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CONTROLLABILITY OF FRACTIONAL INTEGRODIFFERENTIAL SYSTEMS VIA SEMIGROUP THEORY IN BANACH SPACES

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Abstract. This paper focuses on controllability results of fractional integrodifferential systems in Banach spaces. We obtain sufficient conditions for the controllability results by using fractional calculus, semigroup theory and the fixed point theorem.

1. Introduction

Fractional differential equations and control problems appear in many branches of physics and technical sciences. In fact fractional differential equations are considered as an alternative model to nonlinear differential equations [3]. The theory of fractional differential equations has been extensively studied by Delbosco and Rodino [5] and Lakshmikantham et al.[10]–[13]. The concept of controllability plays a major role in both finite and infinite dimensional spaces, that is, systems represented by ordinary differential equations and partial differential equations respectively. So it is natural to extend this concept to dynamical systems represented by fractional differential equations. Many partial fractional differential equations and integrodifferential equations can be expressed as fractional differential and integrodifferential equations in some Banach spaces [6]. Recently many authors established sufficient conditions for the controllability of fractional differential systems in finite dimensional space, see for example [1, 2, 4, 16]. In this paper we give some controllability results for nonlinear fractional integrodifferential systems by using the semigroup theory and a fixed point theorem.

2. Preliminaries

This section is concerned with some notations, definitions, lemmas, theorems, and preliminary facts which are used in what follows.

Definition 1 ([2, 14])**.** A real function $f(t)$ is said to be in the space $C_\alpha, \alpha \in \mathbb{R}$ if there exists a real number $p > \alpha$ such that $f(t) = t^p g(t)$, where $g \in C[0, \infty]$ and it is said to be in the space C_α^m if $f^{(m)} \in C_\alpha, m \in \mathbb{N}$.

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Definition 2 ([2, 14])**.** The Riemann-Liouville fractional integral operator of order-$\beta > 0$ of function $f \in C_\alpha,\ \alpha \geq -1$ is defined as

$$I^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)ds.$$

Definition 3 ([2, 14])**.** If the function $f \in C_{-1}^m$ and m is a positive integer, then we can define the fractional derivative of $f(t)$ in the Caputo sense as

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s)ds,\ m-1 < \alpha \leq m.$$

Let X a Banach space and $B(X)$ be the Banach space of linear bounded operators. If f is an abstract function with values in X, then the integrals and derivatives which appear in Definition 2 and Definition 3 are taken in Bochner's sense.

Definition 4 ([15])**.** A one parameter family $T(t)$, $0 \leq t < \infty$, of bounded linear operators from X into X is a semigroup of bounded linear operators on X if

(i) $T(0) = I$ (the identity operator in X),
(ii) $T(t)T(s) = T(t+s)$, for $t, s \geq 0$ (the semigroup property).

A semigroup of bounded linear operators $T(t)$ is uniformly continuous if

$$\lim_{t \to 0} \|T(t) - I\| = 0.$$

The linear operator A defined by

$$D(A) = \{x \in X \mid \lim_{h \to 0} \frac{T(h)(x) - x}{h} \quad \text{exists}\}$$

and

$$A(x) = \lim_{h \to 0} \frac{T(h)(x) - x}{h}, \quad \text{for} \quad x \in D(A),$$

is the infinitesimal generator of the semigroup $T(t)$, $D(A)$ is the domain of A.

Definition 5 ([15])**.** A one parameter family $T(t)$, $0 \leq t < \infty$, of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators if

$$\lim_{t \to 0} T(t)\, x = x \quad \text{for every} \quad x \in X.$$

A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of class C_0 or simply C_0 semigroup.

Theorem 2.1 ([15]). *Let $T(t)$ be a C_0 semigroup. There exists $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t}, \quad \text{for} \quad 0 \leq t < \infty.$$

If $\omega = 0$, $T(t)$ is called uniformly bounded and if moreover $M = 1$ it is called a C_0 semigroup of contractions.
Also we need the following auxiliary lemma.

