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ON A GENERALIZATION OF CQF-3′ MODULES AND COHEREDITARY TORSION THEORIES

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Throughout this paper we assume that R is a right perfect ring with identity and let Mod-R be the category of right R-modules. Let M be a right R-module. We denote by $0 \to K(M) \to P(M) \to M \to 0$ the projective cover of M. M is called a CQF-3′ module, if $P(M)$ is M-generated, that is, $P(M)$ is isomorphic to a homomorphic image of a direct sum $\oplus M$ of some copies of M.

A subfunctor of the identity functor of Mod-R is called a preradical. For a preradical σ, $\mathcal{T}_\sigma := \{M \in \text{Mod-}R : \sigma(M) = M\}$ is called the class of σ-torsion right R-modules, and $\mathcal{F}_\sigma := \{M \in \text{Mod-}R : \sigma(M) = 0\}$ is called the class of σ-torsionfree right R-modules. A right R-module M is called σ-projective if the functor $\text{Hom}_R(M, -)$ preserves the exactness for any exact sequence $0 \to A \to B \to C \to 0$ with $A \in \mathcal{F}_\sigma$. We put $P_\sigma(M) = P(M)/\sigma(K(M))$ for a module M. We call a right R-module M a σ-CQF-3′ module if $P_\sigma(M)$ is M-generated.

In this paper, we characterize σ-CQF-3′ modules and give some related facts.

1. CQF-3′ modules relative to a cohereditary torsion theories

F. F. Mbuntum and K. Varadarajan defined a CQF-3′ module as a dualization of a QF-3′ module and characterized it in [10]. In this paper we generalize a CQF-3′ module by using an idempotent radical. A preradical σ is idempotent [radical] if $\sigma(\sigma(M)) = \sigma(M)$ $[\sigma(M/\sigma(M)) = 0]$ for a module M, respectively. It is well known that if σ is idempotent preradical, then $\mathcal{F}_\sigma$ is closed under taking extensions. It is also well known that if σ is a radical, then $\mathcal{T}_\sigma$ is closed under taking extensions. A preradical t is called epi-preserving if $t(M/N) = (t(M) + N)/N$ holds for any submodule N of a module M. It holds that any epi-preserving preradical is a radical. For a preradical σ we say that t is σ-epi-preserving if $t(M/N) = (t(M) + N)/N$ holds for any module M and any submodule N of M with $N \in \mathcal{F}_\sigma$. For modules M and N, $t_N(M)$ denote $\sum_{f \in \text{Hom}_R(N,M)} \text{im} f$. It holds that t_N is an idempotent preradical for any module N and that $\mathcal{F}_{t_A} = \{M \in \text{Mod-}R : \text{Hom}_R(A, M) = 0\}$ and $\mathcal{T}_{t_A} = \{M \in \text{Mod-}R : \oplus A \to M \to 0\}$

A short exact sequence $0 \to K(M) \to P(M) \xrightarrow{f} M \to 0$ is called a projective cover of a module M if $P(M)$ is projective and $K(M) := \ker f$ is

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small in $P(M)$. For $X, Y \in \text{Mod-}R$ we call an epimorphism $g \in \mathrm{Hom}_R(X, Y)$ a minimal epimorphism if $g(H) \subsetneqq Y$ holds for any proper submodule H of X. It is well known that a minimal epimorphism is an epimorphism having a small kernel. A short exact sequence $0 \to X \to Y \to M \to 0$ is called σ-projective cover of a module M if Y is σ-projective, X is σ-torsionfree and X is small in Y. If σ is an idempotent preradical, then $P(M)/\sigma(K(M))$ is σ-projective for any module M by Lemma 1.4 in [11]. If σ is a radical, $K(M)/\sigma(K(M)) \in \mathcal{F}_\sigma$. We put $K_\sigma(M) = K(M)/\sigma(K(M))$ and $P_\sigma(M) = P(M)/\sigma(K(M))$ for a preradical σ. Then $K_\sigma(M)$ is small in $P_\sigma(M)$. Thus if σ is an idempotent radical, then a module M has a σ-projective cover and it is given by $0 \to K_\sigma(M) \to P_\sigma(M) \to M \to 0$.

Let σ be a preradical and $\mathcal{C}$ a class of R-modules. We say that $\mathcal{C}$ is closed under taking $\mathcal{F}_\sigma$-extensions if the following condition holds: if N, $M/N \in \mathcal{C}$ and $N \in \mathcal{F}_\sigma$ then $M \in \mathcal{C}$. Next we say that $\mathcal{C}$ is closed under taking $\mathcal{F}_\sigma$-factor modules if : if $M \in \mathcal{C}$ and N is a σ-torsionfree submodule of M then $M/N \in \mathcal{C}$. For a preradical σ we say that M is a σ-coessential extension of X if there exists a minimal epimorphism $h : M \twoheadrightarrow X$ with $\ker h \in \mathcal{F}_\sigma$. We say that $\mathcal{C}$ is closed under taking σ-coessential extensions if : for any minimal epimorphism $f : M \twoheadrightarrow X$ with $\ker f \in \mathcal{F}_\sigma$ if $X \in \mathcal{C}$ then $M \in \mathcal{C}$.

For the sake of simplicity we say that M is a σ-coessential extension of M/N if N is a σ-torsionfree small submodule of M. We say that $\mathcal{C}$ is closed under taking σ-coessential extensions if : if $M/N \in \mathcal{C}$ then $M \in \mathcal{C}$ for any σ-torsion free small submodule N of any module M.

