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EXPONENTIAL GENERALIZED DISTRIBUTIONS

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ABSTRACT. In this paper we generalize the Fourier transform from the space of tempered distributions to a bigger space called exponential generalized distributions. For that purpose we replace the Schwartz space $\mathcal{S}$ by a smaller space $\mathcal{X}_0$ of smooth functions such that, among other properties, decay at infinity faster than any exponential. The construction of $\mathcal{X}_0$ is such that this space of test functions is closed for derivatives, for Fourier transform and for translations. We equip $\mathcal{X}_0$ with an appropriate locally convex topology and we study it's dual $\mathcal{X}_0'$; we call $\mathcal{X}_0'$ the space of exponential generalized distributions. The space $\mathcal{X}_0'$ contains all the Schwartz tempered distributions, is closed for derivatives, and both, translations and Fourier transform, are vector and topological automorphisms in $\mathcal{X}_0'$. As non trivial examples of elements in $\mathcal{X}_0'$, we show that some multipole series appearing in physics are convergent in this space.

1. Introduction

Laurent Schwartz in [18] introduced an important subspace of the space $\mathcal{D}'$ of all distributions in $\mathbb{R}^N$: the space $\mathcal{S}'$ of tempered distributions. This is the strong dual of the Schwartz space $\mathcal{S}$. For the reader's convenience we recall the definiton of $\mathcal{S}$ and certain properties needed later on. The Schwartz test function space $\mathcal{S}$ is the vector space of smooth functions[1] defined by

$$\mathcal{S} = \{\varphi \in C^\infty; \quad \forall \alpha, \beta \in \mathbb{N}^N \quad x^\alpha \partial^\beta \varphi \in L^\infty\} \ . \tag{1.1}$$

The topology of $\mathcal{S}$ is determined by the following family (indexed on the set $\mathbb{N}$ of nonnegative integers) of seminorms:

$$\varphi \quad \longrightarrow \quad \sup_{|\alpha|,|\beta|\leq m} \| \ x^\alpha \partial^\beta \varphi \ \|_{L^\infty} \ , \tag{1.2}$$

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[1]We always consider complex functions defined in $\mathbb{R}^N$.

where, for a bounded continuous function ψ, $\| \psi \|_{L^\infty}$ is the usual sup-norm defined by $\| \psi \|_{L^\infty} = \sup_{x \in \mathbb{R}^N} |\psi(x)|$. The space $\mathcal{S}$ is a locally convex separable (Hausdorff) vector space; it is metrizable, Fréchet, Montel, reflexive, bornological (Mackey) and $\mathcal{D}$ is continuous and densely embedded in $\mathcal{S}$.

The space $\mathcal{S}$ is closed for derivatives and the derivative operators ∂^α are linear continuous from $\mathcal{S}$ into $\mathcal{S}$. The translation operator[2] τ_a is a vector and topological automorphism in $\mathcal{S}$ with inverse τ_{-a}.

For $w \in \mathbb{R} \backslash \{0\}$ we define the Fourier operator $\mathcal{F}_w$ by

$$\mathcal{F}_w \varphi(\xi) = \int_{\mathbb{R}^N} e^{-iwx.\xi} \varphi(x) dx \ , \tag{1.3}$$

where $\varphi \in \mathcal{S}$ and $x.\xi = x_1\xi_1 + \cdots + x_N\xi_N$ is the usual inner product[3] in $\mathbb{R}^N$. The operator $\mathcal{F}_w$ is a vector and topological automorphism in $\mathcal{S}$.

The space of tempered distributions $\mathcal{S}'$ is defined as the strong dual of $\mathcal{S}$. It is a locally convex separable (Hausdorff) vector space, non-metrizable, Montel, reflexive, bornological (Mackey) and is continuous and densely embedded in $\mathcal{D}'$. Moreover, $\mathcal{S}$ is continuous and densely embedded in $\mathcal{S}'$. A distribution $T \in \mathcal{D}'$ is tempered iff T is of the form $\partial^\alpha F$ where F is a continuous function of slow growth (that is, F is bounded by a polynomial).

The operators ∂^α and τ_a are extended to $\mathcal{S}'$ by transposition, in the usual way:

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle \ , \tag{1.4}$$

$$\langle \tau_a T, \varphi \rangle = \langle T, \tau_{-a} \varphi \rangle \ , \tag{1.5}$$

where $T \in \mathcal{S}'$, $\varphi \in \mathcal{S}$ and $\langle . , . \rangle$ denotes the dual product between $\mathcal{S}'$ and $\mathcal{S}$. The derivatives are linear continuous operators from $\mathcal{S}'$ into $\mathcal{S}'$. The translation operator τ_a is an automorphism in $\mathcal{S}'$ with inverse τ_{-a}.

By transposition the operator $\mathcal{F}_w$ can also be extended to $\mathcal{S}'$ in the standard way

$$\langle \mathcal{F}_w T, \varphi \rangle = \langle T, \mathcal{F}_w \varphi \rangle \ . \tag{1.6}$$

The operator $\mathcal{F}_w$ is a vector and topological automorphisms in $\mathcal{S}'$ and we recall (for the proofs see Schwartz [18]) some of its properties:

$$(\mathcal{F}_w)^{-1} = \left(\frac{|w|}{2\pi} \right)^N \mathcal{F}_{-w} \ ; \tag{1.7}$$

$$\mathcal{F}_w \partial^\beta T = (iw\xi)^\beta \mathcal{F}_w T \ ; \tag{1.8}$$

[2]For $a \in \mathbb{R}^N$ and ψ continuous, the translation operator τ_a is defined by $\tau_a \psi(x) = \psi(x - a)$.

[3]We shall use the same notation for the usual inner product in $\mathbb{C}^N$.

$$\partial^\beta \mathcal{F}_w T = \mathcal{F}_w \left[(-iwx)^\beta T \right] \ ; \tag{1.9}$$

$$\mathcal{F}_w \tau_a T = e^{-iwa.\xi} \mathcal{F}_w T \ ; \tag{1.10}$$

$$\mathcal{F}_w \left(e^{iwa.x} T \right) = \tau_a \left(\mathcal{F}_w T \right) \ ; \tag{1.11}$$

$$\mathcal{F}_w 1 = \left(\frac{2\pi}{|w|} \right)^N \delta \ ; \tag{1.12}$$

$$\mathcal{F}_w \partial^\beta \delta = (iw\xi)^\beta \ ; \tag{1.13}$$

$$\mathcal{F}_w \left(\mathcal{F}_w T \right) = \left(\frac{2\pi}{|w|} \right)^N T(-\xi) \, ; \tag{1.14}$$

$$\mathcal{F}_w T = \mathcal{F}_{w_0} T \left(\frac{w}{w_0} \, \xi \right) . \tag{1.15}$$

In formulas (1.12) and (1.13) δ is the Dirac distribution at the point 0; in (1.15) w_0 is a non null real number. We write $\mathcal{F}$ for $\mathcal{F}_1$ and we call $\mathcal{F}\varphi$ the Fourier transform of φ.

We need also the space $\mathcal{G}_c$, where $c \in]0, +\infty[$, defined by

$$\mathcal{G}_c = \{ \varphi \in C^\infty ; \quad \exists p \in \mathcal{P} \quad \varphi(x) = p(x) e^{-\frac{c}{2}|x|^2} \} \ , \tag{1.16}$$

where $\mathcal{P}$ denotes the space of all polynomials in $\mathbb{R}^N$ and $|x| = \sqrt{x_1^2 + \cdots + x_N^2}$. We shall write $\mathcal{G}$ instead of $\mathcal{G}_1$. The space $\mathcal{G}$ is important because it is a dense subset of $\mathcal{S}$. The proof is trivial because $\mathcal{G}$ contains all the Hermite functions and these functions are dense in $\mathcal{S}$; the proof of this last statement is not an easy one and can be found in [22] or, for the one dimensional case, in [6].

