \leq 0$ (cf. Remark (a) under Theorem 2.10), so that the necessity parts of (i) and (ii) follow from Theorems 2.10(ii) and 2.11(ii), respectively.

To show the sufficiency part of (i), suppose $\lambda^\alpha\|\mathfrak{a}_\lambda\| \leq M$ for all $\lambda > 0$. By Lemma 2.6 we have

$$\lambda^\alpha \mathfrak{a}_\lambda = \lambda^{\alpha+\gamma}\int_0^\infty e^{-\lambda s}(k_{\gamma-1} * u)(s)\,ds$$

for all $\lambda > 0$. Since u is positive, it follows that

$$\begin{aligned}\lambda^{\alpha}\mathfrak{a}_{\lambda} &\geq \lambda^{\alpha+\gamma}\int_0^t e^{-\lambda s}(k_{\gamma-1} * u)(s)\,ds \\ &\geq \lambda^{\alpha+\gamma}e^{-\lambda t}(k_{\gamma} * u)(t) = \lambda^{\alpha+\gamma}e^{-\lambda t}k_{\gamma+1}(t)\mathfrak{c}_t^{\gamma} \geq 0 \qquad (t > 0).\end{aligned}$$

Thus $\lambda^{\alpha+\gamma}e^{-\lambda t}k_{\gamma+1}(t)\|\mathfrak{c}_t^{\gamma}\| \leq M$ for all $\lambda > 0$ and $t > 0$. Fix any $t > 0$ and let $\lambda = 1/t$. Then we have that $t^{-\alpha}\|\mathfrak{c}_t^{\gamma}\| \leq Me\,\Gamma(\gamma+1)$ for all $t > 0$, i.e. that $\sup_{t>0}\|t^{-\alpha}\mathfrak{c}_t^{\gamma}\| < \infty$. This proof also shows the sufficiency part of (ii), since $\lambda = 1/t \downarrow 0$ is equivalent to $t \to \infty$. $\square$

Remarks. **(a)** Assumption (B) cannot be replaced with (A) in Theorem 2.13; further, the hypothesis $\gamma \geq 1$ cannot be replaced with $\gamma > 1-\epsilon$, where $0 < \epsilon < 1$. For these and more we refer the reader to [3].

(b) Suppose assumption (B) holds. Let $\gamma \geq 1$ and $\alpha > -1-\gamma$. Then, under the additional assumption that $\sigma(u) < \infty$, we have $\|\mathfrak{c}_t^{\gamma}\| = O(t^{\alpha})$ as $t \to +0$ if and only if $\|\mathfrak{a}_{\lambda}\| = O(\lambda^{-\alpha})$ as $\lambda \uparrow \infty$. This follows easily from Remark under Theorem 2.11 and the argument given in Theorem 2.13. We note that assumption (B) cannot be replaced with (A), here. To see this we give the following example.

Example 9. Let u be the real-valued function on $[0,\infty)$ defined by $u(t) := \sum_{n=1}^{\infty} f_n(t)$, where f_n has the form

$$f_n(t) = a_n\left(\chi_{[b_n-\delta_n,\, b_n)}(t) - \chi_{[b_n,\, b_n+\delta_n)}(t)\right),$$

with a_n, b_n, $\delta_n > 0$ (δ_n is sufficiently small), and satisfies

$$\int_0^{\infty}|f_n(t)|\,dt = a_n(2\delta_n) = n^{-2} \quad (n \geq 1).$$

(b_n and δ_n will be determined below.) Hence $\displaystyle\int_0^{\infty}|u(t)|\,dt \leq \sum_{n=1}^{\infty}\int_0^{\infty}|f_n(t)|\,dt$ $< \infty$, and

$$\mathfrak{c}_{b_n}^1 = \frac{1}{b_n}\int_{b_n-\delta n}^{b_n} f_n(t)\,dt = \frac{n^{-2}}{2b_n}.$$

Thus, if $1 > b_1 > b_2 > \dots$, $\lim_{n\to\infty} n^{-2}/2b_n = \infty$, and $b_n - \delta_n > b_{n+1} + \delta_{n+1}$, then $\limsup_{t\to 0}\mathfrak{c}_t^1 \geq \lim_{n\to\infty}\mathfrak{c}_{b_n}^1 = \infty$.

On the other hand, since $0 < te^{-t} \leq e^{-1}$ for all $t > 0$ and $\lim_{t\to\infty} te^{-t} = 0$, it follows that

$$0 < \lambda\int_0^{\infty} e^{-\lambda t}f_n(t)\,dt \leq \int_{b_{n+1}}^{b_{n-1}} \lambda te^{-\lambda t}\,\frac{1}{t}|f_n(t)|\,dt \to 0$$

as $\lambda \uparrow \infty$. Hence there exists $\lambda_n > 0$ so that if $\lambda \geq \lambda_n$ then

$$0 < \lambda \int_0^\infty e^{-\lambda t} f_n(t)\, dt < n^{-2}.$$

Now, suppose $0 < \lambda < \lambda_n$. Then we have

$$\begin{aligned} \lambda \int_0^\infty e^{-\lambda t} f_n(t)\, dt &\leq \lambda a_n \delta_n \Big(e^{-\lambda(b_n - \delta_n)} - e^{-\lambda(b_n + \delta_n)} \Big) \\ &= \lambda \frac{n^{-2}}{2} e^{-\lambda b_n} (e^{\lambda \delta_n} - e^{-\lambda \delta_n}) \\ &\leq \lambda_n \frac{n^{-2}}{2} (e^{\lambda_n \delta_n} - e^{-\lambda_n \delta_n}) \qquad (\text{because } 0 < \lambda < \lambda_n). \end{aligned}$$

Since $\lim_{\delta \downarrow 0} e^{\lambda_n \delta} - e^{-\lambda_n \delta} = 0$, if $\delta_n > 0$ is sufficiently small, then

$$\lambda_n \frac{n^{-2}}{2} (e^{\lambda_n \delta_n} - e^{-\lambda_n \delta_n}) < n^{-2} \qquad (n \geq 1).$$

Thus we have

$$0 < \lambda \int_0^\infty e^{-\lambda t} f_n(t)\, dt < n^{-2} \quad \text{for all } \lambda > 0 \text{ and } n \geq 1,$$

so that

$$\sup_{\lambda > 0} |\mathfrak{a}_\lambda| = \sup_{\lambda > 0} \lambda \int_0^\infty e^{-\lambda t} u(t)\, dt = \sup_{\lambda > 0} \sum_{n=1}^\infty \lambda \int_0^\infty e^{-\lambda t} f_n(t)\, dt < \infty.$$

Consequently we have: $\sigma(u) \leq 0$, $\|\mathfrak{a}_\lambda\| = O(1)$ as $\lambda \uparrow \infty$, and $\|\mathfrak{c}_t^1\| \neq O(1)$ as $t \to +0$.

(c) Suppose assumption (A) or (B) holds. Assume that $x = \lim_{t\to\infty} \mathfrak{c}_t^\gamma$ (mod dt) exists for some $\gamma \geq 0$. Then

(i) $\lim_{t\to\infty} \mathfrak{c}_t^{\gamma+\beta} = x$ (mod dt) for all $\beta > 0$;

(ii) $\lim_{\lambda\downarrow 0} \mathfrak{a}_\lambda = x$ whenever $\sigma(u) \leq 0$.

For, by considering the function $t \mapsto u(t) - x$, it is sufficient to check the case where $\|\mathfrak{c}_t^\gamma\| = o(1)$ (mod dt) as $t \to \infty$. Then the proof of Theorem 2.11 with $\alpha = 0$ shows that $\|\mathfrak{c}_t^{\gamma+\beta}\| = o(1)$ (mod dt) as $t \to \infty$ for all $\beta > 0$, and $\|\mathfrak{a}_\lambda\| = o(1)$ as $\lambda \downarrow 0$. (For these and more we refer the reder to [2], [5]–[6], [12].)

Here, if the assumption "$x = \lim_{t\to\infty} \mathfrak{c}_t^\gamma$ (mod dt)" is replaced with the local assumption "$x = \lim_{t\to+0} \mathfrak{c}_t^\gamma$" (mod dt), then the same proof together with Remark under Theorem 2.11 yields that

(iii) $\lim_{t\to+0} \mathfrak{c}_t^{\gamma+\beta} = x$ (mod dt) for all $\beta > 0$;

(iv) $\lim_{\lambda\uparrow\infty} \mathfrak{a}_\lambda = x$ whenever $\sigma(u) < \infty$.

Corollary 2.14. (Cf. Corollary 2.10 of [3].) *Suppose assumption (B) holds. Then the following hold.*

(i) *If* $\gamma \geq 1$ *and* $\alpha > -2$, *then* $\sup_{t>0} \|t^{-\alpha}\mathfrak{c}_t^{\gamma}\| < \infty \Leftrightarrow \sup_{t>0} \|t^{-\alpha}\mathfrak{c}_t^{1}\| < \infty \Leftrightarrow \sigma(u) \leq 0$ *and* $\sup_{\lambda>0} \|\lambda^{\alpha}\mathfrak{a}_\lambda\| < \infty$.

(ii) *If* $u \neq 0$ *and* $\sigma(u) \leq 0$, *then the function* $\gamma \mapsto \alpha_0(\mathfrak{c}_\cdot^{\gamma})$ *is decreasing on* $(0,\infty)$ *and satisfies* $\alpha_0(\mathfrak{c}_\cdot^{\gamma}) = \alpha_0(\mathfrak{a}_\cdot) \geq -1$ *for all* $\gamma \geq 1$.

Proof. (i) This is direct from Theorem 2.13(i).

(ii) For $\lambda > 0$ and $K > 0$ we have

$$\mathfrak{a}_\lambda = \lambda \int_0^\infty e^{-\lambda t} u(t)\,dt \geq \lambda \int_0^K e^{-\lambda t} u(t)\,dt \geq \lambda e^{-\lambda K} \int_0^K u(t)\,dt \geq 0.$$

By the hypothesis $u \neq 0$,

$$\lim_{\lambda \downarrow 0} e^{-\lambda K} \int_0^K u(t)\,dt = \int_0^K u(t)\,dt > 0$$

for some $K > 0$. It follows that $\alpha_0(\mathfrak{a}_\cdot) \geq -1$, whence $\alpha_0(\mathfrak{c}_\cdot^{\gamma}) = \alpha_0(\mathfrak{a}_\cdot) \geq -1$ for all $\gamma \geq 1$ by Theorem 2.13(ii). In particular, $\alpha_0(\mathfrak{c}_\cdot^{1}) \geq -1$. Thus, by Corollary 2.12(i), $\alpha_0(\mathfrak{c}_\cdot^{\gamma}) \geq \alpha_0(\mathfrak{c}_\cdot^{\gamma'}) \geq \alpha_0(\mathfrak{c}_\cdot^{1}) \geq -1$ if $0 < \gamma < \gamma' < 1$.

This completes the proof. □

Remark. The following examples explain the behaviour of the function $\gamma \mapsto \alpha_0(\mathfrak{c}_\cdot^{\gamma})$ on $[0,\infty)$ for $u \in L_1^{improp.\,loc}([0,\infty), X^+)$ with X a Banach lattice. (Cf. Corollary 2.14(ii).)

Example 10. The nonnegative real-valued function u on $[0,\infty)$ defined by

$$u(t) := \begin{cases} 1 & \text{if } t \in \cup_{n=1}^{\infty} [n^n - 1,\, n^n], \\ 0 & \text{otherwise.} \end{cases}$$

satisfies $\sigma(u) = 0$, $\alpha_0(\mathfrak{c}_\cdot^{\gamma}) = -\gamma$ for $0 \leq \gamma \leq 1$, and $\alpha_0(\mathfrak{c}_\cdot^{\gamma}) = -1 = \alpha_0(\mathfrak{a}_\cdot)$ for all $\gamma \geq 1$. (Indeed, it is clear that $u \in L_1^{loc}([0,\infty), \mathbb{R}^+)$, $\sigma(u) = 0$, and $\alpha_0(u) = \alpha_0(\mathfrak{c}_\cdot^{0}) = 0$. If $0 < \gamma \leq 1$, then, using the relation $\int_0^1 s^{\gamma-1}\,ds = \gamma^{-1} \geq \int_a^{a+1} s^{\gamma-1}\,ds$ for all $a \geq 0$, we obtain that, for all $n^n < t \leq (n+1)^{n+1}$,

$$\begin{aligned} \mathfrak{c}_t^{\gamma} &= \frac{\gamma}{t^\gamma} \int_0^t (t-s)^{\gamma-1} u(s)\,ds \\ &= \frac{\gamma}{t^\gamma} \left(\sum_{k=1}^{n} \int_{k^k-1}^{k^k} (t-s)^{\gamma-1}\,ds + \int_{(n+1)^{n+1}-1}^{t} (t-s)^{\gamma-1}\,ds \right) \\ &\leq \frac{1}{t^\gamma}(n+1) \leq \frac{2}{t^\gamma} \cdot n \leq \frac{2}{t^\gamma} \cdot t^{1/n}, \end{aligned}$$

and

$$\mathfrak{c}_{n^n}^{\gamma} = \frac{\gamma}{(n^n)^\gamma} \int_0^{n^n} (n^n - s)^{\gamma-1} u(s)\,ds \geq \frac{1}{(n^n)^\gamma}.$$

It follows that $\alpha_0(\mathfrak{c}^\gamma_\cdot) = -\gamma$ if $0 < \gamma \leq 1$. Hence, by Corollary 2.14(ii), $\alpha_0(\mathfrak{c}^\gamma_\cdot) = -1 = \alpha_0(\mathfrak{a}_\cdot)$ for all $\gamma \geq 1$.)

Example 11. The nonnegative real-valued function u on $[0, \infty)$ defined by $u(t) := \chi_{[0,1]}(t)$ satisfies $\alpha_0(u) = \alpha_0(\mathfrak{c}^0_\cdot) = -\infty$ and $\alpha_0(\mathfrak{c}^\gamma_\cdot) = -1 = \alpha_0(\mathfrak{a}_\cdot)$ for all $\gamma > 0$. (Indeed, it is clear that $\alpha_0(u) = -\infty$. If $\gamma > 0$, $t > 1$ and $\lambda > 0$, then

$$\mathfrak{c}^\gamma_t = \frac{\gamma}{t^\gamma} \int_0^1 (t-s)^{\gamma-1}\, ds = \frac{\gamma}{t} \int_0^1 \Big(1 - \frac{s}{t}\Big)^{\gamma-1} ds = \frac{\gamma}{t}\,(1 + o(1))$$

as $t \to \infty$, and

$$\mathfrak{a}_\lambda = \lambda \int_0^1 e^{-\lambda s}\, ds = \lambda\, \frac{1 - e^{-\lambda}}{\lambda} = \lambda(1 + o(1))$$

as $\lambda \downarrow 0$. It follows that $\alpha_0(\mathfrak{c}^\gamma_\cdot) = -1 = \alpha_0(\mathfrak{a}_\cdot)$ for all $\gamma > 0$.)

Finally, we characterize the condition $\sup_{t>0} \|\mathfrak{c}^1_t\| < \infty$ under assumption (A) by using the *partial Abel means* $\lambda \int_0^b e^{-\lambda t} u(t)\, dt$ $(\lambda,\ b > 0)$ as follows.

Theorem 2.15. *Suppose assumption (A) holds. Let $M > 0$. Then the following hold.*

(i) *If* $\sup_{t>0} \|\mathfrak{c}^1_t\| \leq M$, *then* $\sup_{\lambda,\, b>0} \|\lambda \int_0^b e^{-\lambda t} u(t)\, dt\| \leq M$.

(ii) *If* $\sup_{\lambda,\, b>0} \|\lambda \int_0^b e^{-\lambda t} u(t)\, dt\| \leq M$, *then* $\sup_{t>0} \|\mathfrak{c}^1_t\| \leq (2e-1)M$

Proof. (i) Let $b > 0$. Then the function $u_b(t) = \chi_{[0,\, b]}(t) u(t)$ satisfies $\|\mathfrak{c}^1_t(u_b)\| \leq M$ for all $t > 0$. Since $\lambda \int_0^b e^{-\lambda t} u(t)\, dt = \mathfrak{a}_\lambda(u_b)$ for all $\lambda > 0$, it now follows from Theorem 2.10(ii) (with $\alpha = 0$) that $\|\lambda \int_0^b e^{-\lambda t} u(t)\, dt\| \leq M$ for all $\lambda > 0$.

(ii) Let λ, $t > 0$. Integration by parts gives

$$\begin{aligned}
\int_0^t u(s)\, ds &= \int_0^t e^{\lambda s} e^{-\lambda s} u(s)\, ds \\
&= e^{\lambda t} \int_0^t e^{-\lambda s} u(s)\, ds - \lambda \int_0^t e^{\lambda s} \Big(\int_0^s e^{-\lambda r} u(r)\, dr \Big) ds \\
&= \frac{e^{\lambda t}}{\lambda}\, \lambda \int_0^t e^{-\lambda s} u(s)\, ds - \int_0^t e^{\lambda s} \Big(\lambda \int_0^s e^{-\lambda r} u(r)\, dr \Big) ds,
\end{aligned}$$

so that

$$\Big\| \int_0^t u(s)\, ds \Big\| \leq \Big(\frac{e^{\lambda t}}{\lambda} + \int_0^t e^{\lambda s}\, ds \Big) M = \frac{2e^{\lambda t} - 1}{\lambda}\, M.$$

Here, putting $\lambda = 1/t$, we obtain $\|t^{-1} \int_0^t u(s)\, ds\| \leq (2e-1)M$ for all $t > 0$.

