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FP-GR-INJECTIVE MODULES

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Abstract. In this paper, we give some characterizations of FP-gr-injective R-modules and graded right R-modules of FP-gr-injective dimension at most n. We study the existence of FP-gr-injective envelopes and FP-gr-injective covers. We also prove that (1) $({}^{\perp}\text{gr-}\mathcal{FI}, \text{gr-}\mathcal{FI})$ is a hereditary cotorsion theory if and only if R is a left gr-coherent ring, (2) If R is right gr-coherent with FP-gr-id$(R_R) \leq n$, then $(\text{gr-}\mathcal{FI}_n, \text{gr-}\mathcal{FI}_n^{\perp})$ is a perfect cotorsion theory, (3) $({}^{\perp}\text{gr-}\mathcal{FI}_n, \text{gr-}\mathcal{FI}_n)$ is a cotorsion theory, where gr-$\mathcal{FI}$ denotes the class of all FP-gr-injective left R-modules, gr-$\mathcal{FI}_n$ is the class of all graded right R-modules of FP-gr-injective dimension at most n. Some applications are given.

1. Introduction.

All rings considered are associative with identity element and the R-modules are unital. By R-Mod we will denote the Grothendieck category of all left R-modules. Let G be a multiplicative group with neutral element e. A graded ring R is a ring with identity 1 together with a direct decomposition $R = \bigoplus_{\sigma \in G} R_\sigma$ (as additive subgroups) such that $R_\sigma R_\tau \subseteq R_{\sigma\tau}$ for all $\sigma, \tau \in G$. Thus R_e is a subring of R, $1 \in R_e$ and R_σ is an R_e-bimodule for every $\sigma \in G$. A graded left R-module is a left R-module M endowed with an internal direct sum decomposition $M = \bigoplus_{\sigma \in G} M_\sigma$, where each M_σ is a subgroup of the additive group of M satisfying $R_\sigma M_\tau \subseteq M_{\sigma\tau}$ for all $\sigma, \tau \in G$. For graded left R-modules M and N, we put

$$\mathrm{Hom}_{R\text{-gr}}(M, N) = \{f : M \to N | f \text{ is } R\text{-linear and } f(M_\sigma) \subseteq N_\sigma \ \ \forall \sigma \in G\}$$

is the group of all morphisms from M to N in the category R-gr of all graded left R-modules. It is well known that R-gr is a Grothendieck category. An R-linear map $f : M \to N$ is said to be a graded morphism of degree τ, $\tau \in G$ if $f(M_\sigma) \subseteq M_{\sigma\tau}$ for all $\sigma \in G$. Graded morphisms of degree σ build an additive subgroup $\mathrm{HOM}_R(M, N)_\sigma$ of $\mathrm{Hom}_R(M, N)$. Then $\mathrm{HOM}_R(M, N) = \bigoplus_{\sigma \in G} \mathrm{HOM}_R(M, N)_\sigma$ is a graded abelian group of type G. We will denote $\mathrm{Ext}^i_{R\text{-gr}}$ and EXT^i_R as the right derived functors of $\mathrm{Hom}_{R\text{-gr}}$ and HOM_R.

Let M be a graded right R-module and N a graded left R-module. The abelian group $M \otimes_R N$ may be graded by putting $(M \otimes_R N)_\sigma$, $\sigma \in G$, equal

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to the additive subgroup generated by elements $x \otimes y$ with $x \in M_\alpha$, $y \in N_\beta$ such that $\alpha\beta = \sigma$. The object of $\mathbb{Z}$-gr thus defined will be called the graded tensor product of M and N.

If $M = \bigoplus_{\sigma \in G} M_\sigma$ is a graded left R-module and $\sigma \in G$, then $M(\sigma)$ is the graded left R-module obtained by putting $M(\sigma)_\tau = M_{\tau\sigma}$ for all $\tau \in G$; the graded module $M(\sigma)$ is called the σ-suspension of M. We can see the σ-suspension as an isomorphism of categories $T_\sigma : R\text{-gr} \to R\text{-gr}$, given on objects as $T_\sigma(M) = M(\sigma)$ for $M \in R$-gr.

For any element $m = \sum_{\sigma \in G} m_\sigma$ of R, $\mathrm{Supp}(m) = \{\sigma \in G | m_\sigma \neq 0\}$. Consider $\{M_i | i \in I\}$ a set of graded left R-modules and let $\{\prod_{i\in I} M_i, \pi_i\}$ be the direct product in R-Mod of the underlying left R-modules M_i, where $\pi_j : \prod_{i \in I} M_i \to M_j$ denotes the j-th canonical projection for each $j \in I$. Given $m \in \prod_{i\in I} M_i$, define $\mathrm{SUPP}(m) = \bigcup_{i\in I} \mathrm{Supp}(\pi_i(m)) \subset G$. We can define $\prod_{i\in I}^{R\text{-gr}} M_i = \{m \in \prod_{i\in I} M_i | \mathrm{SUPP}(m) \text{ is finite}\}$. Then $\{\prod_{i\in I}^{R\text{-gr}} M_i, \pi_i\}$ is the direct product of the graded left R-modules $\{M_i | i \in I\}$. It is a graded left R-module, where $(\prod_{i\in I}^{R\text{-gr}} M_i)_\sigma = \{m \in \prod_{i\in I}^{R\text{-gr}} M_i | \mathrm{SUPP}(m) \subset \{\sigma\}\}$. Observe that, as R_e-modules $(\prod_{i\in I}^{R\text{-gr}} M_i)_\sigma \cong \prod_{i\in I}(M_i)_\sigma$ for any $\sigma \in G$.

Given a graded left R-module M, we can define the graded character module of M as $M^+ = \mathrm{HOM}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. We note then that it can be seen as $M^+ = \bigoplus_{\sigma \in G} \mathrm{Hom}_{\mathbb{Z}}(M_{\sigma^{-1}}, \mathbb{Q}/\mathbb{Z})$.

The injective objects of R-gr will be called gr-injective modules. Projective (resp. flat) objects of R-gr will be called projective (resp. flat) graded modules because M is gr-projective (resp. gr-flat) if and only if it is a projective (resp. flat) graded module. We will denote the gr-injective dimension of a graded module M by gr-id$M$ and fd$M$ will denote the flat dimension of M. We will denote the gr-injective envelope of M by $E^g(M)$. We will call FP-gr-injective module to those graded R-module M such that $\mathrm{EXT}^1_R(N, M) = 0$ for any finitely presented graded R-module N. It can be proved that if R is gr-noetherian, M is gr-injective if and only if M is FP-gr-injective and that in the case that R is gr-coherent, i.e. a graded ring R such that given a family of graded flat R-modules $\{F_i\}_{i\in I}$, then the graded R-module $\prod_{i\in I}^{R\text{-gr}} F_i$ is flat, M is FP-gr-injective if and only if M^+ is flat. The FP-gr-injective dimension of a graded R-module M will be the least integer n such that $\mathrm{EXT}^{n+1}_R(N, M) = 0$ for any finitely presented graded R-module N.

The forgetful functor $U : R\text{-gr} \to R\text{-Mod}$ associates to M the underlying ungraded R-module. This functor has a right adjoint F which associated to $M \in R$-Mod the graded R-module $F(M) = \bigoplus_{\sigma\in G}({}^\sigma M)$, where each ${}^\sigma M$ is a copy of M written $\{{}^\sigma x : x \in M\}$ with R-module structure defined by $r * {}^\tau x = {}^{\sigma\tau}(rx)$ for each $r \in R_\sigma$. If $f : M \to N$ is R-linear, then $F(f) : F(M) \to F(N)$ is a graded morphism given by $F(f)({}^\sigma x) = {}^\sigma f(x)$.

