Math. J. Okayama Univ. **54** (2012), 53–63

ON A GENERALIZATION OF QF-3′ MODULES AND HEREDITARY TORSION THEORIES

Yasuhiko TAKEHANA

Let R be a ring with identity, and let Mod-R be the category of right R-modules. Let M be a right R-module. We denote by $E(M)$ the injective hull of M. M is called QF-3′ module, if $E(M)$ is M-torsionless, that is, $E(M)$ is isomorphic to a submodule of a direct product ΠM of some copies of M.

A subfunctor of the identity functor of Mod-R is called a preradical. For a preradical σ, $\mathcal{T}_\sigma := \{M \in \text{Mod-}R : \sigma(M) = M\}$ is the class of σ-torsion right R-modules, and $\mathcal{F}_\sigma := \{M \in \text{Mod-}R : \sigma(M) = 0\}$ is the class of σ-torsionfree right R-modules. A right R-module M is called σ-injective if the functor $\text{Hom}_R(-, M)$ preserves the exactness for any exact sequence $0 \to A \to B \to C \to 0$ with $C \in \mathcal{T}_\sigma$. A right R-module M is called σ-QF-3′ module if $E_\sigma(M)$ is M-torsionless, where $E_\sigma(M)$ is defined by $E_\sigma(M)/M := \sigma(E(M)/M)$.

In this paper, we characterize σ-QF-3′ modules and give some related facts.

1. QF-3′ modules relative to torsion theories

In [8], Y. Kurata and H. Katayama characterize QF-3′ modules by using torsion theories. In this section we generalize QF-3′ modules by using an idempotent radical. A preradical σ is idempotent[radical] if $\sigma(\sigma(M)) = \sigma(M)[\sigma(M/\sigma(M)) = 0]$ for a module M, respectively. A subclass $\mathcal{C}$ of Mod-R is closed under taking extensions if $M \in \mathcal{C}$ holds for any module M and any submodule N such that $N \in \mathcal{C}$ and $M/N \in \mathcal{C}$. It is well known that if σ is idempotent preradical then $\mathcal{F}_\sigma$ is closed under taking extensions and that if σ is a radical then $\mathcal{T}_\sigma$ is closed under taking extensions. It is well known, too, that a preradical σ is idempotent if σ is left exact. For a module M, $E_\sigma(M)$ is the same as in the above introduction. If a preradical σ is a radical, then $E(M)/E_\sigma(M) \in \mathcal{F}_\sigma$, and so $E_\sigma(M)$ is σ-injective for any module M by Lemma 2.4 in [9]. $E_\sigma(M)$ is called the σ-injective hull of M. For a module M and N, $k_N(M)$ denote $\cap\{\ker f : f \in \text{Hom}_R(M, N)\}$. It is well known that k_N is a radical for any module N and that $\mathcal{T}_{k_N} = \{M \in \text{Mod-}R : \text{Hom}_R(M, N) = 0\}$ and $\mathcal{F}_{k_N} = \{M \in \text{Mod-}R : M \hookrightarrow \Pi N\}$. For a preradical σ, N is called to be a σ-dense submodule of a module M if $M/N \in \mathcal{T}_\sigma$. For a preradical σ and t, we call t σ-left exact if $t(N) = N \cap t(M)$ holds for any σ-dense submodule N of a module M. A subclass $\mathcal{C}$ of Mod-R

Key words and phrases. QF-3′, hereditary.

is called to be closed under taking σ-extensions if $M \in \mathcal{C}$ holds for any module M and any submodule N such that $N \in \mathcal{C}$ and $M/N \in \mathcal{C} \cap \mathcal{T}_\sigma$. For a module M and a submodule N of M, N is called to be a σ-essential extension of M if N is essential in M and is a σ-dense submodule of M.

Theorem 1. *Let A be a nonzero module and σ a preradical. Then the following conditions (1), (2) and (3) are equivalent. If σ is an idempotent radical, then (1), (2), (3) and (4) are equivalent. Moreover if σ is a left exact radical and A is σ-torsion, then all conditions are equivalent.*

(1) A is a σ-QF-3′ module, that is, it holds that $E_\sigma(A) \hookrightarrow \Pi A$.

(2) $k_A(E_\sigma(A)) = 0$.

(3) $k_A(-) = k_{E_\sigma(A)}(-)$.

(4) k_A is a σ-left exact preradical.

(5) Let $0 \to N \xrightarrow{f} M \to L \to 0$ be an exact sequence such that L is σ-torsion. If $\mathrm{Hom}_R(f, A) = 0$, then $\mathrm{Hom}_R(N, A) = 0$.

(6) (i) $\mathcal{T}_{k_A}$ is closed under taking σ-dense submodules.

(ii) $\mathcal{F}_{k_A}$ is closed under σ-extensions.

(7) $\mathcal{F}_{k_A}$ is closed under taking σ-injective hulls.

(8) $\mathcal{F}_{k_A}$ is closed under taking σ-essential extensions.