Lemma 2.2. *Let $T(t)$ be a C_0 semigroup for $t \geq 0$ and be compact for $t > 0$ (so $T(t)$ is continuous in the uniform operator topology for $t > 0$). Then*

$$\lim_{t \to 0} \| (r+t)^{\beta} T(t+\varepsilon) - r^{\beta} T(\varepsilon) \| = 0,$$

for $\beta = \alpha - 1$ and $\varepsilon > 0$.

Proof. let $r, t \geq 0$ and $\varepsilon > 0$, we have

$$\begin{aligned} &\| (r+t)^{\beta} T(t+\varepsilon) - r^{\beta} T(\varepsilon) \| \\ &\qquad = \| \left(r^{\beta} + \beta r^{\beta-1} t + o(t) \right) T(t+\varepsilon) - r^{\beta} T(\varepsilon) \| \\ &\qquad \leq r^{\beta} \|T(t+\varepsilon) - T(\varepsilon)\| + \beta r^{\beta-1} t \|T(t+\varepsilon)\| + o(t) \end{aligned}$$

and the lemma follows from

$$\lim_{t \to 0} \|T(t+\varepsilon) - T(\varepsilon)\| = 0.$$

As a key tool for developing the controllability in this work, the fixed point theorem will be introduced as follows.

Theorem 2.3 ([7, 8]). *Let E be a Banach space, C a closed convex subset of E, V an open subset of C and $0 \in V$. If we suppose that $F : \overline{V} \to C$ is a continuous, completely continuous (that is, $F(\overline{V})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{V}$, or*
(ii) *there is a $x \in \partial V$ (the boundary of V in C) and $\lambda \in (0,1)$ with $x = \lambda F(x)$.*

Now consider the following system represented by the fractional integrodifferential equation of the form

$$\frac{d^{\alpha} x(t)}{dt^{\alpha}} = Ax(t) + f\left(t, x(t), \int_0^t h(t,s,x(s)) ds\right) + Bu(t), \quad t \in J := [0,b], \tag{2.1}$$

$$x(0) = x_0, \tag{2.2}$$

where $0 < \alpha < 1$, $h : \Delta \times X \to X$, $\Delta = \{(t,s) \,|\, 0 \leq s \leq t \leq b\}$, $f : J \times X \times X \to X$ are given function, $x_0 \in X$, A is the infinitesimal generator of a strongly continuous semigroup $T(t),\ t \geq 0$, and the state $x(\cdot)$ takes values in the real Banach space X. Also the control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space. Finally, B is a bounded linear operator from U to X.

Definition 6 ([2, 9])**.** A continuous solution of the integral equation

$$x(t) = T(t)\,x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, T(t-s) \Big[Bu(s) + f\Big(s, x(s), \int_0^s h(s,\tau,x(\tau))\, d\tau \Big) \Big] ds,\ t \in J,$$

is said to be a mild solution of the problem (2.1)–(2.2) on J.

Definition 7 ([2, 4])**.** The system (2.1)–(2.2) is said to be controllable on the interval J, if for every $x_0,\ x_1 \in X$ there is exists a control $u \in L^2(J,U)$ such that the mild solution $x(t)$ of (2.1)–(2.2) satisfies $x(b) = x_1$.

3. Main results

To investigate the controllability of the system (2.1)–(2.2), we assume the following conditions:

(H1) A is the infinitesimal generator of a strongly continuous semigroup of a bounded linear operators $T(t),\ t \geq 0$ in X, which is compact for $t > 0$, and there exist constant $M \geq 1$ such that $\|T(t)\|_{B(X)} \leq M,\ t \geq 0$;

(H2) The linear operator $W : L^2(J,U) \to X$, defined by

$$Wu = \frac{1}{\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1}\, T(b-s)\, Bu(s)\, ds,$$

has a bounded invertible operator $W^{-1} : X \to L^2(J,U)$ and there exist positive constants $M_1,\ M_2$ such that $\|B\| \leq M_1$ and $\| W^{-1} \| \leq M_2$;

(H3) h satisfies Caratheodory condition, i.e.
- for each $(t,s) \in \Delta$, the function $h(t,s,\cdot) : X \to X$ is continuous, and
- for each $x \in X,\ h(\cdot,\cdot,x) : \Delta \to X$ is strongly measurable;

