Math. J. Okayama Univ. **53** (2011), 101–109

# ON ALMOST $N$-SIMPLE-PROJECTIVES

Yoshitomo BABA and Takeshi YAMAZAKI

Abstract. The concept of almost $N$-projectivity is defined in [5] by M. Harada and A. Tozaki to translate the concept "lifting module" in terms of homomorphisms. In [6, Theorem 1] M. Harada defined a little weaker condition "almost $N$-simple-projecive" and gave the following relationship between them:

*For a semiperfect ring $R$ and $R$-modules $M$ and $N$ of finite length, $M$ is almost $N$-projective if and only if $M$ is almost $N$-simple-projective.*

We remove the assumption "of finite length" and give the result in Theorem 5 as follows:

*For a semiperfect ring $R$, a finitely generated right $R$-module $M$ and an indecomposable right $R$-module $N$ of finite Loewy length, $M$ is almost $N$-projective if and only if $M$ is almost $N$-simple-projective.*

We also see that, for a semiperfect ring $R$, a finitely generated $R$-module $M$ and an $R$-module $N$ of finite Loewy length, $M$ is $N$-simple-projective if and only if $M$ is $N$-projective.

Throughout this paper, we let $R$ be a semiperfect ring unless otherwise stated and $R$-modules unitary. For an $R$-module $M$, we denote the Loewy length and the composition length of $M$ by $L(M)$ and $|\,M\,|$, respectively.

Let $M$ and $N$ be $R$-modules. We say that $M$ is *$N$-projective* if, for any submodule $L$ of $N$ and an $R$-homomorphism $\varphi : M \to N/L$, there exists an $R$-homomorphism $\tilde{\varphi} : M \to N$ with $\nu\tilde{\varphi} = \varphi$, where $\nu : N \to N/L$ is the natural epimorphism. If, in this definition, we only consider the $R$-homomorphisms $\varphi$ with simple images, $M$ is said to be *$N$-simple-projective.*

First we give a lemma in which $N$-simple-projectivity is investigated for an $R$-homomorphism with its image semisimple artinian.

**Lemma 1.** *Let $R$ be a ring, $M$ and $N$ $R$-modules, $L$ a submodule of $N$ and $\varphi : M \to N/L$ an $R$-homomorphism with $\operatorname{Im}\varphi$ semisimple artinian. If $M$ is $N$-simple-projective, then there exists an $R$-homomorphism $\tilde{\varphi} : M \to N$ with $\nu\tilde{\varphi} = \varphi$, where $\nu : N \to N/L$ is the natural epimorphism.*

---

*Mathematics Subject Classification.* Primary 16D40; Secondary 16D10.

*Key words and phrases.* ring, module, almot projective, almost simple-projective.

The first author was supported by Grant-in-Aid for Scientific Research (No.19540029), JSPS.

*Proof.* Let $\operatorname{Im}\varphi = N'/L$, where $N'$ is a submodule of $N$. Then $M$ is $N'$-simple-projective by assumption. So the statement follows from [6, Lemma 1]. $\square$

Using Lemma 1 we obtain the following result which is a generalization of [6, Lemma 1]. And we also note that, in [2, Proposition 2], Baba and Oshiro gave the dual result which played an important role to characterize Fuller's theorem for injective modules.

**Proposition 2.** *Let $M$ be a finitely generated right $R$-module and $N$ a right $R$-module with $L(N) < \infty$. If $M$ is $N$-simple-projective, then $M$ is $N$-projective.*

*Proof.* Let $L$ be a submodule of $N$, $\varphi : M \to N/L$ an $R$-homomorphism and $\nu : N \to N/L$ the natural epimorphism. Since $L(N) < \infty$, there exists $n_1 \in \mathbb{N}$ such that $\operatorname{Im}\varphi \subseteq (N/L)J^{n_1-1}$ but $\operatorname{Im}\varphi \nsubseteq (N/L)J^{n_1}$. Then $(\operatorname{Im}\varphi + (N/L)J^{n_1})/(N/L)J^{n_1}$ is semisimple artinian since $R$ is semiperfect and $M$ is finitely generated. So, by Lemma 1, there exists an $R$-homomorphism $\tilde{\varphi}_1 : M \to N$ with $\nu_1\nu\tilde{\varphi}_1 = \nu_1\varphi$, where $\nu_1 : N/L \to (N/L)/(N/L)J^{n_1}$ is the natural epimorphism.