Proof. (1)→(3): It is clear that $k_A(M) \supseteq k_{E_\sigma(A)}(M)$ for a module M. We will show that $k_A(M) \subseteq k_{E_\sigma(A)}(M)$. Let m be a nonzero element in $k_A(M)$. Assume that $f(m) \neq 0$ for an $f \in \mathrm{Hom}_R(M, E_\sigma(A))$. Since $E_\sigma(A)$ is A-torsionless, there exists a $g \in \mathrm{Hom}_R(E_\sigma(A), A)$ such that $g(f(m)) \neq 0$. It is a contradiction for $gf \in \mathrm{Hom}_R(M, A)$ and $m \in k_A(M)$. Thus it holds that $k_A(M) \subseteq k_{E_\sigma(A)}(M)$ as desired.

(3)→(2): This is clear, for $0 = k_{E_\sigma(A)}(E_\sigma(A)) = k_A(E_\sigma(A))$.

(2)→(1): Let ϕ be a homomorphism from $E_\sigma(A)$ to $\prod_{f_i \in I} A_{f_i}$

($I = \mathrm{Hom}_R(E_\sigma(A), A)$, $x \in E_\sigma(A) \Rightarrow \phi(x) = \prod_{f_i}(f_i(x))$). By the assumption ϕ is a monomorphism.

(3)→(4): Suppose that σ is an idempotent radical. Let N be a submodule of a module M such that $M/N \in \mathcal{T}_\sigma$. It is clear that $k_A(N) \subseteq N \cap k_A(M)$ holds. Since σ is a radical, $E_\sigma(A)$ is σ-injective. Thus $k_{E_\sigma(A)}(N) \supseteq N \cap k_{E_\sigma(A)}(M)$ holds, and so by the assumption $k_A(N) \supseteq N \cap k_A(M)$ holds, as desired.

(4)→(2): As σ is an idempotent preradical, it follows that $E_\sigma(A)/A \in \mathcal{T}_\sigma$. Thus by the assumption $A \cap k_A(E_\sigma(A)) = k_A(A) = 0$. Since $E_\sigma(A)$ is essential in A, $k_A(E_\sigma(A)) = 0$, as desired.

For the rest of the proof we assume that σ is a left exact radical and $A \in \mathcal{T}_\sigma$.

(1)→(5): Let $0 \to N \xrightarrow{f} M \to L \to 0$ be an exact sequence such that L is σ-torsion. If $\mathrm{Hom}_R(N, A) \ni g \neq 0$, g is extended to $g' \in \mathrm{Hom}_R(M, E_\sigma(A))$ such that $g'f = ig$, where i is a inclusion from A to $E_\sigma(A)$. Since $ig \neq 0$ and $E_\sigma(A) \subseteq \Pi A$, there exists a $p \in \mathrm{Hom}_R(E_\sigma(A), A)$ such that $pig \neq 0$. Then $0 \neq pig = pg'f \in \mathrm{Hom}_R(f, A) = 0$, this is a contradiction, and so $\mathrm{Hom}_R(N, A) = 0$ holds.

(5)→(2): We put $N = k_A(E_\sigma(A))$. Since $\mathcal{T}_\sigma$ is closed under taking extensions, $E_\sigma(A) \in \mathcal{T}_\sigma$. As $\mathcal{T}_\sigma$ is closed under taking factor modules, $E_\sigma(A)/N \in \mathcal{T}_\sigma$. Consider the exact sequence $0 \to N \xrightarrow{f} E_\sigma(A) \to E_\sigma(A)/N \to 0$.

It follows that $\mathrm{Hom}_R(E_\sigma(A), A) \stackrel{\mathrm{Hom}_R(f,A)}{\to} \mathrm{Hom}_R(N, A)$.Then it holds that $\mathrm{Hom}_R(f, A) = 0$, since N is $k_A(E_\sigma(A))$. By the assumption, it holds that $\mathrm{Hom}_R(N, A) = 0$.

Next we will show that $N = 0$. Assume that $N \neq 0$. Since A is essential in $E_\sigma(A)$, $N \cap A \neq 0$. It follows that $N/(N \cap A) \simeq (N + A)/A \subseteq E_\sigma(A)/A \in \mathcal{T}_\sigma$. Consider the sequence $0 \to N \cap A \xrightarrow{g} N \to N/(N \cap A) \to 0$. Since $\mathrm{Hom}_R(N \cap A, A) \neq 0$, $\mathrm{Hom}_R(g, A) \neq 0$. However this is a contradiction to the fact that $\mathrm{Hom}_R(N, A) = 0$. Thus $N = 0$, as desired.

(4)→(8): Let $N \in \mathcal{F}_{k_A}$ be an essential submodule of a module M with $M/N \in \mathcal{T}_\sigma$. Then by the assumption $0 = k_A(N) = N \cap k_A(M)$, and so $k_A(M) = 0$ since N is essential in M.

(8)→(7): It is clear, since $E_\sigma(M)$ is σ-essential extension of M for any module M.

(7)→(6): (i) Let N be a submodule of $M \in \mathcal{T}_{k_A}$ such that $M/N \in \mathcal{T}_\sigma$. Consider the following diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{i} & M & \longrightarrow & M/N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & & & \\
0 & \longrightarrow & N/k_A(N) & \xrightarrow{j} & E_\sigma(N/k_A(N)), & & & &
\end{array}$$

where i and j are the inclusion maps, h is the canonical epimorphism and f is a homomorphism induced by the σ-injectivity of $E_\sigma(N/k_A(N))$.