(H4) f satisfies Caratheodory condition, i.e.
- for each $t \in J$, the function $f(t,\cdot,\cdot) : X \times X \to X$ is continuous, and
- for each $x, y \in X,\ f(\cdot,x,y) : J \to X$ is strongly measurable;

(H5) For every positive integer k, there exists $\varphi_k \in L^1(J)$ such that for a.e. $t \in J$ and $x \in C(J, X)$

$$\sup_{\|x\| \leq k} \left\| f\left(t, x(t), \int_0^t h(t, s, x(s)) ds\right) \right\| \leq \varphi_k(t),$$

where $\|x\| = \sup_{t\in J} \|x(t)\|$;

(H6) There exists $q \in L^1(J, \mathbb{R}_+)$ such that

$$\|h(t, s, u)\| \leq q(t)\,\Psi(\|u\|), \quad (t, s) \in \Delta, \quad u \in X,$$

where $\Psi : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function;

(H7) There exists $p \in L^1(J, \mathbb{R}_+)$ such that

$$\|f(t, u, v)\| \leq p(t)\,\Psi(\|u\|) + \|v\| \quad \textit{for each} \ (t, u, v) \in J \times X \times X$$

and there exists a constant $M_* > 0$ with

$$\frac{M_*}{C_1 + \left(C_2 + \frac{\mathrm{b}^{\alpha-1}}{\Gamma(\alpha)} M\right) \Psi(M_*) \int_0^b [p(s) + q(s)]\, ds} > 1,$$

where

$$C_1 = M\|x_0\| + \frac{b^\alpha}{\Gamma(\alpha+1)} M M_1 M_2 (\|x_1\| + M\|x_0\|)$$

and

$$C_2 = \frac{b^{2\alpha-1}}{\Gamma(\alpha)\Gamma(\alpha+1)} M^2 M_1 M_2.$$

Theorem 3.1. *If the hypotheses* (H1)–(H7) are satisfied, system (2.1)–(2.2) is controllable on J.

Proof. In view of the hypothesis (H2) for an arbitrary function $x(t)$, the control is defined as follows

$$u_x(t) = W^{-1}\Bigg[x_1 - T(b)\,x_0 - \frac{1}{\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} T(b-s)\, f\left(s, x(s), \int_0^s h(s, \tau, x(\tau))\, d\tau\right) ds\Bigg](t).$$

In what follows, it suffices to show that when using this control the operator $\Phi : C(J, X) \to C(J, X)$ defined by

$$\Phi x(t) = T(t)\,x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) \Bigg[B u_x(s) + f\left(s, x(s), \int_0^s h(s, \tau, x(\tau))\, d\tau\right)\Bigg] ds, \ t \in J,$$

has a fixed point $x(\cdot)$ from which it follows that this fixed point is a mild solution of the system (2.1)-(2.2). Clearly $\Phi x(b) = x_1$, from which we conclude that the system is controllable.

As in ([7]), we first prove that Φ is continuous and completely continuous. For convenience we let

$$G(\eta) = BW^{-1}\left[x_1 - T(b)x_0 - \frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1} T(b-s)\hat{f}(s)\,ds\right](\eta),$$

where $\hat{f}(s) = f\left(s, x(s), \int_0^s h(s,\tau,x(\tau))d\tau\right)$.
Let $B_k = \{x \in C(J,X) \mid \|x\| \le k\}$ for some $k \ge 1$. It is clear for $x \in B_k$ that

$$\|G(\eta)\| \le M_1M_2\left[\|x_1\| + M\|x_0\| + \frac{b^{\alpha-1}}{\Gamma(\alpha)}M\int_0^b \varphi_k(s)\,ds\right] := G_0$$

and

$$\|\Phi x(t)\| \le M\|x_0\| + \frac{b^{\alpha}}{\Gamma(\alpha+1)}MG_0 + \frac{b^{\alpha-1}}{\Gamma(\alpha)}M\|\varphi_k\|_{L^1};$$

here φ_k satisfies the hypothesis (H5). Let $\epsilon > 0$ be given. Now let $\tau_1, \tau_2 \in J$ with $\tau_2 > \tau_1$. We consider two cases $\tau_1 > \epsilon$ and $\tau_1 \le \epsilon$.