Theorem 1. *Let σ be a preradical. We consider the following conditions.*

(1) A is a σ-CQF-3$'$ module.

(2) $t_A(P_\sigma(A)) = P_\sigma(A)$

(3) $t_A(M) = t_{P_\sigma(A)}(M)$ for any module M.

(4) $t_A(-)$ is σ-epi-preserving.

(5) (a) $\mathcal{T}_{t_A}$ is closed under taking $\mathcal{F}_\sigma$-extensions.

(b) $\mathcal{F}_{t_A}$ is closed under taking $\mathcal{F}_\sigma$-factor modules.

(6) $\mathcal{T}_{t_A}$ is closed under taking σ-projective covers.

(7) $\mathcal{T}_{t_A}$ is closed under taking σ-coessential extensions.

(8) If $\mathrm{Hom}_R(A, f) = 0$, then $\mathrm{Hom}_R(A, M/N) = 0$ holds for any submodule $N \in \mathcal{F}_\sigma$ of a module M, where f is the canonical epimorphism $f : M \to M/N$

Then we have implications (1)→(3)→(2)→(1) and (4)→(5).

If σ is idempotent, then (3)→(4), (1)→(8) and (6)→(5), (7) hold.

If σ is a radical, then (7)→(6), (4)→(2), (6) hold.

If σ is an epi-preserving radical and A is in $\mathcal{F}_\sigma$, then (8)→(5) holds, moreover if σ is idempotent then (5)→(2) hold.

Thus if σ is an epi-preserving idempotent radical and A is in $\mathcal{F}_\sigma$, all conditions are equivalent.

Proof. (1)→(3): Let M be a module in Mod-R. By the assumption there exists an exact sequence $\oplus A \to P_\sigma(A) \to 0$, and hence $t_A(M)$ contains $t_{P_\sigma(A)}(M)$. Since $P_\sigma(A) \to A \to 0$ is exact, $t_A(M)$ is contained in $t_{P_\sigma(A)}(M)$. Thus it follows that $t_A(M) = t_{P_\sigma(A)}(M)$ for any module M.

(3)→(2): It is clear, for $t_A(P_\sigma(A)) = t_{P_\sigma(A)}(P_\sigma(A)) = P_\sigma(A)$.

(2)→(1): It is clear, for $P_\sigma(A) = t_A(P_\sigma(A))$ is a homomorphic image of direct sums of copies of A.

(3)→(4): Suppose that σ is an idempotent preradical. Then $P_\sigma(A)$ is σ-projective. Let $N \in \mathcal{F}_\sigma$ be a submodule of a module M. Consider the following diagram.

$$\begin{tikzcd} & & & P_\sigma(A) \arrow[dl, "h"'] \arrow[d, "f"] & \\ 0 \arrow[r] & N \arrow[r] & M \arrow[r, "g"'] & M/N \arrow[r] & 0, \end{tikzcd}$$

where g is the canonical epimorphism, f is any homomorphism from $P_\sigma(A)$ to M/N and $h \in \mathrm{Hom}_R(P_\sigma(A), M)$ is induced by the σ-projectivity of $P_\sigma(A)$ such that $f = gh$.

Thus $t_{P_\sigma(A)}(M/N) \subseteqq (t_{P_\sigma(A)}(M) + N)/N$. By the assumption, it holds that $t_A(M/N) \subseteqq (t_A(M) + N)/N$. Since $t_A(-)$ is a preradical, $t_A(M/N) \supseteqq (t_A(M) + N)/N$ holds, and so $t_A(-)$ is a σ-epi-preserving preradical.

(4)→(2): Here we assume that σ is a radical.

Then it holds that $K_\sigma(A) = K(A)/\sigma(K(A)) \in \mathcal{F}_\sigma$. Thus it holds $(t_A(P_\sigma(A)) + K_\sigma(A))/K_\sigma(A) = t_A(P_\sigma(A)/K_\sigma(A))$. Since $t_A(A) = A$ and $A \simeq P_\sigma(A)/K_\sigma(A)$, it follows that $t_A(P_\sigma(A)/K_\sigma(A)) = P_\sigma(A)/K_\sigma(A)$. Thus $t_A(P_\sigma(A)) + K_\sigma(A) = P_\sigma(A)$ holds. Consequently $P_\sigma(A) = t_A(P_\sigma(A))$, for $K_\sigma(A)$ is small in $P_\sigma(A)$.

(4)→(5)(a): Let N be a submodule of a module M such that $N \in \mathcal{F}_\sigma \cap \mathcal{T}_{t_A}$ and $M/N \in \mathcal{T}_{t_A}$, then $N = t_A(N) \subseteqq t_A(M)$ and $t_A(M/N) = M/N$. By the assumption $t_A(M/N) = (t_A(M) + N)/N$, and so $M = t_A(M) + N = t_A(M)$, as desired.

(b): Let $N \in \mathcal{F}_\sigma$ be a submodule of a module $M \in \mathcal{F}_{t_A}$, then we have the equation $t_A(M/N) = (t_A(M) + N)/N = N/N = 0$, as desired.