Let $\mathcal{F}$ be a class of graded R-modules for a graded ring R. If $\varphi : C \to M$ is a graded morphism, where $C \in \mathcal{F}$ and $M \in R$-gr, then $\varphi : C \to M$ is called an $\mathcal{F}$-precover of M if $\mathrm{Hom}_{R\text{-gr}}(C', C) \to \mathrm{Hom}_{R\text{-gr}}(C', M) \to 0$ is exact for all $C' \in \mathcal{F}$. Moreover, if whenever a graded morphism $f : C \to C$ such that $\varphi \circ f = \varphi$ is an automorphism of C, then $\varphi : C \to M$ is called an $\mathcal{F}$-cover of M. $\mathcal{F}$-envelope and $\mathcal{F}$-preenvelope are defined dually. Let $\varphi : C \to M$ be an $\mathcal{F}$-cover of M. If for any graded morphism $f : C' \to M$ with $C' \in \mathcal{F}$, there is a unique graded morphism $g : C' \to C$ such that $\varphi g = f$, then we say that φ has the unique mapping property. Dually we have the definition of an $\mathcal{F}$-envelope has the unique mapping property.

2. FP-gr-injective envelopes of graded modules.

In this section, we give some characterizations of FP-gr-injective modules and prove that $({}^{\perp}\text{gr-}\mathcal{FI}, \text{gr-}\mathcal{FI})$ is a hereditary cotorsion theory if and only if R is a left gr-coherent ring, where gr-$\mathcal{FI}$ denotes the class of all FP-gr-injective left R-modules.

An exact sequence $0 \to M' \to M \to M'' \to 0$ in R-gr is said to be gr-pure if for any $N \in$ gr-R, the sequence $0 \to N \otimes_R M' \to N \otimes_R M \to N \otimes_R M'' \to 0$ is exact in $\mathbb{Z}$-gr.

Proposition 2.1. *Let R be a ring graded by a group G. Then the following are equivalent for a graded left R-module M:*

(1) M is FP-gr-injective;

(2) The functor $HOM_R(-, M)$ is exact with respect to every exact sequence $0 \to A \to B \to C \to 0$ in R-gr with C finitely presented;

(3) $M(\sigma)$ is FP-gr-injective for all $\sigma \in G$;

(4) $M(\sigma)$ is gr-injective with respect to every exact sequence $0 \to A \to B \to C \to 0$ in R-gr with C finitely presented for all $\sigma \in G$;

(5) M is gr-pure in every graded left R-module that contains it;

(6) M is gr-pure in every gr-injective left R-module that contains it;

(7) M is gr-pure in $E^g(M)$.

Proof. (1) $\Leftrightarrow$ (2) is clear by definition. (3) $\Rightarrow$ (1) and (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are obvious.

(2) $\Rightarrow$ (3) Let $0 \to A \to B \to C \to 0$ be exact in R-gr with C finitely presented. Then

$$0 \longrightarrow \mathrm{HOM}_R(C, M)_\sigma \longrightarrow \mathrm{HOM}_R(B, M)_\sigma \longrightarrow \mathrm{HOM}_R(A, M)_\sigma \longrightarrow 0$$

is exact for all $\sigma \in G$. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{HOM}_R(C,M)_{\sigma\tau} \arrow[r] \arrow[d, "\cong"] & \mathrm{HOM}_R(B,M)_{\sigma\tau} \arrow[r] \arrow[d, "\cong"] & \mathrm{HOM}_R(A,M)_{\sigma\tau} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{R\text{-gr}}(C,M(\sigma\tau)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{R\text{-gr}}(B,M(\sigma\tau)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{R\text{-gr}}(A,M(\sigma\tau)) \arrow[d, "\cong"] & \\
0 \arrow[r] & \mathrm{HOM}_R(C,M(\sigma))_\tau \arrow[r] & \mathrm{HOM}_R(B,M(\sigma))_\tau \arrow[r] & \mathrm{HOM}_R(A,M(\sigma))_\tau &
\end{tikzcd}$$

with the upper row exact for every $\tau \in G$. So

$$0 \longrightarrow \mathrm{HOM}_R(C,M(\sigma)) \longrightarrow \mathrm{HOM}_R(B,M(\sigma)) \longrightarrow \mathrm{HOM}_R(A,M(\sigma)) \longrightarrow 0$$

is exact, which means that $M(\sigma)$ is FP-gr-injective for all $\sigma \in G$.

(2) $\Leftrightarrow$ (4) By $\mathrm{HOM}_R(-,M)_\sigma = \mathrm{Hom}_{R\text{-gr}}(-,M(\sigma))$ for every $\sigma \in G$.

(1) $\Rightarrow$ (5) Let $0 \to M \to L \to L/M \to 0$ be exact, N a finitely presented graded left R-module. Then

$$\begin{aligned} 0 \longrightarrow &\mathrm{HOM}_R(N,M) \longrightarrow \mathrm{HOM}_R(N,L) \\ &\longrightarrow \mathrm{HOM}_R(N,L/M) \longrightarrow \mathrm{EXT}^1_R(N,M) = 0 \end{aligned}$$

is exact. So M is gr-pure in L by [9, Proposition 3.1].

(7) $\Rightarrow$ (1) Let N be any finitely presented graded left R-module. Then

$$\begin{aligned} 0 \longrightarrow &\mathrm{HOM}_R(N,M) \longrightarrow \mathrm{HOM}_R(N,E^g(M)) \\ &\longrightarrow \mathrm{HOM}_R(N,E^g(M)/M) \longrightarrow 0 \end{aligned}$$

is exact, and so $\mathrm{EXT}^1_R(N,M) = 0$, which implies that M is FP-gr-injective. $\square$

Remark 2.2. *By the definition and Proposition 2.1, we see that the class of all FP-gr-injective R-modules is closed under graded direct summands, graded direct products and graded pure submodules.*

Lemma 2.3. *Let R be a ring graded by a group G. If M is an FP-injective left R-module, then $F(M)$ is FP-gr-injective.*

Proof. Let $0 \to A \xrightarrow{f} B \to C \to 0$ be exact in R-gr with C finitely presented, $g : A \to F(M)$ a graded morphism. Since F is a right adjoint functor of the forgetful functor, we have the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[dl] & C \arrow[r] & 0 \\
& M & & &
\end{tikzcd}$$

Now, again by the adjoint situation between the forgetful functor and F we have a graded morphism $B \to F(M)$ such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[dl] & C \arrow[r] & 0 \\
& F(M) & & &
\end{tikzcd}$$

which shows that $F(M)$ is gr-injective with respect to the exact sequence $0 \to A \xrightarrow{f} B \to C \to 0$. Let $\sigma \in G$ and $g : A \to F(M)(\sigma)$ be a graded morphism. Since $0 \to A(\sigma^{-1}) \xrightarrow{T_{\sigma^{-1}}(f)} B(\sigma^{-1}) \to C(\sigma^{-1}) \to 0$ is exact and $C(\sigma^{-1})$ is finitely presented, there exists a graded morphism $h : B(\sigma^{-1}) \to F(M)$ such that $hT_{\sigma^{-1}}(f) = T_{\sigma^{-1}}(g)$, and so $T_\sigma(h)f = g$ for $T_\sigma(h) : B \to F(M)(\sigma)$, which gives that $F(M)(\sigma)$ is gr-injective with respect to the exact sequence $0 \to A \to B \to C \to 0$ for all $\sigma \in G$. Therefore $F(M)$ is FP-gr-injective by Proposition 2.1. $\square$

Corollary 2.4. *Let R be a ring graded by a finite group G and $M \in R$-gr. Then M is FP-gr-injective if and only if M is an FP-injective left R-module.*

Proof. "$\Leftarrow$" By Lemma 2.3, $F(M)$ is FP-gr-injective, and so M is FP-gr-injective since M is a direct summand of $F(M)$.