This completes the proof. ☐

Remark. Suppose assumption (A) holds. Let $\alpha+1\geq 0$. Then the next results follow easily from the proofs of Theorems 2.10, 2.11 and 2.15. (We omit the details.)

(i) $\sup_{t>0}\|t^{-\alpha}\mathfrak{c}_t^1\|<\infty$ if and only if $\sup_{\lambda>0}\|\lambda^{\alpha+1}\int_0^b e^{-\lambda t}u(t)\,dt\|<\infty$.

(ii) $\|\mathfrak{c}_t^1\|=O(t^\alpha)$ as $t\to\infty$ if and only if $\sup_{b>0}\|\lambda\int_0^b e^{-\lambda t}u(t)\,dt\|=O(\lambda^{-\alpha})$ as $\lambda\downarrow 0$.

(iii) $\|\mathfrak{c}_t^1\|=O(t^\alpha)$ as $t\to+0$ if and only if there exist constants $K,\ M>0$ such that $\sup_{0<b<K}\|\lambda\int_0^b e^{-\lambda t}u(t)\,dt\|\leq M\lambda^{-\alpha}$ for all $\lambda>0$. (The latter condition is equivalent to $\sup_{0<b<K}\|\lambda\int_0^b e^{-\lambda t}u(t)\,dt\|=O(\lambda^{-\alpha})$ as $\lambda\uparrow\infty$, since $\alpha+1\geq 0$.)

3. Operator-valued functions

In this section we consider strongly measurable operator-valued functions $T:[0,\infty)\to\boldsymbol{B}(X)$, where $\boldsymbol{B}(X)$ denotes the Banach algebra of all bounded linear operators on a Banach space X, with the usual operator norm. We assume that T is *strongly locally integrable.* This means by definition that $\int_0^b\|T(t)x\|\,dt<\infty$ for all $x\in X$ and $0<b<\infty$. When X is a Banach lattice and T is positive (i.e. $T(t)$ is a positive operator on X for all $t\geq 0$), we also assume that T is *improperly strongly locally integrable.* This means by definition that for all $x\in X$ and $0<a<b<\infty$, $\int_a^b\|T(t)x\|\,dt<\infty$ and $\int_0^b T(t)x\,dt:=\lim_{a\downarrow 0}\int_a^b T(t)x\,dt$ exists. Thus, unless the contrary is mentioned explicitely, we will assume below that

(OA) T is strongly locally integrable, or

(OB) X is a Banach lattice and T is positive and improperly strongly locally integrable.

Under assumption (OA) or (OB), $\int_0^b T(t)\,dt$ denotes the operator $x\mapsto\int_0^b T(t)x\,dt$ on X. Similarly, for $\gamma\geq 1$ and $t>0$, $C_t^\gamma=C_t^\gamma(T)$ denotes the operator $x\mapsto\mathfrak{c}_t^\gamma(u_x)=(k_{\gamma+1}(t))^{-1}(k_\gamma*u_x)(t)$ on X, where $u_x(s):=T(s)x$. C_t^γ is called the γ*-th order Cesàro mean* of T over $[0,\,t]$. We note that if $0<\gamma<1$, then the integral $(k_\gamma*u_x)(t)=\int_0^t(t-s)^{\gamma-1}T(s)x\,ds$ may fail to exist for some $x\in X$ and $t>0$, so that the γ-th order Cesàro mean C_t^γ cannot be defined as an operator on X in general. But, if we assume that T satisfies $\sup\{\|T(s)\|\ :\ a\leq s\leq b\}<\infty$ for all $0<a<b<\infty$, then $\mathfrak{c}_t^\gamma(u_x)$ exists for all $x\in X$ and $t>0$. Thus in this case the operator $C_t^\gamma:x\mapsto\mathfrak{c}_t^\gamma(u_x)$ can be defined as an operator on X for all $\gamma>0$ and $t>0$.

In the following we set $C_t^0:=T(t)$ for $t>0$, and $C_0^\gamma:=T(0)$ for $\gamma\geq 0$.

We recall (see §1) that

(20)
$$
\begin{aligned}
w_0(T) &:= \inf\{w \in \mathbb{R} \,:\, \|T(t)\| = O(e^{wt}) \ (\mathrm{mod}\ dt) \ \text{as} \ t \to \infty\},\\
\sigma(T) &:= \inf\left\{ \mathrm{Re}\lambda \,\middle|\, \begin{array}{l} \int_0^\infty e^{-\lambda s}T(s)x\,ds := \lim_{t\to\infty}\int_0^t e^{-\lambda s}T(s)x\,ds \\ \text{exists for all } x \in X \end{array} \right\}\\
&= \sup\{\sigma(u_x) \,:\, x \in X\}.
\end{aligned}
$$

If T is strongly continuous on $(0,\infty)$ and strongly locally bounded (i.e., the function $t \mapsto T(t)x$ is continuous on $(0,\infty)$ and satisfies $\sup_{0\le t\le 1}\|T(t)x\| < \infty$ for all $x \in X$), then, by the uniform boundedness principle, $\sup_{0\le t\le b}\|T(t)\| < \infty$ for all $b > 0$. In this case we have

$$
\begin{aligned}
w_0(u_x) &= \sup\{w \in \mathbb{R} \,:\, \sup_{t\ge 0}\|e^{-wt}T(t)x\| < \infty\},\\
w_0(T) &= \inf\{w \in \mathbb{R} \,:\, \sup_{t\ge 0}\|e^{-wt}T(t)\| < \infty\}\\
&= \sup\{w_0(u_x) \,:\, x \in X\}.
\end{aligned} \tag{21}
$$

Let $\lambda \in \mathbb{C}$. If $\mathrm{Re}\lambda > \sigma(T)$, then $\int_0^\infty e^{-\lambda s}T(s)\,ds$ is defined as the operator $x \mapsto \int_0^\infty e^{-\lambda s}T(s)x\,ds$ on X; in particular, if $\mathrm{Re}\lambda > \max\{\sigma(T),\,0\}$, then the *Abel mean* $A_\lambda := A_\lambda(T) = \lambda\int_0^\infty e^{-\lambda s}T(s)\,ds$ of T is defined as $A_\lambda x = \lambda\int_0^\infty e^{-\lambda s}T(s)x\,ds$ for all $x \in X$.

We recall that if $\sigma(T) \le 0$, then the *growth order* $\alpha_0(A.)$ of $A.$ (at $\lambda = 0$) is defined by the right-hand side of (4) with A_λ instead of $\mathfrak{a}_\lambda$. Similarly, the *polynomial growth order* $\alpha_0(T)$ of T (at ∞) is defined by the right-hand side of (5) with $T(t)$ instead of $u(t)$.

It is easily seen (and well-known) that if X is a Banach lattice and $P : X \to X$ is a positive linear operator on X, then $P \in \boldsymbol{B}(X)$. But, if X is a general Banach space with $\dim X = \infty$, then there are many linear operators on X which are not bounded. Therefore the next lemma is of some interest by itself.

Lemma 3.1. *Let $T : [0,\,1] \to \boldsymbol{B}(X)$ be strongly integrable. Then the operator $\int_0^1 T(t)\,dt$ is in $\boldsymbol{B}(X)$.*

Proof. Let $x_n \in X$ and $\lim_{n\to\infty}\|x_n\| = 0$. For the proof it is sufficient to show that

$$\lim_{n\to\infty}\left\|\int_0^1 T(t)x_n\,dt\right\| = 0.$$

To do this we note that, since the functions $t \mapsto T(t)x_n$ are strongly measurable on $[0,\,1]$, there exists a separable closed subspace Y of X and a Lebesgue measurable set $D \subset [0,\,1]$ with Lebesgue measure zero such that $x_n,\, T(t)x_n \in Y$ for all $n \ge 1$ and $t \in [0,\,1] \setminus D$. Choose a countable set $\{y_n \,:\, n \ge 1\} \subset Y$, with $\|y_n\| = 1$ for all $n \ge 1$, such that it is dense in

$\{y \in Y \;:\; \|y\| = 1\}$. Define

$$\|T(t)\|_Y := \sup\{\|T(t)y\| \;:\; y \in Y,\ \|y\| = 1\}.$$

Since $\|T(t)\|_Y = \sup\{\|T(t)y_n\| : n \geq 1\}$ for $t \in [0, 1]$, it follows that the function $t \mapsto \|T(t)\|_Y$ is Lebesgue measurable on $[0, 1]$. Thus $[0, 1]$ can be written as a countable union of Lebesgue measurable subsets E_n, where $E_n := \{t \in [0, 1] \;:\; \|T(t)\|_Y \leq n\}$. Then, for $n \geq 1$ and $y \in Y$, define

$$S_n y := \int_{E_n} T(t)y\, dt.$$

It follows that each S_n is a bounded linear operator from Y to X, with $\|S_n\| \leq n$. Since $\lim_{n\to\infty} S_n y = \int_0^1 T(t)y\, dt$ for $y \in Y$, we then apply the uniform boundedness principle to infer that $M := \sup_{n\geq 1} \|S_n\| < \infty$. Hence $\|\int_0^1 T(t)y\, dt\| \leq M\|y\|$ for $y \in Y$, so that $\lim_{n\to\infty} \|\int_0^1 T(t)x_n\, dt\| \leq \lim_{n\to\infty} M\|x_n\| = 0$. This completes the proof. □

It follows from Lemma 3.1 and the uniform boundedness principle that, under assumption (OA) or (OB), $\|\int_0^t T(s)\, ds\| < \infty$ and $\|\int_0^t e^{-\lambda s}T(s)\, ds\| < \infty$ for all $t > 0$. Further, we have $\sup_{0\leq t\leq b} \|\int_0^t T(s)\, ds\| < \infty$ for all $0 < b < \infty$, and $\|\int_0^\infty e^{-\lambda s}T(s)\, ds\| < \infty$ whenever $\mathrm{Re}\lambda > \sigma(T)$.

Theorem 3.2. *Suppose assumption (OA) or (OB) holds. Let* $S(t) := \int_0^t T(s)\, ds$ *and* S_∞ *be the strong limit of* $S(t)$ *as* $t \to \infty$ *if it exists, and* $S_\infty := 0$ *otherwise. Then*

(i) $\sigma(T) = \inf\left\{\lambda \in \mathbb{R} \;:\; \sup_{t>0} \|\int_0^t e^{-\lambda s}T(s)x\, ds\| < \infty \ \textit{for all}\ x \in X\right\}$
$= w_0(S - S_\infty)$;

(ii) $\lim_{t\to\infty} \|\int_0^\infty e^{-\lambda s}T(s)\, ds - \int_0^t e^{-\lambda s}T(s)\, ds\| = 0$ *whenever* $\mathrm{Re}\lambda > \sigma(T)$.

Proof. This is an adaptation of the argument in [1, Proposition 1.4.5].

(i) Suppose $\lambda_0 > \sigma(T)$ and $x \in X$. Since $\lambda_0 > \sigma(T) \geq \sigma(u_x)$, it follows that $\int_0^\infty e^{-\lambda_0 s}T(s)x\, ds = \lim_{t\to\infty} \int_0^t e^{-\lambda_0 s}T(s)x\, ds$ exists, so that

$$\sup_{t>0} \left\|\int_0^t e^{-\lambda_0 s}T(s)x\, ds\right\| < \infty.$$

Conversely, suppose $\lambda_0 \in \mathbb{R}$ satisfies $\sup_{t>0} \|\int_0^t e^{-\lambda_0 s}T(s)x\, ds\| < \infty$ for all $x \in X$. Then put, for $x \in X$,

$$G_x(t) := \int_0^t e^{-\lambda_0 s}T(s)x\, ds.$$

Integration by parts gives

$$(22)\quad \int_0^t e^{-\lambda s}T(s)x\,ds = \int_0^t e^{-(\lambda-\lambda_0)s}e^{-\lambda_0 s}T(s)x\,ds = e^{-(\lambda-\lambda_0)t}G_x(t) + (\lambda-\lambda_0)\int_0^t e^{-(\lambda-\lambda_0)s}G_x(s)\,ds$$

for all $\lambda \in \mathbb{C}$ and $t > 0$. Since $\sup_{t>0}\|G_x(t)\| < \infty$, it follows that $\int_0^\infty e^{-\lambda s}T(s)x\,ds = \lim_{t\to\infty}\int_0^t e^{-\lambda s}T(s)x\,ds$ exists if $\lambda > \lambda_0$. It follows that $\lambda_0 \geq \sigma(u_x)$ for all $x \in X$, and hence $\lambda_0 \geq \sigma(T)$. This proves the first equality in (i).

To prove the second equality $\sigma(T) = w_0(S-S_\infty)$, let $\widetilde{v}_x(t) := S(t)x - S_\infty x$, and

$$v_x(t) := \begin{cases} S(t)x - \lim_{s\to\infty} S(s)x & \text{if the limit exists,} \\ S(t)x & \text{otherwise.} \end{cases}$$

First assume that the strong limit S_∞ exists. Then $v_x = \widetilde{v}_x$ and $\sigma(u_x) = \sigma(T(\cdot)x) = w_0(v_x)$ by Theorem 1.4.3 of [1] for all $x \in X$. Since $S - S_\infty$ is strongly continuous on $[0,\infty)$ and hence strongly locally bounded, we may apply (21) for $S - S_\infty$ instead of T to obtain that

$$\sigma(T) = \sup\{\sigma(u_x) : x \in X\} = \sup\{w_0(v_x) : x \in X\} = \sup\{w_0(\widetilde{v}_x) : x \in X\} = w_0(S - S_\infty).$$

Next assume that $S(t)$ does not converge strongly on X as $t \to \infty$. Then $S_\infty = 0$ by definition, and the set $E := \{x \in X : \lim_{t\to\infty} S(t)x \text{ does not exist}\}$ is not empty. If $x \in E$, then $v_x = \widetilde{v}_x$ and $\sigma(u_x) = w_0(v_x) \geq 0$. If $x \notin E$, then $\sigma(u_x) = w_0(v_x) \leq 0$ (by Theorem 1.4.3 of [1]), and thus $w_0(\widetilde{v}_x) \leq 0$ (since $v_x(t) = \widetilde{v}_x(t) - \lim_{s\to\infty} S(s)x$ for all $t \geq 0$). It follows that

$$\begin{aligned} 0 &\leq \sigma(T) = \sup\{\sigma(u_x) : x \in E\} = \sup\{w_0(v_x) : x \in E\} \\ &= \sup\{w_0(\widetilde{v}_x) : x \in E\} = \sup\{w_0(\widetilde{v}_x) : x \in X\} = w_0(S - S_\infty). \end{aligned}$$

(ii) Let $\mathrm{Re}\lambda > \lambda_0 > \sigma(T)$. Since the above-defined function $G_x(\cdot)$ is bounded on $(0,\infty)$ for all $x \in X$, the uniform boundedness principle implies

$$(23)\qquad M := \sup_{t>0}\left\|\int_0^t e^{-\lambda_0 s}T(s)\,ds\right\| < \infty.$$

Hence (ii) follows immediately from the following formula (see (22)):

$$(24)\quad \left\|\int_0^\infty e^{-\lambda s}T(s)x\,ds - \int_0^t e^{-\lambda s}T(s)x\,ds\right\| \leq \left\| e^{-(\lambda-\lambda_0)t}\int_0^t e^{-\lambda_0 s}T(s)x\,ds\right\|$$

$$
\begin{aligned}
& + \ \Big\|(\lambda-\lambda_0)\int_t^\infty e^{-(\lambda-\lambda_0)s}\left(\int_0^s e^{-\lambda_0 r}T(r)x\,dr\right)ds\Big\| \\
& \leq \ e^{-(\mathrm{Re}\lambda-\lambda_0)t}M\|x\| + \frac{|\lambda-\lambda_0|}{|\mathrm{Re}\lambda-\lambda_0|}e^{-(\mathrm{Re}\lambda-\lambda_0)t}M\|x\| \qquad (x\in X).
\end{aligned}
$$

This completes the proof. □

We next investigate the analytic behaviour of the operator-valued function $\lambda \mapsto \widehat{T}(\lambda) := \int_0^\infty e^{-\lambda s}T(s)\,ds$. Thus we must assume that X is a *complex* Banach space. When X is a *real* Banach space, we complexificate X as follows (cf. [13, Chapter II, §11]). Set $X_{\mathbb{C}} := \{x+iy \ : \ x,\, y\in X\}$, and

$$\|x+iy\| := \sup\big\{\|(\cos\theta)x+(\sin\theta)y\| \ : \ 0\leq\theta<2\pi\big\}.$$

Under the usual operations

$$
\begin{aligned}
(a+ib)(x+iy) &:= (ax-by)+i(ay+bx),\\
(x+iy)+(x'+iy') &:= (x+x')+i(y+y'),
\end{aligned}
$$

where $a,\, b\in\mathbb{R}$, $X_{\mathbb{C}}$ becomes a complex Banach space. Regarding $x = x+i0$ for $x\in X$, we may consider X to be a subset of $X_{\mathbb{C}}$. When $Q\in\boldsymbol{B}(X)$, it can be extended canonically to $Q_{\mathbb{C}}\in\boldsymbol{B}(X_{\mathbb{C}})$ by setting $Q_{\mathbb{C}}(x+iy) := Qx+iQy$. Since

$$
\begin{aligned}
\|Q_{\mathbb{C}}(x+iy)\| &= \sup\{\|Q((\cos\theta)x+(\sin\theta)y)\| \ : \ 0\leq\theta<2\pi\}\\
&\leq \|Q\|\,\|x+iy\|,
\end{aligned}
$$

it follows that $\|Q_{\mathbb{C}}\| = \|Q\|$. We will use below the original symbol Q to denote $Q_{\mathbb{C}}$.