Incidentally we mention that the space $\mathcal{G}$ has been studied in [13]. In that paper a locally convex topology is introduced in $\mathcal{G}$ and the dual $\mathcal{G}'$ is studied in detail. For our purposes we do not need to introduce a topology in $\mathcal{G}$ because we just use the property that the set $\mathcal{G}$ is dense in $\mathcal{S}$, and this requires only the topology of $\mathcal{S}$.

The aim of this paper is to construct an extension of the space $\mathcal{S}'$, using the same duality method used by L. Schwartz in his distribution theory. For that purpose, we construct a space of exponential generalized distributions $\mathcal{X}_0'$ which is closed for derivatives, translations and Fourier transform. These operators are linear and continuous of $\mathcal{X}_0'$ into itself. The translations and

the Fourier transform define a vector and topological automorphism in $\mathcal{X}_0'$, generalizing the beautiful result of Schwartz for tempered distributions.

This paper is organized as follows: In Section 2 we construct the space $\mathcal{D}_{e^-}$ of all C^∞ functions φ defined on $\mathbb{R}^N$, decaying exponentially at infinity as well as their derivatives: $|\partial^\alpha \varphi(x)| \leq Ce^{-k|x|}$. We show that this space is closed for derivatives, for translations and for the product by functions of slow exponential growth. We equip it with a locally convex structure and we study some of their topological properties.

In Section 3 we study the space of exponential distributions, that is the dual space $\mathcal{D}_{e^-}'$ of $\mathcal{D}_{e^-}$. The space $\mathcal{D}_{e^-}'$ is closed for derivatives and translations. We also study the topological properties of $\mathcal{D}_{e^-}'$ and prove that the space $\mathcal{S}'$ is a continuous and dense subspace of $\mathcal{D}_{e^-}'$.

Since $\mathcal{D}_{e^-}$ is not closed for the Fourier operator, in Section 4 we construct the test function space $\mathcal{X}_0$ of all complex-valued functions φ defined on $\mathbb{R}^N$ such that both φ and $\mathcal{F}_\omega \varphi$ belong to $\mathcal{D}_{e^-}$. This space is closed for derivatives, translations and Fourier transform. We also prove that any function of $\mathcal{X}_0$ may be extended to $\mathbb{C}^N$ as an entire function. We introduce a locally convex structure in $\mathcal{X}_0$, we show that $\mathcal{X}_0$ is continuous and densely embedded in $\mathcal{S}$ and we prove the continuity of the derivative, translation and Fourier operators in $\mathcal{X}_0$.

In Section 5 we study the strong dual $\mathcal{X}_0'$ of $\mathcal{X}_0$, that we call the space of exponential generalized distributions. The space $\mathcal{X}_0'$ is closed for the above mentioned operators, and both translations and Fourier transform are vector and topological automorphism in $\mathcal{X}_0'$. The space $\mathcal{S}'$ is continuous and densely embedded in $\mathcal{X}_0'$. In Theorem 5.3 we prove that all continuous function with exponential growth are in $\mathcal{X}_0'$. A structure theorem for the space $\mathcal{X}_0'$ remains an open problem but we hope to solve it in a near future. To close the section we give some examples of convergent multipole series in $\mathcal{X}_0'$ appearing as solutions of ODE.

2. The test function space $\mathcal{D}_{e^-}$

We say that a continuous function φ defined on $\mathbb{R}^N$ decays exponentially at infinity iff, for each $k \in \mathbb{N}$, there is an upper bound of the type $|\varphi(x)| \leq Ce^{-k|x|}$, where $C > 0$ is a real number depending on k. We denote by $\mathcal{D}_{e^-}$ the vector space of all C^∞ functions defined on $\mathbb{R}^N$, decaying exponentially at infinity as well as their derivatives:

(2.1)

$$\mathcal{D}_{e^-} = \{\varphi \in C^\infty;\ \forall \alpha \in \mathbb{N}^N\ \forall k \in \mathbb{N}\ \exists C > 0\ \forall x \in \mathbb{R}^N\ \ |\partial^\alpha \varphi(x)| \leq Ce^{-k|x|}\}\ .$$

Notice that the space $\mathcal{G}_c$ defined in (1.16) is included in $\mathcal{D}_{e^-}$.

We say that a continuous function f defined on $\mathbb{R}^N$ has an exponential growth iff there is an upper bound of the type $|f(x)| \leq Ce^{k|x|}$, where $C > 0$ and $k \in \mathbb{N}$. We denote by $\mathcal{D}_{e^+}$ the vector space of all C^∞ functions f defined on $\mathbb{R}^N$, such that f and its derivatives have an exponential growth:

(2.2)
$$\mathcal{D}_{e^+} = \{f \in C^\infty; \quad \forall \alpha \in \mathbb{N}^N \ \ \exists C > 0 \ \ \exists k \in \mathbb{N} \ \ \forall x \in \mathbb{R}^N \quad |\partial^\alpha f(x)| \leq Ce^{k|x|}\} \ .$$

If $f \in \mathcal{D}_{e^+}$ we say that f has a slow exponential growth.

It is obvious that the exponentials $e^{z.x}$, with $z \in \mathbb{C}^N$, and the polynomials are functions of $\mathcal{D}_{e^+}$ and that $\mathcal{D}_{e^-}$ is a vector subspace of $\mathcal{D}_{e^+}$.

The following theorem collects some properties of the space $\mathcal{D}_{e^-}$.

Theorem 2.1. *(i) The space $\mathcal{D}_{e^-}$ is a dense subspace of the Schwartz space $\mathcal{S}$.*
(ii) For all $a \in \mathbb{R}^N$ we have $\tau_a(\mathcal{D}_{e^-}) = \mathcal{D}_{e^-}$.
(iii) For all φ in $\mathcal{D}_{e^-}$ and all $\alpha \in \mathbb{N}^N$ we have $\partial^\alpha \varphi$ in $\mathcal{D}_{e^-}$.
(iv) For all φ in $\mathcal{D}_{e^-}$ and all f in $\mathcal{D}_{e^+}$ we have $f\varphi$ in $\mathcal{D}_{e^-}$.
(v) For all φ_1 and φ_2 in $\mathcal{D}_{e^-}$ we have $\varphi_1\varphi_2$ in $\mathcal{D}_{e^-}$.
(vi) For all φ in $\mathcal{D}_{e^-}$ and all exponential $e^{z.x}$, with $z \in \mathbb{C}^N$, we have $e^{z.x}\varphi$ in $\mathcal{D}_{e^-}$.
(vii) For all φ in $\mathcal{D}_{e^-}$ and all polynomial p we have $p\varphi$ in $\mathcal{D}_{e^-}$.
(viii) For any φ in $\mathcal{D}_{e^-}$, we have $\mathcal{F}_w\varphi$ in $\mathcal{S}$ and it can be extended to $\mathbb{C}^N$ as an entire function.
(ix) For all φ in $\mathcal{D}_{e^-}$ and all $\lambda \in \mathbb{R} \setminus \{0\}$ we have $\varphi(\lambda x)$ in $\mathcal{D}_{e^-}$.

Proof. Statement (i) follows from the fact that $\mathcal{G}$ is included in $\mathcal{D}_{e^-}$ and $\mathcal{G}$ is dense in $\mathcal{S}$. The proofs of statements (ii) to (vii) and (ix) are straightforward.