(1)→(8): Suppose that σ is idempotent. Then $P_\sigma(A)$ is σ-projective. Let N be a submodule of a module M such that $N \in \mathcal{F}_\sigma$. Since A is σ-CQF-3′, there exists an epimorphism $\oplus A_i \overset{(\varphi_i)}{\twoheadrightarrow} P_\sigma(A)$, defined by $(\varphi_i)(a_i) = \sum_i \varphi_i(a_i)$ for $(a_i) \in \oplus A_i$, $\varphi_i \in \mathrm{Hom}_R(A_i, P_\sigma(A))$, where $A_i \cong A$.

We will show that if $\mathrm{Hom}_R(A,f)=0$ then $\mathrm{Hom}_R(A,M/N)=0$. Suppose that $\mathrm{Hom}_R(A,M/N)\neq 0$. Then there exists a nonzero element j in $\mathrm{Hom}_R(A,M/N)$.

Let $f : M \to M/N$ be the canonical epimorphism, $g : P_\sigma(A)\to A$ a homomorphism associated with the σ-projectivity of A and $h : P_\sigma(A)\to M$ a homomorphism induced by the σ-projectivity of $P_\sigma(A)$ such that $jg=fh$.

Consider the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
 & & & & P_\sigma(A) & \xrightarrow{g} & A & \longrightarrow & 0\\
 & & & & \big\downarrow h & & \big\downarrow j & & \\
0 & \longrightarrow & N & \longrightarrow & M & \xrightarrow[f]{} & M/N & \longrightarrow & 0
\end{array}$$

There exists a nonzero element $x\in A$ such that $j(x)\neq 0$. Then there exists a nonzero element $y\in P_\sigma(A)$ such that $y=\sum\limits_i \varphi_i(a_i)$ and $x=g(y)=g(\sum\limits_i\varphi_i(a_i))=\sum\limits_i g(\varphi_i(a_i))$. Therefore it holds that $0\neq j(x)=j(g(y))=\sum\limits_i j(g(\varphi_i(a_i)))$, and so there exists some a_i in A and some φ_i in $\mathrm{Hom}_R(A,P_\sigma(A))$ such that $j(g(\varphi_i(a_i)))\neq 0$. Then it holds that $0\neq j(g(\varphi_i(a_i)))=f(h(\varphi_i(a_i)))$ for $jg=fh$. Since $h\varphi_i\in\mathrm{Hom}_R(A,M)$, it holds that $0\neq fh\varphi_i=\mathrm{Hom}(A,f)(h\varphi_i)$. This is a contradiction, and so $\mathrm{Hom}_R(A,M/N)=0$, as desired.

(8)$\to$(5): Here we assume that σ is an epi-preserving preradical and $A\in\mathcal{F}_\sigma$.

(a): We show the stronger condition that $\mathcal{T}_{t_A}$ is closed under taking extensions. Let N be a submodule of a module M such that $M/N\in\mathcal{T}_{t_A}$ and $N\in\mathcal{T}_{t_A}$. Since $t_A(M)$ is a homomorphic image of a direct sum of copies of $A\in\mathcal{F}_\sigma$, it follows that $t_A(M)\in\mathcal{F}_\sigma$. Consider the following sequence. $\mathcal{F}_\sigma\ni t_A(M)\hookrightarrow M \underset{f}{\twoheadrightarrow} M/t_A(M)$. By the definition of $t_A(M)$ it follows that $\mathrm{Hom}_R(A,f)=0$. Cosequently $\mathrm{Hom}_R(A,M/t_A(M))=0$ by the assumption, and so $M/t_A(M)\in\mathcal{F}_{t_A}$.

Since $N\in\mathcal{T}_{t_A}$, $N=t_A(N)\subsetneqq t_A(M)$. Thus $M/t_A(M)$ is a factor module of $M/N\in\mathcal{T}_{t_A}$, and so $M/t_A(M)\in\mathcal{T}_{t_A}$.

Consequently it follows that $M/t_A(M)=0$, as desired.

(b): Let $N\in\mathcal{F}_\sigma$ be a submodule of a module $M\in\mathcal{F}_{t_A}$. Consider the exact sequence $0\to N\to M\xrightarrow{f} M/N\to 0$. Since $M\in\mathcal{F}_{t_A}$, $\mathrm{Hom}_R(A,f)=0$. Thus by the assumption $\mathrm{Hom}_R(A,M/N)=0$, and so $M/N\in\mathcal{F}_{t_A}$.

(5)$\to$(2): Let σ be an epi-preserving idempotent radical and $A\in\mathcal{F}_\sigma$. Since $\mathcal{F}_\sigma$ is closed under taking extensions and $K_\sigma(A)\in\mathcal{F}_\sigma$, it follows that $P_\sigma(A)\in\mathcal{F}_\sigma$ and so $t_A(P_\sigma(A))\in\mathcal{F}_\sigma$ since $\mathcal{F}_\sigma$ is closed under taking submodules. We put $K=t_A(P_\sigma(A))$. We will show that $K=P_\sigma(A)$.

