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THE UNIFORM EXPONENTIAL STABILITY OF LINEAR SKEW-PRODUCT SEMIFLOWS ON REAL HILBERT SPACE

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Abstract. The goal of the paper is to present some characterizations for the uniform exponential stability of linear skew-product semiflows on real Hilbert space.

1. Introduction:

In recent years, the classical ideas of exponential stability and other asymptotic properties concerning evolution equations in infinite dimensional Banach space have witnessed significant development. The techniques used in studying became very various and complex: input-output of characterization relative to integral equations and to difference equations have been obtained in [1], [4], [9], ... ; discrete-time methods have been developed in [6], ... ; and also the theory of linear skew-product semiflow (LSPS) on function spaces has been found.

This paper considers the concepts of LSPS and give conditions about the uniform exponential stability of LSPS on real Hilbert space. Let X be the Banach space, let $(\circleddash, d)$ be the metric space. In what follows, we denote by $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators acting on X, $\mathbb{R}_+ := [0, \infty)$, $\mathbb{N} := \{0, 1, 2, ...\}$.

Definition 1.1. Continuous mapping $\sigma : \circleddash \times \mathbb{R}_+ \to \circleddash$ is called a semiflow on $\circleddash$ if $\sigma(\theta, 0) = \theta$ and $\sigma(\theta, t+s) = \sigma(\sigma(\theta, s), t)$, for all $(\theta, s, t) \in \circleddash \times \mathbb{R}_+^2$.

Definition 1.2. $\pi = (\Phi, \sigma)$ is called a linear skew-product semiflow on $\mathcal{E} = X \times \circleddash$ if σ is a semiflow on $\circleddash$ and $\Phi : \circleddash \times \mathbb{R}_+ \to \mathcal{L}(X)$ satisfies the following conditions:

(1) $\Phi(\theta, 0) = I$, the identity operator on X, for all $\theta \in \circleddash$.
(2) $\Phi(\theta, t+s) = \Phi(\sigma(\theta, t), s)\Phi(\theta, t)$, for all $(\theta, t, s) \in \circleddash \times \mathbb{R}_+^2$.
(3) $\lim\limits_{t \to 0^+} \Phi(\theta, t)x = x$, uniformly in θ.

Remark (See 11). If $\pi = (\Phi, \sigma)$ is a linear skew-product semiflow then there are M, ω such that:
$$\|\Phi(\theta, t)x\| \leq Me^{\omega t}\|x\|,$$

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for all $(\theta, t, x) \in \ominus \times \mathbb{R}_+ \times X$.

The mapping Φ given Definition 1.2 is called the cocycle associated to the linear skew-product semiflow π.

Example 1.1. It is easy to prove that C_0-semigroups, evolution families are particular cases of linear skew-product semiflows.

Example 1.2 (See 7). Let σ be a semiflow on the compact Hausdorff space $\ominus$ and $\{T(t)\}_{t\geq 0}$ a C_0-semigroup on the Banach space X. For every strongly continuous mapping:

$$F : \ominus \; \to \; \mathcal{L}(X),$$

there is a linear skew-product semiflow $\pi_F = (\Phi_F, \sigma)$ on $\mathcal{E} = X \times \ominus$ such that:

$$\Phi_F(\theta, t)x \; = \; T(t)x \; + \int_0^t T(t-s)F(\sigma(\theta, s))\Phi_F(\theta, s)x \, ds,$$

The linear skew-product semiflow $\pi_F = (\Phi_F, \sigma)$ is called the linear skew-product semiflow generated by the triplet (T, F, σ).

Classical examples of cocycles appear as operator solutions for variational equations.

Example 1.3. Let $\ominus$ be a compact metric space, σ a semiflow and $A : \ominus \to \mathcal{L}(X)$ a continuous map. If $\Phi(\theta, t)x$ is the solution of the anstract Cauchy problem:

$$\begin{cases} u'(t) \; = \; A(\sigma(\theta, t))u(t), \\ u(0) \; = \; x, \end{cases}$$

then the pair $\pi = (\sigma, \Phi)$ is a linear skew-product semiflow.