To prove (viii) let φ in $\mathcal{D}_{e^-}$. We have $\mathcal{F}_w\varphi \in \mathcal{S}$ because $\varphi \in \mathcal{S}$ and the operator $\mathcal{F}_w$ is a vector and topological automorphism in $\mathcal{S}$. Furthermore, for each $j = 1, \ldots, N$ and each $z = (z_1, \ldots, z_N)$ in $\mathbb{C}^N$, the integrals

$$\int_{\mathbb{R}^N} e^{-iwx.z}\varphi(x)dx \qquad \text{and} \qquad \int_{\mathbb{R}^N} \partial_{z_j}\left(e^{-iwx.z}\varphi(x)\right)dx$$

are absolutely convergent; this proves that $\mathcal{F}_w\varphi$ can be extended to $\mathbb{C}^N$ as an entire function. □

We introduce a locally convex structure in $\mathcal{D}_{e^-}$ with the family (indexed in $\mathbb{N}$) of seminorms defined by

$$\nu_k(\varphi) = \sup_{|\alpha| \leq k} \| \, e^{k|x|}\partial^\alpha \varphi \, \|_{L^\infty} \ . \tag{2.3}$$

Notice that a sequence (φ_n) in $\mathcal{D}_{e^-}$ is bounded iff

$$\forall \alpha \in \mathbb{N}^N \quad \forall k \in \mathbb{N} \quad \exists C \geq 0 \quad \forall n \in \mathbb{N} \quad \forall x \in \mathbb{R}^N \qquad e^{k|x|}|\partial^\alpha \varphi_n(x)| \leq C \tag{2.4}$$

and converges to zero iff

$$\forall \alpha \in \mathbb{N}^N \quad \forall k \in \mathbb{N} \quad e^{k|x|}|\partial^\alpha \varphi_n| \to 0 \ \text{ uniformly in } \mathbb{R}^N. \tag{2.5}$$

The space $\mathcal{D}$ is clearly included in $\mathcal{D}_{e^-}$. The inclusion is continuous because a filter (φ_j) converges to 0 in $\mathcal{D}$ iff there exists a compact subset K of $\mathbb{R}^N$ such that all the φ_j have their supports in K and, for each $\alpha \in \mathbb{N}^N$, $\partial^\alpha \varphi_j \to 0$ uniformly in $\mathbb{R}^N$. This implies that, for each $k \in \mathbb{N}$, $e^{k|x|}\partial^\alpha \varphi_j \to 0$ uniformly in $\mathbb{R}^N$, that is (φ_j) converges to 0 in $\mathcal{D}_{e^-}$.

The space $\mathcal{D}$ is dense in $\mathcal{D}_{e^-}$. To prove this, we fix a function ψ in $\mathcal{D}$ such that $0 \le \psi \le 1$, $\psi(x) = 1$ if $|x| \le 1$ and $\psi(x) = 0$ if $|x| \ge 2$ and, for each natural n, we write $\psi_n(x) = \psi(\frac{1}{n}\, x)$. Given a function φ in $\mathcal{D}_{e^-}$, it is easily seen that the sequence $(\psi_n \varphi)$ is in $\mathcal{D}$ and converges to φ in $\mathcal{D}_{e^-}$.

We saw in Theorem 2.1 that $\mathcal{D}_{e^-}$ is a dense subspace of $\mathcal{S}$. From the seminorms (1.2) and (2.3) we conclude that the injection of $\mathcal{D}_{e^-}$ into $\mathcal{S}$ is continuous. Thus, we have[4]:

$$\mathcal{D} \underset{d}{\hookrightarrow} \mathcal{D}_{e^-} \underset{d}{\hookrightarrow} \mathcal{S}\ . \tag{2.6}$$

In order to prove the main theorem of this section, Theorem 2.2 below, we need the following useful lemma.

Lemma 1. *Let (φ_n) be a bounded sequence in $\mathcal{D}_{e^-}$ and let $\varphi \in \mathcal{S}$ be such that $\varphi_n \to \varphi$ in $\mathcal{S}$. Then $\varphi \in \mathcal{D}_{e^-}$ and $\varphi_n \to \varphi$ in $\mathcal{D}_{e^-}$.*

Proof. We first prove that φ is in $\mathcal{D}_{e^-}$. On the one hand, the boundedness of (φ_n) allow us to use (2.4). On the other hand, as $\varphi_n \to \varphi$ in $\mathcal{S}$, we know that, for each $\alpha \in \mathbb{N}^N$, $(\partial^\alpha \varphi_n)$ converges uniformly to $\partial^\alpha \varphi$ in $\mathbb{R}^N$. Fix $\alpha \in \mathbb{N}^N$ and $k \in \mathbb{N}$, and let $n \to +\infty$ in (2.4):

$$e^{k|x|}|\partial^\alpha \varphi(x)| \le C\ .$$

As α and k are arbitrary, this means that φ is in $\mathcal{D}_{e^-}$.

It remains to prove that $\varphi_n \to \varphi$ in $\mathcal{D}_{e^-}$. We write $\psi_n = \varphi_n - \varphi$ and we prove that $\psi_n \to 0$ in $\mathcal{D}_{e^-}$. Fix $\alpha \in \mathbb{N}^N$ and $k \in \mathbb{N}$; from the boundedness of (φ_n) in $\mathcal{D}_{e^-}$, we know that (ψ_n) is bounded in $\mathcal{D}_{e^-}$ and, by (2.4), this implies the existence of a real number $C \ge 0$ such that, for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}^N$,

$$e^{(k+1)|x|}\, |\partial^\alpha \psi_n(x)| \le C\ .$$

We may rewrite this inequality in the form

$$e^{k|x|}\, |\partial^\alpha \psi_n(x)| \le C e^{-|x|}\ . \tag{2.7}$$

The sequence $(\partial^\alpha \psi_n)$ converges to zero uniformly in $\mathbb{R}^N$, because it converges to zero in $\mathcal{S}$. Therefore, the sequence $e^{k|x|}\, |\partial^\alpha \psi_n|$ converges to zero uniformly

[4]We use $\hookrightarrow$ for continuous injection and $\underset{d}{\hookrightarrow}$ for continuous and dense injection.

in the compact subsets of $\mathbb{R}^N$. The function $Ce^{-|x|}$ tends to zero at infinity. From these two facts we conclude, by (2.7), that $e^{k|x|}\,|\partial^\alpha \psi_n|$ converges to zero uniformly in $\mathbb{R}^N$. This means, by (2.5), that $\psi_n \to 0$ in $\mathcal{D}_{e^-}$. □

Now we are ready to state the main theorem of this section:

Theorem 2.2. *The space $\mathcal{D}_{e^-}$ is: (i) Hausdorff; (ii) metrizable; (iii) Fréchet; (iv) Montel; (v) reflexive; (vi) Mackey (bornological).*

Proof. (i) $\mathcal{D}_{e^-}$ is obviously Hausdorff because, for all $\varphi \in \mathcal{D}_{e^-} \setminus \{0\}$, there exists $a \in \mathbb{R}^N$ such that $\varphi(a) \neq 0$, thus

$$\nu_0(\varphi) \;=\; \|\varphi\|_{L^\infty} \;\geq\; |\varphi(a)| \;>\; 0 \quad .$$

(ii) $\mathcal{D}_{e^-}$ is metrizable because the familly of seminorms (2.3) is countable.

(iii) We just have to prove that $\mathcal{D}_{e^-}$ is complete. Let (φ_n) be a Cauchy sequence in $\mathcal{D}_{e^-}$. By (2.6), (φ_n) is also a Cauchy sequence in $\mathcal{S}$ and, as we know that $\mathcal{S}$ is complete, there exists φ in $\mathcal{S}$ such that φ_n converges to φ in $\mathcal{S}$. By Lemma 1, we see that φ belongs to $\mathcal{D}_{e^-}$ and that φ_n converges to φ in $\mathcal{D}_{e^-}$. This proves that $\mathcal{D}_{e^-}$ is complete.

(iv) The space $\mathcal{D}_{e^-}$ is barrelled because it is a Frćhet space. Thus, we just have to prove that, if L is a bounded subset of $\mathcal{D}_{e^-}$, then L is relatively compact in $\mathcal{D}_{e^-}$, or, equivalently, that every sequence (φ_n) in L has a convergent subsequence (in the $\mathcal{D}_{e^-}$ topology).