Suppose $K \subsetneqq P_\sigma(A)$. Since $K_\sigma(A)$ is small in $P_\sigma(A)$, $K + K_\sigma(A) \subsetneqq P_\sigma(A)$. Since $A \simeq P_\sigma(A)/K_\sigma(A) \twoheadrightarrow P_\sigma(A)/(K_\sigma(A) + K) \neq 0$, it follows that $\mathrm{Hom}_R(A, P_\sigma(A)/(K_\sigma(A) + K)) \neq 0$, and so $P_\sigma(A)/(K_\sigma(A) + K)) \notin \mathcal{F}_{t_A}$. As $(K_\sigma(A) + K)/K$ is an epimorphic image of $K_\sigma(A) \in \mathcal{F}_\sigma$, it follows that $(K_\sigma(A) + K)/K \in \mathcal{F}_\sigma$ since $\mathcal{F}_\sigma$ is closed under taking factor modules. Consider the exact sequence $0 \to (K_\sigma(A) + K)/K \to P_\sigma(A)/K \to P_\sigma(A)/(K_\sigma(A) + K) \to 0$. By the assumption (b), it follows that $(P_\sigma(A)/K) \notin \mathcal{F}_{t_A}$. We put $X/K = t_A(P_\sigma(A)/K)(\neq 0)$. Consider the exact sequence $0 \to K \to X \to X/K \to 0$. As $K = t_A(P_\sigma(A)) \in \mathcal{F}_\sigma$, $K \in \mathcal{F}_\sigma \cap \mathcal{T}_{t_A}$. Since $X/K \in \mathcal{T}_{t_A}$, it follows that $X \in \mathcal{T}_{t_A}$ by the assumption (a). As $X \subseteqq P_\sigma(A)$, $X = t_A(X) \subseteqq t_A(P_\sigma(A)) = K$. Thus it follows that $X = K$. But this is a contradiction, for $X/K = t_A(P_\sigma(A)/K) \neq 0$. It concludes that $t_A(P_\sigma(A)) = K = P_\sigma(A)$, as desired.

(4)→(6): We assume that σ is a radical. Then $K_\sigma(X) \in \mathcal{F}_\sigma$ for any module X. Let $M \in \mathcal{T}_{t_A}$. Consider the exact sequence $0 \to K_\sigma(M) \to P_\sigma(M) \to P_\sigma(M)/K_\sigma(M) \to 0$. Since $K_\sigma(M) \in \mathcal{F}_\sigma$ and $P_\sigma(M)/K_\sigma(M) \simeq M \in \mathcal{T}_{t_A}$, it follows that $P_\sigma(M)/K_\sigma(M) = t_A(P_\sigma(M)/K_\sigma(M)) = (t_A(P_\sigma(M)) + K_\sigma(M))/K_\sigma(M)$. Thus it follows that $P_\sigma(M) = t_A(P_\sigma(M)) + K_\sigma(M)$. As $K_\sigma(M)$ is small in $P_\sigma(M)$, it follows that $P_\sigma(M) = t_A(P_\sigma(M)) \in \mathcal{T}_{t_A}$, as desired.

(6)→(5): We assume that σ is idempotent. Then $P_\sigma(X)$ is σ-projective for any module X.

(a): Let $N \in \mathcal{F}_\sigma \cap \mathcal{T}_{t_A}$ be a submodule of a module M such that $M/N \in \mathcal{T}_{t_A}$. Consider the following diagram.

$$\begin{array}{ccccccccc} & & & & & & P_\sigma(M/N) & & \\ & & & & & f\swarrow & \big\downarrow g & & \\ 0 & \longrightarrow & N & \longrightarrow & M & \underset{h}{\longrightarrow} & M/N & \longrightarrow & 0, \end{array}$$

where g is an epimorphism associated with the σ-projective cover of M/N, h is the canonical epimorphism and f is a homomorphism induced by the σ-projectivity of $P_\sigma(M/N)$. By the assumption it follows that $P_\sigma(M/N) \in \mathcal{T}_{t_A}$. Thus it follows that $f(P_\sigma(M/N)) = f(t_A(P_\sigma(M/N))) \subseteq t_A(M)$. Since $N \in \mathcal{T}_{t_A}$, $N = t_A(N) \subseteq t_A(M)$. Then the following equalities hold. $M/N = g(P_\sigma(M/N)) = h(f(P_\sigma(M/N))) = (f(P_\sigma(M/N)) + N)/N \subseteq (t_A(M) + N)/N = t_A(M)/N \subseteq t_A(M/N) = M/N$. Thus we conclude that $M = t_A(M)$, as desired.

(b): Let $N \in \mathcal{F}_\sigma$ be a submodule of a module $M \in \mathcal{F}_{t_A}$. Consider the following diagram.

$$\begin{array}{ccccccccc} & & & & P_\sigma(t_A(M/N)) & \xrightarrow{g} & t_A(M/N) & \longrightarrow & 0 \\ & & & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle i} & & \\ 0 & \longrightarrow & N & \longrightarrow & M & \xrightarrow[h]{} & M/N & \longrightarrow & 0, \end{array}$$

where g is an epimorphim associated with the σ-projective cover of $t_A(M/N)$, i is the canonical monomorphism and f is a homomorphism induced by σ-projectivity of $P_\sigma(t_A(M/N))$.

By the assumption $P_\sigma(t_A(M/N)) \in \mathcal{T}_{t_A}$. Since $M \in \mathcal{F}_{t_A}$, it follows that $f = 0$, and so $ig = 0$. Hence $i = 0$, and so we conclude that $t_A(M/N) = 0$, as desired.

(7)→(6): We assume that σ is a radical, and then $K_\sigma(M) \in \mathcal{F}_\sigma$. Thus it is clear, for $P_\sigma(M)$ is a σ-coessential extension of M.