The well-known theorem of Lyapunov states that if A is an $n \times n$ complex matrix then A has all its characteristic roots with real parts negative if and only if for any positive definite Hermitian H there exists an unique definite Hermitian matrix B satisfying

$$A^*B + BA = -H, \qquad (L).$$

Following Lyapunov's idea, the paper extends in a natural way to linear skew-product semiflows. Indeed, from the equation (L), we have:

$$< A(\sigma(\theta, t))x, Wx > + < Wx, A(\sigma(\theta, t))x >= - \|x\|^2 , \;\; (L^*)$$

Assume that (L^*) holds for some conditions, let f be the function defined by

$$f(t) =< W\Phi(\theta, t)x, \Phi(\theta, t)x > .$$

One can easily see that $f'(t) = -\left\|\Phi(\theta,t)x\right\|^2$. Integrating with respect to τ on $[s,t]$, we have

$$< W\Phi(\theta,t)x, \Phi(\theta,t)x > - < W\Phi(\theta,s)x, \Phi(\theta,s)x >= -\int_s^t \left\|\Phi(\theta,\tau)x\right\|^2 \, d\tau,$$

which implies

$$\Phi^*(\theta,t)W\Phi(\theta,t)x + \int_s^t \Phi^*(\theta,\tau)\Phi(\theta,\tau)x \, d\tau = \Phi^*(\theta,s)W\Phi(\theta,s)x.$$

In next section, we establish the uniform exponential stability of linear skew-product semiflows and some equation.

Definition 1.3. A linear skew-product semiflow $\pi = (\Phi,\sigma)$ is said to be uniformly exponentially stable if and only if there exist $K,\nu > 0$ such that:

$$\left\|\Phi(\theta,t)x\right\| \ \leq \ Ke^{-\nu t}\left\|x\right\|,$$

for all $(\theta,t) \in \ominus \times \mathbb{R}_+$

2. Main Results:

2.1. Discrete characterizations for the uniform exponential stability of linear skew-product semiflows.

Definition 2.1. A map $H : \ominus \times \mathbb{N} \to \mathcal{L}(X)$ is called positive if

$$< H(\theta,m)x, x > \geq \ 0,$$

for all $m \in \mathbb{N}$ and $x \in X$; $\theta \in \ominus$.

Let $\mathcal{M}$ be the set of all positive maps H defined in Definition 2.1 with the property

$$\sup_{\theta \in \ominus} \left\|H(\theta,0)\right\| \ < \ \infty.$$

Definition 2.2. A map $H : \ominus \times \mathbb{N} \to \mathcal{L}(X)$ is called uniformly positive if there exists the constant $a > 0$ such that

$$< H(\theta,m)x, x > \geq \ a\left\|x\right\|^2,$$

for all $m \in \mathbb{N}$ and $x \in X$; $\theta \in \ominus$

Let $\mathcal{M}^*$ be the set of all positive maps H defined in Definition 2.2.

Lemma 2.1. *If there are $t_0 \ > \ 0$ and $c \in (0,1)$ such that:*

$$\sup_{\theta \in \ominus} \left\|\Phi(\theta,t_0)\right\| \ \leq \ c,$$

then linear skew-product semiflow $\pi = (\Phi, \sigma)$ is uniformly exponentially stable.

Proof: For every $t \in \mathbb{R}_+$, there are $k \in \mathbb{N}$ and $s \in [0, t_0)$ such that $t = kt_0 + s$. An easy computation shows that

$$\Phi(\theta, kt_0) = \prod_{i=1}^{k} \Phi(\sigma(\theta, (i-1)t_0), t_0),$$

$$\begin{aligned}\Phi(\theta, kt_0 + s) &= \Phi(\sigma(\theta, s), kt_0)\Phi(\theta, s) \\ &= \prod_{i=1}^{k} \Phi(\sigma(\sigma(\theta, s), (i-1)t_0), t_0)\Phi(\theta, s).\end{aligned}$$

Hence it follows that π is uniformly exponentially stable with $\nu = -\dfrac{|lnc|}{t_0}$ and $K = e^{\omega t_0 - lnc}M$ from

$$\begin{aligned}\|\Phi(\theta, kt_0 + s)x\| &\leq c^k M e^{\omega s} \|x\| \\ &< e^{klnc} M e^{\omega t_0} \|x\| \\ &< e^{(\frac{t}{t_0} - 1)lnc} M e^{\omega t_0} \|x\| .\end{aligned}$$

Lemma is proved. $\square$

Lemma 2.2. *If there exists $C > 0$ such that*

$$\sum_{k=0}^{\infty} \|\Phi(\theta, k)x\|^2 \leq C \|x\|^2 < \infty,$$

for all $\theta \in \ominus$, then there is n_0 such that

$$\|\Phi(\theta, n_0)\| \leq \frac{1}{2},$$

for all $\theta \in \ominus$.