Theorem 3.3. *Suppose assumption (OA) or (OB) holds. Then the operator-valued function $\lambda\mapsto\widehat{T}(\lambda) = \int_0^\infty e^{-\lambda s}T(s)\,ds$ is analytic on the domain $\{\lambda\in\mathbb{C} \ : \ \mathrm{Re}\lambda>\sigma(T)\}$ and, for $n\in\mathbb{N}$ and $\mathrm{Re}\lambda>\sigma(T)$, we have*

$$\widehat{T}^{(n)}(\lambda) = \int_0^\infty e^{-\lambda s}(-s)^n T(s)\,ds. \tag{25}$$

Proof. From (23) and (24) it follows that

$$\lim_{t\to\infty}\Big\|\widehat{T}(\lambda)-\int_0^t e^{-\lambda s}T(s)\,ds\Big\| = 0$$

uniformly on compact subsets of $\{\lambda\in\mathbb{C} \ : \ \mathrm{Re}\lambda>\sigma(T)\}$. Thus by the Weierstrass convergence theorem it is sufficient to check that the functions $Q_k : \mathbb{C}\to\boldsymbol{B}(X)$ $(k\in\mathbb{N})$ defined by

$$Q_k(\lambda) := \int_0^k e^{-\lambda s}T(s)\,ds = \lim_{N\to\infty}\sum_{j=0}^N\frac{\lambda^j}{j!}\int_0^k(-s)^jT(s)\,ds \tag{26}$$

are analytic on $\mathbb{C}$, and $Q_k^{(n)}(\lambda) = \int_0^k e^{-\lambda s}(-s)^n T(s)\,ds$ for all $n \in \mathbb{N}$.

To do this, fix any real $G > 0$. For $\lambda \in \mathbb{C}$ with $|\lambda| \le G$, define

$$Q_{k,N}(\lambda) := \sum_{j=0}^{N} \frac{\lambda^j}{j!} \int_0^k (-s)^j T(s)\,ds.$$

We will prove that

$$\lim_{N\to\infty} \|Q_k(\lambda) - Q_{k,N}(\lambda)\| = 0 \quad \text{uniformly on} \quad |\lambda| \le G. \tag{27}$$

Note that

$$Q_k(\lambda) - Q_{k,N}(\lambda) = \int_0^k \left(\sum_{j=N+1}^{\infty} \frac{(-\lambda s)^j}{j!} \right) T(s)\,ds,$$

and

$$\left| \sum_{j=N+1}^{\infty} \frac{(-\lambda s)^j}{j!} \right| \le \sum_{j=N+1}^{\infty} \frac{(Gk)^j}{j!} =: M(N) \to 0 \quad (\text{as } N \to \infty) \tag{28}$$

for all $|\lambda| \le G$ and $0 \le s \le k$.

First suppose assumption (OA) holds. Define for $f \in L_\infty([0,k],\mathbb{C})$ an operator T_f in $\boldsymbol{B}(X)$ by

$$T_f x := \int_0^k f(s)T(s)x\,ds \quad (x \in X).$$

Since $\|T_f x\| \le \int_0^k |f(s)|\|T(s)x\|\,ds \le \|f\|_\infty \int_0^k \|T(s)x\|\,ds < \infty$ for all $x \in X$, the uniform boundedness principle implies that

$$K := \sup\{\|T_f\| \ : \ \|f\|_\infty \le 1\} < \infty,$$

whence by (28) we obtain that

$$\|Q_k(\lambda) - Q_{k,N}(\lambda)\| \le M(N)K \to 0 \quad (\text{as } N \to \infty)$$

for all $|\lambda| \le G$.

Next suppose assumption (OB) holds. Letting

$$a(s) := \ \mathrm{Re}\left(\sum_{j=N+1}^{\infty} \frac{(-\lambda s)^j}{j!} \right), \quad \text{and} \quad b(s) := \ \mathrm{Im}\left(\sum_{j=N+1}^{\infty} \frac{(-\lambda s)^j}{j!} \right),$$

we have $|a(s)|$, $|b(s)| \le M(N)$ for all $|\lambda| \le G$ and $0 \le s \le k$, and

$$\big(Q_k(\lambda) - Q_{k,N}(\lambda)\big)x = \int_0^k (a(s) + ib(s))T(s)x\,ds$$

for $x \in X_{\mathbb{C}}$ (= the compexification of X). (Note that $T(s)$ denotes $T(s)_{\mathbb{C}}$, as was remarked above.) Here, if $x \in X^+$ ($\subset X \subset X_{\mathbb{C}}$), then, since $-M(N)T(s)x \le a(s)T(s)x \le M(N)T(s)x$ for all $0 \le s \le k$, we have

$$-M(N)\int_0^k T(s)x\,ds \le \int_0^k a(s)T(s)x\,ds \le M(N)\int_0^k T(s)x\,ds.$$

Since X is a Banach lattice, it follows that

$$\Big\|\int_0^k a(s)T(s)x\,ds\Big\| \le M(N)\Big\|\int_0^k T(s)x\,ds\Big\| \le M(N)\Big\|\int_0^k T(s)\,ds\Big\|\|x\|.$$

Similarly

$$\Big\|\int_0^k b(s)T(s)x\,ds\Big\| \le M(N)\Big\|\int_0^k T(s)\,ds\Big\|\|x\|.$$

It follows that

$$\|\big(Q_k(\lambda) - Q_{k,N}(\lambda)\big)x\| \le 2M(N)\Big\|\int_0^k T(s)\,ds\Big\|\|x\|. \tag{29}$$

By a similar argument we see that (29) holds for $x \in X$ ($\subset X_{\mathbb{C}}$). Then, for $x + iy \in X_{\mathbb{C}}$, we have

$$\begin{aligned} \|(Q_k(\lambda) - Q_{k,N}(\lambda))(x+iy)\| &\le 2M(N)\Big\|\int_0^k T(s)\,ds\Big\|(\|x\| + \|y\|) \\ &\le 4M(N)\Big\|\int_0^k T(s)\,ds\Big\|\|x+iy\|, \end{aligned}$$

which implies $\|Q_k(\lambda) - Q_{k,N}(\lambda)\| \le 4M(N)\|\int_0^k T(s)\,ds\| \to 0$ (as $N \to \infty$) for all $|\lambda| \le G$.

Thus, in either case, we have proved that $\lim_{N\to\infty}\|Q_k(\lambda) - Q_{k,N}(\lambda)\| = 0$ uniformly on compact subsets of $\mathbb{C}$. Then, by the Weierstrass convergence theorem, Q_k is analytic on $\mathbb{C}$, and $Q_k^{(n)}(\lambda) = \lim_{N\to\infty} Q_{k,N}^{(n)}(\lambda) = \int_0^k e^{-\lambda s}(-s)^n T(s)\,ds$ for all $n \in \mathbb{N}$. This completes the proof. □

Remarks. (a) The following example shows that there exists a strongly measurable positive semigroup $T = (T(t))_{t\ge 0}$ of bounded lnear operators on a Banach lattice X such that it is not strongly locally integrable, but improperly strongly locally integable.

Example 12. For $n \in \mathbb{N}$, let $X_n := L_1([0,\infty), f_n ds)$, where

$$f_n(t) := 1 + n(n+1)\chi_{[1/(n+1),1/n]}(t),$$

and $\|x_n\| := \int_0^\infty |x_n(s)|f_n(s)ds$ for $x_n \in X_n$. Next, let

$$X := \{(x_n)_{n=1}^\infty : x_n \in X_n\,,\ \lim_{n\to\infty}\|x_n\| = 0\},$$

and

$$\|(x_n)_{n=1}^{\infty}\| := \sup_{n\geq 1} \|x_n\| \quad \text{for} \quad (x_n)_{n=1}^{\infty} \in X.$$

X becomes a Banach lattice with the usual operations. For $t \geq 0$ and $x = (x_n)_{n=1}^{\infty} \in X$, define

$$T(t)x := (T(t)x_n)_{n=1}^{\infty},$$

where $[T(t)x_n](s) := x_n(s-t)$ for $n \in \mathbb{N}$. Thus $T(t)x_n \in X_n$ and, by the definition of $T(t)$, if $t > 1/n$ then $\|T(t)x_n\| \leq \|x_n\|$. It follows that $T(t)x = (T(t)x_n)_{n=1}^{\infty} \in X$ for all $x \in X$. It is obvious that $T := (T(t))_{t\geq 0}$ becomes a strongly measurable semigroup of positive linear operators on X.

To prove the existence of $\int_0^1 T(t)x\,dt = \lim_{a\downarrow 0} \int_a^1 T(t)x\,dt$ for all $x \in X$, we may assume without of generality that $x = (x_n)_{n=1}^{\infty} \in X^+$, which is equivalent to $x_n \in L_1^+([0,\infty), f_n ds)$ for all $n \in \mathbb{N}$. Since

$$\int_a^1 T(t)x\,dt = \left(\int_a^1 T(t)x_n\,dt\right)_{n=1}^{\infty},$$

$$\lim_{a\downarrow 0} \int_a^1 T(t)x_n\,dt = \int_0^1 T(t)x_n\,dt \in X_n \quad (\text{in } X_n\text{-norm}),$$

and since

$$\begin{aligned}
(30) \quad \Big\| \int_0^1 T(t)x_n\,dt \Big\| &= \int_0^1 \left\{ \int_0^{\infty} x_n(s-t) f_n(s) ds \right\} dt \\
&= \int_0^1 \left\{ \int_0^{\infty} x_n(s-t)\,ds \right\} dt + \int_0^1 \left\{ \int_{1/(n+1)}^{1/n} x_n(s-t)n(n+1)\,ds \right\} dt \\
&\leq \|x_n\| + \int_{1/(n+1)}^{1/n} \left\{ n(n+1) \int_0^1 x_n(s-t)\,dt \right\} ds \quad (\text{by Fubini's theorem}) \\
&\leq \|x_n\| + \frac{n(n+1)}{n(n+1)} \|x_n\| = 2\|x_n\|,
\end{aligned}$$

it follows that if we set $y := (y_n)_{n=1}^{\infty}$, where $y_n := \int_0^1 T(t)x_n\,dt$ for each $n \in \mathbb{N}$, then $y \in X$ and

$$\int_0^1 T(t)x\,dt = \lim_{a\downarrow 0} \int_a^1 T(t)x\,dt = y \quad (\text{in } X\text{-norm}).$$

We next prove that $\int_0^1 \|T(t)x\|\,dt = \infty$ for some $x \in X^+$. To do this, choose b_n, α_n, $\beta_n > 0$ for each $n \in \mathbb{N}$ such that

$$0 < b_n + \frac{1}{\alpha_n} < \frac{1}{n+1} < b_n + \frac{1}{\alpha_n} + \frac{1}{n+1} < \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n}\right),$$

$$0<\beta_n<\alpha_n\,,\quad \frac{\beta_n}{\alpha_n}\downarrow 0 \quad \text{as} \quad n\to\infty, \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{\beta_n}{\alpha_n}=\infty.$$

Define an element x_n of X_n^+ for each $n\in\mathbb{N}$ by

$$x_n := \beta_n \chi_{[b_n,\, b_n+(1/\alpha_n)]}.$$

Since $\|x_n\| = \int_0^\infty x_n(s) f_n(s)\,ds = \beta_n/\alpha_n$, it follows that $x := (x_n)_{n=1}^\infty$ is in X^+. Further we see that if

$$\frac{1}{n+1} < t < \frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n}\right),$$

then

$$\frac{1}{n+1} < t + b_n + (1/\alpha_n) < \frac{1}{n} \quad \text{and} \quad T(t)x_n = \beta_n \chi_{[t+b_n,\, t+b_n+(1/\alpha_n)]},$$

so that

$$\begin{aligned} \|T(t)x_n\| &= \int_0^\infty [T(t)x_n](s) f_n(s)\,ds = \int_{1/(n+1)}^{1/n} [T(t)x_n](s) f_n(s)\,ds \\ &= (1+n(n+1))\,\frac{\beta_n}{\alpha_n} \qquad (n\in\mathbb{N}). \end{aligned}$$

It follows that

$$\int_{1/(n+1)}^{1/n} \|T(t)x\|\,dt \ \geq \int_{1/(n+1)}^{2^{-1}(1/(n+1)+1/n)} \|T(t)x_n\|\,dt \ \geq\ 2^{-1}\,\frac{\beta_n}{\alpha_n} \qquad (n\in\mathbb{N}).$$

Therefore

$$\int_0^1 \|T(t)x\|\,dt \ = \sum_{n=1}^\infty \int_{1/(n+1)}^{1/n} \|T(t)x\|\,dt \ \geq\ 2^{-1}\sum_{n=1}^\infty \frac{\beta_n}{\alpha_n} \ = \ \infty,$$

which is the desired result.

(b) The following example shows that a strongly locally integrable positive semigroup $T = (T(t))_{t\geq 0}$ of bounded linear operators on a Banach lattice X may satisfy $\lim_{b\downarrow 0}\left\| \int_0^b T(t)\,dt\right\| = 0$ and $\lim_{t\downarrow 0}\|T(t)\| = \infty$.

Example 13. Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of real numbers such that

$$0<\alpha_{n+1}<\alpha_n<\frac{1}{n(n+1)} \quad \text{and} \quad \sum_{n=1}^\infty n\alpha_n = 1.$$

Define a nonnegative real-valued integrable function f on $[0,\infty)$ by

$$f(s) := 1 + \sum_{n=1}^\infty n\chi_{((1/n)-\alpha_n,\, 1/n]}(s),$$

and $X := L_1([0,\infty), f\,ds)$. Let $[T(t)x](s) := x(s-t)$ for $t \geq 0$ and $x \in X$. It is clear that if $t > 1/n$, then $\|T(t)x\| \leq n\|x\|$ for all $x \in X$, so that $T(t)$ is a positive linear operator on X with $\|T(t)\| \leq n$. It follows that $T := (T(t))_{t\geq 0}$ becomes a strongly measurable semigroup of positive linear operators on X satisfying $\lim_{t\downarrow 0} \|T(t)\| = \infty$, since $\|T(t)\| \geq n$ for $t > 0$ with $(n+1)^{-1} < t \leq n^{-1}$ by the definitions of $T(t)$ and X.

To see that $\int_0^b \|T(t)x\|\,dt < \infty$ for all $x \in X$ and $b > 0$, we may assume that $x \in X^+ = L_1^+([0,\infty), f\,ds)$. Given an $\epsilon > 0$, choose $\eta > 0$ so that $\int_0^\eta f(s)\,ds < \epsilon$. Then

$$\begin{aligned} \int_0^b \|T(t)x\|\,dt &= \int_0^b \int_0^\infty x(s-t)f(s)\,dsdt \\ &= \int_0^b \left(\int_0^\eta + \int_\eta^\infty\right) x(s-t)f(s)\,dsdt =: I + II, \end{aligned}$$

where

$$\begin{aligned} I &= \int_0^b \int_0^\eta x(s-t)f(s)\,dsdt = \int_0^\eta \left(\int_0^b x(s-t)\,dt\right) f(s)\,ds \\ &\leq \int_0^\eta \|x\| f(s)\,ds = \|x\| \int_0^\eta f(s)\,ds = \epsilon\|x\|. \end{aligned}$$

Let $N(\eta) := \min\{n \geq 1 \ :\ 1/n < \eta\}$. From the definition of f we see that $f(s) \leq N(\eta)$ for all $s \geq \eta$. Therefore

$$\begin{aligned} II &= \int_0^b \int_\eta^\infty x(s-t)f(s)\,dsdt \leq \int_0^b \int_\eta^\infty x(s-t)N(\eta)\,dsdt \\ &\leq \int_0^b N(\eta)\|x\|\,dt = bN(\eta)\|x\|. \end{aligned}$$

Hence $\int_0^b \|T(t)x\|\,dt = I + II \leq (\epsilon + bN(\eta))\|x\| < \infty$. Thus if $b > 0$ satisfies $\epsilon > bN(\eta)$, then $\int_0^b \|T(t)x\|\,dt \leq 2\epsilon\|x\|$ for all $x \in X^+$. Therefore $\|\int_0^b T(t)\,dt\| \leq 2\epsilon$, and $\lim_{b\downarrow 0} \|\int_0^b T(t)\,dt\| = 0$.

(c) The following example shows that there exists a Banach lattice X and a strongly integrable positive operator-valued function $T : [0,\infty) \to \boldsymbol{B}(X)$ such that $\|\int_0^\infty T(t)\,dt\| = 1 = \|\int_0^b T(t)\,dt\|$ for all $b > 0$.