We assume that $\varphi \neq \nu\tilde{\varphi}_1$. Since $\operatorname{Im}(\varphi - \nu\tilde{\varphi}_1) \subseteq (N/L)J^{n_1}$, there exists $n_2 \in \mathbb{N}$ with $n_2 > n_1$, $\operatorname{Im}(\varphi - \nu\tilde{\varphi}_1) \subseteq (N/L)J^{n_2-1}$ but $\operatorname{Im}(\varphi - \nu\tilde{\varphi}_1) \nsubseteq (N/L)J^{n_2}$. Then $(\operatorname{Im}(\varphi - \nu\tilde{\varphi}_1) + (N/L)J^{n_2})/(N/L)J^{n_2}$ is semisimple artinian. So, by Lemma 1, there exists an $R$-homomorphism $\tilde{\varphi}_2 : M \to N$ with $\nu_2\nu\tilde{\varphi}_2 = \nu_2(\varphi - \nu\tilde{\varphi}_1)$, where $\nu_2 : N/L \to (N/L)/(N/L)J^{n_2}$ is the natural epimorphism.

We assume that $\varphi \neq \nu(\tilde{\varphi}_1 + \tilde{\varphi}_2)$. Since $\operatorname{Im}(\varphi - \nu(\tilde{\varphi}_1 + \tilde{\varphi}_2)) \subseteq (N/L)J^{n_2}$, we have $n_3 \in \mathbb{N}$ such that $n_3 > n_2$, $\operatorname{Im}(\varphi - \nu(\tilde{\varphi}_1 + \tilde{\varphi}_2)) \subseteq (N/L)J^{n_3-1}$ but $\operatorname{Im}(\varphi - \nu(\tilde{\varphi}_1 + \tilde{\varphi}_2)) \nsubseteq (N/L)J^{n_3}$.

Continuing this argument, we have $k \in \mathbb{N}$ with $\varphi = \nu(\tilde{\varphi}_1 + \cdots + \tilde{\varphi}_k)$ because $L(N) < \infty$. $\square$

Now we define "almost $N$-projective" and "almost $N$-simple-projective". Let $M$ and $N$ be $R$-modules. We say that $M$ is *almost $N$-projective* if, for any submodule $L$ of $N$ and an $R$-homomorphism $\varphi : M \to N/L$, letting $\nu : N \to N/L$ be the natural epimorphism, either the following (I) or (II) holds:

(I) There exists an $R$-homomorphism $\tilde{\varphi} : M \to N$ with $\nu\tilde{\varphi} = \varphi$.
(II) There exist a non-zero direct summand $N'$ of $N$ and an $R$- homomorphism $\tilde{\psi} : N' \to M$ with $\phi\tilde{\psi} = \nu|_{N'}$.

If, in this definition, we only consider the $R$-homomorphisms $\varphi$ with simple images, $M$ is said to be *almost $N$-simple-projective*.

We note that, in these definitions, if $N$ is indecomposable, the condition (II) is as follows:

(II$'$) There exists an $R$-homomorphism $\tilde{\psi} : N \to M$ with $\phi\tilde{\psi} = \nu$.

In this paper, we consider the case that $N$ is indecomposable.

The following in which almost $N$-simple projective is investigated for an $R$-homomorphism with its image semisimple artinian is the first step to prove Theorem 5.

**Lemma 3.** *Let $M$ be an $R$-module, $N$ an indecomposable $R$-module, $L$ a submodule of $N$ and $\varphi : M \to N/L$ an $R$-homomorphism with* $\operatorname{Im}\varphi$ *semisimple artinian. We consider the following three conditions:*

(1) *$\varphi$ is not epic.*
(2) $|\operatorname{Im}\varphi| \geq 2$.
(3) *$L \not\ll N$.*

*If $M$ is almost $N$-simple-projective and, at least, one of the above three conditions holds, then there exists an $R$-homomorphism $\tilde{\varphi} : M \to N$ with $\nu\tilde{\varphi} = \varphi$, where $\nu : N \to N/L$ is the natural epimorphism.*