"$\Rightarrow$" Let $0 \to A \to B \to C \to 0$ be exact in R-Mod with C finitely presented. Then $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact in R-gr and $F(C)$ is finitely presented since G is finite. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{R\text{-gr}}(F(C), M) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{R\text{-gr}}(F(B), M) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{R\text{-gr}}(F(A), M) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(C, M) \arrow[r] & \mathrm{Hom}_R(B, M) \arrow[r] & \mathrm{Hom}_R(A, M) &
\end{tikzcd}$$

with the upper row exact. Therefore M is an FP-injective left R-module. $\square$

Theorem 2.5. *Let R be a ring graded by a group G. Then every graded R-module has an FP-gr-injective preenvelope.*

Proof. Let M be a graded R-module. We take $\mathcal{N}_\beta$ an infinite cardinal number such that $\mathrm{Card}(M)\mathrm{Card}(R)\mathrm{Card}(G) \le \mathcal{N}_\beta$. Set

$$Y = \{A | A \text{ is an FP-gr-injective } R\text{-module and } \mathrm{Card}(A) \le \mathcal{N}_\beta\}.$$

Let $\{A_i\}_{i \in I}$ be a family of representatives of this class with the index set I. Let $H_i = \mathrm{Hom}_{R\text{-gr}}(M, A_i)$ for every $i \in I$ and let $B = \prod_{i\in I}^{R\text{-gr}}(\prod_{j\in H_i}^{R\text{-gr}}(A_i)_j)$, where $(A_i)_j = A_i$ for each $j \in H_i$. Then B is FP-gr-injective. Define $\varphi : M \to B$ so that the composition of φ with the projective map $B \to \prod_{j\in H_i}^{R\text{-gr}}(A_i)_j$ maps $x \in B_\sigma$ to $(h(x))_{h\in H_i}$ for any $\sigma \in G$. Then φ is a graded morphism. We claim that $\varphi : M \to B$ is an FP-gr-injective preenvelope. Let $\varphi' : M \to B'$ with B' an FP-gr-injective R-module. By [9, Lemma 2.3], the graded submodule $\varphi'(M)$ can be enlarged to a graded pure submodule $\varphi'(M)^* \subseteq B'$ with $\mathrm{Card}(\varphi'(M)^*) \leq \mathcal{N}_\beta$ and $\varphi'(M)^*$ is FP-gr-injective by Remark 2.2. Thus $\varphi'(M)^*$ is isomorphic to one of the A_i. By the construction of the map φ, it is easy to see that φ' can be factored through φ. □

Definition 2.6. *([9]) A pair $(\mathcal{F},\mathcal{C})$ of classes of graded R-modules is a cotorsion theory in R-gr if the following properties are satisfied:*

$Ext^1_{R\text{-}gr}(F,C) = 0$ for every $F \in \mathcal{F}$, $C \in \mathcal{C}$.

$Ext^1_{R\text{-}gr}(F,C) = 0$ for every $F \in \mathcal{F}$, implies $C \in \mathcal{C}$.

$Ext^1_{R\text{-}gr}(F,C) = 0$ for every $C \in \mathcal{C}$, implies $F \in \mathcal{F}$.

A cotorsion theory $(\mathcal{F},\mathcal{C})$ is called hereditary if whenever $0 \to F' \to F \to F'' \to 0$ is exact in R-gr with F, $F'' \in \mathcal{F}$, then F' is also in $\mathcal{F}$. A cotorsion theory $(\mathcal{F},\mathcal{C})$ is said to be perfect if every graded R-module has an $\mathcal{F}$-cover and an $\mathcal{C}$-envelope.

Let $\mathcal{FI}$ denote the class of all FP-injective left R-modules. It is well known that $({}^\perp\mathcal{FI},\mathcal{FI})$ is a hereditary cotorsion theory if and only if R is a left coherent ring. Here we have a graded version.

Theorem 2.7. *Let gr-$\mathcal{FI}$ denote the class of all FP-gr-injective left R-modules. Then $({}^\perp gr\text{-}\mathcal{FI}, gr\text{-}\mathcal{FI})$ is a hereditary cotorsion theory if and only if R is a left gr-coherent ring.*

Proof. "⇒" Let I be a finitely generated graded left ideal of R, N an FP-injective left R-module and let $0 \to N \to E \to C \to 0$ be exact in R-Mod with E injective. Then $0 \to F(N) \to F(E) \to F(C) \to 0$ is exact in R-gr with $F(E)$ gr-injective, and so $F(C)$ is FP-gr-injective by Lemma 2.3 and hypothesis. Hence

$$\mathrm{Ext}^1_{R\text{-gr}}(I,F(N)) \cong \mathrm{Ext}^2_{R\text{-gr}}(R/I,F(N)) \cong \mathrm{Ext}^1_{R\text{-gr}}(R/I,F(C)) = 0.$$

Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{R\text{-gr}}(I,F(N)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{R\text{-gr}}(I,F(E)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{R\text{-gr}}(I,F(C)) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(I,N) \arrow[r] & \mathrm{Hom}_R(I,E) \arrow[r] & \mathrm{Hom}_R(I,C) &
\end{tikzcd}$$

with the upper row exact. Thus $\mathrm{Ext}^1_R(I,N)=0$, which means that I is finitely presented.

"$\Leftarrow$" Let $X \in {}^{\perp}\text{gr-}\mathcal{FI}$. Then $X(\sigma) \in {}^{\perp}\text{gr-}\mathcal{FI}$ for all $\sigma \in G$ by a proof dual to that of Lemma 2.3. Let $M \in ({}^{\perp}\text{gr-}\mathcal{FI})^{\perp}$ and N be a finitely presented graded left R-module. Then $N \in {}^{\perp}\text{gr-}\mathcal{FI}$ and $M(\sigma) \in ({}^{\perp}\text{gr-}\mathcal{FI})^{\perp}$ for all $\sigma \in G$ by analogy with the proof of Lemma 2.3. Thus $\mathrm{EXT}^1_R(N,M)_\sigma = \mathrm{Ext}^1_{R\text{-gr}}(N, M(\sigma)) = 0$, and so $\mathrm{EXT}^1_R(N,M)=0$, which implies that $M \in \text{gr-}\mathcal{FI}$. Let $0 \to A \to B \to C \to 0$ be exact in R-gr with A and B FP-gr-injective. Then $0 \to C^+ \to B^+ \to A^+ \to 0$ is exact and A^+, B^+ are flat, and so C^+ is flat. Hence C is FP-gr-injective. It follows that $({}^{\perp}\text{gr-}\mathcal{FI}, \text{gr-}\mathcal{FI})$ is a hereditary cotorsion theory. □

3. FP-gr-injective covers of graded modules.

In this section, we give some characterizations of gr-coherent rings and prove that if R is left gr-coherent, then every graded left R-module has an FP-gr-injective cover. Some applications are given.