Since $N/k_A(N) \in \mathcal{F}_{k_A}$, it holds that $E_\sigma(N/k_A(N)) \in \mathcal{F}_{k_A}$ by the assumption. Since $M \in \mathcal{T}_{k_A}$, it follows that $f = 0$, and so $h = 0$. As h is onto, it follows that $N/k_A(N) = 0$, as desired.

(ii): Let $N \in \mathcal{F}_{k_A}$ be a submodule of a module M such that $M/N \in \mathcal{F}_{k_A} \cap \mathcal{T}_\sigma$. By σ-injectivity of $E_\sigma(N)$, the inclusion map i from N to $E_\sigma(N)$ is extended to $f \in \mathrm{Hom}_R(M, E_\sigma(N))$. By the assumption, it follows that $E_\sigma(N) \in \mathcal{F}_{k_A}$, and so $f(k_A(M)) \subseteq k_A(E_\sigma(N)) = 0$. Since $M/N \in \mathcal{F}_{k_A}$, $0 = k_A(M/N) \supseteq (k_A(M) + N)/N$, and so $N \supseteq k_A(M)$. Thus $0 = f(k_A(M)) = i(k_A(M)) = k_A(M)$, and so $M \in \mathcal{F}_{k_A}$.

(6)→(1): First we will show that $k_A(E_\sigma(A)) \subsetneqq E_\sigma(A)$. If $E_\sigma(A) \in \mathcal{T}_{k_A}$, then $A \in \mathcal{T}_{k_A}$ holds by (i) for $E_\sigma(A)/A \in \mathcal{T}_\sigma$. However $A \in \mathcal{F}_{k_A}$ holds, and so $A = 0$. This is a contradiction. Thus it follows that $E_\sigma(A) \notin \mathcal{T}_{k_A}$, and then $k_A(E_\sigma(A)) \subsetneqq E_\sigma(A)$ holds.

Next we will show that $k_A(E_\sigma(A)) = 0$. We put $K = k_A(E_\sigma(A))$. If $K \neq 0$, then $A \cap K \neq 0$ holds since A is essential in $E_\sigma(A)$. As $\mathrm{Hom}_R(A \cap K, A) \neq 0$, it follows that $A \cap K \notin \mathcal{T}_{k_A}$. Since $K/(A \cap K) \simeq (A + K)/A \subseteq E_\sigma(A)/A \in \mathcal{T}_\sigma$, it follows that $K \notin \mathcal{T}_{k_A}$, and so $k_A(K) \subsetneqq K$. We put $K' = k_A(K)$. Since $A \in \mathcal{T}_\sigma$ and $E_\sigma(A)/A \in \mathcal{T}_\sigma$, it follows that $E_\sigma(A) \in \mathcal{T}_\sigma$. Thus $E_\sigma(A)/K \in \mathcal{T}_\sigma \cap \mathcal{F}_{k_A}$. As $K/K' \in \mathcal{F}_{k_A}$, it follows that $E_\sigma(A)/K' \in \mathcal{F}_{k_A}$ by (ii). Then $K = k_A(E_\sigma(A)) \subseteq K'$ holds. This is a contradiction to the fact that $K' = k_A(K) \subsetneqq K$. Thus $K = 0$, as desired.

□

If σ is identity functor, then σ is a left exact radical and A is σ-torsion. Thus then σ-QF-3$'$ modules are QF-3$'$ modules.

Next let $\sigma = k_{E(R_R)}(-)$. Then it is well known that σ is a left exact radical. The torsion theory $(\mathcal{T}_\sigma, \mathcal{F}_\sigma)$ is called the Lambek torsion theory. The localization of R_R with respect to $(\mathcal{T}_\sigma, \mathcal{F}_\sigma)$ is known as the right maximal quotient ring. Let Q be the right maximal quotient ring. Then since $Q = E_\sigma(R)$, we have the following result as an application of (1), (2), (3) and (4) of Theorem 1.

Corollary 2. *Let Q be a maximal right quotient ring. Then the following conditions are equivalent.*

(1) Q is torsionless(i.e. $Q \hookrightarrow \Pi R$)

(2) $k_R(Q) = 0$

(3) $k_R(-) = k_Q(-)$

(4) $k_R(N) = N \cap k_R(M)$ holds for a module M and any submodule N of M such that $\mathrm{Hom}_R(M/N, E(R)) = 0$.

Proposition 3. *Suppose that σ is a left exact radical, then the following conditions are equivalent.*

(1) $\mathcal{T}_{k_A}$ is closed under taking σ-dense submodules.

(2) $\mathcal{T}_{k_A} = \mathcal{T}_{k_{E_\sigma(A)}}$

Proof. (2)→(1): Let N be a submodule of a module $M \in \mathcal{T}_{k_A}$ such that $M/N \in \mathcal{T}_\sigma$. We will show that $N \in \mathcal{T}_{k_A}$. Since $\mathcal{T}_{k_{E_\sigma(A)}}$ is closed under taking σ-dense submodules and $M \in \mathcal{T}_{k_A} = \mathcal{T}_{k_{E_\sigma(A)}}$, it follows that $N \in \mathcal{T}_{k_{E_\sigma(A)}}$ for $M/N \in \mathcal{T}_\sigma$. As $\mathcal{T}_{k_A} = \mathcal{T}_{k_{E_\sigma(A)}}$, it follows that $N \in \mathcal{T}_{k_A}$, as desired.