(6)→(7): We assume that σ is idempotent, and then $P_\sigma(X)$ is σ-projective for any module X. Let N be a small submodule of a module M such that $M/N \in \mathcal{T}_{t_A}$ and $N \in \mathcal{F}_\sigma$. Consider the following diagram.

$$\begin{array}{ccccccccc} & & & & & & P_\sigma(M/N) & & \\ & & & & & {\scriptstyle h}\swarrow & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & N & \longrightarrow & M & \xrightarrow[g]{} & M/N & \longrightarrow & 0, \end{array}$$

where f is an epimorphism associated with the σ-projective cover of M/N, g is the canonical epimorphism and h is a homomorphism induced by the σ-projectivity of $P_\sigma(M/N)$. Since g is a minimal epimorphism and f is an epimorphism, it follows that h is also an epimorphism. By the assumption, $M/N \in \mathcal{T}_{t_A}$ implies $P_\sigma(M/N) \in \mathcal{T}_{t_A}$. Since h is an epimorphism, it follows that $M \in \mathcal{T}_{t_A}$. □

If σ is zero functor, then σ is an epi-preserving idempotent radical and A is σ-torsionfree. Thus then σ-CQF-3′ modules are CQF-3′ modules.

Proposition 2. *Let σ be an epi-preserving idempotent radical. Then the following conditions on a module A are equivalent.*

(1) $\mathcal{F}_{t_A}$ is closed under taking $\mathcal{F}_\sigma$-factor modules.

(2) $\mathcal{F}_{t_A} = \mathcal{F}_{t_{P_\sigma(A)}}$

Proof. (1)→(2): Since $P_\sigma(A) \to A$ is surjective, it follows that $\mathcal{F}_{t_{P_\sigma(A)}} \subseteq \mathcal{F}_{t_A}$. Next we will show that $\mathcal{F}_{t_{P_\sigma(A)}} \supseteq \mathcal{F}_{t_A}$. Let M be in $\mathcal{F}_{t_A}$. We will show that $M \in \mathcal{F}_{t_{P_\sigma(A)}}$. Suppose that $M \notin \mathcal{F}_{t_{P_\sigma(A)}}$. Then it holds that $\mathrm{Hom}_R(P_\sigma(A), M) \neq 0$, and there exists $0 \neq f \in \mathrm{Hom}_R(P_\sigma(A), M)$. Since $\ker f \subsetneqq P_\sigma(A)$ and $K_\sigma(A)$ is small in $P_\sigma(A)$, it follows that $\ker f + K_\sigma(A) \subsetneqq P_\sigma(A)$. Since σ is an epi-preserving preradical, $\mathcal{F}_\sigma$ is closed under taking factor modules. Thus it follows that $(K_\sigma(A) + \ker f)/\ker f \in \mathcal{F}_\sigma$. Since

$P_\sigma(A)/\ker f \subseteq M \in \mathcal{F}_{t_A}$, $P_\sigma(A)/\ker f \in \mathcal{F}_{t_A}$. Consider the exact sequence $0 \to (K_\sigma(A)+\ker f)/\ker f \to P_\sigma(A)/\ker f \to P_\sigma(A)/(K_\sigma(A)+\ker f) \to 0$. By the assumption it follows that $P_\sigma(A)/(K_\sigma(A)+\ker f) \in \mathcal{F}_{t_A}$. Since $A \in \mathcal{T}_{t_A}$ and $A \simeq P_\sigma(A)/K_\sigma(A) \twoheadrightarrow P_\sigma(A)/(K_\sigma(A)+\ker f)$, $P_\sigma(A)/(K_\sigma(A)+\ker f) \in \mathcal{T}_{t_A} \cap \mathcal{F}_{t_A}$. Thus $P_\sigma(A)/(K_\sigma(A)+\ker f) = 0$, this is a contradiction. Hence it follows that $M \in \mathcal{F}_{t_{P_\sigma(A)}}$.

(2)→(1): By the assumption, it is sufficient to prove that $\mathcal{F}_{t_{P_\sigma(A)}}$ is closed under taking $\mathcal{F}_\sigma$-factor modules. Let $N \in \mathcal{F}_\sigma$ be a submodule of a module $M \in \mathcal{F}_{t_{P_\sigma(A)}}$. Suppose that $M/N \notin \mathcal{F}_{t_{P_\sigma(A)}}$, then there exists $0 \neq f \in \mathrm{Hom}_R(P_\sigma(A), M/N)$. Since $P_\sigma(A)$ is σ-projective, there exists an $h \in \mathrm{Hom}_R(P_\sigma(A), M)$ such that $gh = f$. Since $M \in \mathcal{F}_{t_{P_\sigma(A)}}$, it follows that $h = 0$, and then $f = 0$. This is a contradiction, and so $M/N \in \mathcal{F}_{t_{P_\sigma(A)}}$, as desired. □

Lemma 3. *Let σ be an idempotent radical. For a module M and its submodule N, consider the following diagram with exact rows.*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_\sigma(M) & \longrightarrow & P_\sigma(M) & \xrightarrow{f} & M & \longrightarrow & 0 \\
 & & & & & & \big\downarrow{\scriptstyle j} & & \\
0 & \longrightarrow & K_\sigma(M/N) & \longrightarrow & P_\sigma(M/N) & \xrightarrow[g]{} & M/N & \longrightarrow & 0,
\end{array}$$

where f and g are epimorphisms associated with the σ-projective covers and j is the canonical epimorphism. Then there exists a homomorphism $h : P_\sigma(M) \to P_\sigma(M/N)$ induced by the σ-projectivity of $P_\sigma(M)$ such that $jf = gh$.

Then the following conditions hold.