Lemma 3.1. *Let R be a graded ring, A a finitely generated graded left R-module. Then A is finitely presented if and only if $Hom_{R\text{-}gr}(A, \varinjlim M_i) \cong \varinjlim Hom_{R\text{-}gr}(A, M_i)$, where $\{M_i\}_{i\in I}$ is a family of gr-injective left R-modules.*

Proof. "$\Rightarrow$" By [15, Chap.V, Proposition 3.4].

"$\Leftarrow$" Let E be a gr-injective cogenerator of R-gr. Define $H : R\text{-gr} \to R\text{-gr}$ as follows. Let $H(N) = \prod^{R\text{-gr}}_{i\in I_N} E_i$, where $E_i = E$ and $I_N = \mathrm{Hom}_{R\text{-gr}}(N,E)$. If $\alpha \in \mathrm{Hom}_{R\text{-gr}}(N_1,N_2)$, let $\alpha^* : \mathrm{Hom}_{R\text{-gr}}(N_2,E) \to \mathrm{Hom}_{R\text{-gr}}(N_1,E)$ be canonical. Then $H(\alpha) : H(N_1) \to H(N_2)$ via $\beta \mapsto \beta \cdot \alpha^*$. Note that $H(N)$ is gr-injective. The evaluation map $h_N : N \to H(N)$ yields a natural transformation.

Let (X_i, φ_{ji}) be a direct system of graded R-modules. Then $(H(X_i), H(\varphi_{ji}))$ is a direct system and

$$0 \to \varinjlim X_i \to \varinjlim H(X_i) \to \varinjlim H(X_i)/X_i \to 0$$

is exact. So we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{R\text{-gr}}(A, \varinjlim X_i) \arrow[r] \arrow[d, "\alpha"] & \mathrm{Hom}_{R\text{-gr}}(A, \varinjlim H(X_i)) \arrow[r] \arrow[d, "\beta"] & \mathrm{Hom}_{R\text{-gr}}(A, \varinjlim H(X_i)/X_i) \arrow[d, "\gamma"] \\
0 \arrow[r] & \varinjlim \mathrm{Hom}_{R\text{-gr}}(A, X_i) \arrow[r] & \varinjlim \mathrm{Hom}_{R\text{-gr}}(A, H(X_i)) \arrow[r] & \varinjlim \mathrm{Hom}_{R\text{-gr}}(A, H(X_i)/X_i)
\end{tikzcd}$$

Since β is an isomorphism, α is monic. Similarly, we have γ is monic. So α is an isomorphism, which implies that A is finitely presented by [15, Chap.V, Proposition 3.4]. □

Theorem 3.2. *The following are equivalent for a ring R graded by a group G:*

(1) R is left gr-coherent;

(2) Any direct limit of FP-gr-injective left R-modules is FP-gr-injective;

(3) $EXT^1_R(N, \varinjlim M_i) \to \varinjlim EXT^1_R(N, M_i)$ is an isomorphism for any finitely presented graded left R-module N and direct system $\{M_i\}_{i\in\Lambda}$ of graded left R-modules;

(4) $EXT^2_R(N, M) = 0$ for any finitely presented graded left R-module N and FP-gr-injective left R-module M.

Proof. (1) ⇒ (4) and (3) ⇒ (2) are obvious.

(1) ⇒ (3) Let N be any finitely presented graded left R-module. Then there exists an exact sequence $0 \to K \to P \to N \to 0$ in R-gr with P finitely generated projective and K finitely generated . Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{HOM}_R(P, \varinjlim M_i) \arrow[r] \arrow[d, "\cong"] & \mathrm{HOM}_R(K, \varinjlim M_i) \arrow[r] \arrow[d, "\cong"] & \mathrm{EXT}^1_R(N, \varinjlim M_i) \arrow[r] \arrow[d] & 0 \\
\varinjlim \mathrm{HOM}_R(P, M_i) \arrow[r] & \varinjlim \mathrm{HOM}_R(K, M_i) \arrow[r] & \varinjlim \mathrm{EXT}^1_R(N, M_i) \arrow[r] & 0
\end{tikzcd}$$

So $\mathrm{EXT}^1_R(N, \varinjlim M_i) \to \varinjlim \mathrm{EXT}^1_R(N, M_i)$ is an isomorphism.

(2) ⇒ (1) Let I be a finitely generated graded left ideal of R and $\{M_i\}_{i\in\Lambda}$ be a family of gr-injective left R-modules. Then $\varinjlim M_i$ is FP-gr-injective, and so $\mathrm{EXT}^1_R(R/I, \varinjlim M_i) = 0$. Thus we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{R\text{-gr}}(R/I, \varinjlim M_i) \arrow[r] \arrow[d, "\alpha"] & \mathrm{Hom}_{R\text{-gr}}(R, \varinjlim M_i) \arrow[r] \arrow[d, "\beta"] & \mathrm{Hom}_{R\text{-gr}}(I, \varinjlim M_i) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \varinjlim \mathrm{Hom}_{R\text{-gr}}(R/I, M_i) \arrow[r] & \varinjlim \mathrm{Hom}_{R\text{-gr}}(R, M_i) \arrow[r] & \varinjlim \mathrm{Hom}_{R\text{-gr}}(I, M_i) \arrow[r] & 0
\end{tikzcd}$$

Since α, β are isomorphisms, then γ is an isomorphism, and so I is finitely presented by Lemma 3.1.

(4) ⇒ (1) By analogy with the proof of Theorem 2.7. □

Theorem 3.3. *Let R be left gr-coherent. Then every graded left R-module has an FP-gr-injective cover.*

Proof. Let M be any graded left R-module and $A \to M$ be any graded morphism with A FP-gr-injective. We want to show that $A \to M$ can be factored through an FP-gr-injective left R-module B with $\mathrm{Card}(B) \le \mathcal{N}_\beta$ for some cardinal number $\mathcal{N}_\beta$. If $\mathrm{Card}(A) \le \mathcal{N}_\beta$, set $A = B$. So suppose that

$\mathrm{Card}(A) > \mathcal{N}_\beta$. Consider a graded submodule $S \subseteq A$ maximal with respect to the two properties that S is gr-pure in A and that $S \subseteq \mathrm{Ker}(A \to M)$. Let $B = A/S$. Then B is FP-gr-injective by Remark 2.2 and Theorem 2.7. We wish to argue that $\mathrm{Card}(B) \leq \mathcal{N}_\beta$. Consider a submodule $S' \subseteq A$ maximal with respect to the two properties that S' is pure in A and that $S' \subseteq \mathrm{Ker}(A \to M)$. Then $S' \subseteq S$ and $\mathrm{Card}(A/S') \leq \mathcal{N}_\beta$ by the proof of [14, Lemma 2.5]. Since $0 \to S/S' \to A/S' \to A/S \to 0$ is exact, we have $\mathrm{Card}(B) \leq \mathcal{N}_\beta$.

