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ON SELF MAPS OF $\mathbb{H}\mathrm{P}^n$ FOR $n = 4$ AND 5.

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Abstract. We determine the cardinality of the set of the homotopy classes of self maps of $\mathbb{H}\mathrm{P}^4$ with degree 0. And we shall determine the nilpotency of $\mathbb{H}\mathrm{P}^5$.

1. Main results

Let $\mathbb{H}\mathrm{P}^n$ be the quaternionic projective space of dimension n. We shall denote by $j_n : S^4 \to \mathbb{H}\mathrm{P}^n$ the inclusion. Especially, we put $j = j_\infty$.

We shall prove the following theorem:

Theorem 1. *The cardinality of* $\mathscr{H}_0(\mathbb{H}\mathrm{P}^4)$ *is* 2.

Here, for each integer k, $\mathscr{H}_k(\mathbb{H}\mathrm{P}^n)$ is the totalities of the homotopy classes $[f]$ of maps $f : \mathbb{H}\mathrm{P}^n \to \mathbb{H}\mathrm{P}^n$ such that $f \circ j_n : S^4 \to \mathbb{H}\mathrm{P}^n$ has degree k.

For $n = 2, 3$, the cardinalities $K(n, k)$ of $\mathscr{H}_k(\mathbb{H}\mathrm{P}^n)$, if it is not 0, are determined in [2] so as to $K(2, 2k) = 1$, $K(2, 2k+1) = 2$, $K(3, 2k) = 2$ and $K(3, 2k+1) = 4$.

Next, for all based spaces X , we shall denote by $Z_\infty(X)$ (c.f., [1]) the totalities of the homotopy classes $\alpha \in [X, X]$ with the property that $\pi_n(\alpha) = 0 : \pi_n(X) \to \pi_n(X)$ for all integer $n \geqslant 0$. Clearly, $Z_\infty(\mathbb{H}\mathrm{P}^n) \subseteq \mathscr{H}_0(\mathbb{H}\mathrm{P}^n)$ holds.

In [1], the nilpotency $t_\infty(X)$ of a based space X is defined to be the least natural number k such that $x_1 \circ \cdots \circ x_k = 0$ holds for all $x_1, \cdots, x_k \in Z_\infty(X)$, if such k exists. If not, we put $t_\infty(X) = \infty$. It was proved in [1] that $t_\infty(\mathbb{H}\mathrm{P}^i) = 1$ for $i = 1, 2, 3$ and $t_\infty(\mathbb{H}\mathrm{P}^4) = 2$. We shall prove:

Theorem 2. $t_\infty(\mathbb{H}\mathrm{P}^5) = 2$

2. Proof of the theorems

We shall consider in the category $\mathscr{T}_0$ of based topological spaces and based maps. We denote by $[f]$ the homotopy class of each map f in $\mathscr{T}_0$. We denote by 0 the homotopy class of any trivial maps. We put $\Sigma^n X = S^n \wedge X$.

By $h_n : S^{4n+3} \to \mathbb{H}\mathrm{P}^n$, denotes the Hopf fiber map, and by $q_n : \mathbb{H}\mathrm{P}^n \to S^{4n}$ the canonical quotient map. We shall put $r_n = \Sigma^{n-4}(q_1 \circ h_1) : S^{n+3} \to S^n$. We shall put, for each m, n, k with $0 < k \leqslant m \leqslant n \leqslant \infty$, $\mathbb{H}\mathrm{P}^n_k := \mathbb{H}\mathrm{P}^n / \mathbb{H}\mathrm{P}^{k-1}$, $q^n_k : \mathbb{H}\mathrm{P}^n \to \mathbb{H}\mathrm{P}^n_k$ to be the quotient map and $i^{m,n}_k : \mathbb{H}\mathrm{P}^m_k \to$

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$\mathbb{H}\mathrm{P}^n_k$ induced from the inclusion $i^{m,n} : \mathbb{H}\mathrm{P}^m \to \mathbb{H}\mathrm{P}^n$. Especially, we shall put $i_n := i^{n,n+1}$.

For all spaces $X, Y \in \mathscr{T}_0$, the symbol $\mathbf{F}(X, Y)$ denotes the totalities of the maps $X \to Y$ in $\mathscr{T}_0$, endowed with the compact open topology and with the trivial map as its base point. We shall denote by $\mathbf{F}_0(\mathbb{H}\mathrm{P}^n, \mathrm{B}S^3)$ the totality of the maps $\mathbb{H}\mathrm{P}^n \to \mathrm{B}S^3$ with degree 0 endowed with the relative topology induced from $\mathbf{F}(\mathbb{H}\mathrm{P}^n, \mathrm{B}S^3)$.