Proof: By hypothesis, we get that

$$\|\Phi(\theta, k)\|^2 \leq C,$$

for every $\theta \in \ominus$ and $k \in \mathbb{N}$. From this inequality, we obtain

$$\begin{aligned}(n+1)\|\Phi(\theta, n)x\|^2 &= \sum_{k=0}^{n} \|\Phi(\theta, n)x\|^2 \\ &= \sum_{k=0}^{n} \|\Phi(\sigma(\theta, k), n-k)\Phi(\theta, k)x\|^2\end{aligned}$$

$$\begin{aligned} &\leq C\sum_{k=0}^{n}\|\Phi(\theta,k)x\|^2 \\ &\leq C\sum_{k=0}^{\infty}\|\Phi(\theta,k)x\|^2 \\ &\leq C^2\|x\|^2. \end{aligned}$$

Hence, it follows

$$\|\Phi(\theta,n)\| \leq \frac{C}{\sqrt{n+1}}.$$

On the other hand

$$\lim_{n\to+\infty}\frac{C}{\sqrt{n+1}} = 0.$$

Then there exists $n_0 \in \mathbb{N}$ such that

$$\|\Phi(\theta,n_0)\| \leq \frac{1}{2},$$

for all $\theta \in \ominus$. Lemma is proved. $\square$

Theorem 2.3. *$\pi = (\Phi,\sigma)$ is uniformly exponentially stable if and only if there exist $H \in \mathcal{M}^*$ and $W \in \mathcal{M}$ such that*

$$\begin{aligned} 0 &= W(\theta,0)x - \Phi^*(\theta,n)W(\theta,n)\Phi(\theta,n)x \\ &- \sum_{k=0}^{n-1}\Phi^*(\theta,k)H(\theta,n)\Phi(\theta,k)x, \qquad (\mathbb{L}) \end{aligned}$$

for every $n \geq 1$, x, θ.

Proof:
Sufficiency:
Definition 1.3 guarantees that there are $K,\nu > 0$ such that

$$\|\Phi(\theta,t)\| \leq Ke^{-\nu t}.$$

Let $H,W : \ominus \times \mathbb{N} \to \mathcal{L}(X)$ be respectively given by

$$H(\theta,m) = I,$$

$$W(\theta,m) = \sum_{k=0}^{\infty}\Phi^*(\sigma(\theta,m),k)\Phi(\sigma(\theta,m),k).$$

Here, it follows that $W(\theta,m)$ is well defined from the uniform convergence with respect to $\theta \in \ominus$ and $m \in \mathbb{N}$:

$$\|W(\theta,0)\| \leq \sum_{k=0}^{\infty}\|\Phi(\theta,k)\|^2$$

$$\begin{aligned} &\leq \sum_{k=0}^{\infty} K^2 e^{-2\nu k} \\ &= \frac{K^2}{1-e^{-2\nu}}. \end{aligned}$$

It is easy to check that

$$< H(\theta, m)x, x > \geq \|x\|^2 .$$

This says that

$$\sup_{\theta \in \ominus} \|W(\theta, 0)\| < \infty.$$

The uniform positivity of H and the positivity of W follow directly from their individual definitions. Therefore, we can conclude that $H \in \mathcal{M}^*$ and $W \in \mathcal{M}$.
Necessity:
Assume that $H \in \mathcal{M}^*$ and $W \in \mathcal{M}$ satisfying $(\mathbb{L})$. From the condition $W \in \mathcal{M}$, we can put

$$K := \sup_{\theta \in \ominus} \|W(\theta, 0)\| .$$

Now, let a be the constant defined in Definition 2.2, this means:

$$< H(\theta, m)x, x > \geq a \|x\|^2 ,$$

for all $m \in \mathbb{N},\ x \in X,\ \theta \in \ominus$. Using the uniform positivity of H, the equation $(\mathbb{L})$ and the uniform boundedness of $W(.,0)$, we obtain:

$$\begin{aligned} &\sum_{k=0}^{n-1} \|\Phi(\theta, k)x\|^2 \\ \leq\ & \sum_{k=0}^{n-1} \frac{1}{a} < H(\theta, n)\Phi(\theta, k)x, \Phi(\theta, k)x > \\ =\ & \frac{1}{a} < \sum_{k=0}^{n-1} \Phi^*(\theta, k)H(\theta, n)\Phi(\theta, k)x, x > \\ =\ & \frac{1}{a} < W(\theta, 0)x, x > \\ -\ & \frac{1}{a} < \Phi^*(\theta, n)W(\theta, n)\Phi(\theta, n)x, x > \\ \leq\ & \frac{1}{a} < W(\theta, 0)x, x > \leq \frac{K}{a} \|x\|^2 . \end{aligned}$$