The set L is bounded in $\mathcal{D}_{e^-}$ and, by (2.6), L is also bounded in $\mathcal{S}$. But $\mathcal{S}$ is a Montel space, so there exists φ in $\mathcal{S}$ and a subsequence (φ_{m_n}) of the sequence (φ_n) such that $\varphi_{m_n} \to \varphi$ in $\mathcal{S}$. By Lemma 1 we see that φ is in $\mathcal{D}_{e^-}$ and $\varphi_{m_n} \to \varphi$ in $\mathcal{D}_{e^-}$. This proves that $\mathcal{D}_{e^-}$ is a Montel space.

(v) The space $\mathcal{D}_{e^-}$ is reflexive because it is a Montel space.

(vi) The space $\mathcal{D}_{e^-}$ is a Mackey (or bornological) space because all metrizable locally convex spaces are bornological. □

Remark 1. Using the seminorms (2.3) it is obvious that the derivative operators ∂^α and the translation operators τ_a are linear continuous from $\mathcal{D}_{e^-}$ into $\mathcal{D}_{e^-}$. Moreover, as τ_a is bijective, with inverse τ_{-a}, we see that the translations are vector and topological automorphisms in $\mathcal{D}_{e^-}$.

3. Exponential distributions

The dual $\mathcal{D}'_{e^-}$ of $\mathcal{D}_{e^-}$ is called the space of exponential distributions. We consider in $\mathcal{D}'_{e^-}$ the strong dual topology. From (2.6) we have the inclusions

$$\mathcal{S}' \hookrightarrow \mathcal{D}'_{e^-} \hookrightarrow \mathcal{D}' \; .$$

Obviously $\mathcal{D}'_{e^-}$ is dense in $\mathcal{D}'$ because $\mathcal{D}$ is included in $\mathcal{D}'_{e^-}$ and it is well known that $\mathcal{D}$ is dense in $\mathcal{D}'$.

The space $\mathcal{D}_{e^-}$ is continuously embedded in $\mathcal{S}$ and, as we know that $\mathcal{S}$ is continuously embedded in $\mathcal{S}'$, we see that $\mathcal{D}_{e^-}$ is continuously embedded in $\mathcal{D}'_{e^-}$. Next, we prove that $\mathcal{D}_{e^-}$ is dense in $\mathcal{D}'_{e^-}$. It is sufficient to prove that every continuous linear form f on $\mathcal{D}'_{e^-}$ that is null on $\mathcal{D}_{e^-}$ is identically null. Taking such a form f, we have $f \in (\mathcal{D}'_{e^-})'$ and, as $\mathcal{D}_{e^-}$ is reflexive, $f \in \mathcal{D}_{e^-}$; this implies that

$$\langle f, f\rangle = \int_{\mathbb{R}^N} |f(x)|^2 dx = 0$$

and this shows that f is the null form.

Collecting all the previous results, we have the following chain of continuous and dense embeddings:

$$\mathcal{D} \underset{d}{\hookrightarrow} \mathcal{D}_{e^-} \underset{d}{\hookrightarrow} \mathcal{S} \underset{d}{\hookrightarrow} \mathcal{S}' \underset{d}{\hookrightarrow} \mathcal{D}'_{e^-} \underset{d}{\hookrightarrow} \mathcal{D}' \ . \tag{3.1}$$

Theorem 3.1. *The space $\mathcal{D}'_{e^-}$ is: (i) Hausdorff; (ii) complete; (iii) Montel; (iv) reflexive.*

Proof. (i) The strong dual topology (as well as the weak dual topology) is always separated.

(ii) The space $\mathcal{D}'_{e^-}$ is complete because it is the dual of a separated Mackey space.

(iii) The dual of a Montel space is always a Montel space.

(iv) All Montel spaces are reflexive. □

Similarly to what has been done in the tempered distributions, the operators ∂^α and τ_a can be extended to $\mathcal{D}'_{e^-}$ by transposition, using formulas (1.4) and (1.5), where now $T \in \mathcal{D}'_{e^-}$, $\varphi \in \mathcal{D}_{e^-}$ and $\langle .\,, .\rangle$ denotes the dual product between $\mathcal{D}'_{e^-}$ and $\mathcal{D}_{e^-}$. These operators are obviously linear and continuous from $\mathcal{D}'_{e^-}$ into $\mathcal{D}'_{e^-}$. The translation operator is a vector and topological automorphism in $\mathcal{D}'_{e^-}$ with inverse τ_{-a}. All these operators are the restriction to $\mathcal{D}'_{e^-}$ of the corresponding operators defined on $\mathcal{D}'$ and are extensions of the operators on $\mathcal{S}'$.

Like the case of tempered distributions, there is a structure result for the space $\mathcal{D}'_{e^-}$. A distribution T in $\mathcal{D}'$ belongs to $\mathcal{D}'_{e^-}$ if and only if

$$T = \partial^\alpha F \quad , \tag{3.2}$$

where $\alpha \in \mathbb{N}^N$ and F is a continuous function of exponential growth. The sufficient condition is trivial; the proof of the necessary condition is based on the structure of the bounded distributions and may be done exactly in the same way as for the case of tempered distributions (see [18], page 240).

4. The test function space $\mathcal{X}_0$

We call $\mathcal{X}_0$ the space of all complex-valued functions φ defined on $\mathbb{R}^N$ such that both φ and $\mathcal{F}\varphi$ belong to $\mathcal{D}_{e^-}$:

$$\mathcal{X}_0 = \{\varphi \in \mathcal{D}_{e^-} \,;\ \mathcal{F}\varphi \in \mathcal{D}_{e^-}\} \quad . \tag{4.1}$$

It is clearly a vector subspace of $\mathcal{D}_{e^-}$ and we have the following results:

Theorem 4.1. *(i) The space $\mathcal{X}_0$ is a dense subspace of the Schwartz space $\mathcal{S}$.*
(ii) Every function φ in $\mathcal{X}_0$ can be extended to $\mathbb{C}^N$ as an entire function.
(iii) For all $a \in \mathbb{R}^N$ we have $\tau_a(\mathcal{X}_0) = \mathcal{X}_0$.
(iv) For all φ in $\mathcal{X}_0$ and all $\alpha \in \mathbb{N}^N$ we have $\partial^\alpha \varphi$ in $\mathcal{X}_0$.
(v) $\mathcal{F}(\mathcal{X}_0) = \mathcal{X}_0$.
(vi) For all φ_1 and φ_2 in $\mathcal{X}_0$ we have $\varphi_1\varphi_2$ in $\mathcal{X}_0$.
(vii) For all φ in $\mathcal{X}_0$ and all polynomial p we have $p\varphi$ in $\mathcal{X}_0$.
(viii) For all φ in $\mathcal{X}_0$ and all $b \in \mathbb{R}^N$ we have $e^{ib.x}\varphi$ in $\mathcal{X}_0$.
(ix) For all ψ in $\mathcal{G}_c$ and all $b \in \mathbb{R}^N$ we have $e^{b.x}\psi$ in $\mathcal{X}_0$.
(x) For all φ in $\mathcal{X}_0$ and all $w \in \mathbb{R} \setminus \{0\}$ we have $\mathcal{F}_w\varphi$ in $\mathcal{X}_0$.

Proof. (i) $\mathcal{X}_0$ is a subspace of the Schwartz space $\mathcal{S}$ because $\mathcal{X}_0 \subset \mathcal{D}_{e^-}$ and $\mathcal{D}_{e^-} \subset \mathcal{S}$. To prove the density, let $c > 0$ and φ in $\mathcal{G}_c$. Then $\varphi \in \mathcal{D}_{e^-}$ because $\mathcal{G}_c$ is included in $\mathcal{D}_{e^-}$. It is easy to verify that $\mathcal{F}\varphi \in \mathcal{G}_{\frac{1}{2c}}$, thus $\mathcal{F}\varphi \in \mathcal{D}_{e^-}$; this means that $\mathcal{G}_c$ is included in $\mathcal{X}_0$. As the set $\mathcal{G}$ is dense in $\mathcal{S}$ and $\mathcal{G} \subset \mathcal{X}_0$ this implies that $\mathcal{X}_0$ is dense in $\mathcal{S}$.