(1) If M is a σ-coessential extension of M/N, then $h : P_\sigma(M) \to P_\sigma(M/N)$ is an isomorphism.

(2) Moreover if σ is epi-preserving and $h : P_\sigma(M) \to P_\sigma(M/N)$ is an isomorphism, then M is a σ-coessential extension of M/N.

Proof. (1): Let $N \in \mathcal{F}_\sigma$ be a small submodule of a module M. Since jf is an epimorphism and g is a minimal epimorphism, h is also an epimorphism. Since $j(f(\ker h)) = g(h(\ker h)) = g(0) = 0$, it follows that $f(\ker h) \subseteq \ker j = N \in \mathcal{F}_\sigma$, and so $f(\ker h) \in \mathcal{F}_\sigma$. Let $f|_{\ker h}$ be the restriction of f to $\ker h$. Then it follows that $\ker(f|_{\ker h}) = \ker h \cap \ker f = \ker h \cap K_\sigma(M) \subseteq K_\sigma(M) \in \mathcal{F}_\sigma$. Consider the exact sequence $0 \to \ker f|_{\ker h} \to \ker h \to f(\ker h) \to 0$. Since $\mathcal{F}_\sigma$ is closed under taking extensions, it follows that $\ker h \in \mathcal{F}_\sigma$. As $P_\sigma(M/N)$ is σ-projective, the exact sequence $0 \to \ker h \to P_\sigma(M) \to P_\sigma(M/N) \to 0$ splits, and so there exists a submodule L of $P_\sigma(M)$ such that $P_\sigma(M) = L \oplus \ker h$. So it follows that $f(P_\sigma(M)) = f(L) + f(\ker h)$. As $f(\ker h) \subseteq N$ and $f(P_\sigma(M)) = M$, $M = f(L) + N$. Since N is small in M, it follows that $M = f(L)$. As f is a minimal epimorphism, it follows that $P_\sigma(M) = L$ and $\ker h = 0$, and so $h : P_\sigma(M) \simeq P_\sigma(M/N)$, as desired.

(2): Suppose that $h : P_\sigma(M) \simeq P_\sigma(M/N)$. By the commutativity of the above diagram and h, it follows that $h(f^{-1}(N)) \subseteq K_\sigma(M/N) \in \mathcal{F}_\sigma$. Since h is an isomorphism, $f^{-1}(N) \in \mathcal{F}_\sigma$. As $f|_{f^{-1}(N)} : f^{-1}(N) \to N \to 0$ and σ is an epi-preserving preradical, it follows that $N \in \mathcal{F}_\sigma$.

Next we will show that N is small in M. Let K be a submodule of M such that $M = N + K$. If $f^{-1}(K) \subsetneqq P_\sigma(M)$, then $h(f^{-1}(K)) \subsetneqq P_\sigma(M/N)$ as h is an isomorphism. Since $g(h(f^{-1}(K))) = j(f(f^{-1}(K))) = j(K) = (K + N)/N = M/N$ and g is a minimal epimorphism, this is a contradiction. Thus it holds that $f^{-1}(K) = P_\sigma(M)$, and so $K = f(f^{-1}(K)) = f(P_\sigma(M)) = M$. Thus it follows that N is small in M. □

Proposition 4. *Let σ be an idempotent radical. The class of σ-CQF-3′ modules is closed under taking σ-coessntial extensions.*

Proof. Let $N \in \mathcal{F}_\sigma$ be a submodule of a module M such that $P_\sigma(M/N)$ is (M/N)-generated. Then by Lemma 3 it follows that $\oplus M \twoheadrightarrow \oplus(M/N) \twoheadrightarrow P_\sigma(M/N) \simeq P_\sigma(M)$. Thus it follows that M is a σ-CQF-3′ module. □

2. σ-epi-preserving preradical and σ-cohereditary torsion theories

In this section we generalize epi-preserving preradicals by using torsion theories. If a module A is σ-CQF-3′ and $t = t_A$, then t is a σ-epi-preserving idempotent preradical by Theorem 1.

Theorem 5. *Let σ be an idempotent radical. Consider the following conditions on a preradical t.*

(1) t is a σ-epi-preserving preradical.

(2) $\mathcal{T}_t$ is closed under taking σ-coessential extensions.

(3) $\mathcal{T}_t$ is closed under taking σ-projective covers.

(4) (i) $\mathcal{F}_t$ is closed under taking $\mathcal{F}_\sigma$-factor modules.

(ii) $\mathcal{T}_t$ is closed under taking $\mathcal{F}_\sigma$-extensions.

Then we have the implications (4)←(1)→(2)⟷(3).

If t is an idempotent preradical, then we have the implication (3)→(1).

If σ is an epi-preserving preradical and t is a radical, then (4)→(1) holds.

Thus if σ is an epi-preserving idempotent radical and t is an idempotent radical, then all conditions are equivalent.

Proof. By the assumption every module has its σ-projective cover.

(1)→(2): Let $N \in \mathcal{F}_\sigma$ be a small submodule of a module M such that $M/N \in \mathcal{T}_t$. By the assumption $M/N = t(M/N) = (t(M) + N)/N$. Thus it follows that $M = t(M) + N$, and so $M = t(M)$, for N is small in M.

(2)→(3): This is clear.