Hence, it folows

$$\sum_{k=0}^{\infty} \|\Phi(\theta, k)x\|^2 \le \frac{K}{a} \|x\|^2 < \infty.$$

Applying Lemma 2.1, 2.2, we get that π is uniformly exponentially stable. □

2.2. **Continuous characterizations for the uniform exponential stability of linear skew-product semiflows.**

Definition 2.3. A map $H : \ominus \times \mathbb{R}_+ \to \mathcal{L}(X)$ is called positive if:

$$< H(\theta, t)x, x > \ge 0,$$

for all $t \in \mathbb{R}_+$ and $x \in X$; $\theta \in \ominus$.

Let $\mathbb{M}$ be the set of positive maps defined in Definition 2.3 with the property

$$\sup_{(\theta,\, t) \in\, \ominus \times \mathbb{R}_+} \|H(\theta, t)\| < \infty.$$

Definition 2.4. A map $H : \ominus \times \mathbb{R}_+ \to \mathcal{L}(X)$ is called uniformly positive if there exists the constant $a > 0$ such that

$$< H(\theta, t)x, x > \ge a \|x\|^2,$$

for all $t \in \mathbb{R}_+$ and $x \in X$; $\theta \in \ominus$

Let $\mathbb{M}^*$ be the set of positve maps H defined in Definition 2.4.

Lemma 2.4. *π is uniformly exponentially stable if and only if there is L such that:*

$$\int_t^{\infty} \|\Phi(\theta, \tau)x\|^2 \, d\tau \le L \|\Phi(\theta, t)x\|^2,$$

for all $\theta \in \ominus$, $x \in X$, $t \in \mathbb{R}_+$.

Proof: The necessity is obvious. Now, we prove the sufficiency. Put

$$\psi(t) = M^2 e^{\omega 2t} ; \ \frac{1}{L_0} = \int_0^1 \frac{1}{\psi(\tau)} \, d\tau.$$

Step 1. We prove there is L_1 such that

$$\|\Phi(\theta, t)x\| \le L_1 \|\Phi(\theta, s)x\|,$$

for all $t \ge s \ge 0$, $x \in X$ and $\theta \in \ominus$. Indeed, we consider two possibilities. If $t \in [s; s+1]$, it follows easily

$$\|\Phi(\theta, t)x\| \le M e^{\omega(t-s)} \|\Phi(\theta, s)x\|$$

$$\leq \quad Me^{\omega} \|\Phi(\theta,s)x\| .$$

If $t \geq s+1$

$$\begin{aligned}
\frac{\|\Phi(\theta,t)x\|^2}{L_0} &= \int_0^1 \frac{\|\Phi(\theta,t)x\|^2}{\psi(\tau)}\, d\tau \\
&= \int_{t-1}^{t} \frac{\|\Phi(\theta,t)x\|^2}{\psi(t-\tau)}\, d\tau \\
&\leq \int_{s}^{t} \frac{\|\Phi(\theta,t)x\|^2}{\psi(t-\tau)}\, d\tau \\
&\leq \int_{s}^{t} \|\Phi(\theta,\tau)x\|^2 \, d\tau \\
&\leq \int_{s}^{\infty} \|\Phi(\theta,\tau)x\|^2 \, d\tau \\
&\leq L. \|\Phi(\theta,s)x\|^2 ,
\end{aligned}$$

$$\|\Phi(\theta,t)x\| \leq \sqrt{L_0 L}\, \|\Phi(\theta,s)x\| .$$

Hence, Step 1 is proved by choosing

$$L_1 := \max\{Me^{\omega} \,;\, \sqrt{L_0 L}\,\}.$$

Step 2 We prove that Lemma 2.1 works. Indeed, notice that

$$\begin{aligned}
(t+1)\|\Phi(\theta,t)x\|^2 &= \int_0^t \|\Phi(\theta,t)x\|^2 \, d\tau + \|\Phi(\theta,t)x\|^2 \\
&\leq L_1^2 \int_0^t \|\Phi(\theta,\tau)x\|^2 \, d\tau + L_1^2 \|x\|^2 \\
&\leq L_1^2 \int_0^\infty \|\Phi(\theta,\tau)x\|^2 \, d\tau + L_1^2 \|x\|^2 \\
&\leq L_1^2 L \|x\|^2 + L_1^2 . \|x\|^2 \\
&= (L_1^2 L + L_1^2)\|x\| ,
\end{aligned}$$