Proof of the theorem 1. The theorem 1 consists of following two lemmas:

Lemma 1. *The cardinality of $Z_\infty(\mathbb{H}\mathrm{P}^4)$ is 2 and the only non-trivial homotopy class of the set is $[j_4] \circ (\Sigma\nu') \circ \mu_7 \circ [q_4]$.*

Lemma 2. *$Z_\infty(\mathbb{H}\mathrm{P}^4) = \mathscr{H}_0(\mathbb{H}\mathrm{P}^4)$ holds.*

Proof of Lemma 1. Consider the following exact sequence:

$$\pi_{16}(\mathbb{H}\mathrm{P}^4) \xrightarrow[q_4^*]{} [\mathbb{H}\mathrm{P}^4, \mathbb{H}\mathrm{P}^4] \xrightarrow[i_3^*]{} [\mathbb{H}\mathrm{P}^3, \mathbb{H}\mathrm{P}^4] \xrightarrow[h_3^*]{} \pi_{15}(\mathbb{H}\mathrm{P}^4)$$

Let $i_* : [\mathbb{H}\mathrm{P}^3, \mathbb{H}\mathrm{P}^3] \to [\mathbb{H}\mathrm{P}^3, \mathbb{H}\mathrm{P}^4]$ be induced from i_3. From Lemma 5.9 and Theorem 5.10 of [1], the image of $Z_\infty(\mathbb{H}\mathrm{P}^4)$ by the map $i_*^{-1} \circ i_3^* : [\mathbb{H}\mathrm{P}^4, \mathbb{H}\mathrm{P}^4] \to [\mathbb{H}\mathrm{P}^3, \mathbb{H}\mathrm{P}^3]$ is contained in $Z_\infty(\mathbb{H}\mathrm{P}^3) = \{0\}$. Hence, $Z_\infty(\mathbb{H}\mathrm{P}^4)$ is contained in the image of the map $q_4^* : \pi_{16}(\mathbb{H}\mathrm{P}^4) \to [\mathbb{H}\mathrm{P}^4, \mathbb{H}\mathrm{P}^4]$.

Conversely, let $\alpha \in \pi_{16}(\mathbb{H}\mathrm{P}^4)$. Then, since $\pi_{16}(\mathbb{H}\mathrm{P}^4) = \mathbb{Z}_2\{[j_4] \circ (\Sigma\nu') \circ \mu_7\} \oplus \mathbb{Z}_2\{[j_4] \circ (\Sigma\nu') \circ \eta_7 \circ \varepsilon_8\}$ holds by taking adjoint of $\pi_{15}(S^3)$ ([6]), $\alpha \circ [q_4] \circ [h_4] = \alpha \circ (4[r_{16}]) = 0$ holds in the group $\pi_{19}(\mathbb{H}\mathrm{P}^4)$, and clearly $\alpha \circ [\,q_4|_{S^4}] = 0$. Hence, from the Proposition 5.4 of [1], $\alpha \circ [q_4] \in Z_\infty(\mathbb{H}\mathrm{P}^4)$. Therefore, $Z_\infty(\mathbb{H}\mathrm{P}^4) = q_4^*(\pi_{16}(\mathbb{H}\mathrm{P}^4))$.

Next, it is well known by [4] that $[j_4] \circ (\Sigma\nu') \circ \mu_7 \circ [q_4] \neq 0$, in the set $[\mathbb{H}\mathrm{P}^4, \mathbb{H}\mathrm{P}^4]$. Hence ([1]) $\mathrm{card}(Z_\infty(\mathbb{H}\mathrm{P}^4)) \geqslant 2$.