case 1. If $\tau_1 > \epsilon$ then

$$\begin{aligned}
&\| \Phi x(\tau_2) - \Phi x(\tau_1) \| \\
&\quad\le \| T(\tau_2)x_0 - T(\tau_1)x_0 \| \\
&\qquad + \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1-\varepsilon} \| (\tau_2-s)^{\alpha-1}T(\tau_2-s) - (\tau_1-s)^{\alpha-1}T(\tau_1-s) \| \\
&\qquad \times \| G(s) + \hat{f}(s) \|\,ds \\
&\qquad + \frac{1}{\Gamma(\alpha)}\int_{\tau_1-\varepsilon}^{\tau_1} \| (\tau_2-s)^{\alpha-1}T(\tau_2-s) - (\tau_1-s)^{\alpha-1}T(\tau_1-s) \| \\
&\qquad \times \| G(s) + \hat{f}(s) \|\,ds \\
&\qquad + \frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} (\tau_2-s)^{\alpha-1} \| T(\tau_2-s) \| \| G(s) + \hat{f}(s) \|\,ds \\
&\quad\le \| T(\tau_2)x_0 - T(\tau_1)x_0 \| \\
&\qquad + \frac{b^{\alpha-1}}{\Gamma(\alpha)}M\Big[\| (\tau_2-s)^{\alpha-1}T(\tau_2-\tau_1+\varepsilon) - (\tau_1-s)^{\alpha-1}T(\varepsilon) \| \\
&\qquad \times \int_0^{\tau_1-\varepsilon} [G_0 + \varphi_k(s)]\,ds\Big]
\end{aligned}$$

$$+\frac{b^{\alpha-1}}{\Gamma(\alpha)}2M\int_{\tau_1-\varepsilon}^{\tau_1}[G_0+\varphi_k(s)]\,ds+\frac{b^{\alpha-1}}{\Gamma(\alpha)}M\int_{\tau_1}^{\tau_2}[G_0+\varphi_k(s)]\,ds,$$

where we have used the semigroup identities $T(\tau_1-s)=T(\tau_1-s-\varepsilon)T(\varepsilon)$ and $T(\tau_2-s)=T(\tau_2-\tau_1+\varepsilon)T(\tau_1-s-\varepsilon)$.

case 2. Let $\tau_1\le\epsilon$. For $\tau_2-\tau_1<\epsilon$ we get

$$\begin{aligned}\|\Phi x(\tau_2)-\Phi x(\tau_1)\| &\le \|\,T(\tau_2)x_0-T(\tau_1)x_0\,\|\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^{\tau_2}(\tau_2-s)^{\alpha-1}\,\|\,T(\tau_2-s)\,\|\,\|\,G(s)+\hat f(s)\,\|\,ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}(\tau_1-s)^{\alpha-1}\,\|\,T(\tau_1-s)\,\|\,\|\,G(s)+\hat f(s)\,\|\,ds\\ &\le\|\,T(\tau_2)x_0-T(\tau_1)x_0\,\|+\frac{b^{\alpha-1}}{\Gamma(\alpha)}M\int_0^{2\varepsilon}[G_0+\varphi_k(s)]\,ds\\ &\quad+\frac{b^{\alpha-1}}{\Gamma(\alpha)}M\int_0^{\varepsilon}[G_0+\varphi_k(s)]\,ds.\end{aligned}$$

The equicontinuity of Φ follows from (i). the lemma 2.2, (ii). $T(t),t\ge0$, is a strongly continuous semigroup and (iii). $T(t),t>0$, is compact (so $T(t)$ is continuous in the uniform operator topology for $t>0$).
Let $0<t\le b$ be fixed and let ε be a real number satisfying $0<\varepsilon<t$. For $x\in B_k$ we define

$$\begin{aligned}\Phi_\varepsilon x(t)&=T(t)x_0+\frac{1}{\Gamma(\alpha)}\int_0^{t-\varepsilon}(t-s)^{\alpha-1}T(t-s)\left[G(s)+\hat f(s)\right]ds\\ &=T(t)x_0+\frac{1}{\Gamma(\alpha)}T(\varepsilon)\int_0^{t-\varepsilon}(t-s)^{\alpha-1}T(t-s-\varepsilon)\left[G(s)+\hat f(s)\right]ds.\end{aligned}$$