(ii) Let us fix φ in $\mathcal{X}_0$. It is clear that $\mathcal{F}\varphi$ belongs to $\mathcal{D}_{e^-}$ and using Theorem 2.1 we may conclude that both $\mathcal{F}_{-1}(\mathcal{F}\varphi)$ and $\frac{1}{(2\pi)^N}\mathcal{F}_{-1}(\mathcal{F}\varphi) = \varphi$ can be extended to $\mathbb{C}^N$ as entire functions.

(iii) Let $a \in \mathbb{R}^N$ and $\varphi \in \mathcal{X}_0$. As the space $\mathcal{D}_{e^-}$ is closed for translations (see Theorem 2.1), then $\tau_a\varphi$ is in $\mathcal{D}_{e^-}$. It remains to show that $\mathcal{F}(\tau_a\varphi) \in \mathcal{D}_{e^-}$. From (1.10) we have $\mathcal{F}(\tau_a\varphi) = e^{-ia.\xi}\mathcal{F}\varphi$. We know that $\mathcal{F}\varphi$ is in $\mathcal{D}_{e^-}$, because φ is in $\mathcal{X}_0$, and, by (vi) in Theorem 2.1, we see that $\mathcal{F}(\tau_a\varphi) \in \mathcal{D}_{e^-}$.

(iv) Since the derivative operators ∂^α are linear continuous from $\mathcal{D}_{e^-}$ into $\mathcal{D}_{e^-}$, then for all φ in $\mathcal{X}_0$ and all $\alpha \in \mathbb{N}^N$ we have $\partial^\alpha\varphi \in \mathcal{D}_{e^-}$. From (1.8) we have $\mathcal{F}(\partial^\alpha\varphi) = (i\xi)^\alpha\mathcal{F}\varphi$. As $(i\xi)^\alpha$ is a polynomial and $\mathcal{F}\varphi$ is in $\mathcal{D}_{e^-}$, then (vii) in Theorem 2.1 shows that $\mathcal{F}(\partial^\alpha\varphi)$ is in $\mathcal{D}_{e^-}$ and consequently $\partial^\alpha\varphi \in \mathcal{X}_0$.

(v) If $\varphi \in \mathcal{X}_0$, then by definition of $\mathcal{X}_0$ (4.1), $\mathcal{F}\varphi$ belongs to $\mathcal{D}_{e^-}$. From (1.14) we have $\mathcal{F}(\mathcal{F}\varphi) = (2\pi)^N \varphi(-\xi)$. Since $\varphi(-\xi) \in \mathcal{D}_{e^-}$, then $\mathcal{F}(\mathcal{F}\varphi)$ also belongs to $\mathcal{D}_{e^-}$. Thus $\mathcal{F}(\varphi) \in \mathcal{X}_0$, that is $\mathcal{F}(\mathcal{X}_0) = \mathcal{X}_0$.

(vi) Since the space $\mathcal{D}_{e^-}$ is closed for the product (cf. Theorem 2.1 (v)), then for all φ_1 and φ_2 in $\mathcal{X}_0$, we obtain $\varphi_1\varphi_2 \in \mathcal{D}_{e^-}$. Now we need to prove

that the Fourier transform of $\varphi_1\varphi_2$ is in $\mathcal{D}_{e^-}$. From Theorem 2.1 (viii), $\mathcal{F}(\varphi_1\varphi_2)$ can be extended to $\mathbb{C}^N$ as an entire function, so $\mathcal{F}(\varphi_1\varphi_2) \in C^\infty$. Fix $\alpha \in \mathbb{N}^N$ and $k \in \mathbb{N}$. Using the classic properties of the convolution we have

$$e^{k|\xi|}\partial^\alpha\left[\mathcal{F}(\varphi_1\varphi_2)\right] = \left(\frac{1}{2\pi}\right)^N e^{k|\xi|}\mathcal{F}\varphi_1 * e^{k|\xi|}\partial^\alpha\mathcal{F}\varphi_2 \ .$$

On the other hand, $e^{k|\xi|}\mathcal{F}\varphi_1$ and $e^{k|\xi|}\partial^\alpha\mathcal{F}\varphi_2$ are in $\mathcal{D}_{e^-}$ (cf. Theorem 2.1). Therefore $e^{k|\xi|}\mathcal{F}\varphi_1, e^{k|\xi|}\partial^\alpha\mathcal{F}\varphi_2 \in L^2$ because $\mathcal{D}_{e^-}$ is continuously embedded in $\mathcal{S}$ and we know that $\mathcal{S}$ is continuously embedded in L^q, for all $q \geq 1$ (see [18]). Thus, we can deduce, from the general theory of the convolution product, that $e^{k|\xi|}\mathcal{F}\varphi_1 * e^{k|\xi|}\partial^\alpha\mathcal{F}\varphi_2$ exists for a.a. ξ in $\mathbb{C}^N$, is a continuous function and

$$\lim_{|\xi|\to+\infty} e^{k|\xi|}\partial^\alpha\left[\mathcal{F}(\varphi_1\varphi_2)\right] = \left(\frac{1}{2\pi}\right)^N \lim_{|\xi|\to+\infty}\left(e^{k|\xi|}\mathcal{F}\varphi_1 * e^{k|\xi|}\partial^\alpha\mathcal{F}\varphi_2\right) = 0$$

which is equivalent to say that $\mathcal{F}(\varphi_1\varphi_2)$ belongs to $\mathcal{D}_{e^-}$.

(vii) From Theorem 2.1 we can say that $p\varphi$ belongs to $\mathcal{D}_{e^-}$, for all φ in $\mathcal{X}_0$ and all polynomial p. It is easy to show that $\mathcal{F}(p\varphi) \in \mathcal{D}_{e^-}$, because for $p = \sum_{j\in\mathbb{N}^N,\,|j|\leq n} c_j x^j$ we have

$$\mathcal{F}(p\varphi) = \sum_{j\in\mathbb{N}^N,\,|j|\leq n} c_j\ i^{|j|}\partial^j\left(\mathcal{F}\varphi\right).$$

(viii) Let $\varphi \in \mathcal{X}_0$ and $b \in \mathbb{R}^N$. From (iii) and (v) above we have $\tau_b(\mathcal{F}\varphi) \in \mathcal{X}_0$. But $\tau_b(\mathcal{F}\varphi) = \mathcal{F}\left(e^{ib.x}\varphi\right)$ (cf. (1.11)). Using again (v) we can say that $\mathcal{F}\left(\mathcal{F}\left(e^{ib.x}\varphi\right)\right)$ belongs to $\mathcal{X}_0$ and by (1.14) finally we obtain $e^{ib.x}\varphi \in \mathcal{X}_0$.

(ix) Let $\psi \in \mathcal{G}_c$ and $b \in \mathbb{R}^N$. Then $\psi = pe^{-\frac{c}{2}|x|^2}$, where $p \in \mathcal{P}$. We know that $e^{-\frac{c}{2}|x|^2} \in \mathcal{X}_0$, thus $\tau_{\frac{b}{c}}\left(e^{-\frac{c}{2}|x|^2}\right) \in \mathcal{X}_0$, by (iii). Since $\mathcal{X}_0$ is closed for the product of polynomials (see (vii) before), we have $pe^{-\frac{c}{2}\left|x-\frac{b}{c}\right|^2} \in \mathcal{X}_0$. As the set $\mathcal{X}_0$ is a vector space, then

$$e^{\frac{b^2}{2c}}pe^{-\frac{c}{2}\left|x-\frac{b}{c}\right|^2} = e^{b.x}\psi \in \mathcal{X}_0 \quad .$$

(x) It results from (1.15) and (ix) of Theorem 2.1. □

Now we introduce in $\mathcal{X}_0$ the following family of seminorms:

$$\begin{aligned}\mu_k(\varphi) &= \nu_k(\varphi) + \nu_k(\mathcal{F}\varphi)\\ &= \sup_{|\alpha|\leq k} \| e^{k|x|}\partial^\alpha\varphi \|_{L^\infty} + \sup_{|\alpha|\leq k} \| e^{k|\xi|}\partial^\alpha\mathcal{F}\varphi \|_{L^\infty} \quad .\end{aligned} \tag{4.2}$$

With the family of seminorms (μ_k) $\mathcal{X}_0$ is a locally convex space.