Set $Y = \{B | B$ is an FP-gr-injective left R-module and $\mathrm{Card}(B) \leq \mathcal{N}_\beta\}$. Let $\{B_i\}_{i \in I}$ be a family of representatives of this class with the index set I. Then $\bigoplus_{i \in I} B_i^{(\mathrm{Hom}_{R\text{-gr}}(B_i, M))} \to M$ is an FP-gr-injective precover by analogy with the proof of [14, Lemma 2.4], which implies that every graded left R-module has an FP-gr-injective cover by Theorem 3.2 and [1, Theorem 2.10]. □

Lemma 3.4. *Let R be a ring graded by a group G. Then $0 \to A \to B \to C \to 0$ is a gr-pure exact sequence in R-gr if and only if $0 \to A(\sigma) \to B(\sigma) \to C(\sigma) \to 0$ is gr-pure exact for all $\sigma \in G$.*

Proof. "$\Rightarrow$" Let M be a graded right R-module and $\sigma \in G$. We have to prove the exactness of

$$0 \longrightarrow M \otimes_R A(\sigma) \longrightarrow M \otimes_R B(\sigma) \longrightarrow M \otimes_R C(\sigma) \longrightarrow 0,$$

which is equivalent to proving the exactness of each of the homogeneous components

$$0 \longrightarrow (M \otimes_R A(\sigma))_\tau \longrightarrow (M \otimes_R B(\sigma))_\tau \longrightarrow (M \otimes_R C(\sigma))_\tau \longrightarrow 0,$$

i.e., the exactness of

$$0 \longrightarrow M_\alpha \otimes_{R_e} A(\sigma)_\beta \longrightarrow M_\alpha \otimes_{R_e} B(\sigma)_\beta \longrightarrow M_\alpha \otimes_{R_e} C(\sigma)_\beta \longrightarrow 0$$

with $\alpha\beta = \tau$. Since $0 \to A \to B \to C \to 0$ is gr-pure exact, we have

$$0 \longrightarrow M_\alpha \otimes_{R_e} A_{\beta\sigma} \longrightarrow M_\alpha \otimes_{R_e} B_{\beta\sigma} \longrightarrow M_\alpha \otimes_{R_e} C_{\beta\sigma} \longrightarrow 0$$

is exact with $\alpha\beta\sigma = \tau\sigma$, which implies that $0 \to A(\sigma) \to B(\sigma) \to C(\sigma) \to 0$ is gr-pure exact.

"$\Leftarrow$" is trivial. □

A graded left R-module Q is called pure gr-injective if for every pure sequence $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ in R-gr and every graded morphism $\varphi : L \to Q$, there exists $\psi : M \to Q$ such that $\psi\alpha = \varphi$.

Lemma 3.5. *Let R be a ring graded by a group G. Then H is a pure gr-injective left R-module if and only if $H(\sigma)$ is pure gr-injective for all $\sigma \in G$.*

Proof. By analogy with the proof of Lemma 2.3. □

Proposition 3.6. *The following are true for any graded ring R of type G:*

(1) A graded left R-module M is FP-gr-injective if and only if for any pure gr-injective left R-module H, every graded morphism $f : M \to H$ factors through a gr-injective left R-module.

(2) If M is a pure gr-injective left R-module and $f : C \to M$ is an FP-gr-injective cover of M, then C is gr-injective.

Proof. (1) "$\Rightarrow$" Consider the exact sequence $0 \to M \to E^g(M) \to C \to 0$. Then the sequence is gr-pure by Proposition 2.1. So there exists a graded morphism $g : E^g(M) \to H$ such that $g|_M = f$, as required.

"$\Leftarrow$" It is enough to show that the exact sequence $0 \to M \to E^g(M) \to C \to 0$ is gr-pure. Let H be a graded right R-module. Then $H^+(\sigma^{-1})$ is pure gr-injective for all $\sigma \in G$ by Lemma 3.5. For any graded morphism $f : M \to H^+(\sigma^{-1})$, there are a gr-injective left R-module E and graded morphisms $g : M \to E$, $h : E \to H^+(\sigma^{-1})$ such that $f = hg$ by hypothesis. Thus there exists a graded morphism $k : E^g(M) \to E$ such that $k|_M = g$, and so $hk|_M = f$. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{R\text{-gr}}(E^g(M), H^+(\sigma^{-1})) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{R\text{-gr}}(M, H^+(\sigma^{-1})) \arrow[r] \arrow[d, "\cong"] & 0 \\
\mathrm{HOM}_R(E^g(M), H^+)_{\sigma^{-1}} \arrow[r] \arrow[d, "\cong"] & \mathrm{HOM}_R(M, H^+)_{\sigma^{-1}} \arrow[d, "\cong"] & \\
\mathrm{Hom}_{\mathbb{Z}}((H \otimes_R E^g(M))_\sigma, \mathbb{Q}/\mathbb{Z}) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}((H \otimes_R M)_\sigma, \mathbb{Q}/\mathbb{Z}) &
\end{tikzcd}$$

with the upper row exact. Then $0 \to (H \otimes_R M)_\sigma \to (H \otimes_R E^g(M))_\sigma$ is exact for all $\sigma \in G$. Therefore $0 \to H \otimes_R M \to H \otimes_R E^g(M) \to H \otimes_R C \to 0$ is exact and M is FP-gr-injective.

(2) By (1), there exist a gr-injective left R-module E and graded morphisms $g : C \to E$, $h : E \to M$ such that $f = hg$, and so there is a graded morphism $k : E \to C$ such that $fk = h$ since f is a cover. Thus $fkg = f$ and kg is an isomorphism, which implies that C is isomorphic to a direct summand of E, and hence C is gr-injective. □

Lemma 3.7. *Let R be left gr-coherent and M a graded left R-module. Then FP-gr-id$M \leq n$ if and only if there is an exact sequence $0 \to M \to E^0 \to \cdots \to E^n \to 0$ in R-gr with each E^i FP-gr-injective.*

Proof. Easy. □

Proposition 3.8. *The following are equivalent for a left gr-coherent ring R of type G:*

(1) ${}_RR$ is FP-gr-injective;

(2) Every (finitely presented) graded left R-module has an epic FP-gr-injective cover;

(3) Every (finitely presented) graded right R-module has a monic gr-flat preenvelope;

(4) Every (finitely presented) graded right R-module is a graded submodule of a gr-flat right R-module.

Proof. (2) $\Rightarrow$ (1) and (3) $\Leftrightarrow$ (4) are obvious.

(1) $\Rightarrow$ (2) Let M be a graded left R-module. Then M has an FP-gr-injective cover $f : C \to M$. On the other hand, there is an exact sequence $\bigoplus_{\sigma\in S} R(\sigma) \to M \to 0$ for some $S \subseteq G$. Let N be any finitely presented graded left R-module and $0 \to K \to P \to N \to 0$ be exact in R-gr, where P is finitely generated projective and K is finitely generated. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{HOM}_R(N,\oplus_{\sigma\in S}R(\sigma)) \arrow[r] \arrow[d, "\cong"] & \mathrm{HOM}_R(P,\oplus_{\sigma\in S}R(\sigma)) \arrow[r] \arrow[d, "\cong"] & \mathrm{HOM}_R(K,\oplus_{\sigma\in S}R(\sigma)) \arrow[d] & \\
\mathrm{Hom}_R(N,\oplus_{\sigma\in S}R(\sigma)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(P,\oplus_{\sigma\in S}R(\sigma)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(K,\oplus_{\sigma\in S}R(\sigma)) \arrow[d] & \\
\oplus_{\sigma\in S}\mathrm{HOM}_R(N,R(\sigma)) \arrow[r] & \oplus_{\sigma\in S}\mathrm{HOM}_R(P,R(\sigma)) \arrow[r] & \oplus_{\sigma\in S}\mathrm{HOM}_R(K,R(\sigma)) \arrow[r] & 0
\end{tikzcd}$$

with the lower row exact. Then the upper row exact. Hence

$$\mathrm{EXT}^1_R(N,\oplus_{\sigma\in S}R(\sigma)) = 0$$

and $\bigoplus_{\sigma\in S} R(\sigma)$ is FP-gr-injective. So f is epic.