(1)→(2): It is clear that $\mathcal{T}_{k_A} \supseteq \mathcal{T}_{k_{E_\sigma(A)}}$. Let M be a module in $\mathcal{T}_{k_A}$. Assume that $M \notin \mathcal{T}_{k_{E_\sigma(A)}}$. Then there exists $0 \neq f \in \mathrm{Hom}_R(M, E_\sigma(A))$, and so $f(M) \neq 0$. Since A is essential in $E_\sigma(A)$, $f(M) \cap A \neq 0$. Let N

denote $f^{-1}(f(M) \cap A)$. Then $N \neq 0$ and $M/N \simeq f(M)/(A \cap f(M)) \simeq (A + f(M)/A \subseteq E_\sigma(A)/A \in \mathcal{T}_\sigma$, and so $M/N \in \mathcal{T}_\sigma$. $0 \neq f|_N : N \twoheadrightarrow f(M) \cap A \subseteq A$, where $f|_N$ is a restriction map of f to N. Thus it follows that $N \notin \mathcal{T}_{k_A}$. By the assumption, $M \notin \mathcal{T}_{k_A}$ holds, but this is a contradiction, and so $M \in \mathcal{T}_{k_{E_\sigma(A)}}$. Thus $\mathcal{T}_{k_A} \subseteq \mathcal{T}_{k_{E_\sigma(A)}}$ holds, as desired. □

For a module M, $Z(M)$ denotes the singular submodule of M. Then the singular functor Z is a left exact preradical. For singular modules, please refer to [5].

Proposition 4. *If σ is a left exact radical and $A \in \mathcal{T}_\sigma \cap \mathcal{F}_Z$, then the following conditions are equivalent.*

(1) $\mathcal{T}_{k_A}$ is closed under taking σ-dense submodules.

(2) A is a σ-QF-3′ module.

Proof. It is sufficient to prove that (1)→(2). We will show that $k_A(E_\sigma(A)) = 0$. Suppose that $k_A(E_\sigma(A)) \neq 0$. Let K denote $k_A(E_\sigma(A))$.

First we will show that $\mathrm{Hom}_R(E_\sigma(K), A) \neq 0$. Assume that $\mathrm{Hom}_R(E_\sigma(K), A) = 0$, then $E_\sigma(K) \in \mathcal{T}_{k_A}$. Since $E_\sigma(K)/(A \cap E_\sigma(K)) \simeq (A + E_\sigma(K))/A \subseteq E_\sigma(A)/A \in \mathcal{T}_\sigma$, $E_\sigma(K)/(A \cap E_\sigma(K)) \in \mathcal{T}_\sigma$ holds, and so $A \cap E_\sigma(K) \in \mathcal{T}_{k_A}$ holds by (1). Thus $\mathrm{Hom}_R(A \cap E_\sigma(K), A) = 0$, and so $A \cap E_\sigma(K) = 0$. Since A is essential in $E_\sigma(A)$, $E_\sigma(K) = 0$. This is a contradiction to the fact that $K \neq 0$, and so $\mathrm{Hom}_R(E_\sigma(K), A) \neq 0$.

Thus there exists an $f \in \mathrm{Hom}_R(E_\sigma(K), A)$ and $0 \neq x \in E_\sigma(K)$ with $f(x) \neq 0$. Since $Z(A) = 0$, $(0 : f(x))$ is not essential in R. Then there exists a nonzero right ideal L of R such that $L \cap (0 : f(x)) = 0$.

Next we will show that $xL \cap K \neq 0$. Suppose that $xL \cap K = 0$. Since $xL \subseteq E_\sigma(K)$ and K is essential in $E_\sigma(K)$, $xL = 0$ holds. Therefore there exists a nonzero element r of L such that $xr = 0$. Then $0 = f(0) = f(xr) = f(x)r$, and so $r \in L \cap (0 : f(x)) = 0$, this is a contradiction to the fact that $r \neq 0$. Thus $xL \cap K \neq 0$ holds.

Thus there exists $0 \neq r \in L$ such that $0 \neq xr \in K$. If $f(xr) = 0$, then $r \in L \cap (0 : f(x)) = 0$, this is a contradiction to $r \neq 0$. Therefore it follows that $f(xr) \neq 0$, and so $f(K) \neq 0$ for $f(K) \ni f(xr)$. Since $A \in \mathcal{T}_\sigma$ and $E_\sigma(A)/A \in \mathcal{T}_\sigma$, it follows that $E_\sigma(A) \in \mathcal{T}_\sigma$. And so $E_\sigma(A)/E_\sigma(K) \in \mathcal{T}_\sigma$. Thus the following exact sequence $0 \to E_\sigma(K) \to E_\sigma(A) \to E_\sigma(A)/E_\sigma(K) \to 0$ splits. Since $E_\sigma(K)$ is a direct summand of $E_\sigma(A)$, $f \in \mathrm{Hom}_R(E_\sigma(K), A)$ can be extended to $g \in \mathrm{Hom}_R(E_\sigma(A), A)$. Therefore $g(K) = f(K) \neq 0$ holds, but this is a contradiction to the fact that $K = k_A(E_\sigma(A))$. Thus $K = 0$, as desired. □

A module M is called a σ-essential extension of N if N is an essential submodule of M such that M/N is σ-torsion, and then N is also called as a σ-essential submodule of M.