Example 14. Let $X := \ell_1 = \{(x_n)_{n=1}^\infty \ :\ x_n \in \mathbb{R},\ \sum_{n=1}^\infty |x_n| < \infty\}$, with the norm $\|(x_n)_{n=1}^\infty\| := \sum_{n=1}^\infty |x_n|$. For $n \geq 1$, let $h_n(t) := n(n+1)\chi_{(1/(n+1),\,1/n]}(t)$. Next, for $t \geq 0$, define a positive linear operator $T(t) : X \to X$ by $T(t)x := (h_n(t)x_n)_{n=1}^\infty$ for $x = (x_n)_{n=1}^\infty \in X$. It is clear that $\|T(t)x\| = n(n+1)|x_n| \leq n(n+1)\|x\|$ for $1/(n+1) < t \leq 1/n$, and that the function $T : t \mapsto T(t) \in \boldsymbol{B}(X)$ is strongly measurable. If $x \in X$, then

$$\begin{aligned}(31)\quad \Big\| \int_0^\infty T(t)x\,dt \Big\| &= \int_0^\infty \|T(t)x\|\,dt = \int_0^1 \|T(t)x\|\,dt \\ &= \sum_{n=1}^\infty \int_{1/(n+1)}^{1/n} \|T(t)x\|\,dt = \sum_{n=1}^\infty \int_{1/(n+1)}^{1/n} n(n+1)|x_n|\,dt = \sum_{n=1}^\infty |x_n| = \|x\|.\end{aligned}$$

Hence $\| \int_0^\infty T(t)\,dt\| = 1$. Next, given $b > 0$, choose $j \geq 1$ so that $1/j < b$. Then, the element $x := (x_n)_{n=1}^\infty \in X$ defined by $x_n = 1$ if $n = j$ and $x_n = 0$ if $n \neq j$ satisfies $\int_0^\infty T(t)x\,dt = \int_0^b T(t)x\,dt = \int_{1/(j+1)}^{1/j} T(t)x\,dt = x$. It follows that $1 = \| \int_0^\infty T(t)\,dt\| \geq \| \int_0^b T(t)\,dt\| \geq 1$ for all $b > 0$.

Here we would like to note that the positive semigroup $T = (T(t))_{t\geq 0}$ in Example 12 satisfies $\lim_{b\downarrow 0} \| \int_0^b T(t)\,dt\| = 1$. Indeed, by a slight modification of (30),

$$\Big\| \int_0^b T(t)x_n\,dt \Big\| \leq (b+1)\|x_n\| \qquad (x_n \in X_n^+,\ n \in \mathbb{N}),$$

so that $\limsup_{b\downarrow 0} \big\| \int_0^b T(t)\,dt \big\| \leq 1$. On the other hand, by the definition of f_n, $\big\| \int_0^{1/n(n+1)} T(t)\,dt \big\| \geq 1$. (To see this, it suffices to estimate $\big\| \int_0^{1/n(n+1)} T(t)x\,dt \big\| / \|x\|$ for elements $x = (x_k)_{k=1}^\infty \in X$ of the form $x_k(s) = 0$ if $k \neq n$, and $x_k(s) = \chi_{[(n+1)^{-1}-\epsilon,\,(n+1)^{-1}]}(s)$ if $k = n$.) Hence $\lim_{b\downarrow 0} \big\| \int_0^b T(t)\,dt \big\| = 1$.

The next theorem is an immdiate consequence of Theorem 2.10.

Theorem 3.4. *Suppose (OA) or (OB) holds. Let $\gamma \geq 1$, $\alpha > -1-\gamma$, and $M > 0$. Assume $\|C_t^\gamma\| \leq Mt^\alpha$ for all $t > 0$. Then the following hold.*

(i) *For all $\beta > 0$ and $t > 0$,*

$$(32)\qquad \|C_t^{\gamma+\beta}\| \leq M\frac{\Gamma(\gamma+\alpha+1)}{\Gamma(\gamma+1)} \frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma+\alpha+\beta+1)} t^\alpha,$$

where the right-hand side of (32) is less than or equal to Mt^α when $\alpha \geq 0$.

(ii) *If $\sigma(T) \leq 0$, then, for all $\lambda > 0$,*

$$(33)\qquad \|A_\lambda\| \leq M\frac{\Gamma(\gamma+\alpha+1)}{\Gamma(\gamma+1)}\lambda^{-\alpha}.$$

Theorem 3.5. *Suppose (OA) or (OB) holds. Let $\gamma \geq 1$, $\alpha \geq -1-\gamma$. Assume that $\|C_t^\gamma\| = O(t^\alpha)$ as $t \to \infty$. Then the following hold.*

(i) *For all $\beta > 0$, $\|C_t^{\gamma+\beta}\| = O(t^\alpha)$ as $t \to \infty$.*

(ii) *If $\sigma(T) \leq 0$, then $\|A_\lambda\| = O(\lambda^{-\alpha})$ as $\lambda \downarrow 0$.*

Proof. (i) By the assumption there exist $M >$ and $K > 0$ such that $\|C_t^\gamma\| \le Mt^\alpha$ for all $t \ge K$. Then $\|C_t^\gamma x\| \le Mt^\alpha$ for all $x \in X$ with $\|x\| = 1$, and $t \ge K$. By the proof of Theorem 2.11(i) we have

$$\|C_t^{\gamma+\beta}x\| \le \frac{M_x}{t^{1+\gamma}} + M\frac{\Gamma(\gamma+\alpha+1)}{\Gamma(\gamma+1)}\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma+\alpha+\beta+1)}t^\alpha$$

for all $x \in X$ with $\|x\| = 1$, and $t \ge 2K$, where $M_x > 0$ is a positive constant depending only on $x \in X$ with $\|x\| = 1$. Since $\alpha > -1-\gamma$, it follows that

$$\sup_{t\ge 2K} \|C_t^{\gamma+\beta}x\|t^{-\alpha} < \infty$$

for all $x \in X$ with $\|x\| = 1$. Thus, by the uniform boundedness principle, $\sup_{t\ge 2K} \|C_t^{\gamma+\beta}\|t^{-\alpha} < \infty$.

(ii) This follows similarly by using the proof of Theorem 2.11(ii). □

Remark. If T satisfies $\operatorname{ess\,sup}_{0<a<s<b} \|T(s)\| < \infty$ for all $0 < a < b < \infty$, then C_t^γ can be defined as an operator on X for all $\gamma \ge 0$ and $t > 0$, and the function $t \mapsto C_t^\beta$ becomes strongly continuous on $(0,\infty)$ for all $\beta > 0$. Thus, in this case, the hypothesis $\gamma \ge 1$ in Theorems 3.4 and 3.5 can be replaced with $\gamma \ge 0$, where in particular, if $\gamma = 0$, then the assumptions "$\|T(t)\| \le Mt^\alpha$ for all $t > 0$" in Theorem 3.4 and "$\|T(t)\| = O(t^\alpha)$ as $t \to \infty$" in Theorem 3.5 may be replaced with "$\|T(t)\| \le Mt^\alpha$ for dt-almost all $t > 0$" and "$\|T(t)\| = O(t^\alpha)$ (mod dt) as $t \to \infty$", respectively.

Theorem 3.6. *Suppose (OB) holds. Let $\gamma \ge 1$ and $\alpha > -1-\gamma$. Then*

(i) $\sup_{t>0} \|t^{-\alpha}C_t^\gamma\| < \infty$ *if and only if* $\sigma(T) \le 0$ *and* $\sup_{\lambda>0} \|\lambda^\alpha A_\lambda\| < \infty$*;*

(ii) $\|C_t^\gamma\| = O(t^\alpha)$ *as* $t \to \infty$ *if and only if* $\sigma(T) \le 0$ *and* $\|A_\lambda\| = O(\lambda^{-\alpha})$ *as* $\lambda \downarrow 0$.

(iii) *If* $\int_0^b T(t)\,dt \ne 0$ *for some* $b > 0$ *and* $\sigma(T) \le 0$, *then* $\alpha_0(C_\cdot^\beta) = \alpha_0(A_\cdot) \ge -1$ *for all* $\beta \ge 1$.

Proof. Since T is positive, each of the first conditions of (i) and (ii) implies $\sigma(T) \le 0$ (cf. Remark (a) under Theorem 2.10). Then the necessity parts of (i) and (ii) follow from Theorem 3.4(ii) and Theorem 3.5(ii), respectively. Also the sufficiency parts of (i) and (ii) follow from the proof of Theorem 2.13, because we may only consider the case $x \in X^+$ with $\|x\| = 1$.

To prove (iii), note that, by the hypotheses of (iii), there exists $x \in X^+$, with $\|x\| = 1$, such that $\int_0^b T(t)x\,dt > 0$; then for all $\lambda > 0$

$$\begin{aligned}\|\lambda^{-1}A_\lambda\| &\ge \|\lambda^{-1}A_\lambda x\,\| \ge \left\|\int_0^b e^{-\lambda t}T(t)x\,dt\right\| \\ &\ge e^{-\lambda b}\left\|\int_0^b T(t)x\,dt\right\| \to \left\|\int_0^b T(t)x\,dt\right\| > 0\end{aligned}$$

as $\lambda \downarrow 0$. It follows that $\alpha_0(A.) \geq -1$. Hence, by (ii), $\alpha_0(C^{\beta}_{\cdot}) = \alpha_0(A.) \geq -1$ for all $\beta \geq 1$. This completes the proof. $\square$

Remarks. (a) Assumption (OB) cannot be replaced with (OA) in Theorem 3.6; further, the hypothesis $\gamma \geq 1$ cannot be replaced with $\gamma > 1 - \epsilon$, where $0 < \epsilon < 1$. (See [3].)

(b) The following example shows that there exists a Banach lattice X and a strongly integrable positive operator-valued function $T : [0, \infty) \to \boldsymbol{B}(X)$ such that $T(t) \neq 0$ for all $t \geq 0$ and $\int_0^\infty T(t)\, dt = 0$.

Example 15. Let $X := \Big\{(a_\iota)_{\iota \in \mathbb{R}} \ : \ a_\iota \in \mathbb{R}$ for all $\iota \in \mathbb{R}$, and $\sum_{\iota \in \mathbb{R}} |a_\iota| < \infty\Big\}$. With the usual operations and the norm $\|(a_\iota)\| := \sum_{\iota \in \mathbb{R}} |a_\iota|$, X becomes a Banach lattice. For $t \geq 0$, define a non-zero positive linear operator $T(t) : X \to X$ by $T(t)(a_\iota) := (b_\iota)$, where $b_\iota = a_t$ if $\iota = t$, and $b_\iota = 0$ if $\iota \neq t$. Since $\{\iota : a_\iota \neq 0\}$ is countable for all $(a_\iota) \in X$, the operator-valued function $T : t \mapsto T(t)$ becomes strongly integrable on $[0, \infty)$ and satisfies $\int_0^\infty T(t)(a_\iota)\, dt = 0$ for all $(a_\iota) \in X$.

The next theorem is an immediate consequence of Theorem 2.15.

Theorem 3.7. *Suppose (OA) holds. Then* $\sup_{t>0} \|C^1_t\| < \infty$ *if and only if* $\sup_{\lambda, b>0} \|\lambda \int_0^b e^{-\lambda t} T(t)\, dt\| < \infty$.

4. The discrete case

In this section we consider a sequence $\{x_n\} = \{x_n\}_{n=0}^\infty$ in a Banach space X. If $\gamma \geq 0$, we define the *γ-th order Cesàro mean* $\mathfrak{c}_k^\gamma$ of the sequence over $\{0, 1, \ldots, k\}$ as

$$\mathfrak{c}_k^\gamma = \mathfrak{c}_k^\gamma(\{x_n\}) := \frac{1}{j_{\gamma+1}(k)} \sum_{l=0}^k j_\gamma(k-l) x_l, \tag{34}$$

where

$$j_\gamma(n) := \begin{cases} 1 & \text{if } n = 0, \\ \frac{\gamma(\gamma+1)\ldots(\gamma+n-1)}{n!} & \text{if } n \geq 1. \end{cases} \tag{35}$$

Thus, in particular, we have $\mathfrak{c}_k^0 = x_k$ and $\mathfrak{c}_k^1 = (k+1)^{-1} \sum_{l=0}^k x_l$ for all $k \in \mathbb{N}_0$. It follows from (35) that

$$\frac{1}{(1-t)^\gamma} = \sum_{n=0}^\infty j_\gamma(n) t^n \tag{36}$$

for all $t \in \mathbb{C}$ with $|t| < 1$. We note that the function $j_\gamma : \mathbb{N}_0 \to \mathbb{R}$ can be defined for all $\gamma \in \mathbb{R}$ by (35), and then (36) holds for all j_γ with $\gamma \in \mathbb{R}$. It then follows that

$$j_{\gamma+\beta}(k) = \sum_{l=0}^{k} j_\gamma(k-l) j_\beta(l) = (j_\gamma * j_\beta)(k) \qquad (\gamma,\, \beta \in \mathbb{R},\;\; k \in \mathbb{N}_0), \tag{37}$$

where $j_\gamma * j_\beta$ denotes the convolution of j_γ and j_β. It is known (cf. [17, p. 77]) that if $\beta \notin \{0, -1, -2, \ldots\}$, then

$$\lim_{n\to\infty} \frac{j_\beta(n)}{n^{\beta-1}} = \frac{1}{\Gamma(\beta)}. \tag{38}$$

Therefore there exist constants $B_\beta > A_\beta > 0$ such that

$$A_\beta \cdot n^{\beta-1} \le |j_\beta(n)| \le B_\beta \cdot n^{\beta-1} \qquad (n \in \mathbb{N}_0), \tag{39}$$

where we let $0^{\beta-1} := 1$ for convenience sake. It is possible to define the γ-th order Cesàro means $\mathfrak{c}_k^\gamma$ for all $\gamma \notin \{-1, -2, \ldots\}$ by (34), because $j_{\gamma+1}(k) \neq 0$ for all $k \ge 0$ whenever $\gamma \notin \{-1, -2, \ldots\}$. But the author thinks that to treat the Cesàro means of a sequence it would be natural to consider the case where the terms $j_\gamma(k-l)$ in (34) are all nonnegative. (Indeed, there is a pathological phenomenon when we consider the case $-1 < \gamma < 0$. See, for example, Theorem 4.1 of Li-Sato-Shaw [7].) So in this paper we restrict ourselves to the case $\gamma \ge 0$. It should be mentioned here that Shaw and Chen considered Cesàro means $\mathfrak{c}_n^\gamma$ for $\gamma \notin \{-1, -2, \ldots\}$ and obtained some results (see [14]).

The *exponential growth order* $w_0(\{x_n\})$ of $\{x_n\}$ is defined as

$$w_0(\{x_n\}) := \inf\{w \in \mathbb{R} \;:\; \|x_n\| = O(e^{wn})\}. \tag{40}$$

If $w_0(\{x_n\}) < \infty$, then $\{x_n\}$ is said to be *exponentially bounded.*

Let

$$\mathrm{rad}(\{x_n\}) := \frac{1}{\limsup_{n\to\infty} \|x_n\|^{1/n}}\,. \tag{41}$$

It follows easily that

$$\mathrm{rad}(\{x_n\}) = \sup\Big\{r \ge 0 \;:\; \sum_{n=0}^{\infty} r^n \|x_n\| < \infty\Big\} = \sup\Big\{|r| \;:\; \sum_{n=0}^{\infty} r^n x_n \text{ converges}\Big\},$$

from which we see that $\mathrm{rad}(\{x_n\}) \ge e^{-w_0(\{x_n\})}$. If $r \in \mathbb{C}$ satisfies $0 < |r| < \min\{\mathrm{rad}(\{x_n\}), 1\}$, then the *Abel mean* $\mathfrak{a}_r$ of the sequence is defined as

$$\mathfrak{a}_r = \mathfrak{a}_r(\{x_n\}) := (1-r) \sum_{n=0}^{\infty} r^n x_n \tag{42}$$

and, when $\operatorname{rad}(\{x_n\}) \geq 1$, the *growth order* $\alpha_0(\mathfrak{a}.)$ of $\mathfrak{a}.$ (at $r = 1$) is defined as

$$\alpha_0(\mathfrak{a}.) := \inf\{\alpha \in \mathbb{R} \ : \ \|\mathfrak{a}_r\| = O((1-r)^{-\alpha}) \quad \text{as} \quad 0 < r \uparrow 1\}. \tag{43}$$

Similarly, the *polynomial growth order* $\alpha_0(\{x_n\})$ of $\{x_n\}$ (at $n = \infty$) is defined as

$$\alpha_0(\{x_n\}) := \inf\{\alpha \in \mathbb{R} \ : \ \|x_n\| = O(n^{\alpha})\}. \tag{44}$$

If $\alpha_0(\{x_n\}) < \infty$, then $\{x_n\}$ is said to be *polynomially bounded.*

In the following the sequence $\{x_n\}_{n=0}^{\infty}$ will be considered to be the function $u : n \mapsto x_n$ from $\mathbb{N}_0$ to X. We define the convolution $j_\gamma * u$ of j_γ and u as

$$(j_\gamma * u)(k) := \sum_{l=0}^{k} j_\gamma(k-l)u(l) \qquad (k \in \mathbb{N}_0).$$

Thus we have $\mathfrak{c}_k^\gamma = \mathfrak{c}_k^\gamma(u) = (j_{\gamma+1}(k))^{-1}(j_\gamma * u)(k)$ by (34). It follows from (37) that $j_\beta * (j_\gamma * u) = (j_\beta * j_\gamma) * u = j_{\gamma+\beta} * u$ for all $\gamma,\ \beta \in \mathbb{R}$.