(1) $\Rightarrow$ (3) Let E be any gr-injective right R-module. Then there exists an exact sequence $\bigoplus_{\sigma\in S} R(\sigma) \to E^+ \to 0$ for some $S \subseteq G$, and hence $0 \to E^{++} \to (\bigoplus_{\sigma\in S} R(\sigma))^+$ is exact. Since E^{++} is gr-injective and $(\bigoplus_{\sigma\in S} R(\sigma))^+$ is flat, we have E^{++} is flat, and so E is flat and (3) follows.

(3) $\Rightarrow$ (1) Since $({}_RR)^+$ has a monic gr-flat preenvelope, $({}_RR)^+$ is flat, and so ${}_RR$ is FP-gr-injective. $\square$

We denote l.FP-gr-dim$R = \sup\{$FP-gr-id$M | M$ is a graded left R-module$\}$.

Proposition 3.9. *The following are equivalent for a left gr-coherent ring R of type G:*

(1) l.FP-gr-dim$R \leq 1$;

(2) Every graded left R-module has a monic FP-gr-injective cover;

(3) Every graded right R-module has an epic gr-flat preenvelope;

(4) The kernel of any FP-gr-injective (pre)cover of a graded left R-module is FP-gr-injective;

(5) The cokernel of any FP-gr-injective preenvelope of a graded left R-module is FP-gr-injective;

(6) The cokernel of any gr-flat preenvelope of a graded right R-module is gr-flat;

(7) The kernel of any gr-flat (pre)cover of a graded right R-module is gr-flat.

Proof. (1) $\Rightarrow$ (2) Let M be any graded left R-module. Then M has an FP-gr-injective cover $f : C \to M$. Since $0 \to \mathrm{Ker}f \to C \to \mathrm{Im}f \to 0$ is exact, we have $\mathrm{Im}f$ is FP-gr-injective by Lemma 3.7. So the inclusion $\mathrm{Im}f \to M$ is a monic FP-gr-injective cover.

(2) $\Rightarrow$ (4) Let $f : L \to M$ be an FP-gr-injective precover of a graded left R-module M and $K = \mathrm{Ker}f$ and let $g : C \to M$ be a monic FP-gr-injective cover. Consider the pullback of f and g:

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, "\beta"] & C \arrow[d, "g"] \\ L \arrow[r, "f"] & M \end{tikzcd}$$

By the definition of precover, there is a factorization $C \to L \to M$ of the graded morphism $C \to M$. This means that there is a graded morphism $\gamma : C \to P$ such that $\alpha\gamma = 1_C$, and so $P \cong K \oplus C$ since $\mathrm{Ker}\alpha \cong K$. Similarly $P \cong L$. Thus $K \oplus C \cong L$, which gives that K is FP-gr-injective.

(4) $\Rightarrow$ (1) It is enough to show that any quotient of an FP-gr-injective left R-module is FP-gr-injective. Let M be a quotient of an FP-gr-injective left R-module. Note that M has an FP-gr-injective cover $f : C \to M$. Then f is an epimorphism. Since $\mathrm{Ker}f$ is FP-gr-injective, we have $\mathrm{Ker}f^+$ and C^+ are flat, and so M^+ is flat. Thus M is FP-gr-injective since R is left gr-coherent.

(1) $\Rightarrow$ (3) Let M be a graded right R-module. Then M has a gr-flat preenvelope $f : M \to L$. Consider the exact sequence $0 \to \mathrm{Im}f \to L \to L/\mathrm{Im}f \to 0$. Then $0 \to (L/\mathrm{Im}f)^+ \to L^+ \to \mathrm{Im}f^+ \to 0$ is exact in R-gr and L^+ is FP-gr-injective, and hence $\mathrm{Im}f^+$ is FP-gr-injective by Lemma 3.7. Therefore $f : M \to \mathrm{Im}f$ is an epic gr-flat preenvelope.

(3) $\Rightarrow$ (6) The proof is dual to that of (2) $\Rightarrow$ (4).

(6) $\Rightarrow$ (1) By a proof dual to that of (4) $\Rightarrow$ (1), we can show that any graded submodule of a gr-flat right R-module is gr-flat. Let M be any graded left R-module. Then FP-gr-id$M$= fd$M^+ \leq 1$, and hence l.FP-gr-dim$R \leq 1$.

(1) $\Leftrightarrow$ (5) is obvious.

(1) $\Leftrightarrow$ (7) By analogy with the proof of (1) $\Leftrightarrow$ (6). $\square$

Proposition 3.10. *The following are equivalent for a graded ring R of type G:*

(1) R is left gr-coherent and l.FP-gr-dim$R \leq 2$;

(2) Every graded left R-module has an FP-gr-injective cover with the unique mapping property.

Proof. (1) $\Rightarrow$ (2) Let M be any graded left R-module. Then M has an FP-gr-injective cover $f : C \to M$ by (1). It is enough to show that, for any FP-gr-injective left R-module A and any graded morphism $g : A \to C$ such that $fg = 0$, we have $g = 0$. In fact, there is a morphism in R-Mod $\beta : C/\mathrm{Img} \to M$ such that $\beta\eta = f$, where $\eta : C \to C/\mathrm{Img}$ is the natural map, and so there exists a graded morphism $\beta' : C/\mathrm{Img} \to M$ such that $\beta'\eta = f$ by [15, Lemma I.2.1]. Since l.FP-gr-dim$R \leq 2$, C/Img is FP-gr-injective. Thus there exists a graded morphism $\alpha : C/\mathrm{Img} \to C$ such that $\beta' = f\alpha$, which gives the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}g \arrow[r, "i"] & A \arrow[r, "g"] \arrow[dr, "0"'] & C \arrow[r, "\eta"] \arrow[d, "f"] & C/\mathrm{Img} \arrow[r] \arrow[dl, "\beta'"] & 0 \\
& & & M & &
\end{tikzcd}$$

Thus $f\alpha\eta = f$, and hence $\alpha\eta$ is an isomorphism. It follows that η is monic, and so $g = 0$.

(2) $\Rightarrow$ (1) We first prove that R is left gr-coherent. Let $\{C_i, \varphi^i_j\}$ be a direct system with each C_i FP-gr-injective. Then $\varinjlim C_i$ has an FP-gr-injective cover $\alpha : E \to \varinjlim C_i$ with the unique mapping property. Let $\alpha_i : C_i \to \varinjlim C_i$ satisfy $\alpha_i = \alpha_j\varphi^i_j$ whenever $i \leq j$. Then there is a graded morphism $f_i : C_i \to E$ such that $\alpha_i = \alpha f_i$ for any i. It follows that $\alpha f_i = \alpha f_j \varphi^i_j$, and so $f_i = f_j\varphi^i_j$ whenever $i \leq j$. Therefore, by the definition of direct limits and [15, Lemma I.2.1], there exists a graded morphism $\beta : \varinjlim C_i \to E$ such that the following diagram is commutative:

$$\begin{tikzcd}
C_i \arrow[r, "\alpha_i"] \arrow[d, "\varphi^i_j"'] & \varinjlim C_i \arrow[d, "\beta"] \\
C_j \arrow[r, "f_j"] & E
\end{tikzcd}$$

Thus $(\alpha\beta)\alpha_i = \alpha f_i = \alpha_i$ for any i, which means that $\alpha\beta = 1_{\varinjlim C_i}$ by the definition of direct limits, and so $\varinjlim C_i$ is a direct summand of E. Hence $\varinjlim C_i$ is FP-gr-injective, it follows that R is left gr-coherent by Theorem 3.2.