Lemma 5. *Let σ be an idempotent radical. If M is a σ-essential extension of a module N, then $E_\sigma(M) = E_\sigma(N)$ holds. The converse holds if σ is a left exact radical.*

Proof. Let N be an σ-essential submodule of a module M. Consider the exact sequence $0 \to M/N \to E_\sigma(M)/N \to E_\sigma(M)/M \to 0$. Since $\mathcal{T}_\sigma$ is closed under taking extensions, $E_\sigma(M)/N \in \mathcal{T}_\sigma$. As $\mathcal{T}_\sigma$ is closed under taking factor modules, $E_\sigma(M)/E_\sigma(N) \in \mathcal{T}_\sigma$. Thus there exists a submodule K of $E_\sigma(M)$ such that $E_\sigma(M) = E_\sigma(N) \oplus K$. Since N is essential in M and M is essential in $E_\sigma(M)$, N is essential in $E_\sigma(M)$. As $E_\sigma(N) \cap K = 0$, $N \cap K = 0$, and so $K = 0$ holds. Thus $E_\sigma(M) = E_\sigma(N)$ holds.

The converse is clear since $M/N \hookrightarrow E_\sigma(M)/N = E_\sigma(N)/N \in \mathcal{T}_\sigma$. □

Proposition 6. *Let σ be an idempotent radical. Then the class of σ-QF-3′ modules is closed under taking σ-essential extensions.*

Proof. Let N be a σ-QF-3′ module and suppose that N is a σ-essential submodule of M. Then $E_\sigma(N) = E_\sigma(M)$ holds by Lemma 5, and so $E_\sigma(M) = E_\sigma(N) \hookrightarrow \Pi N \hookrightarrow \Pi M$, as desired. □

2. σ-left exact preradical and σ-hereditary torsion theories

It is well known that preradical t is left exact iff $t(N) = N \cap t(M)$ holds for any module M and any submodule N of M. In this section we generalize left exact preradicals by using torsion theories.

Let σ be a preradical. We call a preradical t σ-left exact if $t(N) = N \cap t(M)$ holds for any module M and any σ-dense submodule N of M. If a module A is σ-QF-3′ and $t = k_A$, then t is a σ-left exact radical. Now we characterize σ-left exact preradicals.

Lemma 7. *For a preradical t and an idempotent radical σ, let $t_\sigma(M)$ denote $M \cap t(E_\sigma(M))$ for any module M. Then $t_\sigma(M)$ is uniquely determined for any choice of $E(M)$.*

Proof. Let E_1 and E_2 be σ-injective hulls of a module M. Then there exists isomorphisms $g : E_1 \to E_2$ and $h : E_2 \to E_1$ such that $gh = 1_{E_2}$ and $hg = 1_{E_1}$ and $h|_M = g|_M = 1_M$. Now we get the following equation $M \cap t(E_1) = g(M \cap t(E_1)) \subseteq g(M) \cap g(t(E_1)) \subseteq M \cap t(E_2)$. By the same way $M \cap t(E_2) \subseteq M \cap t(E_1)$ holds. Therefore we conclude that $M \cap t(E_2) = M \cap t(E_1)$. □

Lemma 8. *Let t be a preradical and σ an idempotent radical. Then t_σ is a σ-left exact preradical.*

Proof. Let N be a submodule of a module M such that $M/N \in \mathcal{T}_\sigma$. Since M/N and $E_\sigma(M)/M$ is σ-torsion, it follows that $E_\sigma(M)/N \in \mathcal{T}_\sigma$, and so

$E_\sigma(M)/E_\sigma(N) \in \mathcal{T}_\sigma$. Since $E_\sigma(N)$ is σ-injective, there exists a submodule K of $E_\sigma(M)$ such that $E_\sigma(M) = E_\sigma(N) \oplus K$. Then $E_\sigma(N) \cap t(E_\sigma(M)) = E_\sigma(N) \cap \{t(E_\sigma(N)) \oplus t(K)\} = t(E_\sigma(N)) \oplus (E_\sigma(N) \cap t(K)) = t(E_\sigma(N))$ by modular law. Therefore $t_\sigma(N) = N \cap t(E_\sigma(N)) = N \cap E_\sigma(N) \cap t(E_\sigma(M)) = N \cap t(E_\sigma(M)) = N \cap M \cap t(E_\sigma(M)) = N \cap t_\sigma(M)$. □

Theorem 9. *Let σ be an idempotent radical. We consider the following conditions on a preradical t. Then the implications (5)←(1)⇔(2)→(3)⇔(4) hold. If t is a radical, then (4)→(1) holds. If t is an idempotent preradical and σ is left exact, then(5)(i)→(1)holds. Thus if t is an idempotent radical and σ is a left exact radical, then all conditions are equivalent.*

(1) t is a σ-left exact preradical.

(2) $t(M) = M \cap t(E_\sigma(M))$ holds for any module M.

(3) $\mathcal{F}_t$ is closed under taking σ-essential extensions.

(4) $\mathcal{F}_t$ is closed under taking σ-injective hulls.

(5) (i) $\mathcal{T}_t$ is closed under taking σ-dense submodules.

(ii) $\mathcal{F}_t$ is closed under taking σ-extensions.

Proof. (1)→(2): It is clear, by $E_\sigma(M)/M \in \mathcal{T}_\sigma$.