(3)→(2): Let $N \in \mathcal{F}_\sigma$ be a small submodule of a module M such that $M/N \in \mathcal{T}_t$. Consider the following commutative diagram.

$$\begin{tikzcd}
 & & & P_\sigma(M/N) \arrow[dl, "h"'] \arrow[d, "f"] & \\
0 \arrow[r] & N \arrow[r] & M \arrow[r, "g"'] & M/N \arrow[r] & 0,
\end{tikzcd}$$

where f is an epimorphism associated with the σ-projective cover of M/N, g is the canonical epimorphism and h is a homomorphism induced by the σ-projectivity of $P_\sigma(M/N)$.

Since f is an epimorphism and g is a minimal epimorphism, it follows that h is an epimorphism. By the assumption it holds that $P_\sigma(M/N) \in \mathcal{T}_t$, and so $M \in \mathcal{T}_t$, as desired.

(1)→(4): This is almost the same as (4)→(5) in Theorem 1.

(3)→(1): Let $N \in \mathcal{F}_\sigma$ be a submodule of a module M and t an idempotent preradical. Consider the following diagram.

$$\begin{tikzcd}
 & & & P_\sigma(t(M/N)) \arrow[d, "f"] & \\
 & & t(M) \arrow[r] \arrow[d, "j"] & t(M/N) \arrow[d, "i"] & \\
0 \arrow[r] & N \arrow[r, "u"'] & M \arrow[r, "g"'] & M/N \arrow[r] & 0,
\end{tikzcd}$$

where i, j and u are the inclusions, f is an epimorphism associated with the σ-projective cover of $t(M/N)$ and g is the canonical epimorphism from M to M/N. By the assumption $P_\sigma(t(M/N)) \in \mathcal{T}_t$. Since $N \in \mathcal{F}_\sigma$, there exists an $h \in \mathrm{Hom}_R(P_\sigma(t(M/N)), M)$ such that $if = gh$ by the σ-projectivity of $P_\sigma(t(M/N))$. Since $h(P_\sigma(t(M/N))) = h(t(P_\sigma(t(M/N)))) \subseteq t(M)$, $h \in \mathrm{Hom}_R(P_\sigma(t(M/N)), t(M))$. Since $g(t(M)) \subseteq t(M/N)$, g induces $g' \in \mathrm{Hom}_R(t(M), t(M/N))$ such that $f = g'h$. As f is an epimorphism, g' is also an epimorphism. Thus $(t(M) + N)/N = g'(t(M)) = t(M/N)$, as desired.

(4)→(1): Let $N \in \mathcal{F}_\sigma$ be a submodule of a module M, t a radical and σ an epi-preserving preradical. Then $(N + t(M))/t(M) \simeq N/(N \cap t(M)) \twoheadleftarrow N \in \mathcal{F}_\sigma$. Consider the exact sequence $0 \to (N + t(M))/t(M) \to M/t(M) \to M/(N + t(M)) \to 0$. Since $M/t(M) \in \mathcal{F}_t$, it follows that $M/(N + t(M)) \in \mathcal{F}_t$ by the assumption (i). Hence $(M/N)/((N + t(M))/N) \in \mathcal{F}_t$, and so $t(M/N) \subseteqq (N + t(M))/N$. Since t is a preradical, it follows that $t(M/N) \supseteqq (N + t(M))/N$, and so $t(M/N) = (N + t(M))/N$ holds. □

Proposition 6. *Let σ be an epi-preserving radical and t a preradical. Then the following conditions are equivalent.*

(1) Let N be a submodule of a module M such that $M \supseteq N \supseteq t(M)$. If $N/t(M) \in \mathcal{F}_\sigma$, then $M/N \in \mathcal{F}_t$.

(2) t is both a radical and a σ-epi-preserving preradical.

Proof. (1)→(2): We use $t(M)$ instead of N. Then it follows that $M/t(M) \in \mathcal{F}_t$, and so t is a radical.

Next we will show that if $N \in \mathcal{F}_\sigma$, then $t(M/N) = (t(M) + N)/N$. We use $N+t(M)$ instead of N. Consider the sequence $0 \to (N+t(M))/t(M) \to M/t(M) \to M/(N+t(M) \to 0$. Since $(N+t(M))/t(M) \simeq N/(N \cap t(M)) \twoheadleftarrow N \in \mathcal{F}_\sigma$, $(N + t(M))/t(M) \in \mathcal{F}_\sigma$. It holds that $(M/(N + t(M)) \in \mathcal{F}_t$, and so $(M/N)/((N + t(M))/N) \in \mathcal{F}_t$. Thus $t(M/N) \subseteq (N + t(M))/N$. Since t is a preradical, $t(M/N) \supseteq (N + t(M))/N$, and so it follows that $t(M/N) = (N + t(M))/N$.

(2)→(1): Let N be a submodule of a module M such that $M \supseteq N \supseteq t(M)$ and $N/t(M) \in \mathcal{F}_\sigma$. Consider the sequence $0 \to N/t(M) \to M/t(M) \to M/N \to 0$. Since t is an σ-epi-preserving preradical and a radical,

$\{t(M/t(M)) + N/t(M)\}/(N/t(M)) \simeq t(M/N)$, and so $0 \simeq t(M/N)$, as desired. □

A torsion theory for Mod-R is a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects of Mod-R such that

(i) $\mathrm{Hom}_R(T, F) = 0$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$

(ii) If $\mathrm{Hom}_R(M, F) = 0$ for all $F \in \mathcal{F}$, then $M \in \mathcal{T}$

(iii) If $\mathrm{Hom}_R(T, N) = 0$ for all $T \in \mathcal{T}$, then $N \in \mathcal{F}$

We put $t(M) = \sum_{\mathcal{T} \ni N \subset M} N \ (= \underset{M/N \in \mathcal{F}}{\cap N})$, then $\mathcal{T} = \mathcal{T}_t$ and $\mathcal{F} = \mathcal{F}_t$ hold.