*Proof.* First we consider the case that either the condition (1) or (2) holds. Let $\operatorname{Im}\varphi = \overline{S}_1 \oplus \cdots \oplus \overline{S}_n$, where $\overline{S}_i$ is simple for any $i = 1, \ldots, n$. Further, for each $i = 1, \ldots, n$, we let $\pi_i : \oplus_{j=1}^{n} \overline{S}_j \to \overline{S}_i$ be the projection and $\iota_i : \overline{S}_i \to N/L$ the injection. Then $\operatorname{Im}\iota_i\pi_i\varphi$ is simple and a proper submodule of $N/L$ by the condition (1) or (2). So, because $M$ is almost $N$-simple-projective, there exists an $R$-homomorphism $\tilde{\varphi}_i : M \to N$ with $\nu\tilde{\varphi}_i = \iota_i\pi_i\varphi$. Put $\tilde{\varphi} := \tilde{\varphi}_1 + \cdots + \tilde{\varphi}_n$. Then $\nu\tilde{\varphi} = \varphi$.

Next we consider the case that the condition (3) holds. Since $L \ll N$, there exists a proper submodule $L'$ of $N$ with $L + L' = N$. We consider an $R$-isomorphism $\eta : N/L = (L+L')/L \to L'/(L\cap L')$ naturally. Let $\nu' : N \to N/(L\cap L')$ be the natural epimorphism and $\iota : L'/(L\cap L') \to N/(L\cap L')$ the inclusion map. The condition (1) holds for $\iota\eta\varphi$, and so there exists an $R$-homomorphism $\tilde{\varphi}' : M \to N$ with $\nu'\tilde{\varphi}' = \iota\eta\varphi$. Then $\operatorname{Im}\tilde{\varphi}' \subseteq L'$. Hence $\nu\tilde{\varphi}' = \varphi$ since $\nu|_{L'} = \eta^{-1}\nu'|_{L'}$. $\square$

Using Lemma 3, we obtain the following.

**Lemma 4.** *Let $M$ be a finitely generated right $R$-module, $N$ an indecomposable right $R$-module with $L(N) < \infty$, $L$ a proper submodule of $N$ and $\varphi : M \to N/L$ an $R$-homomorphism. Suppose that $M$ is almost $N$-simple-projective and let $\nu : N \to N/L$ be the natural epimorphism.*

(1) *If $\varphi$ is not epic, then there exists an $R$-homomorphism $\tilde{\varphi} : M \to N$ with $\nu\tilde{\varphi} = \varphi$.*

(2) *Suppose that there exist a proper submodule $N'/L$ of $N/L$ and an $R$-homomorphism $\tilde{\varphi}' : M \to N$ with $(\nu'\nu)\tilde{\varphi}' = \nu'\varphi$, where $\nu' : N/L \to N/N'$ is the natural epimorphism. Then there exists an $R$-homomorphism $\tilde{\varphi}'' : M \to N$ with $\nu\tilde{\varphi}'' = \varphi$.*

*Proof.* (1) Since $L(N) < \infty$, there exists $n_1 \in \mathbb{N}$ such that $\operatorname{Im}\varphi \nsubseteq (NJ^{n_1} + L)/L$ but $\operatorname{Im}\varphi \subseteq (NJ^{n_1-1} + L)/L$. Let $\nu_1 : N/L \to N/(NJ^{n_1} + L)$ be the natural epimorphism and let $\operatorname{Im}\varphi = L_0'/L$, where $L_0'$ is a submodule of $N$. Then $\operatorname{Im}\nu_1\varphi = (L_0' + NJ^{n_1})/(NJ^{n_1} + L)$ and it is semisimple artinian because $M$ is finitely generated. Hence we claim that there exists an $R$-homomorphism $\tilde{\varphi}_1 : M \to N$ with $\nu_1\nu\tilde{\varphi}_1 = \nu_1\varphi$. If $\nu_1\varphi$ is not epic, then the condition (1) in Lemma 3 holds. Assume that $\nu_1\varphi$ is epic and, further, $NJ^{n_1} + L \ll N$, i.e., the condition (3) in Lemma 3 does not hold for $\nu_1\varphi$. Then $\operatorname{Ker}\nu_1 = (NJ^{n_1} + L)/L \ll N/L$. Since $\nu_1\varphi$ is epic, we see that $\varphi$ is also epic, a contradiction. In consequence, either the condition (1) or (3) in Lemma 3 holds for $\nu_1\varphi$ and we obtain the desired $\tilde{\varphi}_1$.