(2)→(1): Let N be a σ-dense submodule of a module M. Since $M/N \in \mathcal{T}_\sigma$ and $E_\sigma(M)/M \in \mathcal{T}_\sigma$, it follows that $E_\sigma(M)/N \in \mathcal{T}_\sigma$, and so

$E_\sigma(M)/E_\sigma(N) \in \mathcal{T}_\sigma$. Thus there exists a submodule K such that

$E_\sigma(M) = E_\sigma(N) \oplus K$. Then $E_\sigma(N) \cap t(E_\sigma(M)) = E_\sigma(N) \cap \{t(E_\sigma(N) \oplus t(K)\} = t(E_\sigma(N)) \oplus \{E_\sigma(N) \cap t(K)\} = t(E_\sigma(N))$ by modular law. Then $t(N) = N \cap t(E_\sigma(N)) = N \cap E_\sigma(N) \cap t(E_\sigma(M)) = N \cap t(E_\sigma(M)) = N \cap M \cap t(E_\sigma(M)) = N \cap t(M)$

(1)→(3): Let $N \in \mathcal{F}_t$ be a σ-essential submodule of a module M. Then $0 = t(N) = N \cap t(M)$, and so $t(M) = 0$, as desired.

(3)→(4): This is clear, since M is σ-essential in $E_\sigma(M)$ for any module M.

(4)→(3): Let $N \in \mathcal{F}_t$ be a σ-essential submodule of a module M. It holds that $E_\sigma(N) \in \mathcal{F}_t$ by the assumption and that $E_\sigma(M) = E_\sigma(N)$ by Lemma 5. Thus it follows that $E_\sigma(M) \in \mathcal{F}_t$. Therefore $M \in \mathcal{F}_t$ since $\mathcal{F}_t$ is closed under taking submodules.

(1)→(5):(i) Let $M \in \mathcal{T}_t$ and N a σ-dense submodule of M. Then $t(N) = N \cap t(M) = N \cap M = N$, as desired.

(ii) Assume that $N \in \mathcal{F}_t$ and $M/N \in \mathcal{F}_t \cap \mathcal{T}_\sigma$. Then $0 = t(M/N) \supseteq (t(M) + N)/N$, and so $N \supseteq t(M)$. By the assumption $0 = t(N) = N \cap t(M) = t(M)$, as desired.

(4)→(1): We assume that t is a radical. Let N be a σ-dense submodule of M. Consider the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{g} & M & \longrightarrow & M/N & \longrightarrow & 0 \\ & & \Big\downarrow j & & \Big\downarrow f & & & & \\ 0 & \longrightarrow & N/t(N) & \xrightarrow{i} & E_\sigma(N/t(N)), & & & & \end{array}$$

where g and i are the inclusion maps, j is the canonical homomorphism and f is a homomorphism determined by the σ-injectivity of $E_\sigma(N/t(N))$.

Since t is a radical, $N/t(N) \in \mathcal{F}_t$. By the assumption $E_\sigma(N/t(N)) \in \mathcal{F}_t$. Then it follows that $f(t(M)) \subseteq t(E_\sigma(N/t(N))) = 0$, and so $t(M) \subseteq \ker f$. Let $f|_N$ be a restriction map of f to N. Then it follows that $t(N) = \ker j = \ker f|_N = N \cap \ker f \supseteq N \cap t(M) \supseteq t(N)$, and so $t(N) = N \cap t(M)$, as desired.

(5)→(1): We assume that t is an idempotent radical and σ is left exact. We know that $\mathcal{F}_t$ is closed under taking extensions since t is an idempotent preradical. And so we use the condition (i) only. Let N be a σ-dense submodule of M. Since $t(M)/(N \cap t(M)) \simeq (t(M) + N)/N \subseteq M/N \in \mathcal{T}_\sigma$, $N \cap t(M)$ is a σ-dense submodule of $t(M) \in \mathcal{T}_t$. Therefore $N \cap t(M) \in \mathcal{T}_t$ holds. Thus it follows that $t(N) \subseteq N \cap t(M) = t(N \cap t(M)) \subseteq t(N)$, and so $t(N) = N \cap t(M)$, as desired.

□

Proposition 10. *Let σ be a left exact preradical and t a preradical. Then the following conditions are equivalent.*

(1) For any submodule N of any module M such that $t(M) \supseteq N$ and $t(M)/N \in \mathcal{T}_\sigma$, it follows that N is in $\mathcal{T}_t$.

(2) t is an idempotent preradical and a σ-left exact preradical.

Proof. (1)→(2): In (1) we use $t(M)$ instead of N, then it is concluded that t is idempotent preradical. Next in (1) we use $N \cap t(M)$ instead of N, for $t(M)/(N \cap t(M)) \simeq (N + t(M))/N \subseteq M/N \in \mathcal{T}_\sigma$. Thus $N \cap t(M) \in \mathcal{T}_t$ holds, and so $t(N) \supseteq t(N \cap t(M)) = N \cap t(M) \supseteq t(N)$. Therefore $t(N) = N \cap t(M)$ holds.