Note that

$$\left\|\int_0^{t-\varepsilon}T(t-s-\varepsilon)\left[G(s)+\hat f(s)\right]ds\right\|\le M\int_0^{t-\varepsilon}[G_0+\varphi_k(s)]\,ds$$

and now since $T(t)$ is a compact operator for $t>0$, the set $Y_\varepsilon(t)=\{\Phi_\varepsilon x(t)\mid x\in B_k\}$ is relatively compact in X for every ε, $0<\varepsilon<t$. Moreover, for every $x\in B_k$ we have

$$\|\Phi x(t)-\Phi_\varepsilon x(t)\|\le\frac{b^{\alpha-1}}{\Gamma(\alpha)}M\int_{t-\varepsilon}^{t}[G_0+\varphi_k(s)]\,ds.$$

Therefore, the set $Y(t)=\{\Phi x(t)\mid x\in B_k\}$ is totally bounded. Hence $Y(t)$ is relatively compact in X and as a consequence from the Arzelá–Ascoli theorem we can conclude that $\Phi:C(J,X)\to C(J,X)$ is completely continuous.

Let $\{x_n\}_0^\infty \subseteq C(J,X)$ with $x_n \to x$ in $C(J,X)$. Then, there exists an integer k such that $\|x_n(t)\| \le k$ for all $n \in \mathbb{N}$ and $t \in J$, so $x_n \in B_k$ and $x \in B_k$. For brevity we let

$$G_n(\eta) = BW^{-1}\left[x_1 - T(b)x_0 - \frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1} T(b-s)\hat{f}_n(s)\,ds\right](\eta),$$

where $\hat{f}_n(s) = f\left(s, x_n(s), \int_0^s h(s,\tau,x_n(\tau))d\tau\right)$. By (H3) and (H4),

$$\hat{f}_n(t) \to \hat{f}(t) \quad \text{as } n \to \infty,$$

for each $t \in J$, since

$$\| \hat{f}_n(t) - \hat{f}(t) \| \le 2\varphi_k(t)$$

and

$$\begin{aligned}\| \Phi x_n(t) - \Phi x(t) \| \le{}& \frac{b^{\alpha-1}}{\Gamma(\alpha)} M \int_0^t \| G_n(s) - G(s) \|\,ds \\ &+ \frac{b^{\alpha-1}}{\Gamma(\alpha)} M \int_0^t \| \hat{f}_n(s) - \hat{f}(s) \|\,ds,\end{aligned}$$

where

$$\| G_n(s) - G(s) \| \le M_1 M_2 \frac{b^{\alpha-1}}{\Gamma(\alpha)} M \int_0^b \| \hat{f}_n(s) - \hat{f}(s) \|\,ds,$$

by using the dominated convergence theorem, we get

$$\| \Phi x_n - \Phi x \| \to 0 \quad \text{as } n \to \infty.$$

Thus, Φ is continuous and completely continuous.

Now let $\lambda \in (0,1)$ and let $x = \lambda \Phi x$. Then for $t \in J$

$$\begin{aligned}x(t) ={}& \lambda T(t)x_0 + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} T(t-s) B u_x(s)\,ds \\ &+ \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} T(t-s)\hat{f}(s)\,ds,\end{aligned}$$

where

$$u_x(s) = W^{-1}\left[x_1 - T(b)x_0 - \frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1} T(b-s)\hat{f}(s)ds\right](s)$$

and we have

$$\begin{aligned}\|x(t)\| \le{}& M\|x_0\| + \frac{b^\alpha}{\Gamma(\alpha+1)} M M_1 M_2 \\ &\times \left[\|x_1\| + M\|x_0\| + \frac{b^{\alpha-1}}{\Gamma(\alpha)} M \int_0^b [p(s) + q(s)]\,\Psi(\|x(s)\|)\,ds\right]\end{aligned}$$

$$
\begin{aligned}
&+\frac{\mathrm{b}^{\alpha-1}}{\Gamma(\alpha)}M\int_0^t [p(s)+q(s)]\,\Psi(\|x(s)\|)\,ds\\
&\le C_1 + C_2\Psi(\|x\|)\int_0^b [p(s)+q(s)]\,ds\\
&+\frac{\mathrm{b}^{\alpha-1}}{\Gamma(\alpha)}M\Psi(\|x\|)\int_0^b [p(s)+q(s)]\,ds.
\end{aligned}
$$