Assume that $\nu\tilde{\varphi}_1 \neq \varphi$. There exists $n_2 \in \mathbb{N}$ such that $n_2 > n_1$, $\operatorname{Im}(\varphi - \nu\tilde{\varphi}_1) \nsubseteq (NJ^{n_2} + L)/L$ but $\operatorname{Im}(\varphi - \nu\tilde{\varphi}_1) \subseteq (NJ^{n_2-1} + L)/L$. Let $\nu_2 : N/L \to N/(NJ^{n_2} + L)$ be the natural epimorphism. Then, since $\operatorname{Im}(\varphi - \nu\tilde{\varphi}_1) \subseteq (NJ^{n_1} + L)/L < N/L$, there exists an $R$-homomorphism $\tilde{\varphi}_2 : M \to N$ with $\nu_2\nu\tilde{\varphi}_2 = \nu_2(\varphi - \nu\tilde{\varphi}_1)$ by Lemma 3.

Assume that $\nu(\tilde{\varphi}_1 + \tilde{\varphi}_2) \neq \varphi$. Then there exists $n_3 \in \mathbb{N}$ such that $n_3 > n_2$, $\operatorname{Im}(\varphi - \nu(\tilde{\varphi}_1 + \tilde{\varphi}_2)) \nsubseteq (NJ^{n_3} + L)/L$ but $\operatorname{Im}(\varphi - \nu(\tilde{\varphi}_1 + \tilde{\varphi}_2)) \subseteq (NJ^{n_3-1} + L)/L$. Using this procedure finite times, since $L(N) < \infty$, we have $m \in \mathbb{N}$ with $\nu(\tilde{\varphi}_1 + \cdots + \tilde{\varphi}_m) = \varphi$.

(2) $\nu'(\varphi - \nu\tilde{\varphi}') = 0$. So $\operatorname{Im}(\varphi - \nu\tilde{\varphi}') \leq \operatorname{Ker}\nu' = N'/L < N/L$. Therefore, from (1) which we already show, there exists an $R$-homomorphism $\tilde{\varphi} : M \to N$ with $\nu\tilde{\varphi} = \varphi - \nu\tilde{\varphi}'$. Hence $\nu(\tilde{\varphi} + \tilde{\varphi}') = \varphi$. □

Now we give a theorem which is a generalization of [6, Theorem 1].

**Theorem 5.** *Let $M$ be a finitely generated right $R$-module and $N$ an indecomposable right $R$-module with $L(N) < \infty$. Suppose that $M$ is almost $N$-simple-projective. Then $M$ is almost $N$-projective.*

*Proof.* We consider the following diagram:

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow \varphi & & \\ N & \xrightarrow{\nu} & N/L & \to & 0\,, \end{array}$$

where $L$ is a proper submodule of $N$ and $\nu$ is the natural epimorphism. If $\varphi$ is not epic, then, by Lemma 4 (1), there exists an $R$-homomorphism $\tilde{\varphi} : M \to N$ with $\nu\tilde{\varphi} = \varphi$. So we may assume that $\varphi$ is epic.

First we consider the case that $L \not\ll N$. Then there exists a proper submodule $N'$ of $N$ with $N = N' + L$. So we can define an $R$-isomorphism $\eta : N/(L \cap N') \to (N/L) \oplus (N/N')$ naturally. Further define an $R$-homomorphism $\varphi' : M \to (N/L) \oplus (N/N')$ by $\varphi(m) = (\varphi(m), \overline{0})$ for any $m \in M$ and let $\nu_1 : N \to N/(L \cap N')$ be the natural epimorphism. Since $\varphi'$ is not epic, by Lemma 4 (1), there exists an $R$-homomorphism $\tilde{\varphi} : M \to N$ with $\eta\nu_1\tilde{\varphi} = \varphi'$. Then, for any $m \in M$, $(\varphi(m), \overline{0}) = \varphi'(m) = \eta\nu_1\tilde{\varphi}(m) = (\overline{\tilde{\varphi}(m)}, \overline{\tilde{\varphi}(m)})$. So $\varphi(m) = \nu\tilde{\varphi}(m)$. Hence $\nu\tilde{\varphi} = \varphi$.