(2)→(1): Consider the exact sequence $0 \to N \to t(M) \to t(M)/N \to 0$, where $t(M)/N \in \mathcal{T}_\sigma$. By the assumption $t(N) = N \cap t(t(M)) = N \cap t(M) = N$, as desired. □

A torsion theory for $\mathcal{C}$ is a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects of $\mathcal{C}$ such that
(i) $\mathrm{Hom}_R(T, F) = 0$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$
(ii) $\mathrm{Hom}_R(M, F) = 0$ for all $F \in \mathcal{F}$, then $M \in \mathcal{T}$
(iii) $\mathrm{Hom}_R(T, N) = 0$ for all $T \in \mathcal{T}$, then $N \in \mathcal{F}$
We put $t(M) = \sum_{\mathcal{T} \ni N \subset M} N$ $(= \cap_{M/N \in \mathcal{F}} N)$, then $\mathcal{T} = \mathcal{T}_t$ and $\mathcal{F} = \mathcal{F}_t$ hold.

For a torsion theory $(\mathcal{T}, \mathcal{F})$, if $\mathcal{T}$ is closed under taking submodules, then $(\mathcal{T}, \mathcal{F})$ is called a hereditary torsion theory. It is well known that $\mathcal{T}$ is closed under taking submodules if and only if $\mathcal{F}$ is closed under taking injective hulls. Now we call $(\mathcal{T},\mathcal{F})$ a σ-hereditary torsion theory if $\mathcal{T}$ is closed under taking σ-dense submodules, where σ is a preradical. If σ is a left exact radical, $\mathcal{T}$ is closed under taking σ-dense submodules if and only if $\mathcal{F}$ is closed under taking σ-injective hulls by Theorem 9.

Proposition 11. *Let t be an idempotent preradical and σ a radical such that $\mathcal{F}_\sigma \subseteq \mathcal{F}_t$. If $\mathcal{F}_t$ is closed under taking σ-injective hulls, then $\mathcal{F}_t$ is closed under taking injective hulls.*

Proof. Let M be a module in $\mathcal{F}_t$. Then it follows that $E(M)/E_\sigma(M) \simeq (E(M)/M)/\sigma(E(M)/M) \in \mathcal{F}_\sigma \subseteq \mathcal{F}_t$, and so $0 = t(E(M)/E_\sigma(M)) \supseteq (t(E(M)) + E_\sigma(M))/E_\sigma(M)$. Thus $t(E(M)) \subseteq E_\sigma(M) \in \mathcal{F}_t$, and so $0 = t(t(E(M))) = t(E(M))$. Therefore it follows that $E(M) \in \mathcal{F}_t$. □

Proposition 12. *If $\sigma(M) \supseteq Z(M)$ for any module M, then a σ-left exact preradical is left exact, where σ is a preradical.*

Proof. Let t be a σ-left exact preradical. Since M is essential in $E(M)$ for a module M, it follows that $E(M)/M = Z(E(M)/M) \subseteq \sigma(E(M)/M)$. So it holds that $E(M)/M \in \mathcal{T}_\sigma$. Thus $t(M) = M \cap t(E(M))$ holds since t is σ-left exact. If we use Lemma 8 for $\sigma = 1$, we find that t is a left exact preradical. □

Theorem 13. *Let σ be a left exact radical and $(\mathcal{T},\mathcal{F})$ a torsion theory. Suppose that there exists $Q \in \mathcal{F}$ such that $\mathcal{T} = \{M \in \text{Mod-}R : \text{Hom}_R(M, Q) = 0\}$. Then $(\mathcal{T},\mathcal{F})$ is σ-hereditary if and only if $\mathcal{T} = \{M \in \text{Mod-}R : \text{Hom}_R(M, E_\sigma(Q)) = 0\}$*

Proof. Suppose that $\mathcal{T} = \{M \in \text{Mod-}R : \text{Hom}_R(M, E_\sigma(Q)) = 0\}$. Since it is easily verified that $\mathcal{T}$ is closed under taking factor modules, direct sums and extensions, $\mathcal{T}$ is a torsion part of some torsion theory. Thus it is sufficient to be proved that $\mathcal{T}$ is closed under taking σ-dense submodules. Let M be a module in $\mathcal{T}$ and N be a σ-dense submodule of M. Consider the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N & \longrightarrow & 0 \\ & & {\scriptstyle f}\big\downarrow & & & & & & \\ & & E_\sigma(Q) & \subseteq & \Pi Q & \longrightarrow & Q & & \end{array}$$

For any nonzero homomorphism f from N to $E_\sigma(Q)$, f is extended to a nonzero homomorphism g from M to $E_\sigma(Q)$. Since $E_\sigma(Q)$ is Q-torsionless, there exists a nonzero homomorphism h from $E_\sigma(Q)$ to Q such that hg is a nonzero homomorphism from M to Q, which is a contradiction. Thus $\text{Hom}_R(N, E_\sigma(Q)) = 0$ and so $N \in \mathcal{T}$. Therefore $\mathcal{T}$ is closed under taking σ-dense submodules.

Conversely suppose that $\mathcal{T}$ is closed under taking σ-dense submodules. Let t be a σ-left exact idempotent radical associated with $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{T} = \mathcal{T}_t$ and $\mathcal{F} = \mathcal{F}_t$. By Theorem 9, $\mathcal{F}$ is closed under taking σ-injective hulls if and only if $\mathcal{T}$ is closed under taking σ-dense submodules. Since $Q \in \mathcal{F}$ and $\mathcal{F}$ is closed under taking σ-injective hulls, it follows that $E_\sigma(Q) \in \mathcal{F}$.