We call a torsion theory $(\mathcal{T}, \mathcal{F})$ σ-cohereditary if $\mathcal{F}$ is closed under taking $\mathcal{F}_\sigma$-factor modules for an idempotent radical σ.

Proposition 7. *Let t be a radical and σ an idempotent preradical such that $\mathcal{T}_\sigma \subseteq \mathcal{T}_t$. If $\mathcal{T}_t$ is closed under taking σ-projective covers, then $\mathcal{T}_t$ is closed under taking projective covers.*

Proof. For $M \in \mathcal{T}_t$ it holds that $P_\sigma(M) \in \mathcal{T}_t$ by the assumption. It holds that $\sigma(K(M)) \in \mathcal{T}_\sigma$ since σ is idempotent. As $\mathcal{T}_\sigma \subseteq \mathcal{T}_t$, it follows that $\sigma(K(M)) \in \mathcal{T}_t$. Consider the exact sequence $0 \to \sigma(K(M)) \to P(M) \to P_\sigma(M) \to 0$. Since t is a radical, $\mathcal{T}_t$ is closed under taking extensions. Therefore it follows that $P(M) \in \mathcal{T}_t$. □

Theorem 8. *Let σ be an epi-preserving idempotent radical. Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory. Suppose that there exists $Q \in \mathcal{T}$ such that $\mathcal{F} = \{M_R : \mathrm{Hom}_R(Q, M) = 0\}$. Then $(\mathcal{T}, \mathcal{F})$ is σ-cohereditary if and only if $\mathcal{F} = \{M_R : \mathrm{Hom}_R(P_\sigma(Q), M) = 0\}$.*

Proof. Let $\mathcal{F} = \{M_R : \mathrm{Hom}_R(P_\sigma(Q), M) = 0\}$. Since it is easily verified that $\mathcal{F}$ is closed under taking submodules, direct sums, and extensions by routine caluculations, $\mathcal{F}$ is a torsion free part of some torsion theory. Thus it is sufficient to prove that $\mathcal{F}$ is closed under taking $\mathcal{F}_\sigma$-factor modules.

Let M be a module in $\mathcal{F}$ and N a σ-torsion free submodule of M. Suppose that $\mathrm{Hom}_R(P_\sigma(Q), M/N) \neq 0$. Consider the following diagram.

$$\begin{array}{ccccccccc} & & & & & & P_\sigma(Q) & & \\ & & & & & & \big\downarrow f & & \\ 0 & \longrightarrow & N & \longrightarrow & M & \underset{h}{\longrightarrow} & M/N & \longrightarrow & 0, \end{array}$$

where f is a nonzero homomorphism from $P_\sigma(Q)$ to M/N and h is the canonical epimorphism from M to M/N.

Then there exists a homomorphism i from $P_\sigma(Q)$ to M induced by the σ-projectivity of $P_\sigma(Q)$ such that $f = hi$. Since $hi \neq 0$, $i \neq 0$ for h is an epimorphism. Since $P_\sigma(Q)$ is Q-generated by the assumption, there exists a homomorphism $k : Q \to P_\sigma(Q)$ such that $0 \neq ik \in \mathrm{Hom}_R(Q, M)$. Thus it follows that $\mathrm{Hom}_R(Q, M) \neq 0$. This is a contradiction to the fact that $M \in \mathcal{F}$. Thus $\mathrm{Hom}_R(Q, M/N) = 0$, and so $M/N \in \mathcal{F}$.

Conversely suppose that $\mathcal{F}$ is closed under taking $\mathcal{F}_\sigma$-factor modules. Let t be a σ-epi-preserving idempotent radical associated with $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{T} = \mathcal{T}_t$ and $\mathcal{F} = \mathcal{F}_t$. By Theorem 5, $\mathcal{F}$ is closed under taking $\mathcal{F}_\sigma$-factor modules if and only if $\mathcal{T}$ is closed under taking σ-projective covers. Since $\mathcal{T}$ is closed under taking σ-projective covers, it follows that $P_\sigma(Q) \in \mathcal{T}$.

Next we show that $\mathcal{F} = \{M : \mathrm{Hom}_R(P_\sigma(Q), M) = 0\}$.

If $M \in \mathcal{F}$, then $\mathrm{Hom}_R(P_\sigma(Q), M) = 0$ since $P_\sigma(Q) \in \mathcal{T}$. Thus it follows that $\mathcal{F} \subseteq \{M : \mathrm{Hom}_R(P_\sigma(Q), M) = 0\}$.

Conversely suppose that $\mathrm{Hom}_R(P_\sigma(Q), M) = 0$. Since $P_\sigma(Q) \to Q \to 0$, it follows that $0 \to \mathrm{Hom}_R(Q, M) \to \mathrm{Hom}_R(P_\sigma(Q), M)$, and so $\mathrm{Hom}_R(Q, M) = 0$. Thus $\mathcal{F} \supseteq \{M : \mathrm{Hom}_R(Q, M) = 0\}$. Therefore it follows that $\mathcal{F} = \{M : \mathrm{Hom}_R(P_\sigma(Q), M) = 0\}$. □

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