Next we consider the case that $L \ll N$. Suppose that $N$ is not local. Then there exist proper submodules $N'$ and $N''$ of $N$ such that they contain $NJ$, $N'$ is a maximal submodule of $N$ and $N/NJ = (N'/NJ) \oplus (N''/NJ)$. Let $\nu' : N/L \to N/NJ$ be the natural epimorphism, $\pi : N/NJ \to N''/NJ$ the projection and $\iota : N''/NJ \to N/NJ$ the injection. Then $\iota\pi\nu'\varphi : M \to N/NJ$ and $\operatorname{Im} \iota\pi\nu'\varphi$ is a simple proper submodule of $N/NJ$. So, by assumption, there exists an $R$-homomorphism $\tilde{\varphi}' : M \to N$ with $\nu'\nu\tilde{\varphi}' = \iota\pi\nu'\varphi$. Then, letting $\nu'' : N/L \to N/N'$ be the natural epimorphism, $\nu''\nu\tilde{\varphi}' = \nu''\varphi$. Hence, by Lemma 4 (2), there exists an $R$-homomorphism $\tilde{\varphi} : M \to N$ with $\nu\tilde{\varphi} = \varphi$.

Therefore suppose that $L \ll N$ and $N$ is local. We may assume that $N = eR/A$ and $N/L = eR/B$, where $e$ is a primitive idempotent in $R$ and $A$ and $B$ are submodules of $eR_R$ with $A < B$. Let $\nu_0 : eR/B \to eR/eJ$ be the natural epimorphism. By assumption either the following (I) or (II) holds.

(I) There exists an $R$-homomorphism $\tilde{\varphi}_1 : M \to eR/A$ with $\nu_0\nu\tilde{\varphi}_1 = \nu_0\varphi$.

(II) There exists an $R$-homomorphism $\tilde{\psi}' : eR/A \to M$ with $\nu_0\varphi\tilde{\psi}' = \nu_0\nu$.

In the case (I), we obtain an $R$-homomorphism $\tilde{\varphi} : M \to eR/A$ with $\nu\tilde{\varphi} = \varphi$ from Lemma 4 (2). So we consider the case (II). Put $m_1 := \tilde{\psi}'(\overline{e})$. Since $M$ is finitely generated, we have $m_2, \dots, m_n \in M$ such that $M = m_1R + m_2R + \cdots + m_nR$ but $m_1 \notin m_2R + \cdots + m_nR$. Further we let $\varphi(m_1) = \overline{u}$, where $u \in eRe$. Then $e - u \in eJe$ because $\nu_0\nu = \nu_0\varphi\tilde{\psi}'$. Therefore $u^{-1} - e \in eJe$. Let $u^{-1} = e + j$, where $j \in eJe$. Then the following claim holds.

**Claim.** *There exists an $R$-homomorphism $\tilde{\zeta} : M \to eR/A$ with $\tilde{\zeta}(m_1) = \overline{j}$.*

*Proof of Claim.* When $j \in A$, $\tilde{\zeta} = 0$ is the desired map. So we assume that $j \notin A$. Then we can define an $R$-homomorphism $\zeta_1 : M \to eR/(jJ + A)$ by $\zeta_1(m_1r_1 + m_2r_2 + \cdots + m_nr_n) = \overline{jr_1}$ since $m_1 \notin m_2R + \cdots + m_nR$ and $m_1e = m_1$. And $\operatorname{Im} \zeta_1$ is a simple proper submodule of $eR/(jJ + A)$. So, by assumption, there exists an $R$-homomorphism $\tilde{\zeta}_1 : M \to eR/A$ with $\nu_1'\tilde{\zeta}_1 = \zeta_1$, where $\nu_1' : eR/A \to eR/(jJ + A)$ is the natural epimorphism. Let

$\tilde{\zeta}_1(m_1) = \overline{\tilde{j}_1}$, where $\tilde{j}_1 \in eRe$. Then $j - \tilde{j}_1 \in jJ + A$ since $\nu'_1\tilde{\zeta}_1 = \zeta_1$. Put $j_2 + a_2 := j - \tilde{j}_1$, where $j_2 \in jJ$ and $a_2 \in A$. Then we note that $j_2 \in J^2$.