Lemma 4.1. *Let $u : \mathbb{N}_0 \to X$ be a sequence. Define $\widetilde{u} \in L_1^{loc}([0,\infty), X)$ by*

$$\widetilde{u}(t) := u([t]) \qquad \text{for } t \geq 0, \tag{45}$$

where $[t]$ denotes the largest integer not exceeding t. Then the following hold.

(i) *$\sum_{n=0}^{\infty} e^{-\lambda n} u(n)$ converges if and only if $\int_0^\infty e^{-\lambda t}\, \widetilde{u}(t)\, dt$ converges for all $\lambda \in \mathbb{C}$.*

(ii) *$\operatorname{rad}(\{u(n)\}) = e^{-\sigma(\widetilde{u})}$, $w_0(\{u(n)\}) = w_0(\widetilde{u})$, and $w_0(\{(j_1 * u)(n) - F_\infty\}) = w_0((1 * \widetilde{u}) - F_\infty)$, where*

$$F_\infty := \begin{cases} \sum_{n=0}^{\infty} u(n) & \text{if it converges,} \\ 0 & \text{otherwise.} \end{cases} \tag{46}$$

Proof. (i) Since the case $\lambda = 0$ is immediate, we consider the case $\lambda \neq 0$. Then for $t > 0$

$$\begin{aligned} \int_0^t e^{-\lambda s} \widetilde{u}(s)\, ds - \sum_{n=0}^{[t]-1} e^{-\lambda n} u(n) &= \sum_{n=0}^{[t]-1} \int_0^1 \left(e^{-\lambda(n+s)} - e^{-\lambda n} \right) u(n)\, ds \\ &\quad + e^{-\lambda[t]} \int_0^{t-[t]} e^{-\lambda s} u([t])\, ds =: I + II, \end{aligned} \tag{47}$$

where

$$I = \sum_{n=0}^{[t]-1} e^{-\lambda n} \left(\int_0^1 \left(e^{-\lambda s} - 1 \right) ds \right) u(n) = \left(\frac{1 - e^{-\lambda}}{\lambda} - 1 \right) \sum_{n=0}^{[t]-1} e^{-\lambda n} u(n), \tag{48}$$

and

$$II = \frac{1 - e^{-\lambda(t-[t])}}{\lambda} e^{-\lambda[t]} u([t]). \tag{49}$$

Therefore if $\sum_0^\infty e^{-\lambda n} u(n)$ converges, then so does $\int_0^\infty e^{-\lambda t} \widetilde{u}(s)\, ds$.

Conversely suppose $\int_0^\infty e^{-\lambda s}\, \widetilde{u}(s)\, ds$ converges. Then we apply (47)–(49) with $t = n$, and see that $\sum_0^\infty e^{-\lambda n} u(n)$ converges and further that

$$\lambda \int_0^\infty e^{-\lambda s}\, \widetilde{u}(s)\, ds = (1 - e^{-\lambda}) \sum_0^\infty e^{-\lambda n} u(n). \tag{50}$$

(ii) Using (i) we have

$$\begin{aligned}
\mathrm{rad}(\{u(n)\}) &= \sup\left\{ e^{-\lambda} \ :\ \lambda \in \mathbb{R},\ \sum_{n=0}^\infty e^{-\lambda n} u(n) \text{ converges} \right\} \\
&= \sup\left\{ e^{-\lambda} \ :\ \lambda \in \mathbb{R},\ \int_0^\infty e^{-\lambda s}\, \widetilde{u}(s)\, ds \text{ converges} \right\} \\
&= \exp\left(-\inf\left\{ \lambda \in \mathbb{R} \ :\ \int_0^\infty e^{-\lambda s}\, \widetilde{u}(s)\, ds \text{ converges} \right\} \right) \\
&= e^{-\sigma(\widetilde{u})},
\end{aligned}$$

whence the first equality in (ii) follows. The second equality $w_0(\{u(n)\}) = w_0(\widetilde{u})$ is obvious from the definition of $\widetilde{u}$.

To prove the third equality we note that

$$\begin{aligned}
&((1 * \widetilde{u})(t) - F_\infty) - ((j_1 * u)([t] - 1) - F_\infty) \\
&\qquad = \int_0^t \widetilde{u}(s)\, ds - \sum_{n=0}^{[t]-1} u(n) = (t - [t]) u([t])
\end{aligned}$$

for all $t > 0$. Thus the relation

$$u([t]) = (j_1 * u)([t]) - (j_1 * u)([t] - 1)$$

implies

$$\begin{aligned}
\|(1 * \widetilde{u})(t) - F_\infty\| \le \|(j_1 * u)([t] - 1) - F_\infty\| &+ \|(j_1 * u)([t]) - F_\infty\| \\
&+ \|(j_1 * u)([t] - 1) - F_\infty\|.
\end{aligned}$$

Similarly the relation

$$(t - [t]) u([t]) = (1 * \widetilde{u})(t) - (1 * \widetilde{u})([t])$$

implies

$$\begin{aligned}
\|(j_1 * u)([t] - 1) - F_\infty\| \le \|(1 * \widetilde{u})(t) - F_\infty\| &+ \|(1 * \widetilde{u})(t) - F_\infty\| \\
&+ \|(1 * \widetilde{u})([t]) - F_\infty\|.
\end{aligned}$$

Therefore it follows that $\|(1 * \widetilde{u})(t) - F_\infty)\| = O(e^{wt})$ if and only if $\|(j_1 * u)(n) - F_\infty\| = O(e^{wn})$ for any $w \in \mathbb{R}$. Consequently we have $w_0(\{(j_1 * u)(n) - F_\infty\}) = w_0((1 * \widetilde{u}) - F_\infty)$, completing the proof. $\square$

Theorem 4.2. *Let $u : \mathbb{N}_0 \to X$ be a sequence. Then*

$$\operatorname{rad}(\{u(n)\}) = \exp\big(- w_0(\{(j_1 * u)(n) - F_\infty\})\big).$$

Consequently, $\min\{\operatorname{rad}(\{u(n)\}),\, 1\} = \exp\big(- \max\big\{w_0(\{(j_1 * u)(n)\}),\, 0\big\}\big)$.

Proof. By Lemma 4.1, together with Theorem 1.4.3 of [1], we have

$$\begin{aligned}\operatorname{rad}(\{u(n)\}) &= \exp(-\sigma(\widetilde{u})) = \exp(-w_0((1 * \widetilde{u}) - F_\infty)) \\ &= \exp(-w_0(\{(j_1 * u)(n) - F_\infty\})).\end{aligned}$$

Hence

$$\begin{aligned}\min\{\operatorname{rad}(\{u(n)\}), 1\} &= \exp(- \max\{w_0(\{(j_1 * u)(n) - F_\infty\}), 0\}) \\ &= \exp(- \max\{w_0(\{(j_1 * u)(n)\}), 0\}),\end{aligned}$$

where the last equality comes from the fact that if $F_\infty \neq 0$ then $w_0(\{j_1 * u)(n) - F_\infty\}) \le 0$ and $w_0(\{j_1 * u)(n)\}) = 0$. This completes the proof. $\square$

Theorem 4.3. *Let $u : \mathbb{N}_0 \to X$ be a sequence. Suppose $\gamma \ge 0$ and $\beta > 0$. Then:*

(i) $\|\mathfrak{c}_n^{\gamma+\beta}\| \le \max\{\|\mathfrak{c}_k^\gamma\| \;:\; 0 \le k \le n\}$ *for all* $n \in \mathbb{N}_0$*;*

(ii) *if* $\|\mathfrak{c}_n^\gamma\| \le Me^{wn}$ *for some* $M > 0$ *and* $w \ge 0$ *and all* $n \in \mathbb{N}_0$*, then* $\|\mathfrak{c}_n^{\gamma+\beta}\| \le Me^{wn}$ *for all* $n \in \mathbb{N}_0$*;*

(iii) $\max\{w_0(\{\mathfrak{c}_n^\beta\}), 0\} = \max\{w_0(\{(j_\beta * u)(n)\}), 0\} = \max\{w_0(\{u(n)\}), 0\}$.

Proof. By using the relations

$$\begin{aligned}(51)\quad \mathfrak{c}_n^{\gamma+\beta} &= (j_{\gamma+\beta+1}(n))^{-1}(j_{\gamma+\beta} * u)(n) = (j_{\gamma+\beta+1}(n))^{-1}(j_\beta * (j_\gamma * u))(n) \\ &= (j_{\gamma+\beta+1}(n))^{-1}(j_\beta * (j_{\gamma+1}\mathfrak{c}^\gamma_\cdot))(n),\end{aligned}$$

(i) and (ii) follow similarly as (i) and (ii) of Theorem 2.4. Hence we may omit the details.

To prove (iii), first we see from (ii) that

$$\max\{w_0(\{\mathfrak{c}_n^{\gamma+\beta}\}), 0\} \le \max\{w_0(\{\mathfrak{c}_n^\gamma\}), 0\}. \tag{52}$$

Hence, with $\gamma = 0$, we have $\max\{w_0(\{\mathfrak{c}_n^\beta\}), 0\} \le \max\{w_0(\{u(n)\}), 0\}$. To prove the reverse inequality, suppose $\max\{w_0(\{\mathfrak{c}_n^1\}), 0\} < w < \infty$. Since

$\max\{w_0(\{j_\beta * u)(n)\}), 0\} = \max\{w_0(\{\mathfrak{c}_n^\beta\}), 0\}$ by (34) and (38), it then follows (with $\beta = 1$) that

$$\|(j_1 * u)(n)\| = \left\|\sum_{k=0}^{n} u(k)\right\| = o(e^{wn}). \tag{53}$$

Now, put $\mathbb{N}_0(w) := \{n \in \mathbb{N}_0 \ : \ \|u(n)\| > e^{wn}\}$. If $\mathbb{N}_0(w)$ is finite, then it is obvious that $w_0(\{u(n)\}) \leq w$. If $\mathbb{N}_0(w)$ is infinite, then define $n_G := \min\{n \in \mathbb{N}_0(w) \ : \ n > G\}$ for $G > 0$. Thus $\lim_{G\to\infty} n_G = \infty$, and by (53) we have

$$\|(j_1 * u)(n_G - 1)\| = o(e^{w(n_G-1)}) = o(e^{wn_G}) \quad \text{as } G \to \infty, \tag{54}$$

so that

$$\begin{aligned}\|(j_1 * u)(n_G)\| &\geq \|u(n_G)\| - \|(j_1 * u)(n_G - 1)\| \\ &\geq e^{wn_G} - o(e^{wn_G}) = (1 - o(1))e^{wn_G} \quad \text{as } G \to \infty,\end{aligned}$$

which is a contradiction, because $\|(j_1 * u)(n_G)\| = o(e^{wn_G})$ as $G \to \infty$ by (53). It follows that $w_0(\{u(n)\}) \leq w$, and thus $w_0(\{u(n)\}) \leq \max\{w_0(\{\mathfrak{c}_n^1\}), 0\}$. Consequently $\max\{w_0(\{u(n)\}), 0\} = \max\{w_0(\{j_1 * u)(n)\}), 0\}$, and by an induction argument, $\max\{w_0(\{u(n)\}), 0\} = \max\{w_0(\{j_k * u)(n)\}), 0\}$ for all $k \in \mathbb{N}$. From (52) we observe that that if $\beta \leq k \in \mathbb{N}$, then

$$\begin{aligned}\max\{w_0(\{u(n)\}), 0\} &\geq \max\{w_0(\{\mathfrak{c}_n^\beta\}), 0\} \geq \max\{w_0(\{(\mathfrak{c}_n^k\}), 0\} \\ &= \max\{w_0(\{j_k * u)(n)\}), 0\} = \max\{w_0(\{u(n)\}), 0\}.\end{aligned}$$

The proof is complete. □

Theorem 4.4. *Let $u : \mathbb{N}_0 \to X$ be a sequence. Then*

(i) *for all $\gamma > 0$,*

(55)

$$\min\{\mathrm{rad}(\{\mathfrak{c}_n^\gamma\}), 1\} = \min\{\mathrm{rad}(\{(j_\gamma * u)(n)\}), 1\} = \min\{\mathrm{rad}(\{u(n)\}), 1\};$$

(ii) *for all $r \in \mathbb{C}$ with $0 < |r| < \min\{\mathrm{rad}(\{u(n)\}), 1\}$ and $\gamma \geq 0$,*

$$\mathfrak{a}_r = (1-r)^{\gamma+1} \sum_{n=0}^{\infty} r^n (j_\gamma * u)(n) = (1-r)^{\gamma+1} \sum_{n=0}^{\infty} r^n j_{\gamma+1}(n) \mathfrak{c}_n^\gamma . \tag{56}$$

Proof. (i) It follows from (38) and (41) that

$$\min\{\mathrm{rad}(\{\mathfrak{c}_n^\gamma\}), 1\} = \min\{\mathrm{rad}(\{(j_\gamma * u)(n)\}), 1\}.$$

On the other hand, by Theorem 4.2 and Theorem 4.3(iii), we have

$$\begin{aligned}\min\{\mathrm{rad}(\{(j_\gamma * u)(n)\}), 1\} &= \exp\big(-\max\big\{w_0(\{(j_{\gamma+1} * u)(n)\}), 0\big\}\big) \\ &= \exp\big(-\max\big\{w_0(\{(j_1 * u)(n)\}), 0\big\}\big) \\ &= \min\{\mathrm{rad}(\{u(n)\}), 1\},\end{aligned}$$

which proves the second equality in (55).

(ii) The case $\gamma = 0$ is trivial. So we consider the case $\gamma > 0$. Let $r \in \mathbb{C}$ be such that $0 < |r| < \min\{\mathrm{rad}(\{u(n)\}), 1\}$. Then, since $\sum_{n=0}^{\infty} |r|^n \|u(n)\| < \infty$, it follows from (36) that

$$\begin{aligned}\mathfrak{a}_r(\{u(n)\}) &= (1-r)\sum_{n=0}^{\infty} r^n u(n) = (1-r)^{\gamma+1}\Big(\sum_{n=0}^{\infty} r^n j_\gamma(n)\Big)\sum_{n=0}^{\infty} r^n u(n)\\ &= (1-r)^{\gamma+1}\sum_{n=0}^{\infty} r^n (j_\gamma * u)(n) = (1-r)^{\gamma+1}\sum_{n=0}^{\infty} r^n j_{\gamma+1}(n)\mathfrak{c}_n^{\gamma},\end{aligned}$$

whence the proof is complete. □

Remarks. (a) The equality $\min\{\mathrm{rad}(\{j_\gamma * u)(n)\}, 1\} = \min\{\mathrm{rad}(\{u(n)\}), 1\}$ holds for all $\gamma \in \mathbb{R}$. In fact, the proof of Theorem 4.4(ii) shows that the inequality $\min\{\mathrm{rad}(\{(j_\gamma * u)(n)\}), 1\} \geq \min\{\mathrm{rad}(\{u(n)\}), 1\}$ holds not only for $\gamma \geq 0$ but also for $\gamma \in \mathbb{R}$. Then, by (37), we have $\min\{\mathrm{rad}(\{u(n)\}), 1\} = \min\{\mathrm{rad}(j_{-\gamma} * (j_\gamma * u))(n), 1\} \geq \min\{(j_\gamma * u)(n)\}), 1\}$. (Note that this also follows from Theorem 4.4(i), by using (37).)