From the seminorms (2.3) and (4.2) we get a continuous injection of $\mathcal{X}_0$ into $\mathcal{D}_{e^-}$. Having in mind (2.6) we have the following chain of continuous injections:

$$\mathcal{X}_0 \hookrightarrow \mathcal{D}_{e^-} \hookrightarrow \mathcal{S}. \tag{4.3}$$

Moreover, since $\mathcal{X}_0$ is a dense subspace of $\mathcal{S}$, cf. Theorem 4.1, then

$$\mathcal{X}_0 \underset{d}{\hookrightarrow} \mathcal{S}\ . \tag{4.4}$$

Before we state the main theorem (Theorem 4.2 below) for the test function space $\mathcal{X}_0$ we need the following lemma:

Lemma 2. *Let (φ_n) be a bounded sequence in $\mathcal{X}_0$ and let $\varphi \in \mathcal{D}_{e^-}$ be such that $\varphi_n \to \varphi$ in $\mathcal{D}_{e^-}$. Then $\varphi \in \mathcal{X}_0$ and $\varphi_n \to \varphi$ in $\mathcal{X}_0$.*

Proof. First we prove that φ is in $\mathcal{X}_0$. We have to show that $\mathcal{F}\varphi$ belongs to $\mathcal{D}_{e^-}$. The hypothesis of boundedness of the sequence (φ_n) implies, in particular, that

$$\forall \alpha \in \mathbb{N}^N \quad \forall k \in \mathbb{N} \quad \exists C \geq 0 \quad \forall n \in \mathbb{N} \quad \forall \xi \in \mathbb{R}^N \qquad e^{k|\xi|}|\partial^\alpha \mathcal{F}\varphi_n(\xi)| \leq C\ . \tag{4.5}$$

Using the dominated convergence theorem and the hypothesis $\varphi_n \to \varphi$ in $\mathcal{D}_{e^-}$, we may deduce that the sequence $(\partial^\alpha \mathcal{F}\varphi_n)$ converges uniformly in $\mathbb{R}^N$ to $\partial^\alpha \mathcal{F}\varphi$. Fix $\alpha \in \mathbb{N}^N$ and $k \in \mathbb{N}$ and let $n \to +\infty$ in (4.5); then

$$e^{k|\xi|}|\partial^\alpha \mathcal{F}\varphi(\xi)| \leq C\ .$$

As α and k are arbitrary, this implies that $\mathcal{F}\varphi$ is in $\mathcal{D}_{e^-}$.

It remains to prove that $\varphi_n \to \varphi$ in $\mathcal{X}_0$. We write $\psi_n = \varphi_n - \varphi$ and prove that $\psi_n \to 0$ in $\mathcal{X}_0$. Fix $\alpha \in \mathbb{N}^N$ and $k \in \mathbb{N}$; from the boundedness of (φ_n) in $\mathcal{X}_0$, we know that (ψ_n) is bounded in $\mathcal{X}_0$ and, by (4.5), this implies the existence of a real number $C \geq 0$ such that, for all $n \in \mathbb{N}$ and all $\xi \in \mathbb{R}^N$,

$$e^{k|\xi|}\,|\partial^\alpha \mathcal{F}\psi_n\,(\xi)| \leq Ce^{-|\xi|}\ . \tag{4.6}$$

As we know that $\partial^\alpha \mathcal{F}\varphi_n \to \partial^\alpha \mathcal{F}\varphi$ uniformly in $\mathbb{R}^N$, we see that

$$\partial^\alpha \mathcal{F}\psi_n \to 0 \quad \text{uniformly in } \mathbb{R}^N\ .$$

Therefore, the sequence $e^{k|\xi|}\,|\partial^\alpha \mathcal{F}\psi_n|$ converges to zero uniformly in the compact subsets of $\mathbb{R}^N$. The function $Ce^{-|\xi|}$ tends to zero at infinity. From these two facts we conclude, by (4.6), that $e^{k|\xi|}\,|\partial^\alpha \mathcal{F}\psi_n|$ converges to zero uniformly in $\mathbb{R}^N$. On the other hand, we know that $\varphi_n \to \varphi$ in $\mathcal{D}_{e^-}$, that is $\psi_n \to 0$ in $\mathcal{D}_{e^-}$; this implies

$$e^{k|x|}\,|\partial^\alpha \psi_n| \to 0 \quad \text{uniformly in } \mathbb{R}^N.$$

This prove that $\psi_n \to 0$ in $\mathcal{X}_0$. □

Theorem 4.2. *The space $\mathcal{X}_0$ is: (i) Hausdorff; (ii) metrizable; (iii) Fréchet; (iv) Montel; (v) reflexive; (vi) bornological (Mackey).*

Proof. (i) Let φ in $\mathcal{X}_0$ such that $\varphi \neq 0$. From Theorem 2.2 (i) we see that $\nu_0(\varphi) \neq 0$. Then $\mu_0(\varphi) = \nu_0(\varphi) + \nu_0(\mathcal{F}\varphi) \neq 0$ and this implies that $\mathcal{X}_0$ is a Hausdorff space.

(ii) $\mathcal{X}_0$ is metrizable because the familly of seminorms (4.2) is countable.

(iii) We just have to prove that $\mathcal{X}_0$ is complete. Let (φ_n) be a Cauchy sequence in $\mathcal{X}_0$. By (4.3), (φ_n) is also a Cauchy sequence in $\mathcal{D}_{e^-}$ and, applying Theorem 2.2 (iii), we may conclude that there exists φ in $\mathcal{D}_{e^-}$ such that φ_n converges to φ in $\mathcal{D}_{e^-}$. Since (φ_n) is a bounded sequence in $\mathcal{X}_0$, then φ belongs to $\mathcal{X}_0$ and φ_n converges to φ in $\mathcal{X}_0$ (cf. Lemma 2).

(iv) The space $\mathcal{X}_0$ is barrelled because it is a Fréchet space. Therefore, we just have to prove that, if L is a bounded subset of $\mathcal{X}_0$, then L is relatively compact in $\mathcal{X}_0$, or equivalently, that every sequence (φ_n) in L has a convergent subsequence (in the $\mathcal{X}_0$ topology).

The set L is bounded in $\mathcal{X}_0$ and, by (4.3), L is also bounded in $\mathcal{D}_{e^-}$. From Theorem 2.2, $\mathcal{D}_{e^-}$ is a metrizable and a Montel space, so there exists φ in $\mathcal{D}_{e^-}$ and a subsequence (φ_{m_n}) of the sequence (φ_n) such that $\varphi_{m_n} \to \varphi$ in $\mathcal{D}_{e^-}$. By Lemma 2 we see that φ is in $\mathcal{X}_0$ and $\varphi_{m_n} \to \varphi$ in $\mathcal{X}_0$. This proves that $\mathcal{X}_0$ is a Montel space.

(v) $\mathcal{X}_0$ is a reflexive space because it is a Montel space.