If $j_2 \in A$, then we put $\tilde{\zeta} := \tilde{\zeta}_1$, and this $\tilde{\zeta}$ is the desired map. So assume that $j_2 \notin A$. We define an $R$-homomorphism $\zeta_2 : M \to eR/(j_2J + A)$ by $\zeta_2(m_1r_1+m_2r_2+\cdots+m_nr_n) = \overline{j_2r_1}$. Then $\operatorname{Im}\zeta_2$ is a simple proper submodule of $eR/(j_2J + A)$. So, by assumption, there exists an $R$-homomorphism $\tilde{\zeta}_2 : M \to eR/A$ with $\nu'_2\tilde{\zeta}_2 = \zeta_2$, where $\nu'_2 : eR/A \to eR/(j_2J + A)$ is the natural epimorphism. We let $\tilde{\zeta}_2(m_1) = \overline{\tilde{j}_2}$, where $\tilde{j}_2 \in eRe$. Then $j_2 - \tilde{j}_2 \in j_2J + A$ since $\zeta_2 = \nu'_2\tilde{\zeta}_2$. Put $j_3 + a_3 := j_2 - \tilde{j}_2$, where $j_3 \in j_2J$ and $a_3 \in A$. Then we note that $j_3 \in J^3$.

Since $L(eR/A) < \infty$, this procedure finitely terminates and there exists $s \in \mathbb{N}$ with $j_s - \tilde{j}_s \in A$, i.e., we may let $j_{s+1} = 0$ and $a_{s+1} = j_s - \tilde{j}_s$. Then we put $\tilde{\zeta} := \tilde{\zeta}_1 + \cdots + \tilde{\zeta}_s$, and $\tilde{\zeta}(m_1) = \tilde{\zeta}_1(m_1) + \tilde{\zeta}_2(m_1) + \cdots + \tilde{\zeta}_s(m_1) = \overline{\tilde{j}_1}+\overline{\tilde{j}_2}+\cdots+\overline{\tilde{j}_s} = \overline{(j - j_2 - a_2)}+\overline{(j_2 - j_3 - a_3)}+\cdots+\overline{(j_s - j_{s+1} - a_{s+1})} = \overline{j}$. Hence this $\tilde{\zeta}$ is the desired map. Cliam is shown.

Therefore we put $\tilde{\psi} := (1_M + \tilde{\psi}'\tilde{\zeta})\tilde{\psi}' : eR/A \to M$, and $\varphi\tilde{\psi}(\overline{e}) = \varphi(1_M + \tilde{\psi}'\tilde{\zeta})\tilde{\psi}'(\overline{e}) = \varphi(1_M + \tilde{\psi}'\tilde{\zeta})(m_1) = \varphi(m_1 + \tilde{\psi}'(\overline{j})) = \varphi(m_1 + m_1j) = \varphi(m_1)(e + j) = \varphi(m_1)u^{-1} = \overline{uu^{-1}} = \overline{e} = \nu(\overline{e})$. Hence $\varphi\tilde{\psi} = \nu$. □

We say that $M$ has the *lifting property of simple module modulo radical* (abbriviated *LPSM*) if, for any simple submodule $\overline{S}$ of $M/\operatorname{Rad}(M)$, there exists a decomposition $M = M_1 \oplus M_2$ such that $(M_1 + \operatorname{Rad}(M))/\operatorname{Rad}(M) = \overline{S}$.

Further, for $R$-modules $M$ and $N$ and an $R$-homomophism $\varphi : M \to N$, we represent a submodule $\{\, m + \varphi(m) \mid m \in M \,\}$ of $M \oplus N$ by $M(\varphi)$.

Relationship between almost $N$-projectivity and LPSM was given in [4, Proposition 2] by M. Harada and T. Mabuchi as follows:

> *For a semiperfect ring $R$, a primitive idempotent $e$ in $R$ and submodules $A$ and $B$ of $eR$ with either $eR/A$ or $eR/B$ noetherian, $eR/A$ is almost $eR/B$-projective if and only if $eR/A \oplus eR/B$ has LPSM and $eJeA \leq B$*

Further in [7, Corollary 9.7] M. Harada showed the following:

> *Let $e$ be a primitive idempotent in a ring $R$ with $eRe$ a local ring and let $A$ and $B$ be submodules of $eR_R$ with $|\, eR/A \,|$, $|\, eR/B \,| < \infty$. Then the following are equivalent:*
> $(a)$ *$eR/A$ is almost $eR/B$-projective.*
> $(b)$ $(i)$ *$eR/A \oplus eR/B$ has LPSM.*

(*ii*) *$eR/A$ is $C/B$-projective for any proper submodule $C$ of $eR_R$ with $C > B$.*

As an application of Proposition 2 and Theorem 5, last we give a corollary.