(b) Let $u : \mathbb{N}_0 \to X$ be a (non-zero) sequence. Then Theorem 4.4(i) implies that if $\mathrm{rad}(\{u(n)\}) < 1$, then $\mathrm{rad}(\{(j_\gamma * u)(n)\}) = \mathrm{rad}(\{u(n)\}) < 1$ for all $\gamma > 0$. On the other hand, if $\mathrm{rad}(\{u(n)\}) \geq 1$, then the set $\{\gamma > 0 : \mathrm{rad}(\{(j_\gamma * u)(n)\}) \neq 1\}$ is finite. (This can be proved by using arguments similar to those in Remark (a) under Theorem 2.5. We may omit the details.) Further we note that for any $0 < \gamma \notin \mathbb{N}$ there exists a sequence $u : \mathbb{N}_0 \to \mathbb{R}$ such that $\mathrm{rad}(\{u(n)\}) = 1 < \mathrm{rad}(\{j_\gamma * u)(n)\}) = \infty$. For example, let $u : \mathbb{N}_0 \to \mathbb{R}$ be the sequence defined by the equation $(1-t)^\gamma = \sum_{n=0}^{\infty} u(n)t^n$ (i.e., $u(n) := j_{-\gamma}(n)$ for $n \in \mathbb{N}_0$). Then the hypothesis $0 < \gamma \notin \mathbb{N}$ implies $\mathrm{rad}(\{u(n)\}) = 1$ (see (38)), and for all $\beta \geq 0$

$$\sum_{n=0}^{\infty} t^n (j_\beta * u)(n) = \sum_{n=0}^{\infty} t^n j_\beta(n) \sum_{n=0}^{\infty} t^n u(n) = (1-t)^{\gamma-\beta} \quad (|t| < 1).$$

Hence

$$\mathrm{rad}(\{j_\beta * u)(n)\}) = \begin{cases} \infty & \text{if } \beta \in \{\gamma, \gamma-1, \ldots, \gamma-[\gamma]\},\\ 1 & \text{if } \beta \in [0,\infty) \setminus \{\gamma, \gamma-1, \ldots, \gamma-[\gamma]\}.\end{cases}$$

Lemma 4.5. *Let X be a Banach lattice, and $u : \mathbb{N}_0 \to X^+$ be a positive X-valued sequence. Let $0 < r < 1$, $\gamma > 0$ and $x \in X$. Then*

$$\sum_{n=0}^{\infty} r^n u(n) = x \quad \textit{if and only if} \quad (1-r)^\gamma \sum_{n=0}^{\infty} r^n (j_\gamma * u)(n) = x. \tag{57}$$

Proof. Suppose $\sum_{n=0}^{\infty} r^n u(n) = x$. We first prove that

$$\text{weak-}\lim_{k\to\infty} (1-r)^\gamma \sum_{n=0}^{k} r^n (j_\gamma * u)(n) = x.$$

For this purpose, since X is a Banach lattice, it is enough to show that

$$(1-r)^\gamma \sum_{n=0}^{\infty} r^n \langle (j_\gamma * u)(n),\, x^*\rangle = \lim_{k\to\infty} (1-r)^\gamma \sum_{n=0}^{k} r^n \langle (j_\gamma * u)(n),\, x^*\rangle = \langle x,\, x^*\rangle$$

for all $x^* \in X^*$ which is a positive linear functional on X. Then, since $\langle u(n),\, x^*\rangle \geq 0$ for all $n \in \mathbb{N}_0$ and $\sum_{n=0}^{\infty} r^n \langle u(n),\, x^*\rangle = \langle \sum_{n=0}^{\infty} r^n u(n),\, x^*\rangle = \langle x,\, x^*\rangle$, it follows from Fubini's theorem together with (36) that

$$\begin{aligned}
(1-r)^\gamma \sum_{n=0}^{\infty} r^n \langle (j_\gamma * u)(n),\, x^*\rangle &= (1-r)^\gamma \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} r^{n-k} j_\gamma(n-k)\, r^k \langle u(k),\, x^*\rangle \right) \\
&= (1-r)^\gamma \left(\sum_{n=0}^{\infty} r^n j_\gamma(n) \right) \sum_{k=0}^{\infty} r^k \langle u(k),\, x^*\rangle \\
&= (1-r)^\gamma (1-r)^{-\gamma} \sum_{k=0}^{\infty} r^k \langle u(k),\, x^*\rangle = \langle x,\, x^*\rangle.
\end{aligned}$$

Hence $(1-r)^\gamma \sum_{n=0}^{k} r^n (j_\gamma * u)(n) \leq x$ for all $k \geq 0$, and

$$\text{weak-}\lim_{k\to\infty} (1-r)^\gamma \sum_{n=0}^{k} r^n (j_\gamma * u)(n) = x.$$

By this together with the Corollary of [13, Theorem II.5.9] we see that

$$\lim_{k\to\infty} \left\| (1-r)^\gamma \sum_{n=0}^{k} r^n (j_\gamma * u)(n) - x \right\| = 0,$$

whence $(1-r)^\gamma \sum_{n=0}^{\infty} r^n (j_\gamma * u)(n) = x$.

The converse implication is proved by the same argument, and hence we may omit the details. □

Remark. Let X be a Banach lattice, and $u : \mathbb{N}_0 \to X^+$ be a (non-zero) positive sequence. If $\mathrm{rad}(\{u(n)\}) < 1$, then $\mathrm{rad}(\{j_\gamma * u)(n)\} = \mathrm{rad}(\{u(n)\}$ for all $\gamma > 0$ by Lemma 4.5. On the other hand, if $\mathrm{rad}(\{u(n)\}) \geq 1$, then

$$\lim_{r\uparrow 1} \left\| \sum_{n=0}^{\infty} r^n (j_\gamma * u)(n) \right\| = \lim_{r\uparrow 1} (1-r)^{-\gamma} \left\| \sum_{n=0}^{\infty} r^n u(n) \right\| = \infty$$

for all $\gamma > 0$. It follows that $\mathrm{rad}(\{j_\gamma * u)(n)\}) = 1$ for all $\gamma > 0$. Hence the function $\gamma \mapsto \mathrm{rad}(\{(j_\gamma * u)(n)\})$ is discontinuous at 0 if $\mathrm{rad}(\{u(n)\}) > 1$. This is the discrete version of Theorem 2.7.

Proposition 4.6. (N.H. Abel) *Let $u : \mathbb{N}_0 \to X$ be a sequence. Assume that $\sum_{n=0}^{\infty} u(n)$ converges. Then, for any $0 < \delta < \pi/2$, $\sum_{n=0}^{\infty} e^{-\lambda n} u(n)$ converges uniformly for all λ in $D(0;\delta)$. Hence the function $\lambda \mapsto \sum_{n=0}^{\infty} e^{\lambda n} u(n)$ is continuous on $\{0\} \cup D(0;\delta)$.*

Proof. By hypothesis, for any $\epsilon > 0$ there exists $K \geq 1$ such that $n \geq m \geq K$ implies $\| \sum_{k=m}^{n} u(k) \| < \epsilon$. Suppose $\lambda \in D(0;\delta)$. Then we have

$$\sum_{n=0}^{\infty} e^{-\lambda n} u(n) = (1 - e^{-\lambda}) \sum_{n=0}^{\infty} e^{-\lambda n} \sum_{k=0}^{n} u(k)$$

by Theorem 4.4(ii). Let $n \geq K$. Applying the above equality to the sequence $u_n : \mathbb{N}_0 \to X$ defined by $u_n(k) := 0$ if $0 \leq k < n$ and $u_n(k) := u(k)$ otherwise, we have

$$\sum_{k=n}^{\infty} e^{-\lambda k} u(k) = (1 - e^{-\lambda}) \sum_{k=n}^{\infty} e^{-\lambda k} \sum_{l=n}^{k} u(l).$$

Hence

$$\begin{aligned}
\Big\| \sum_{k=n}^{\infty} e^{-\lambda k} u(k) \Big\| &\leq |1 - e^{-\lambda}| \sum_{k=n}^{\infty} e^{-(\mathrm{Re}\lambda)k} \| \sum_{l=n}^{k} u(l) \| \\
&\leq |1 - e^{-\lambda}| \sum_{k=0}^{\infty} e^{-(\mathrm{Re}\lambda)k} \epsilon \leq \frac{|1 - e^{-\lambda}|}{1 - e^{-\mathrm{Re}\lambda}} \epsilon \\
&= \frac{|1 - e^{-\lambda}|}{|\lambda|} \frac{\mathrm{Re}\lambda}{|1 - e^{-\mathrm{Re}\lambda}|} \frac{|\lambda|}{\mathrm{Re}\lambda} \epsilon \leq M_\delta \frac{|\lambda|}{\mathrm{Re}\lambda} \epsilon < \frac{M_\delta}{\cos\delta} \epsilon,
\end{aligned}$$

where

$$M_\delta := \sup \left\{ \frac{|1 - e^{-\lambda}|}{|\lambda|} \frac{\mathrm{Re}\lambda}{|1 - e^{-\mathrm{Re}\lambda}|} \, : \, \lambda \in D(0;\delta) \right\} \; \Big(< \infty \Big).$$

This completes the proof. □

Remarks. (a) The following example shows that Proposition 4.6 does not hold when $0 < \delta < \pi/2$ is replaced with $\delta = \pi/2$.

Example 16. (This is an adaptation of Example 6.) Let $X = \{(a_n)_{n=1}^{\infty} : a_n \in \mathbb{C}, \lim_{n\to\infty} a_n = 0\}$ be the same as in Example 6. Choose a real-valued sequence $f : \mathbb{N}_0 \to \mathbb{R}$ such that

$$1 = f(0) > f(1) > \ldots > 0, \; \lim_{n\to\infty} f(n) = 0, \; \text{ and } \sum_{n=0}^{\infty} f(n) = \infty. \tag{58}$$

Define a sequence $u : \mathbb{N}_0 \to X$ by

$$u(k) := (u(k)_n)_{n=1}^{\infty}, \quad \text{where } \; u(k)_n := e^{ik\pi/2n} n^{-2} f(k).$$

It follows that

$$\sum_{k=0}^{K} u(k) = \Big(\sum_{k=0}^{K} u(k)_n\Big)_{n=1}^{\infty} \in X \qquad (K \geq 0)$$

and that

$$\begin{aligned}\sum_{k=0}^{K} u(k)_n &= \sum_{k=0}^{K} e^{ik\pi/2n}\, n^{-2} f(k) \\ &= \frac{1}{n^2}\Big(\sum_{k=0}^{K} \cos(k\pi/2n)\, f(k) + i\sum_{k=0}^{K} \sin(k\pi/2n)\, f(k)\Big).\end{aligned}$$

From (58) we easily see that

$$-n \leq \sum_{k=0}^{K} \cos(k\pi/2n)\, f(k) \leq n \tag{59}$$

and that

$$0 \leq \sum_{k=0}^{K} \sin(k\pi/2n)\, f(k) \leq 2n. \tag{60}$$

Thus

$$\Big|\sum_{k=0}^{K} u(k)_n\Big| \leq \frac{1}{n^2}(n+2n) = \frac{3}{n} \qquad (K \geq 0). \tag{61}$$

Further $\sum_{k=0}^{\infty} u(k)_n$ conveges for each $n \geq 1$. Thus

$$\big(\sum_{k=0}^{\infty} u(k)_n\big)_{n=1}^{\infty} \in X,$$

and

$$\begin{aligned}\sum_{k=0}^{\infty} u(k) &= \lim_{K\to\infty} \sum_{k=0}^{K} u(k) = \lim_{K\to\infty} \Big(\sum_{k=0}^{K} u(k)_n\Big)_{n=1}^{\infty} \\ &= \Big(\sum_{k=0}^{\infty} u(k)_n\Big)_{n=1}^{\infty} \qquad \text{(in } X\text{-norm).}\end{aligned}$$

Next, let $\lambda_l := \delta_l + i\pi/2l$ for $l = 1, 2, \dots$, where $\delta_l > 0$ will be determned later. Then by the definition of $u(k)$

$$\begin{aligned}\sum_{k=0}^{\infty} e^{-\lambda_l k} u(k) &= \Big(\sum_{k=0}^{\infty} e^{-\delta_l k}\, e^{-ik\pi/2l} u(k)_n\Big)_{n=1}^{\infty} \\ &= \Big(\sum_{k=0}^{\infty} e^{-\delta_l k}\, e^{-(ik\pi/2l)+(ik\pi/2n)}\, n^{-2} f(k)\Big)_{n=1}^{\infty} \in X,\end{aligned}$$

where, in particular,

$$\sum_{k=0}^{\infty} e^{-\delta_l k}\, e^{-(ik\pi/2l)+(ik\pi/2n)}\, n^{-2} f(k) = \sum_{k=0}^{\infty} e^{-\delta_l k}\, n^{-2} f(k) \qquad \text{when } \ n = l.$$

We now determine $\delta_l > 0$ for each $l \ge 1$ as follows. By (58) there exists δ_l such that $0 < \delta_l < 1/l$ and

$$\frac{1}{l^2} \sum_{k=0}^{\infty} e^{-\delta_l k} f(k) > l.$$

Then we have $\lim_{l\to\infty} \lambda_l = \lim_{l\to\infty} \delta_l + i\pi/2l = 0$, and

$$\Big\| \sum_{k=0}^{\infty} e^{-\lambda_l k} u(k) \Big\| \ge \sum_{k=0}^{\infty} e^{-\delta_l k}\, l^{-2} f(k) > l \qquad (l \ge 1),$$

so that $\lim_{l\to\infty} \sum_{k=0}^{\infty} e^{-\lambda_l k} u(k)$ does not exists in X. Hence the function $\lambda \mapsto \sum_{k=0}^{\infty} e^{-\lambda k} u(k)$ from $\{0\} \cup D(0; \pi/2)$ to X is not continuous at 0, and so the uniform convergence of $\sum_{k=0}^{\infty} e^{-\lambda k} u(k)$ fails to hold on $D(0; \pi/2)$.

(b) The existence of the limit

$$\lim_{D(0;\pi/2)\ni\lambda\to 0} \sum_{0}^{\infty} e^{-\lambda n} u(n)$$

does not imply the convergence of $\sum_{n=0}^{\infty} u(n)$. For example, let $u(n) := (-1)^n$ for $n \ge 0$. Then

$$\sum_{n=0}^{\infty} e^{-\lambda n} u(n) = \sum_{n=0}^{\infty} (-e^{-\lambda})^n = \frac{1}{1 + e^{-\lambda}} \to \frac{1}{2}$$

as $\lambda \to 0$ with $\mathrm{Re}\lambda > 0$. But $\sum_{n=0}^{\infty} u(n) = \sum_{n=0}^{\infty} (-1)^n$ does not converge.

Fact 4.7. *Let $u : \mathbb{N}_0 \to X$ be a sequence. If $0 < \mathrm{rad}(\{u(n)\}) \le 1$, then* $\sup_{0<|r|<K} \|\mathfrak{a}_r\| < \infty$ for all $0 < K < \mathrm{rad}(\{u(n)\})$.

Proof. By (41) and (42)

$$\|\mathfrak{a}_r\| = |1-r|\Big\|\sum_{n=0}^{\infty} r^n u(n)\Big\| \le 2\sum_{n=0}^{\infty} |r|^n \|u(n)\| \le 2\sum_{n=0}^{\infty} K^n \|u(n)\| < \infty$$

for all $r \in \mathbb{C}$ with $0 < |r| < K$. This completes the proof. $\square$

Remark. The hypothesis $0 < K < \operatorname{rad}(\{u(n)\})$ cannot be sharpened as $K = \operatorname{rad}(\{u(n)\})$ in Fact 4.7. To see this, let $0 < r_0 \le 1$, and define a sequence $u : \mathbb{N}_0 \to \mathbb{R}$ by $u(n) := (n+1)r_0^{-n}$ for $n \in \mathbb{N}_0$. Then $\operatorname{rad}(\{u(n)\}) = r_0$, and

$$\mathfrak{a}_r = (1-r)\sum_{n=0}^{\infty}(n+1)(r/r_0)^n = \frac{(1-r)r_0^2}{(r_0-r)^2}$$

for all $0 < r < r_0$. Thus $\lim_{r\uparrow r_0} \mathfrak{a}_r = \infty$.

Theorem 4.8. *Let $u : \mathbb{N}_0 \to X$ be a sequence. Let $\gamma \ge 0$, $\alpha > -1-\gamma$, and $M > 0$. Assume that $\|\mathfrak{c}_n^\gamma\| \le Mn^\alpha$ for all $n \in \mathbb{N}_0$, where $0^\alpha := 1$ as before. Then the following hold.*

(i) *If $\beta > 0$, then there exists $M_\beta > 0$ such that*

$$\|\mathfrak{c}_n^{\gamma+\beta}\| \le M_\beta\, Mn^\alpha \qquad (n \in \mathbb{N}_0), \tag{62}$$

where we may take $M_\beta = 1$ when $\alpha \ge 0$.

(ii) $\operatorname{rad}(\{u(n)\}) \ge 1$, *and there exists $M_\infty > 0$ such that*

$$\|\mathfrak{a}_r\| \le M_\infty\, M(1-r)^{-\alpha} \qquad (0 < r < 1), \tag{63}$$

where we may take $M_\infty = 1$ when $\alpha = 0$.

Proof. (i) Since $\gamma + 1 > 0$ and $\gamma + \alpha + 1 > 0$ by hypothesis, it follows from (39) that

$$j_{\gamma+1}(n)\|\mathfrak{c}_n^\gamma\| \le B_{\gamma+1}n^\gamma\, Mn^\alpha \le B_{\gamma+1}M\,\frac{j_{\gamma+\alpha+1}(n)}{A_{\gamma+\alpha+1}} \tag{64}$$

for all $n \in \mathbb{N}_0$. Therefore

$$\begin{aligned}
\|\mathfrak{c}_n^{\gamma+\beta}\| &= (j_{\gamma+\beta+1}(n))^{-1}\|(j_\beta * (j_{\gamma+1}\mathfrak{c}_\cdot^\gamma)(n)\| && \text{(by (51))}\\
&\le \frac{1}{A_{\gamma+\beta+1}\, n^{\gamma+\beta}}\,\frac{B_{\gamma+1}M}{A_{\gamma+\alpha+1}}\,(j_\beta * j_{\gamma+\alpha+1})(n)\\
&= \frac{1}{A_{\gamma+\beta+1}\, n^{\gamma+\beta}}\,\frac{B_{\gamma+1}M}{A_{\gamma+\alpha+1}}\,j_{\gamma+\alpha+\beta+1}(n)\\
&\le \frac{B_{\gamma+1}M}{A_{\gamma+\beta+1}A_{\gamma+\alpha+1}\, n^{\gamma+\beta}}\,B_{\gamma+\alpha+\beta+1}\, n^{\gamma+\alpha+\beta}\\
&= \frac{B_{\gamma+1}B_{\gamma+\alpha+\beta+1}}{A_{\gamma+\beta+1}A_{\gamma+\alpha+1}}\,Mn^\alpha.
\end{aligned}$$

This proves (62) with $M_\beta = B_{\gamma+1}B_{\gamma+\alpha+\beta+1}(A_{\gamma+\beta+1}A_{\gamma+\alpha+1})^{-1}$.