Next we prove that l.FP-gr-dim$R \leq 2$. Let M be any graded left R-module and

$$\begin{tikzcd} & & & & H \arrow[d, bend right, "\theta"'] & \\ 0 \arrow[r] & M \arrow[r] & E^0 \arrow[r, "\varphi"] & E^1 \arrow[r, "\psi"] \arrow[ur, "\lambda"] & N \arrow[r] \arrow[u, bend right, "\gamma"'] & 0 \end{tikzcd}$$

be exact with E^0 and E^1 gr-injective. Let $\theta : H \to N$ be an FP-gr-injective cover with the unique mapping property. Then there exists a graded morphism $\lambda : E^1 \to H$ such that $\psi = \theta\lambda$. Thus $\theta\lambda\varphi = \psi\varphi = 0 = \theta 0$, and so $\lambda\varphi = 0$, which implies that $\mathrm{Ker}\psi = \mathrm{Im}\varphi \subseteq \mathrm{Ker}\lambda$. Hence there is a graded morphism $\gamma : N \to H$ such that $\gamma\psi = \lambda$ by [15, Lemma I.2.1]. Therefore $\theta\gamma\psi = \psi$, and so $\theta\gamma = 1_N$ since ψ is epic. It follows that N is isomorphic to a direct summand of H, and thus N is FP-gr-injective, that is, l.FP-gr-dim$R \leq 2$. □

A graded ring R of type G is gr-regular if and only if all graded left (right) R-modules are flat by [15, Lemma I.5.4].

Proposition 3.11. *The following are equivalent for a graded ring R of type G:*

(1) R is gr-regular;

(2) Every graded left R-module is FP-gr-injective;

(3) Every finitely presented graded left R-module is projective;

(4) R is left gr-coherent and M is FP-gr-injective for any $M \in {}^{\perp}$gr-$\mathcal{FI}$;

(5) M is projective for any $M \in {}^{\perp}$gr-$\mathcal{FI}$;

(6) M is flat for any $M \in {}^{\perp}$gr-$\mathcal{FI}$;

(7) Every graded left R-module has an FP-gr-injective envelope with the unique mapping property;

(8) R is left gr-coherent and M has an FP-gr-injective envelope with the unique mapping property for any $M \in {}^{\perp}$gr-$\mathcal{FI}$.

Proof. (2) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (7) and (2) $\Rightarrow$ (4) $\Rightarrow$ (8) are obvious.

(4) $\Rightarrow$ (3) Let M be a finitely presented graded left R-module and $0 \to K \to P \to M \to 0$ be exact in R-gr with P finitely generated projective. Then K is finitely presented, and so K is FP-gr-injective by (4), which means that the sequence is split. Thus M is projective.

(1) $\Rightarrow$ (2) Since R is gr-regular, we have R is left gr-coherent by [15, Lemma I.5.4]. Let M be a graded left R-module. Then M^+ is flat, and so M is FP-gr-injective.

(2) $\Rightarrow$ (1) Let M be a graded right R-module. Then M^+ is FP-gr-injective, and so M is flat. Hence R is gr-regular.

(7) $\Rightarrow$ (2) Let M be any graded left R-module and $\mu_M : M \to E(M)$ be an FP-gr-injective envelope with the unique mapping property. Set $L =$

Cokerμ_M. Then L has an FP-gr-injective envelope $\mu_L : L \to E(L)$. Consider the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\mu_M"] \arrow[dr, "0"'] & E(M) \arrow[r, "\eta"] \arrow[d, "\mu_L\eta"'] & L \arrow[r] \arrow[dl, "\mu_L"] & 0 \\ & & E(L) & & \end{tikzcd}$$

Since $\mu_L\eta\mu_M = 0 = 0\mu_M$, then $\mu_L\eta = 0$. Thus $L = \mathrm{Im}\eta \subseteq \mathrm{Ker}\mu_L = 0$, and so M is FP-gr-injective.

(8) $\Rightarrow$ (4) Let $M \in {}^{\perp}$gr-$\mathcal{FI}$ and $\mu_M : M \to E(M)$ an FP-gr-injective envelope with the unique mapping property. Then Coker$\mu_M \in {}^{\perp}$gr-$\mathcal{FI}$. So M is FP-gr-injective by analogy with the proof of (7) $\Rightarrow$ (2). $\square$

4. Relative FP-gr-injective modules.

In this section, we prove that if R is right gr-coherent, then

(1) (gr-$\mathcal{FI}_n$, gr-$\mathcal{FI}_n^{\perp}$) is a perfect cotorsion theory whenever FP-gr-id$(R_R) \leq n$,

(2) (${}^{\perp}$gr-$\mathcal{FI}_n$, gr-$\mathcal{FI}_n$) is a cotorsion theory, where gr-$\mathcal{FI}_n$ is the class of all graded right R-modules of FP-gr-injective dimension at most n.

Lemma 4.1. *Let R be a graded ring and M a graded left R-module. Then fd$M$ = gr-idM^+ = FP-gr-idM^+.*

Proof. By $\mathrm{EXT}_R^i(N, M^+) \cong \mathrm{Tor}_i^R(N, M)^+$ for all $i \geq 1$ and any graded right R-module N. $\square$

Lemma 4.2. *Let R be right gr-coherent and M a graded right R-module. Then fdM^+ = FP-gr-id$M$.*

Proof. By $\mathrm{EXT}_R^i(N, M)^+ \cong \mathrm{Tor}_i^R(N, M^+)$ for all $i \geq 1$ and any finitely presented graded right R-module N. $\square$

For a fixed non-negative integer n, let gr-$\mathcal{FI}_n$ (gr-$\mathcal{F}_n$) be the class of all graded right (left) R-modules of FP-gr-injective (flat) dimension at most n. Now we have the following result.

Theorem 4.3. *Let n be a fixed non-negative integer. Then the following hold:*

(1) If R is right gr-coherent with FP-gr-id$(R_R) \leq n$, then (gr-$\mathcal{FI}_n$, gr-$\mathcal{FI}_n^{\perp}$) is a perfect cotorsion theory.

(2) For any graded ring R, (gr-$\mathcal{F}_n$, gr-$\mathcal{F}_n^{\perp}$) is a perfect hereditary cotorsion theory.