**Corollary 6.** *Let $e$ be a primitive idempotent in $R$ and $A$ and $B$ submodules of $eR_R$. If $L(eR/B) < \infty$, then the following are equivalent.*

(*a*) *$eR/A$ is almost $eR/B$-projective.*
(*b*) *$eR/A$ is almost $eR/B$-simple-projective.*
(*c*) (*i*) *$eR/A \oplus eR/B$ has LPSM.*
(*ii*) *$eR/A$ is $eJ/B$-projective.*
(*d*) (*i*) *$eR/A \oplus eR/B$ has LPSM.*
(*ii*) *$eR/A$ is $eJ/B$-simple-projective.*

*Proof.* (a) $\Leftrightarrow$ (b) This follows from Theorem 5.

(c) $\Leftrightarrow$ (d) This follows from Proposition 2.

(b) $\Rightarrow$ (d) (i) Put $M := (eR/A) \oplus (eR/B)$. Take any simple submodule $S/(eJ \oplus eJ)$ of $(eR/eJ) \oplus (eR/eJ)$. If either $S = eR \oplus eJ$ or $S = eJ \oplus eR$, then $M = (eR/A) \oplus (eR/B)$ is the desired direct decomposition. So we consider the remainder case. Then there exists $\overline{\varphi} \in \mathrm{Aut}(eR/eJ)$ with $S/(eJ \oplus eJ) = (eR/eJ)(\overline{\varphi})$. And we consider the following diagram:

$$\begin{array}{ccccc} & & eR/A & & \\ & & \downarrow \nu & & \\ & & eR/eJ & & \\ & & \downarrow \overline{\varphi} & & \\ eR/B & \xrightarrow{\nu'} & eR/eJ & \rightarrow & 0\ , \end{array}$$

where $\nu$ and $\nu'$ are the natural epimorphisms. By assumption, either the following (I) or (II) holds.

(I) There exists an $R$-homomorphism $\tilde{\varphi} : eR/A \rightarrow eR/B$ such that $\nu'\tilde{\varphi} = \overline{\varphi}\nu$.

(II) There exists an $R$-homomorphism $\tilde{\psi} : eR/B \rightarrow eR/A$ such that $\overline{\varphi}\nu\tilde{\psi} = \nu'$.

In the case (I), $M = (eR/A)(\tilde{\varphi}) \oplus (eR/B)$. And let $X/(A \oplus B) = (eR/A)(\tilde{\varphi})$, where $X$ is a submodule of $eR \oplus eR$. Then $(X + (eJ \oplus eJ))/(eJ \oplus eJ) = S/(eJ \oplus eJ)$.

In the case (II), by the similar argument, we see that $M = (eR/A) \oplus (eR/B)(\tilde{\psi})$ is the desired direct decomposition.

Hence $eR/A \oplus eR/B$ has LPSM.

(ii) We consider the following diagram:

$$\begin{array}{ccccc} & & eR/A & & \\ & & \downarrow \varphi & & \\ eJ/B & \xrightarrow{\nu} & eJ/B' & \rightarrow & 0, \end{array}$$

where $\operatorname{Im}\varphi$ is simple, $B'$ is a submodule of $eJ$ with $B' \geq B$ and $\nu$ is the natural epimorphism. Let $\nu' : eR/B \rightarrow eR/B'$ be the natural epimorphism. From (b), there exists an $R$-homomorphism $\tilde{\varphi} : eR/A \rightarrow eR/B$ with $\nu'\tilde{\varphi} = \varphi$. Then $\operatorname{Im}\tilde{\varphi} \subseteq eJ/B$ since $\operatorname{Im}\varphi \subseteq eJ/B'$. Hence $\nu'\tilde{\varphi} = \varphi$.