(vi) The space $\mathcal{X}_0$ is a Mackey (or bornological) space because all metrizable locally convex spaces are bornological. □

From Theorem 4.1 we can conclude that the derivative operator ∂^α, the translation operator τ_a and the operator $\varphi \to p\varphi$ with $p \in \mathcal{P}$ are linear from $\mathcal{X}_0$ into $\mathcal{X}_0$. We know that the injection of $\mathcal{X}_0$ into $\mathcal{D}_{e^-}$ is continuous (see (4.3)), then using the seminorms (4.2) and the fact that these operators are continuous from $\mathcal{D}_{e^-}$ into $\mathcal{D}_{e^-}$, it is easy to prove the continuity of these operators in $\mathcal{X}_0$. Moreover, as τ_a is bijective, with inverse τ_{-a}, we see that the translations are vector and topological automorphisms in $\mathcal{X}_0$. These results are summarized in the following corollary:

Corollary 4.3. *(i) For each $a \in \mathbb{R}^N$, the translation operator τ_a is a vector and topological automorphism in $\mathcal{X}_0$ with inverse τ_{-a}.*
(ii) For each $\alpha \in \mathbb{N}^N$ the derivative operator ∂^α is linear continuous from $\mathcal{X}_0$ into $\mathcal{X}_0$.
(iii) For each polynomial p, the operator $\varphi \to p\varphi$ is linear continuous from $\mathcal{X}_0$ into $\mathcal{X}_0$.

Theorem 4.4. *The Fourier operator $\mathcal{F}_w$ is a vector and topological automorphism in $\mathcal{X}_0$ with inverse $\left(\frac{|w|}{2\pi}\right)^N \mathcal{F}_{-w}$.*

Proof. Consider first $w = 1$. As the operator $\mathcal{F}$ is the restriction to $\mathcal{X}_0$ of the operator defined in $\mathcal{S}$ (see (1.3)), then Theorem 4.1 allows easily to conclude the linearity of this operator from $\mathcal{X}_0$ into $\mathcal{X}_0$. To prove the continuity it is enough to use the seminorms (4.2). It is simple to show the bijectivity and we see that the inverse operator is $\left(\frac{1}{2\pi}\right)^N \mathcal{F}_{-1}$. As $\mathcal{X}_0$ is a Fréchet space (cf. Theorem 4.2 (iii)), then it follows from the open application theorem that $\mathcal{F}$ is an open application, therefore $\mathcal{F}^{-1}$ is also continuous.

The case $w \neq 1$ is a direct consequence of the formula (1.15) and (ix) in Theorem 2.1. □

5. Exponential generalized distributions

We call the dual $\mathcal{X}_0'$ of $\mathcal{X}_0$ the space of exponential generalized distributions. We consider in $\mathcal{X}_0'$ the strong dual topology.

From (4.4) we have the inclusion

$$\mathcal{S}' \hookrightarrow \mathcal{X}_0' \,.$$

The space $\mathcal{X}_0$ is continuously embedded in $\mathcal{S}$ and, as we know that $\mathcal{S}$ is continuously embedded in $\mathcal{S}'$, then $\mathcal{X}_0$ is also continuously embedded in $\mathcal{X}_0'$. Next, we prove that $\mathcal{X}_0$ is dense in $\mathcal{X}_0'$. It is sufficient to prove that every continuous linear form f on $\mathcal{X}_0'$ that is null on $\mathcal{X}_0$ is identically null. Taking such a form f, we have $f \in (\mathcal{X}_0')'$ and, as $\mathcal{X}_0$ is reflexive, $f \in \mathcal{X}_0$; this implies that

$$\langle f, f\rangle = \int_{\mathbb{R}^N} |f(x)|^2 dx = 0$$

and this shows that f is the null form.

Collecting all the previous results, we have the following chain of continuous and dense embeddings:

$$\mathcal{X}_0 \underset{d}{\hookrightarrow} \mathcal{S} \underset{d}{\hookrightarrow} \mathcal{S}' \underset{d}{\hookrightarrow} \mathcal{X}_0' \,. \tag{5.1}$$

Theorem 5.1. *The space $\mathcal{X}_0'$ is: (i) Hausdorff; (ii) complete; (iii) Montel; (iv) reflexive.*

Proof. (i) The strong dual topology (as well as the weak dual topology) is always separated.

(ii) The space $\mathcal{X}_0'$ is complete because it is the dual of a separated Mackey space.

(iii) The dual of a Montel space is always a Montel space.

(iv) All Montel spaces are reflexive. □

Similarly to what has been done in $\mathcal{D}'_{e^-}$ (see (1.4) and (1.5)) we get, by transposition, the derivative operator ∂^α and the translation operator τ_a in

$\mathcal{X}_0'$, which extend these operators defined in $\mathcal{X}_0$. They are linear and continuous from $\mathcal{X}_0'$ into $\mathcal{X}_0'$. We can still see that τ_a is a vector and topological automorphism in $\mathcal{X}_0'$, with inverse operator τ_{-a}.

Theorem 5.2. *The kernel of the operator ∂^α in $\mathcal{X}_0'$ is the same as its kernel in $\mathcal{D}'$.*

Proof. It is sufficient to prove that, in dimension 1 ($N = 1$), if $\partial T = 0$, then T is constant.

First, it is easy to show the following equality:

$$\partial(\mathcal{X}_0) = \left\{ \psi \in \mathcal{X}_0 \, ; \quad \int_{-\infty}^{+\infty} \psi(x)\, dx = 0 \right\} \quad . \tag{5.2}$$

This shows that $\partial(\mathcal{X}_0)$ is a hyperplane of $\mathcal{X}_0$.

Write

$$\varphi_0 \; = \; \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}} \quad .$$

The function φ_0 belongs to $\mathcal{X}_0$ and

$$\int_{-\infty}^{+\infty} \varphi_0(x)\, dx = 1 \quad , \tag{5.3}$$

thus $\mathcal{X}_0$ is the direct sum of $\partial(\mathcal{X}_0)$ and $\operatorname{span}\{\varphi_0\}$:

$$\mathcal{X}_0 \; = \; \partial(\mathcal{X}_0) \oplus \operatorname{span}\{\varphi_0\} \quad . \tag{5.4}$$

If $\partial T = 0$ then, for all φ in $\mathcal{X}_0$, $\langle \partial T, \varphi \rangle = -\langle T, \partial\varphi \rangle = 0$. This means that

$$\forall \psi \in \partial(\mathcal{X}_0) \qquad \langle T, \psi \rangle = 0 \quad .$$

Using (5.4) we have $\langle T, \varphi \rangle = \langle T, \psi \rangle + \lambda \langle T, \varphi_0 \rangle = \lambda \langle T, \varphi_0 \rangle$, where $\lambda \in \mathbb{C}$, and

$$\int_{-\infty}^{+\infty} \varphi(x)\, dx = \int_{-\infty}^{+\infty} \psi(x)\, dx + \lambda \int_{-\infty}^{+\infty} \varphi_0(x)\, dx = \lambda$$

In the second equality we have used (5.2) and (5.3). Therefore, $\langle T, \varphi \rangle = \langle c, \varphi \rangle$, with $c = \langle T, \varphi_0 \rangle \in \mathbb{C}$, which implies that T is constant. □

As the Fourier transform $\mathcal{F}_w$ is a vector and topological automorphism in $\mathcal{X}_0$, it can be extended this to $\mathcal{X}_0'$, by transposition, in the same way as for $\mathcal{S}'$ (see (1.6)). Moreover, the operator $\mathcal{F}_w$ is a vector and topological automorphism in $\mathcal{X}_0'$ and the properties of $\mathcal{F}_w$ in $\mathcal{X}_0$ are still valid in $\mathcal{X}_0'$, i.e., the properties (1.7) to (1.15).