Next we show that $\mathcal{T} = \{M : \mathrm{Hom}_R(M, E_\sigma(Q)) = 0\}$.

If $M \in \mathcal{T}$, then $\mathrm{Hom}_R(M, E_\sigma(Q)) = 0$ since $E_\sigma(Q) \in \mathcal{F}$. Thus it follows that $\mathcal{T} \subseteq \{M : \mathrm{Hom}_R(M, E_\sigma(Q)) = 0\}$.

Conversely suppose that $\mathrm{Hom}_R(M, E_\sigma(Q)) = 0$. Since $0 \to Q \to E_\sigma(Q)$, it follows that $0 \to \mathrm{Hom}_R(M, Q) \to \mathrm{Hom}_R(M, E_\sigma(Q))$, and so $\mathrm{Hom}_R(M, Q) = 0$. Thus it holds that $M \in \mathcal{T}$. Therefore it follows that $\mathcal{T} = \{M : \mathrm{Hom}_R(M, E_\sigma(Q)) = 0\}$. □

Proposition 14. *Let σ be a left exact radical and $(\mathcal{T}, \mathcal{F})$ be a σ-hereditary torsion theory, where $\mathcal{T} = \{M \in \text{Mod-}R : \mathrm{Hom}_R(M, Q) = 0\}$ for some σ-QF-3′ module Q in $\mathcal{F}$. Let M be in $\mathcal{T}_\sigma$. Then M is in $\mathcal{F}$ if and only if M is contained in a direct product of some copies of Q.*

Proof. Let M be a nonzero module in $\mathcal{F} \cap \mathcal{T}_\sigma$ and x a nonzero element in M. Then xR is in $\mathcal{F}$. If xR is in $\mathcal{T}$, $xR \in \mathcal{F} \cap \mathcal{T} = \{0\}$, a contradiction. Thus it holds that $xR \notin \mathcal{T} = \{M : \mathrm{Hom}_R(M, Q) = 0\}$, and so there exists a nonzero $h \in \mathrm{Hom}_R(xR, Q)$. Consider the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & xR & \xrightarrow{i} & M & \longrightarrow & M/xR & \longrightarrow & 0 \\
 & & h\big\downarrow & & \big\downarrow f_x & & & & \\
0 & \longrightarrow & Q & \xrightarrow{j} & E_\sigma(Q) & \hookrightarrow & \Pi Q & \longrightarrow & Q,
\end{array}$$

where i and j are the inclusion maps f_x is induced by the σ-injectivity of $E_\sigma(Q)$ since $M/xR \in \mathcal{T}_\sigma$. By considering the above diagram we can find that there exists a nonzero $f'_x : M \to Q$ and $s : Q \to Q$ such that $sh(x) = f'_x(x) \neq 0$. Let $g : M \to \prod_{x \in M - \{0\}} Q_x$ be a homomorphism such that $g(y) = (f'_x(y))$. Then clearly $g(y) \neq 0$ if $y \neq 0$. Hence g is a monomorphism. Thus $M \hookrightarrow \Pi Q$.

Conversely If M is contained in a direct product of copies of Q, M is in $\mathcal{F}$, since $Q \in \mathcal{F}$ and $\mathcal{F}$ is closed under taking products and submodules. □

Acknowledgement

The author would like to thank the referee for many valuable suggestions and advices of this paper, especially in Theorem 13.

References

[1] L. Bican, *QF-3′ modules and rings, Commentationes Mathematicae*, Universitatis Carolinae, 14 (1973), 295-301.
[2] R. R. Colby and E. A. Rutter Jr., *Semi-primary QF-3′ rings*, Nagoya Math. J. 32 (1968), 253-258.
[3] S. E. Dickson, *A torsion theory for abelian categories*, Trans. Amer. Math. Soc. 121 (1966), 223-235.
[4] J. S. Golan, *Localization of Noncommutative Rings*, Marcel Dekker, New York, 1975.
[5] K. R. Goodearl, *Ring Theory*, Marcel Dekker, New York, 1976.
[6] J. P. Jans, H. Y. Mochizuki and L. E. T. Wu, *A characterization of QF-3 rings*, Nagoya Math. J. 27 (1966), 7-13.
[7] J. P. Jans, H. Y. Mochizuki, *Torsion associated with duality*, Tohoku Math. J. 32 (1972), 449-452.
[8] Y. Kurata and H. Katayama, *On a generalization of QF-3′ rings*, Osaka Journal of Mathematics, 13 (1976), 407-418.
[9] K. Ohtake, *Colocalization and Localization*, Journal of pure and applied algebra 11 (1977), 217-241.
[10] K. Ohtake, *Equivalence between colocalization and localization in abelian categories with applications to the theory of modules*, Journal of Algebra 79 (1982), 169-205.
[11] Bo Stenstrom, *Rings and Modules of Quotients*, Springer -Verlag, Berlin, 1975.
[12] C. Vinsonhaler, *A note on two generalizations of QF-3*, Pac. J. Math. 40 (1972), 229-233.

Yasuhiko TAKEHANA
General Education
Hakodate National College of Technology
14-1, Tokura-cho, Hakodate-shi, Hokkaido, 042-8501 Japan
e-mail address: takehana@hakodate-ct.ac.jp

(*Received December 11, 2009*)
(*Revised May 21, 2010*)