(d) $\Rightarrow$ (b) We consider a diagram:

$$\begin{array}{ccccc} & & eR/A & & \\ & & \downarrow \varphi & & \\ eR/B & \xrightarrow{\nu} & eR/B' & \rightarrow & 0, \end{array}$$

where $\operatorname{Im}\varphi$ is simple, $B'$ is a submodule of $eR$ with $B' \geq B$ and $\nu$ is the natural epimorphism. When $\varphi$ is not epic, there exists an $R$-homomorphism $\tilde{\varphi} : eR/A \rightarrow eR/B$ with $\nu\tilde{\varphi} = \varphi$ from (d) (ii). So we assume that $\varphi$ is epic. Then $B' = eJ$. Put $M := (eR/A) \oplus (eR/B)$. We consider a submodule

$$N := \{ (\overline{x}_1, \overline{x}_2) \mid \overline{x}_1 \in eR/A,\ \overline{x}_2 \in eR/B,\ \varphi(\overline{x}_1) = \nu(\overline{x}_2) \}$$

of $M$. And we put $M_1 := \{ (\overline{x}_1, \overline{0}) \in M \mid \overline{x}_1 \in eR/A \}$ and $M_2 := \{ (\overline{0}, \overline{x}_2) \in M \mid \overline{x}_2 \in eR/B \}$. Then, by the internal exchange property, either the following (I) or (II) holds:

(I) $M = N \oplus M_1$.

(II) $M = N \oplus M_2$.

First we consider the case (II). Let $\pi_2 : M = N \oplus M_2 \rightarrow M_2$ be the projection and put $\tilde{\varphi} := -\pi_2|_{M_1} : M_1 \rightarrow M_2$. Then we claim that $\nu\tilde{\varphi} = \varphi$. Take any $\overline{x}_1 \in eR/A$. There exist $(\overline{y}_1, \overline{y}_2) \in N$ and $(\overline{0}, \overline{x}_2) \in M_2$ with $(\overline{x}_1, \overline{0}) = (\overline{y}_1, \overline{y}_2) + (\overline{0}, \overline{x}_2)$. Then $\overline{x}_1 = \overline{y}_1$, $\overline{y}_2 = -\overline{x}_2$ and $\varphi(\overline{y}_1) = \nu(\overline{y}_2)$. So $\nu\tilde{\varphi}(\overline{x}_1) = \nu(-\pi_2(\overline{x}_1)) = \nu(\,(\overline{0}, -\overline{x}_2)\,) = \nu(\,(\overline{0}, \overline{y}_2)\,) = \varphi(\overline{y}_1) = \varphi(\overline{x}_1)$.

Next we consider the case (I). Let $\pi'_1 : M = N \oplus M_1 \rightarrow M_1$ be the projection and put $\tilde{\psi} := -\pi'_1|_{M_2} : M_2 \rightarrow M_1$. Then we see, by the same way as the case (II), that $\varphi\tilde{\psi} = \nu$. □

## References


[1] F. W. Anderson and K. R. Fuller, "Rings and categories of modules (second edition)", Graduate Texts in Math. 13, Springer-Verlag (1991)

[2] Y. Baba and K. Oshiro, *On a theorem of Fuller,* J. Algebra **154** (1993), no.1, 86-94.

[3] Y. Baba and K. Oshiro, "Classical artinian rings and related topics", to appear in World Scientific.

[4] M. Harada and T. Mabuchi, *On almost M-projectives*, Osaka J. Math. **26** (1989), 837-848.

[5] M. Harada and A. Tozaki, *Almost $M$-projectives and right Nakayama rings,* J. Alg. **122** (1989), 337-374.
[6] M. Harada, *Note on almost relative projectives and almost relative injectives,* Osaka J. Math. **29** (1992), 91-102.
[7] M. Harada, *Almost relative projective modules and almost relative injective modules,* umpublished.
[8] M. Harada, "Factor categories with applications to direct decomposition of modules," Lecture Note in Pure and Appl. Math., Vol. 88, Dekker, New York, (1983).

Yoshitomo Baba
Department of Mathematics
Osaka-Kyoiku University
Kashiwara, Osaka, 582-8582 Japan
*e-mail address*: ybaba@cc.osaka-kyoiku.ac.jp

Takeshi Yamazaki
Osaka Prefectual Senriseiun Senior High School
Toyonaka, Osaka, 560-0084, Japan
*e-mail address*: take-tdc@ezweb.ne.jp

(*Received April 9, 2009*)