Proof. (1) Let $0 \to A \to B \to C \to 0$ be gr-pure in gr-R with $B \in$ gr-$\mathcal{FI}_n$. Then $0 \to C^+ \to B^+ \to A^+ \to 0$ splits by [9, Proposition 3.1], and hence $A^+, C^+ \in$ gr-$\mathcal{F}_n$, which implies that $A, C \in$ gr-$\mathcal{FI}_n$. Therefore, by [9, Lemma 3.2], if $L \in$ gr-$\mathcal{FI}_n$, then L can be written as the direct union of a continuous chain of graded submodules $(L_\alpha)_{\alpha<\lambda}$ with λ an ordinal number such that $L_0 \in$ gr-$\mathcal{FI}_n$, $L_{\alpha+1}/L_\alpha \in$ gr-$\mathcal{FI}_n$ when $\alpha + 1 < \lambda$ with $\mathrm{Card}(L_0)$, $\mathrm{Card}(L_{\alpha+1}/L_\alpha) \leq \mathrm{Card}(R)\mathrm{Card}(G)$. If N is a graded right R-module such that $\mathrm{Ext}^1_{R\text{-gr}}(L_0, N) = 0$ and $\mathrm{Ext}^1_{R\text{-gr}}(L_{\alpha+1}/L_\alpha, N) = 0$ whenever $\alpha + 1 < \lambda$, then $\mathrm{Ext}^1_{R\text{-gr}}(L, N) = 0$ by the proof of [9, Proposition 3.3]. Thus gr-$\mathcal{FI}_n^\perp = X^\perp$, where X is a set of representatives of all graded modules $H \in$ gr-$\mathcal{FI}_n$ with $\mathrm{Card}(H) \leq \mathrm{Card}(R)\mathrm{Card}(G)$. We note that gr-$\mathcal{FI}_n$ is closed under direct sums, extensions, direct limits since R is right gr-coherent, and contains all gr-projective modules since FP-gr-id$(R_R) \leq n$. Hence (gr-$\mathcal{FI}_n$, gr-$\mathcal{FI}_n^\perp$) is a cotorsion theory by [1, Corollary 2.13]. Since (gr-$\mathcal{FI}_n$, gr-$\mathcal{FI}_n^\perp$) is cogenerated by the set X, (gr-$\mathcal{FI}_n$, gr-$\mathcal{FI}_n^\perp$) is a complete cotorsion theory by [1, Corollary 2.7]. Moreover, (gr-$\mathcal{FI}_n$, gr-$\mathcal{FI}_n^\perp$) is a perfect cotorsion theory since gr-$\mathcal{FI}_n$ is closed under direct limits by Lemma 3.1.

(2) Note that gr-$\mathcal{F}_n$ is closed under direct sums, extensions, direct limits, gr-pure submodules, cokernels of gr-pure monomorphisms and contains all gr-projective modules. An argument similar to that of (1) shows that (gr-$\mathcal{F}_n$, gr-$\mathcal{F}_n^\perp$) is a perfect cotorsion theory. On the other hand, let $0 \to A \to B \to C \to 0$ be exact in R-gr with B, $C \in$ gr-$\mathcal{F}_n$, then $A \in$ gr-$\mathcal{F}_n$. So (gr-$\mathcal{F}_n$, gr-$\mathcal{F}_n^\perp$) is hereditary. $\square$

Lemma 4.4. *Let R be a graded ring of type G. Then M is an FP-gr-injective right R-module if and only if $EXT^1_R(R(\sigma)/A, M) = 0$ for all finitely generated graded submodules A of $R(\sigma)_R$ and all $\sigma \in G$.*

Proof. "⇒" is obvious.

"⇐" Let N be a finitely presented graded right R-module. Then there is an exact sequence $0 \to A \to \bigoplus_{\sigma\in G_0} R(\sigma) \to N \to 0$, where G_0 is a finite subset of G and A is finitely generated. So

$$N \cong (\oplus_{\sigma\in G_0} R(\sigma))/A \cong \oplus_{\sigma\in G_0}(R(\sigma) + A/A) \cong \oplus_{\sigma\in G_0}(R(\sigma)/R(\sigma) \cap A).$$

Consider the sequence $0 \to A \to R(\sigma) + A \to (R(\sigma) + A)/A \to 0$. Since $A, R(\sigma) + A$ are finitely generated, we have $R(\sigma)/R(\sigma) \cap A \cong (R(\sigma) + A)/A$ is finitely presented, and so $R(\sigma) \cap A$ is finitely generated. Thus $\mathrm{EXT}^1_R(N, M) \cong \mathrm{EXT}^1_R(\bigoplus_{\sigma\in G_0}(R(\sigma)/R(\sigma) \cap A), M) = 0$, which implies that M is FP-gr-injective. $\square$

Theorem 4.5. *The following hold for a right gr-coherent ring R of type G and a fixed integer $n \geq 0$:*

(1) Every graded left R-module has a gr-$\mathcal{F}_n$-preenvelope.
(2) $({}^{\perp}$gr-$\mathcal{FI}_n$, gr-$\mathcal{FI}_n)$ is a cotorsion theory.

Proof. (1) Analogous to the ungraded case.

(2) Let M be a graded right R-module. M admits a gr-injective resolution

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^{n-1} \longrightarrow E^n \longrightarrow \cdots .$$

Write $L^n = \mathrm{Im}(E^{n-1} \to E^n)$, $L^0 = M$. Then $M \in$ gr-$\mathcal{FI}_n$ if and only if L^n is FP-gr-injective if and only if $\mathrm{EXT}^1_R(R(\sigma)/A, L^n) = 0$ for all finitely generated graded submodules A of $R(\sigma)_R$ and all $\sigma \in G$ by Lemma 4.4. This means that $\mathrm{EXT}^{n+1}_R(R(\sigma)/A, M) = 0$ for all finitely generated graded submodules A of $R(\sigma)_R$ and all $\sigma \in G$ by dimension shifting. Denote by K_A the n-th syzygy module of the finitely presented graded right R-module $R(\sigma)/A$. Then $\mathrm{EXT}^{n+1}_R(R(\sigma)/A, M) = 0$ if and only if $\mathrm{EXT}^1_R(K_A, M) = 0$. Set $X_\sigma = \bigoplus K_A$, where the sum is over all finitely generated graded submodules A of $R(\sigma)_R$. Let

$$X = \{\oplus_{\sigma \in G_0} X_\sigma \mid G_0 \text{ is a finite subset of } G\}.$$

Then X is a set and gr-$\mathcal{FI}_n = X^{\perp}$. Thus $({}^{\perp}$gr-$\mathcal{FI}_n$, gr-$\mathcal{FI}_n)$ is a cotorsion theory. □

Proposition 4.6. *Let R be a right gr-coherent ring of type G and n a fixed non-negative integer. Then the following are equivalent:*
(1) FP-gr-id$(R_R) \leq n$;
(2) Every graded left R-module has a monic gr-$\mathcal{F}_n$-preenvelope;
(3) Every (FP-) gr-injective left R-module belongs to gr-$\mathcal{F}_n$;
(4) Every graded right R-module has an epic gr-$\mathcal{FI}_n$-cover;
(5) Every gr-flat right R-module belongs to gr-$\mathcal{FI}_n$.

Proof. (1) ⇒ (2) Let M be a graded left R-module. Then M has a gr-$\mathcal{F}_n$-preenvelope $f : M \to L$ by Theorem 4.5. Since there is an exact sequence $0 \to M \to (\bigoplus_{\sigma\in G} R(\sigma))^+$ and $\mathrm{fd}(\bigoplus_{\sigma\in G} R(\sigma))^+ = \text{FP-gr-id}\bigoplus_{\sigma\in G} R(\sigma) \leq n$ by Proposition 2.1 and Lemma 3.7, we see that f is monic.

(2) ⇒ (3) Let M be an FP-gr-injective left R-module. Then there exists a gr-pure exact sequence $0 \to M \to L$ with $L \in$ gr-$\mathcal{F}_n$ by (2) and Proposition 2.1, and hence $L^+ \to M^+ \to 0$ splits. So $M \in$ gr-$\mathcal{F}_n$ by Lemma 4.1.

(3) ⇒ (1) Since $(R_R)^+$ is gr-injective, $\mathrm{fd}(R_R)^+ \leq n$ by (3). Thus FP-gr-id$(R_R) = \mathrm{fd}(R_R)^+ \leq n$.

(1) ⇒ (4) By Theorem 4.3. (4) ⇒ (1) and (5) ⇒ (1) are obvious.

(3) ⇒ (5) Let M be a gr-flat right R-module. Then FP-gr-id$M = \mathrm{fd}M^+ \leq n$ by (3). □

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