Consequently

$$
\frac{\|x\|}{C_1+\left(C_2+\frac{\mathrm{b}^{\alpha-1}}{\Gamma(\alpha)}M\right)\Psi(\|x\|)\int_0^b [p(s)+q(s)]\,ds}\le 1.
$$

Then by (H7) there exists M_* such that $\|x\|\neq M_*$.
Finally set

$$
V\in\{x\in C(J,X)\mid \|x\|<M_*\}.
$$

From the choice of V there is no $x\in\partial V$ such that $x=\lambda\Phi x$ for some $\lambda\in(0,1)$. As a consequence of theorem 2.3 we deduce that Φ has fixed point x in $\overline{V}$ and hence system 2.1–2.2 is controllable on J. This completes the proof.

4. Example

In this section we present an example to illustrate our main results. Let us consider the following nonlinear partial integro-differential equation of the form

(4.1)
$$
\frac{\partial^\alpha}{\partial t^\alpha}\omega(t,y)=\frac{\partial^2}{\partial y^2}\omega(t,y)+\mu(t,y)+P\left(t,\omega(t,y),\int_0^t a(t,s,\omega(s,y))\,ds\right),
$$

(4.2)
$$
\omega(0,y)=\omega_0(y),\quad 0<y<\pi,
$$

(4.3)
$$
\omega(t,0)=\omega(t,\pi),\quad t\in J=[0,1],
$$

where $0<\alpha<1$, $\mu: J\times(0,\pi)\to(0,\pi)$ is continuous.
Let us take $X=U=L^2[0,\pi]$, $u(t)=\mu(t,\cdot)$ and let $A:X\to X$ be defined by

$$
Aw=w'',\quad w\in D(A),
$$

where

$$
D(A)=\{w\in X\mid w,w' \text{ are absolutely continuous, } w''\in X, w(0)=w(\pi)=0\}.
$$

Then

$$
Aw=\sum_{n=1}^{\infty} n^2(w,w_n)\,w_n,\ w\in D(A),
$$

where $w_n(y) = \sqrt{2/\pi}\sin ny,\ n = 1, 2, 3, \ldots$ is the orthogonal set of eigenfunctions of A.
It can be easily shown that A is the infinitesimal generator of an analytic semigroup $T(t),\ t > 0$ in X and is given by

$$T(t)\,w = \sum_{n=1}^{\infty} e^{-n^2 t}(w, w_n)\,w_n,\ w \in X,$$

where $T(t)$ satisfies the hypothesis (H1).
Here $a : \Delta \times \mathbb{R} \to \mathbb{R},\ \Delta = \{(t, s) \,|\, 0 \le s \le t \le 1\}$ and $P : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and we assume that these functions a, P satisfy (H3)–(H7).

Define the function $f : J \times X \times X \to X$ and $h : \Delta \times X \to X$ as follows

$$f(t, x, z)(y) = P(t, x(y), z(y))$$

and

$$h(t, s, x)(y) = a(t, s, x(y)).$$

for $t \in J,\ x, z \in X$ and $0 < y < \pi$.
With the choice of A, B, h, f and $B = I$, the identity operator, we see that Eqs. 2.1–2.2 is the abstract formulation of Eqs. 4.1–4.3.
Now assume that the operator $W : L^2[J, U] \to X$ defined by

$$Wu = \frac{1}{\Gamma(\alpha)} \sum_{n=1}^{\infty} \int_0^1 (1-s)^{\alpha-1} e^{-n^2(1-s)}(u(s), w_n)\,w_n ds.$$

has an inverse operator and satisfies the condition (H2).

Thus all the conditions of the above theorem 3.1 are satisfied. Hence system 4.1–4.3 is controllable on J.

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