for all $t \geq 0,\ x \in X$. Hence, it follows

$$\|\Phi(\theta,t)x\| \leq \frac{L_1\sqrt{L+1}}{\sqrt{t+1}}\|x\|.$$

Since $\lim\limits_{t\to\infty}\dfrac{L_1\sqrt{L+1}}{\sqrt{t+1}} = \infty$, we get that there are t_0, c in Lemma 2.1. Lemma is proved. $\square$

Theorem 2.5. $\pi = (\Phi,\sigma)$ *is uniformly exponentially stable if and only if there exist* $H \in \mathbb{M}^*$ *and* $W \in \mathbb{M}$ *such that*

$$\begin{aligned} 0 &= \Phi^*(\theta,s)W(\theta,s)\Phi(\theta,s)x - \Phi^*(\theta,t)W(\theta,t)\Phi(\theta,t)x \\ &- \int_s^t \Phi^*(\theta,\tau)H(\theta,\tau)\Phi(\theta,\tau)x\,d\tau, \qquad (\mathcal{L}) \end{aligned}$$

for all $\theta \in \Theta$ *and* $t \geq s \geq 0$.

Proof:
Necessity.
From Definition 1.3, we get that there are $K,\nu > 0$ satisfying

$$\|\Phi(\theta,t)\| \leq Ke^{-\nu t},$$

for all $(\theta,t) \in \Theta \times \mathbb{R}_+$. Let $H,W : \Theta \times \mathbb{N} \to \mathcal{L}(X)$ given by

$$H(\theta,t) = I,$$

$$W(\theta,t) = \int_t^\infty \Phi^*(\sigma(\theta,t),\tau-t)\Phi(\sigma(\theta,t),\tau-t)\,d\tau.$$

It is easy to check that H,W satisfying the equation $\mathcal{L}$. On the other hand, from the equality $< H(\theta,t)x, x > = \|x\|^2$, we get the uniform positivity of H. And the uniform boundedness of $W(.,.)$ is gotten from the inequalities

$$\begin{aligned} \|W(\theta,t)\| &\leq \int_t^\infty \|\Phi(\sigma(\theta,t),\tau-t)\|^2\,d\tau \\ &\leq \int_t^\infty K^2e^{-2\nu(\tau-t)}\,d\tau \\ &< \infty. \end{aligned}$$

Sufficiency.
Assume that $H \in \mathbb{M}^*$ and $W \in \mathbb{M}$ satisfying the equation $\mathcal{L}$. Now, define

$$K := \sup_{(\theta,\,t)\in\,\Theta\times\mathbb{R}_+} \|W(\theta,t)\|.$$

Let a be the constant defined in Definition 2.4, this means:

$$< H(\theta,t)x, x > \geq \ a \left\| x \right\|^2 ,$$

for all $t \in \mathbb{R}_+,\ x \in X,\ \theta \in \ominus$. Using the uniform positivity of H, the equation ($\mathcal{L}$) and the uniform boundedness of W, we get the inequalities

$$\begin{aligned} & a \int_s^t \left\| \Phi(\theta,\tau)x \right\|^2 d\tau \\ \leq \quad & < \int_s^t \Phi^*(\theta,\tau) H(\theta,\tau)\Phi(\theta,\tau)x \, d\tau, x > \\ = \quad & < \Phi^*(\theta,s)W(\theta,s)\Phi(\theta,s)x, x > \\ - \quad & < \Phi^*(\theta,t)W(\theta,t)\Phi(\theta,t)x, x > \\ \leq \quad & < \Phi^*(\theta,s)W(\theta,s)\Phi(\theta,s)x, x > \\ \leq \quad & K \left\| \Phi(\theta,s)x \right\|^2 . \end{aligned}$$

Thus

$$\int_s^\infty \left\| \Phi(\theta,\tau)x \right\|^2 d\tau \ \leq \ \frac{K}{a} \ || \ \Phi(\theta,s)x \ ||^2 .$$

Applying Lemma 2.4, we get that π is uniformly exponentially stable.
$\square$

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