We are now able to identify some elements of the space $\mathcal{X}_0'$:

Theorem 5.3. *If f is a continuous function defined on $\mathbb{R}^N$ with exponential growth, then $f \in \mathcal{X}_0'$.*

Proof. Call $\mathcal{C}_{e^+}$ the set of all continuous functions defined on $\mathbb{R}^N$ with exponential growth and define

$$\begin{array}{rcl} \mathcal{T} : \mathcal{C}_{e^+} & \longrightarrow & \mathcal{X}_0' \\ f & \longmapsto & T_f = \mathcal{T}(f) \end{array}$$

by

$$\forall \varphi \in \mathcal{X}_0 \quad \langle T_f, \varphi \rangle = \int_{\mathbb{R}^N} f(x)\, \varphi(x)\, dx \quad .$$

The existence of $\int_{\mathbb{R}^N} f(x)\, \varphi(x)\, dx$ and the linearity of T_f are straightforward. To prove that T_f is in $\mathcal{X}_0'$ we only have to show the continuity of T_f.

Let (φ_n) be a sequence in $\mathcal{X}_0$, such that $\varphi_n \to 0$ in $\mathcal{X}_0$. Using the dominated convergence theorem and the hypothesis $\varphi_n \to 0$ in $\mathcal{X}_0$, we see that $\langle T_f, \varphi_n \rangle$ converges to 0 in $\mathbb{C}$; this shows that T_f belongs to $\mathcal{X}_0'$.

To prove that $\mathcal{T}$ is injective, we have to show that $\operatorname{Ker} \mathcal{T} = \{0\}$. Suppose now that $T_f = 0$; then, for all $\alpha \in \mathbb{N}^N$,

$$m_\alpha \left(f\ e^{-|x|^2} \right) = 0 \quad , \tag{5.5}$$

where $m_\alpha \left(f\ e^{-|x|^2} \right)$ are the moments of order α of the function $f\ e^{-|x|^2}$.

It is easy to see that

$$\partial^\alpha \left[\mathcal{F} \left(f\ e^{-|x|^2} \right) \right] (0) = (-i)^{|\alpha|}\, m_\alpha \left(f\ e^{-|x|^2} \right).$$

Therefore, by (5.5), we have

$$\partial^\alpha \left[\mathcal{F} \left(f\ e^{-|x|^2} \right) \right] (0) = 0 \quad .$$

All the derivatives of $\mathcal{F} \left(f\ e^{-|x|^2} \right)$ are null at the point 0. Since $\mathcal{F} \left(f\ e^{-|x|^2} \right)$ can be extended to $\mathbb{C}^N$ as an entire function (see Theorem 2.1) and the Fourier transform is injective, we can conclude that $f \equiv 0$. This proves the injectivity of $\mathcal{T}$. □

Corollary 5.4. *Let f be an entire function with exponential growth in $\mathbb{C}^N$ and let (p_n) be the corresponding sequence of Mac-Laurin polynomials in $\mathcal{X}_0'$. Then $p_n \to f$ in $\mathcal{X}_0'$.*

Corollary 5.5. *Let $\alpha \in \mathbb{N}^N$ such that $|\alpha| \neq 0$. If the sequence $\left(\sqrt[|\alpha|]{\alpha!\, |b_\alpha|} \right)$ is bounded, then the multipole series $\sum_{\alpha \in \mathbb{N}^N} b_\alpha \partial^\alpha \delta$ is convergent in $\mathcal{X}_0'$.*

Notice that although $e^{-|x|^2}$ may be identify with an element of $\mathcal{X}_0'$ (see Theorem 5.3), the corresponding sequence of Mac-Laurin polynomials do not

converges to $e^{-|x|^2}$ in $\mathcal{X}_0'$. This fact is not in contradiction with Corollary 5.4 because $e^{-|x|^2}$ do not have an exponential growth in $\mathbb{C}^N$.

Next we give some examples of exponential generalized distributions which can be represented by convergent multipole series. Applications to ODE in $\mathcal{X}_0'$ are given.

Example 5.6. It is clear that e^x has exponential growth in $\mathbb{C}$. Then $e^x \in \mathcal{X}_0'$ (cf. Theorem 5.3) and, from Corollary 5.4,

$$\sum_{j=0}^{n} \frac{x^j}{j!} \longrightarrow e^x \text{ in } \mathcal{X}_0' \quad .$$

As the Fourier transform $\mathcal{F}$ is a vector and topological automorphism in $\mathcal{X}_0'$, we have

$$\mathcal{F}\left(e^x\right) = \sum_{j=0}^{\infty} \frac{2\pi i^j}{j!}\, \delta^{(j)} \quad .$$

The previous multipole series is convergent in $\mathcal{X}_0'$ and its sum is the Fourier transform of the exponential function.

Example 5.7. Let $a \in \mathbb{R}$. The function e^{iax} is in $\mathcal{X}_0'$ because it is bounded. Therefore the corresponding sequence of Mac-Laurin polynomials converges in $\mathcal{X}_0'$:

$$\sum_{j=0}^{n} \frac{(ai)^j}{j!} x^j \longrightarrow e^{iax}.$$

Applying Fourier we obtain

$$\mathcal{F}\left(e^{iax}\right) \;=\; 2\pi \sum_{j=0}^{\infty} \frac{(-a)^j}{j!}\, \delta^{(j)} \quad . \tag{5.6}$$

On the other hand, and having in mind the properties of the Fourier transform in $\mathcal{X}_0'$, we have

$$\mathcal{F}\left(e^{iax}\right) = \tau_a\left(\mathcal{F}1\right) = 2\pi\delta_a \quad .$$

Then, from (5.6), we obtain

$$\delta_a = \sum_{j=0}^{\infty} \frac{(-a)^j}{j!}\, \delta^{(j)} \quad . \tag{5.7}$$

We have just proved that the distribution δ_a has the representation (5.7) as a convergent multipole series in $\mathcal{X}_0'$. More generally, for n in $\mathbb{N}$, we have

$$\delta_a^{(n)} \;=\; \sum_{j=0}^{\infty} \frac{(-a)^j}{j!}\, \delta^{(j+n)} \;=\; \sum_{j=n}^{\infty} \frac{(-a)^{j-n}}{(j-n)!}\, \delta^{(j)} \quad . \tag{5.8}$$

Example 5.8. Let us consider the following ODE:

$$x^2T' + T = 0 \quad . \tag{5.9}$$

Let us suppose that T has the form $T = \sum_{j=0}^{\infty} b_j \delta^{(j)}$. From (5.9) we have

$$\sum_{j=0}^{\infty} (b_{j+1}(j+2)(j+1) + b_j)\delta^{(j)} = 0$$

and we conclude that, for all j in $\mathbb{N}$,

$$b_j \;=\; b_0 \, \frac{(-1)^j}{(j+1)!j!} \quad ,$$

with $b_0 \in \mathbb{C}$. Therefore, the solution of (5.9) is

$$T \;=\; b_0 \, \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)!j!} \, \delta^{(j)} \quad . \tag{5.10}$$

Now we show that the multipole series (5.10) is convergent in $\mathcal{X}_0'$. Note that the sequence

$$\left(\sqrt[j]{j! \left| b_0 \frac{(-1)^j}{(j+1)!j!} \right|} \right)_{j \in \mathbb{N}_1}$$

is bounded, because it is convergent. From Corollary 5.5 we can finally deduce the desired result.

In the previous examples we showed that some elements of $\mathcal{X}_0'$ have a representation as a multipole series. Next we show that not all the elements of $\mathcal{X}_0'$ have such representation.

For example e^x belongs to $\mathcal{X}_0'$ but does not have a representation as a multipole series. In fact, e^x is the solution of the following differential equation:

$$T' - T = 0 \quad . \tag{5.11}$$

If $T = \sum_{j=0}^{\infty} b_j \delta^{(j)}$ is a solution of (5.11), then

$$\sum_{j=1}^{\infty} (b_{j-1} - b_j)\delta^{(j)} - b_0\delta = 0 \quad .$$

Thus, for all j in $\mathbb{N}$, we have $b_j = 0$ and consequently $T = 0$.

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