Here if $\alpha \ge 0$, then, since $\|\mathfrak{c}_k^\gamma\| \le Mk^\alpha \le Mn^\alpha$ for all $0 \le k \le n$, it follows that

$$\begin{aligned}\|\mathfrak{c}_n^{\gamma+\beta}\| &= (j_{\gamma+\beta+1}(n))^{-1}\left\|\sum_{k=0}^n j_\beta(n-k)j_{\gamma+1}(k)\mathfrak{c}_k^\gamma\right\| \\ &\le (j_{\gamma+\beta+1}(n))^{-1}Mn^\alpha \sum_{k=0}^n j_\beta(n-k)j_{\gamma+1}(k) \\ &= (j_{\gamma+\beta+1}(n))^{-1}Mn^\alpha\, j_{\gamma+\beta+1}(n) = Mn^\alpha \qquad (n \in \mathbb{N}_0).\end{aligned}$$

Hence (62) holds with $M_\beta = 1$ when $\alpha \ge 0$.

(ii) Since $\|\mathfrak{c}_n^\gamma\|^{1/n} \le (Mn^\alpha)^{1/n} \to 1$ as $n \to \infty$, it follows that $\mathrm{rad}(\{\mathfrak{c}_n^\gamma\}) \ge 1$. Hence $\mathrm{rad}(\{u(n)\}) \ge 1$ by Theorem 4.4(i). Further, by Theorem 4.4(ii),

$$\mathfrak{a}_r = (1-r)^{\gamma+1}\sum_{n=0}^\infty r^n j_{\gamma+1}(n)\mathfrak{c}_n^\gamma$$

for all $0 < r < 1$. Thus

$$\begin{aligned}\|\mathfrak{a}_r\| &\le (1-r)^{\gamma+1}\frac{B_{\gamma+1}M}{A_{\gamma+\alpha+1}}\sum_{n=0}^\infty r^n j_{\gamma+\alpha+1}(n) \quad \text{(by (64))} \\ &= (1-r)^{\gamma+1}\frac{B_{\gamma+1}M}{A_{\gamma+\alpha+1}}\frac{1}{(1-r)^{\gamma+\alpha+1}} = \frac{B_{\gamma+1}M}{A_{\gamma+\alpha+1}}(1-r)^{-\alpha}.\end{aligned}$$

This proves (63) with $M_\infty = B_{\gamma+1}(A_{\gamma+\alpha+1})^{-1}$.

Here if $\alpha = 0$, then $\|\mathfrak{c}_n^\gamma\| \le M$ for all $n \in \mathbb{N}_0$, so that

$$\|\mathfrak{a}_r\| \le (1-r)^{\gamma+1}\sum_{n=0}^\infty r^n j_{\gamma+1}(n)M = M \qquad (0 < r < 1).$$

Hence (63) holds with $M_\infty = 1$ when $\alpha = 0$. □

Theorem 4.9. *Let $u : \mathbb{N}_0 \to X$ be a sequence. Let $\gamma \ge 0$, $\alpha > -1-\gamma$, and $M > 0$. Assume that $\limsup_{n\to\infty}\|n^{-\alpha}\mathfrak{c}_n^\gamma\| < M$. Let M_β and M_∞ be the constants in Theorem 4.8. Then the followin hold.*

(i) *For all $\beta > 0$, $\limsup_{n\to\infty}\|n^{-\alpha}\mathfrak{c}_n^{\gamma+\beta}\| < M_\beta\, M$.*

(ii) $\mathrm{rad}(\{u(n)\}) \ge 1$, *and* $\limsup_{r\uparrow 1}\|(1-r)^\alpha\mathfrak{a}_r\| < M_\infty\, M$.

Proof. (i) By the assumption there exist $\epsilon > 0$ and $K \geq 1$ such that $\|\mathfrak{c}_n^\gamma\| \leq (M-\epsilon)n^\alpha$ for all $n \geq K$. Then

$$\begin{aligned}
\|\mathfrak{c}_n^{\gamma+\beta}\| &= (j_{\gamma+\beta+1}(n))^{-1}\|(j_\beta * (j_\gamma * u))(n)\| \\
&\leq (j_{\gamma+\beta+1}(n))^{-1}\left(\sum_{k=0}^{K-1} + \sum_{k=K}^{n}\right) j_\beta(n-k)\|(j_\gamma * u)(k)\| =: I + II,
\end{aligned}$$

where if $n > 2K$, then by (39)

$$\begin{aligned}
I &\leq \frac{1}{A_{\gamma+\beta+1}\, n^{\gamma+\beta}} \sum_{k=0}^{K-1} B_\beta (n-k)^{\beta-1}\|(j_\gamma * u)(k)\| \\
&= \frac{B_\beta\, n^{\beta-1}}{A_{\gamma+\beta+1}\, n^{\gamma+\beta}} \sum_{k=0}^{K-1} (1-k/n)^{\beta-1}\,\|(j_\gamma * u)(k)\| \\
&\leq \frac{B_\beta}{A_{\gamma+\beta+1}}\, n^{-1-\gamma}\ \max\{(1/2)^{\beta-1}, 1\} \sum_{k=0}^{K-1} \|(j_\gamma * u)(k)\| \\
&= M_{ab}\, n^{-1-\gamma},
\end{aligned}$$

and

$$\begin{aligned}
II &= (j_{\gamma+\beta+1}(n))^{-1} \sum_{k=K}^{n} j_\beta(n-k) j_{\gamma+1}(k)\|\mathfrak{c}_k^\gamma\| \\
&\leq (j_{\gamma+\beta+1}(n))^{-1} \sum_{k=0}^{n} j_\beta(n-k) j_{\gamma+1}(k)(M-\epsilon)k^\alpha \leq M_\beta\,(M-\epsilon)n^\alpha,
\end{aligned}$$

where the last inequality comes from the proof of Theorem 4.8(i) . Since $\lim_{n\to\infty} n^{-1-\gamma-\alpha} = 0$, it follows that $\lim_{n\to\infty} \|n^{-\alpha}\mathfrak{c}_n^{\gamma+\beta}\| < M_\beta\, M$.

(ii) Since $\|\mathfrak{c}_n^\gamma\| = O(n^\alpha)$, we see, as in Theorem 4.8(ii), that $\mathrm{rad}(\{u(n)\}) \geq 1$. Then by Theorem 4.4(ii)

$$\mathfrak{a}_r = (1-r)^{1+\gamma}\sum_{n=0}^{\infty} r^n (j_\gamma * u)(n)$$

for all $0 < r < 1$. Hence

$$\|\mathfrak{a}_r\| \leq (1-r)^{1+\gamma}\left(\sum_{n=0}^{K-1} r^n\|j(_\gamma * u)(n)\| + \sum_{k=K}^{\infty} r^n j_{\gamma+1}(n)\|\mathfrak{c}_n^\gamma\|\right) =: III + IV,$$

where

$$III = (1-r)^{1+\gamma}\sum_{n=0}^{K-1}\|(j_\gamma * u)(n)\| = M_{ab}\,(1-r)^{1+\gamma},$$

and

$$IV \le (1-r)^{1+\gamma} \sum_{n=0}^{\infty} r^n j_{\gamma+1}(n)(M-\epsilon)n^{\alpha} \le M_\infty\,(M-\epsilon)(1-r)^{-\alpha}$$

by the proof of Theorem 4.8(ii). Hence $1+\gamma+\alpha > 0$ implies $\lim_{r\uparrow 1} \|(1-r)^{\alpha}\mathfrak{a}_r\| < M_\infty M$. This completes the proof. $\square$

Corollary 4.10. *Let $u : \mathbb{N}_0 \to X$ be a sequence. Then the following hold.*

(i) *If $\gamma' > \gamma \ge 0$, then $\alpha_0(\{\mathfrak{c}_n^{\gamma'}\}) \le \max\{\alpha_0(\{\mathfrak{c}_n^{\gamma}\}), -1-\gamma\}$.*

(ii) *If $\mathrm{rad}(\{u(n)\}) \ge 1$, then $\alpha_0(\mathfrak{a}_.) \le \max\{\alpha_0(\{\mathfrak{c}_n^{\gamma}\}), -1-\gamma\}$ for all $\gamma \ge 0$.*

Remark. The sequence $u(n) := n^{\lambda_0 - 1}$ for $n \ge 0$, where $\lambda_0 > 0$, satisfies $\lim_{n\to\infty} u(n)/j_{\lambda_0}(n) = \Gamma(\lambda_0) > 0$ by (38). Thus an easy approximation argument implies that

$$\lim_{k\to\infty} \frac{(j_\gamma * u)(k)}{j_{\gamma+\lambda_0}(k)} = \lim_{k\to\infty} \frac{\sum_{l=0}^{k} j_\gamma(k-l)u(l)}{\sum_{l=0}^{k} j_\gamma(k-l)j_{\lambda_0}(l)} = \Gamma(\lambda_0)$$

for all $\gamma > 0$, and that

$$\lim_{r\uparrow 1}\,(1-r)^{\lambda_0} \sum_{n=0}^{\infty} r^n u(n) = \lim_{r\uparrow 1} \frac{\sum_{n=0}^{\infty} r^n u(n)}{\sum_{n=0}^{\infty} r^n j_{\lambda_0}(n)} = \Gamma(\lambda_0) \qquad \text{(cf. (36)).}$$

Hence

$$\begin{aligned}
\lim_{k\to\infty} \frac{\mathfrak{c}_k^{\gamma}(\{u(n)\})}{k^{\lambda_0-1}} &= \lim_{k\to\infty} \frac{(j_\gamma * u)(k)}{k^{\lambda_0-1}\, j_{\gamma+1}(k)} = \lim_{k\to\infty} \frac{j_{\gamma+\lambda_0}(k)\,\Gamma(\lambda_0)}{k^{\lambda_0-1}\, j_{\gamma+1}(k)} \\
&= \lim_{k\to\infty} \frac{\left(k^{\gamma+\lambda_0-1}/\Gamma(\gamma+\lambda_0)\right)\Gamma(\lambda_0)}{k^{\lambda_0-1}\, k^{\gamma}/\Gamma(\gamma+1)} = \frac{\Gamma(\gamma+1)\Gamma(\lambda_0)}{\Gamma(\gamma+\lambda_0)} \\
&= \gamma B(\gamma,\, \lambda_0)
\end{aligned}$$

for all $\gamma > 0$, and

$$\lim_{r\uparrow 1}\,(1-r)^{\lambda_0-1}\mathfrak{a}_r(\{u(n)\}) = \lim_{r\uparrow 1}\,(1-r)^{\lambda_0} \sum_{n=0}^{\infty} r^n u(n) = \Gamma(\lambda_0).$$

It follows that $\alpha_0(\{\mathfrak{c}_n^{\gamma}\}) = \lambda_0 - 1 = \alpha_0(\mathfrak{a}_.)$ for all $\gamma \ge 0$. Of course this is a special case. In general, the function $\gamma \mapsto \alpha_0(\{\mathfrak{c}_n^{\gamma}\})$ is not constant on $[0,\infty)$. To see this we give the following examples.

Example 17. Let $u(n) := (-1)^n$ for $n \ge 0$. Then $\alpha_0(\mathfrak{a}_.) = -1$, and

$$\alpha_0(\{\mathfrak{c}_n^{\gamma}\}) = \begin{cases} -\gamma & \text{if } 0 \le \gamma < 1, \\ -1 & \text{if } \gamma \ge 1. \end{cases}$$

To see this we first note that

$$\mathfrak{a}_r = (1-r)\sum_{n=0}^{\infty}(-r)^n = \frac{1-r}{1+r} \qquad (0<r<1).$$

Hence $\lim_{r\uparrow 1}(1-r)^{-1}\mathfrak{a}_r = \frac{1}{2}$ and $\alpha_0(\mathfrak{a}.) = -1$. Since $\mathfrak{c}_n^1 = (1+(-1)^n)/2(n+1)$ for all $n \geq 0$, we have $\mathfrak{c}_n^1 = O(n^{-1})$ and $\alpha_0(\{\mathfrak{c}_n^1\}) = -1$. By Theorem 4.9 and Corollary 4.10, $\mathfrak{c}_n^\gamma = O(n^{-1})$ and $\alpha_0(\mathfrak{c}_n^\gamma) = -1$ for all $\gamma > 1$. Next we consider the case $0<\gamma<1$. Then by the definition of j_γ (see (35)) we have $j_\gamma(n) > j_\gamma(n+1) > 0$ for all $n \geq 0$, so that $(j_\gamma * u)(n) = \sum_{k=0}^n j_\gamma(n-k)(-1)^k$ satisfies $0 < j_\gamma(0) - j_\gamma(1) \leq (j_\gamma * u)(n) \leq j_\gamma(0)$ for all $n \in \mathbb{N}_0$. Since $\mathfrak{c}_n^\gamma = (j_{\gamma+1}(n))^{-1}(j_\gamma * u)(n)$, it then follows from (38) that $\alpha_0(\{\mathfrak{c}_n^\gamma\}) = -\gamma$. It is clear that $\alpha_0(\{\mathfrak{c}_n^0\}) = \alpha_0(\{u(n)\}) = 0$. (Incidentally, we note that $\mathrm{rad}(\{j_\gamma * u)(n)\})$ $=\mathrm{rad}(\{u(n)\}) = 1$ for all $\gamma \geq 0$. Indeed, if $\gamma > 0$, then by Example 17

$$\begin{aligned}\alpha_0(\{(j_{\gamma+1} * u)(n)\}) &= \alpha_0(\{j_{\gamma+2} * u)(n) \cdot \mathfrak{c}_n^{\gamma+1}\}) \\ &= (\gamma+1) + \alpha_0(\{\mathfrak{c}_n^{\gamma+1}\}) = (\gamma+1) - 1 = \gamma;\end{aligned}$$

it follows that $\lim_{n\to\infty}(j_{\gamma+1} * u)(n)$ does not exist and that $w_0(j_{\gamma+1} * u) = 0$; hence, by Theorem 4.2, $\mathrm{rad}(\{j_\gamma * u)(n)\} = e^{-w_0(\{(j_{\gamma+1}*u)(n)\})} = e^0 = 1$. It is clear that $\mathrm{rad}\{u(n)\}) = 1$.)

Example 18. Let $N \geq 1$ be an integer and $u : \mathbb{N}_0 \to \mathbb{R}$ be the sequence defined by the equation $(1-t)^{N-1} = \sum_{n=0}^\infty u(n)t^n$, (i.e., $u(n) := j_{-N+1}(n)$ for $n \in \mathbb{N}_0$). Then $\alpha_0(\mathfrak{a}.) = -N$, and

$$\alpha_0(\{\mathfrak{c}_n^\gamma\}) = \begin{cases} -\infty & \text{if } \gamma = 0,1,\ldots,N-1, \\ -N & \text{if } \gamma \in [0,\infty)\setminus\{0,1,\ldots,N-1\}. \end{cases}$$

To see this, note that $u(n) = 0$ for all $n \geq N$. It follows that $\alpha_0(\{\mathfrak{c}_n^0\}) = -\infty$. Now suppose $K = 1,2,\ldots,N-1$. Then

$$\mathfrak{c}_n^K = \frac{1}{j_{K+1}(n)}\,(j_K * j_{-N+1})(n) = \frac{1}{j_{K+1}(n)}\,j_{K-N+1}(n),$$

and since $K-N+1 \leq 0$, it follows that $j_{K-N+1}(n) = 0$ for all $n > N-K-1$. Thus $\alpha_0(\{\mathfrak{c}_n^K\}) = -\infty$. Next suppose $\gamma > N-1$. Then by (39)

$$\begin{aligned}\mathfrak{c}_n^\gamma &= \frac{1}{j_{\gamma+1}(n)}\,(j_\gamma * j_{-N+1})(n) = \frac{j_{\gamma-N+1}(n)}{j_{\gamma+1}(n)} \\ &\leq \frac{B_{\gamma-N+1}\,n^{\gamma-N}}{A_{\gamma+1}\,n^\gamma} = \frac{B_{\gamma-N+1}}{A_{\gamma+1}}\,n^{-N}\end{aligned} \tag{65}$$

for all $n \geq 1$. Similarly, $\mathfrak{c}_n^\gamma \geq (A_{\gamma-N+1})(B_{\gamma+1})^{-1}\, n^{-N}$ for all $n \geq 1$. Thus $\alpha_0(\{\mathfrak{c}_n^\gamma\}) = -N$. We apply Theorem 4.4(ii) to see that for all $0 < r < 1$

$$\begin{aligned}\mathfrak{a}_r &= (1-r)^{\gamma+1} \sum_{n=0}^{\infty} r^n j_{\gamma+1}(n)\mathfrak{c}_n^\gamma = (1-r)^{\gamma+1} \sum_{n=0}^{\infty} r^n j_{\gamma-N+1}(n) \text{ (cf. (65))} \\ &= (1-r)^{\gamma+1}(1-r)^{-\gamma+N-1} = (1-r)^N,\end{aligned}$$

which shows that $\alpha_0(\mathfrak{a}_.) = -N$. Finally suppose $\gamma \in [0,\, N-1]\setminus\{0, 1, \ldots, N-1\}$. Then, since $\gamma - N + 1$ is not an integer and $\mathfrak{c}_n^\gamma = j_{\gamma-N+1}(n)/j_{\gamma+1}(n)$ by (65), we may apply (39) again to obtain that $\alpha_0(\{\mathfrak{c}_n^\gamma\}) = -N$.

Theorem 4.11. *Suppose X is a Banach lattice, and $u : \mathbb{N}_0 \to X^+$ is a positive sequence with* $\mathrm{rad}(\{u(n)\}) \geq 1$. *Let $\gamma \geq 1$, $\alpha \in \mathbb{R}$, and $M > 0$. Then the following hold.*

(i) *If* $\sup_{0<r<1} \|(1-r)^\alpha \mathfrak{a}_r\| \leq M$, *then* $\sup_{n\geq 2} \|n^{-\alpha}\mathfrak{c}_n^\gamma\| \leq 4M/A_{\gamma+1}$.

(ii) *If* $\limsup_{r\uparrow 1} \|(1-r)^\alpha \mathfrak{a}_r\| \leq M$, *then* $\limsup_{n\to\infty} \|n^\alpha \mathfrak{c}_n^\gamma\| \leq 4M/A_{\gamma+1}$.

Proof. (i) Since u is positive, it follows from Theorem 4.4(ii) and (39) that

$$\begin{aligned}\mathfrak{a}_r &= (1-r)^\gamma \sum_{n=0}^{\infty} r^n (j_{\gamma-1} * u)(n) \geq (1-r)^\gamma \sum_{n=0}^{N} r^n (j_{\gamma-1} * u)(n) \\ &\geq (1-r)^\gamma\, r^N \sum_{n=0}^{N} (j_{\gamma-1} * u)(n) = (1-r)^\gamma\, r^N (j_\gamma * u)(N) \\ &= (1-r)^\gamma\, r^N\, j_{\gamma+1}(N)\, \mathfrak{c}_N^\gamma \geq (1-r)^\gamma r^N\, A_{\gamma+1} N^\gamma\, \mathfrak{c}_N^\gamma \geq 0\end{aligned}$$

for all $0 < r < 1$. Thus $M \geq \|(1-r)^\alpha \mathfrak{a}_r\| \geq (1-r)^{\alpha+\gamma} r^N A_{\gamma+1} N^\gamma \|\mathfrak{c}_N^\gamma\|$. Letting $r = 1 - 1/N$, with $N \geq 2$, we have

$$M \geq N^{-\alpha-\gamma}(1-1/N)^N A_{\gamma+1} N^\gamma \|\mathfrak{c}_N^\gamma\| = (1-1/N)^N A_{\gamma+1} N^{-\alpha} \|\mathfrak{c}_N^\gamma\|.$$

This proves $\|N^{-\alpha}\mathfrak{c}_N^\gamma\| \leq \big((1-1/N)^{-N}/A_{\gamma+1}\big)M \leq (4/A_{\gamma+1})M$ for all $N \geq 2$.

(ii) Since $r = 1 - 1/N \uparrow 1$ is equivalent to $N \to \infty$, the above proof of (i) can be used to prove (ii). We may omit the details. □

Remarks. **(a)** The hypothesis that u is positive is essential in Theorem 4.11; further, the hypothesis $\gamma \geq 1$ cannot be replaced with $\gamma > 1 - \epsilon$, where $0 < \epsilon < 1$. (For these and more we refer the reader to [7].)

(b) Let $u : \mathbb{N}_0 \to X$ be a sequence. Assume that $x = \lim_{n\to\infty} \mathfrak{c}_n^\gamma$ exists for some $\gamma \geq 0$. Then, applying Theorem 4.9 for the sequence $\{u(n) - x\}_{n=0}^\infty$ with $\alpha = 0$, we have the following well-known results (see e.g. [17, Chapter 3]). (For related topics we refer the reader to [2] and [12].)

(i) $\lim_{n\to\infty} \mathfrak{c}_n^{\gamma+\beta} = x$ for all $\beta > 0$;

(ii) $\operatorname{rad}(\{u(n)\}) \geq 1$ and $\lim_{r\uparrow 1} \mathfrak{a}_r = x$.

Corollary 4.12. *Suppose X is a Banach lattice and $u : \mathbb{N}_0 \to X^+$ is a positive sequence. Then the following hold.*

(i) *If $\gamma \geq 1$ and $\alpha > -2$, then* $\sup_{n\geq 1} \|n^{-\alpha}\mathfrak{c}_n^\gamma\| < \infty \Leftrightarrow \sup_{n\geq 1}\|n^{-\alpha}\mathfrak{c}_n^1\| < \infty \Leftrightarrow$ $\operatorname{rad}(\{u(n)\}) \geq 1$ and $\sup_{0<r<1} \|(1-r)^\alpha \mathfrak{a}_r\| < \infty$.

(ii) *If $u \neq 0$ and* $\operatorname{rad}(\{u(n)\}) \geq 1$, *then the function $\gamma \mapsto \alpha_0(\{\mathfrak{c}_n^\gamma\})$ is decreaing on $(0,\infty)$ and satisfies $\alpha_0(\{\mathfrak{c}_n^\gamma\}) = \alpha_0(\mathfrak{a}.) \geq -1$ for all $\gamma \geq 1$.*

Proof. (i) is direct from Theorem 4.8 and Theorem 4.11(i).

(ii) For all $0 < r < 1$ we have

$$\mathfrak{a}_r = (1-r)\sum_{n=0}^{\infty} r^n u(n) \geq (1-r)\sum_{n=0}^{K} r^n u(n) \geq (1-r)r^K \sum_{n=0}^{K} u(n) \geq 0,$$

and the hypothesis $u \neq 0$ implies

$$\lim_{r\uparrow 1} r^K \sum_{n=0}^{K} u(n) = \sum_{n=0}^{K} u(n) > 0$$

for some $K \in \mathbb{N}_0$. Thus we have $\alpha_0(\mathfrak{a}.) \geq -1$. By this, together with Theorem 4.11(ii) and Theorem 4.9(ii), $\alpha_0(\{\mathfrak{c}_n^\gamma\}) = \alpha_0(\mathfrak{a}.) \geq -1$ for all $\gamma \geq 1$. Since $\alpha_0(\{\mathfrak{c}_n^1\}) \geq -1$, we then apply Corollary 4.10(i) to infer that $0 < \gamma < \gamma' < 1$ implies $\alpha_0(\{\mathfrak{c}_n^\gamma\}) \geq \alpha_0(\{\mathfrak{c}_n^{\gamma'}\}) \geq \alpha_0(\{\mathfrak{c}_n^1\})$.

This completes the proof. □

Remark. There exists a sequence $u : \mathbb{N}_0 \to \mathbb{R}^+$ such that $\alpha_0(\{\mathfrak{c}_n^\gamma\}) = -\gamma$ for all $0 \leq \gamma < 1$ and $\alpha_0(\{\mathfrak{c}_n^\gamma\}) = -1 = \alpha_0(\mathfrak{a}.)$ for all $\gamma \geq 1$. (See Corollary 4.12(ii).) Here is an example.

Example 19. Define a sequence $u : \mathbb{N}_0 \to \mathbb{R}^+$ by

$$u(n) := \begin{cases} 1 & \text{if } n \in \{k^k \,:\, k \geq 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\alpha_0(\{\mathfrak{c}_n^0\}) = 0$. Now suppose $0 < \gamma \leq 1$. Then, since $0 < j_\gamma(n) \leq 1$ for all $n \geq 0$ (cf. (35)), $k^k \leq n < (k+1)^{k+1}$ implies

$$\begin{aligned} \mathfrak{c}_n^\gamma &= (j_{\gamma+1}(n))^{-1}(j_\gamma * u)(n) = (j_{\gamma+1}(n))^{-1}\sum_{j=1}^{k} j_\gamma(n-j^j) \\ &\leq \frac{k}{j_{\gamma+1}(n)} \leq \frac{n^{1/k}}{A_{\gamma+1}}\, n^{-\gamma} = \frac{1}{A_{\gamma+1}}\, n^{-\gamma+k^{-1}} \qquad \text{(cf. (39))}; \end{aligned}$$

and

$$\mathfrak{c}^{\gamma}_{n^n} = (j_{\gamma+1}(n^n))^{-1} \sum_{j=1}^{n} j_\gamma(n^n - j^j) \geq (j_{\gamma+1}(n^n))^{-1} \geq \frac{1}{B_{\gamma+1}}\,(n^n)^{-\gamma}.$$

It follows that $\alpha_0(\{\mathfrak{c}^{\gamma}_n\}) = -\gamma$ for all $0 < \gamma \leq 1$. By this and Corollary 4.12(ii), $\alpha_0(\{\mathfrak{c}^{\gamma}_n\}) = -1 = \alpha_0(\mathfrak{a}_\cdot)$ for all $\gamma \geq 1$.

The next theorem is a discrete version of Theorem 2.15 (for the case where $u(n) = Q^n$, with $Q \in \boldsymbol{B}(X)$, see [4, Theorem 2] and [8, Theorem 3.1]).

Theorem 4.13. *Let $u : \mathbb{N}_0 \to X$ be a sequence. Suppose $M > 0$. Then the following hold.*

(i) *If $\sup_{n\geq 0} \|\mathfrak{c}^1_n\| \leq M$, then $\sup_{0<r<1,\, n\geq 0} \|(1-r)\sum_{k=0}^{n} r^k u(k)\| \leq M$.*

(ii) *If $\sup_{0<r<1,\, n\geq 0} \|(1-r)\sum_{k=0}^{n} r^k u(k)\| \leq M$, then $\sup_{n\geq 0} \|\mathfrak{c}^1_n\| \leq 7M$.*

Proof. (i) This is an immediate consequence of Theorem 4.8(ii) (with $\alpha = 0$) and the argument in Theorem 2.15(i).

(ii) For $0 < r < 1$ and $n \geq 1$ we have, by Abel's partial summation formula, that

$$\begin{aligned} \sum_{k=0}^{n} u(k) &= \sum_{k=0}^{n} r^{-k} r^k u(k) \\ &= r^{-(n+1)} \sum_{k=0}^{n} r^k u(k) - \sum_{k=0}^{n} r^{-(k+1)} \Big((1-r)\sum_{l=0}^{k} r^l u(l)\Big), \end{aligned}$$

whence

$$\Big\|\sum_{k=0}^{n} u(k)\Big\| \leq \Big(\frac{r^{-(n+1)}}{1-r} + \sum_{k=0}^{n} r^{-(k+1)}\Big) M.$$

Putting $r = 1 - (n+1)^{-1}$, we then obtain that

$$\Big\|\sum_{k=0}^{n} u(k)\Big\| \leq (n+1)\Big(2\Big(1 - \frac{1}{n+1}\Big)^{-(n+1)} - 1\Big) M \leq (n+1)7M,$$

which completes the proof. □

In the rest of this section we consider operator-valued sequences $T : \mathbb{N}_0 \to \boldsymbol{B}(X)$. Recall (see (40), (41)) that

$$\begin{aligned} w_0(\{T(n)\}) &= \inf\{w \in \mathbb{R} \,:\, \|T(n)\| = O(e^{wn})\}, \\ \mathrm{rad}(\{T(n)\}) &= \frac{1}{\limsup_{n\to\infty} \|T(n)\|^{1/n}}. \end{aligned}$$

By the uniform boundedness princple we have

$$w_0(\{T(n)\}) = \sup\{w_0(\{T(n)x\}) \,:\, x \in X\}. \tag{66}$$

Theorem 4.14. *Let* $T : \mathbb{N}_0 \to \boldsymbol{B}(X)$ *be an operator-valued sequence. Define* $S(n) := \sum_{k=0}^{n} T(k)$ *for* $n \in \mathbb{N}$*, and let* S_∞ *be the strong limit of* $S(n)$ *as* $n \to \infty$ *if it exists, and* $S_\infty := 0$ *otherwise. Then*

$$\begin{aligned}(67)\quad \operatorname{rad}(\{T(n)\}) &= \sup\Big\{r \geq 0 \,:\, \sup_{n\geq 0}\Big\|\sum_{k=0}^{n} r^k T(k)x\Big\| < \infty \text{ for all } x \in X\Big\} \\ &= \exp\left(-w_0(\{S(n) - S_\infty\})\right).\end{aligned}$$

Consequently, $\min\big\{\operatorname{rad}(\{T(n)\}),\, 1\big\} = \min\big\{e^{-w_0(\{S_n\})},\, 1\big\}$.

Proof. Suppose $r_0 > 0$. If $r_0 < \operatorname{rad}(\{T(n)\})$, then

$$\Big\|\sum_{k=0}^{n} r_0^k T(k)x\Big\| \leq \Big(\sum_{k=0}^{\infty} r_0^k \|T(k)\|\Big)\|x\| < \infty \qquad (x \in X).$$

Conversely if $\sup_{n\geq 0}\Big\|\sum_{k=0}^{n} r_0^k T(k)x\Big\| < \infty$ for all $x \in X$, then, by the uniform boundedness principle,

$$M := \sup_{n\geq 0}\Big\|\sum_{k=0}^{n} r_0^k T(k)\Big\| < \infty;$$

thus if $0 < r < r_0$, then the equation

$$\begin{aligned}\sum_{k=0}^{n} r^k T(k) &= \sum_{k=0}^{n} (r/r_0)^k r_0^k T(k) \\ &= (1 - (r/r_0))\sum_{k=0}^{n-1}(r/r_0)^k \sum_{j=0}^{k} r_0^j T(j) + (r/r_0)^n \sum_{j=0}^{n} r_0^j T(j)\end{aligned}$$

can be used to see that $S_{r,\infty} := \lim_{n\to\infty}\sum_{k=0}^{n} r^k T(k)$ exists in the operator norm topology. It follows that $\operatorname{rad}(\{T(n)\}) \geq r_0$. Hence we have proved the first equality in (67).

To prove the second equality we define, for $x \in X$, $u_x(n) := T(n)x$, $\widetilde{v}_x(n) := S(n)x - S_\infty x$, and

$$v_x(n) := \begin{cases} S(n)x - \lim_{k\to\infty} S(k)x & \text{if the limit exists,} \\ S(n)x & \text{otherwise.}\end{cases}$$

Since

$$\operatorname{rad}(\{u_x(n)\}) = \frac{1}{\limsup_{n\to\infty}\|u_x(n)\|^{1/n}} = \sup\Big\{r \geq 0 \,:\, \sup_{n\geq 0}\Big\|\sum_{k=0}^{n} r^k T(k)x\Big\| < \infty\Big\}$$

for each $x \in X$ (cf. the above proof of the first equality), it follows that

$$\operatorname{rad}(\{T(n)\}) = \inf\left\{\operatorname{rad}(\{u_x(n)\}) \,:\, x \in X\right\}.$$

First assume that the strong limit S_∞ exists. Then $v_x = \widetilde{v}_x$ for all $x \in X$, and by Theorem 4.2, $\mathrm{rad}(\{u_x(n)\}) = \exp(-w_0(\{v_x(n)\}))$. Thus

$$
\begin{aligned}
\mathrm{rad}(\{T(n)\}) &= \inf\{\exp(-w_0(\{v_x(n)\}))\ :\ x \in X\} \\
&= \exp(-\sup\{w_0(\{v_x(n)\})\ :\ x \in X\}) \\
&= \exp(-\sup\{w_0(\{\widetilde{v}_x(n)\})\ :\ x \in X\}) \\
&= \exp(-w_0(\{S(n) - S_\infty\})) \qquad \text{(by (66))}.
\end{aligned} \tag{68}
$$

Next assume that $S(n)$ does not converge strongly as $n \to \infty$. Then $S_\infty = 0$ by definition, and the set $E := \{x \in X\ :\ \lim_{n\to\infty} S(n)x \text{ does not exist}\}$ is not empty. If $x \in E$, then $v_x = \widetilde{v}_x$ and $w_0(\{v_x(n)\}) \geq 0$. On the other hand, if $x \notin E$, then $w_0(\{v_x(n)\}) \leq 0$ and thus $w_0(\{\widetilde{v}_x(n)\}) \leq 0$. It follows (see (68)) that

$$
\begin{aligned}
\mathrm{rad}(\{T(n)\}) &= \exp(-\sup\{w_0(\{v_x(n)\})\ :\ x \in X\}) \\
&= \exp(-\sup\{w_0(\{v_x(n)\})\ :\ x \in E\}) \\
&= \exp(-\sup\{w_0(\{\widetilde{v}_x(n)\})\ :\ x \in X\}) \\
&= \exp(-w_0(\{S(n)\})) = \exp(-w_0(\{S(n) - S_\infty\})).
\end{aligned}
$$

This completes the proof. □

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