We shall express the co-action (c.f., chapter III of [7]) by the element α of $\pi_{16}(\mathbb{H}\mathrm{P}^4)$ on the element x of $[\mathbb{H}\mathrm{P}^4, \mathbb{H}\mathrm{P}^4]$ by the symbol $x \dot{+} \alpha$. Then, from the Puppe theorem, to prove Lemma 1, we have only to prove the following lemma:

Lemma 3. *$[j_4] \circ (\Sigma\nu') \circ \eta_7 \circ \varepsilon_8 \circ [q_4] = 0$ holds in $[\mathbb{H}\mathrm{P}^4, \mathbb{H}\mathrm{P}^4]$.*

Proof. Let $h_3' = q_3 \circ h_3 : S^{15} \to S^{12}$, and $q' : \mathbb{H}\mathrm{P}^4_3 \to S^{16}$ the quotient map. Then, we obtain the following exact sequence:

$$\pi_{13}(\mathrm{B}S^3) \xrightarrow[(\Sigma h_3')^*]{} \pi_{16}(\mathrm{B}S^3) \xrightarrow[q'^*]{} [\mathbb{H}\mathrm{P}^4_3, \mathrm{B}S^3]$$

It is well known that $[h_3'] \equiv \pm 3[r_{12}] \equiv \pm 3\nu_{12}$ (mod. $\Sigma^9\nu' \,(= 2\nu_{12})$) and from [6], $\pi_{12}(S^3) = \mathbb{Z}_2\{\mu_3\} \oplus \mathbb{Z}_2\{\eta_3 \circ \varepsilon_4\}$ holds. Now, since from [5], $\mu_3 \circ \nu_{12} = \nu' \circ \eta_6 \circ \varepsilon_7$ and $\eta_3 \circ \varepsilon_4 \circ \nu_{12} = \varepsilon_3 \circ \eta_{11} \circ \nu_{12} = \varepsilon_3 \circ (\Sigma^8\nu') \circ \eta_{14} = \varepsilon_3 \circ (2\nu_{11}) \circ \eta_{14} = 0$

holds. Therefore, $(\Sigma h_3')^*(\pi_{13}(\mathrm{B}S^3)) = \{0, [j] \circ (\Sigma\nu') \circ \eta_7 \circ \varepsilon_8\}$ holds, hence $[j] \circ (\Sigma\nu') \circ \eta_7 \circ \varepsilon_8 \circ [q_4] = [j] \circ (\Sigma\nu') \circ \eta_7 \circ \varepsilon_8 \circ [q'] \circ [q_3^4] = 0$ holds in the set $[\mathbb{H}P^4, \mathrm{B}S^3]$. □

Proof of Lemma 2. We shall use the notations in [2]: Let $d_{4,0} : \pi_{12}(\mathrm{B}S^3) \to \pi_{15}(\mathrm{B}S^3)$ be the composition of $j_{3,0} : \pi_{12}(\mathrm{B}S^3) \to \pi_0(\mathbf{F}_0(\mathbb{H}P^3, \mathrm{B}S^3))$ and the map $\partial_{4,0} : \pi_0(\mathbf{F}_0(\mathbb{H}P^3, \mathrm{B}S^3)) \to \pi_{15}(\mathrm{B}S^3)$ induced from h_3. As in the proof of Proposition 1.3 of [3], $d_{4,0}(l) = \pm 3l \circ [r_{12}]$ holds for $l \in \pi_{12}(\mathrm{B}S^3)$. Since $\pi_{12}(\mathrm{B}S^3) = \mathbb{Z}_2\{j_*\varepsilon_4\}$ ([6]), $d_{4,0}(j_*\varepsilon_4) = [j] \circ \varepsilon_4 \circ [r_{12}] = [j] \circ \varepsilon_4 \circ \nu_{12} \neq 0$ in the set $\pi_{15}(\mathrm{B}S^3)$.

Finally, from Theorem 2 of [2], the cardinality of the set $\mathscr{H}_0(\mathbb{H}P^3)$ ($\cong \pi_0(\mathbf{F}_0(\mathbb{H}P^3, \mathrm{B}S^3))$) is 2. Therefore, $\partial_{4,0}$ is injection, hence $i_3^*(\mathscr{H}_0(\mathbb{H}P^4)) \subseteq \mathrm{Ker}(\partial_{4,0}) = 0$. Hence $\mathscr{H}_0(\mathbb{H}P^4) \subseteq q_4^*(\pi_{16}(\mathbb{H}P^4)) = Z_\infty(\mathbb{H}P^4)$. This completes the proof of Lemma 2 so that Theorem 1 holds.

Proof of theorem 2. The non trivial class $\xi := [j_4]\circ(\Sigma\nu')\circ\mu_7\circ[q_4] \in Z_\infty(\mathbb{H}P^4)$ is represented by the restriction of a map $f : \mathbb{H}P^5 \to \mathbb{H}P^5$ which represents a non trivial class $\alpha \in [\mathbb{H}P^5, \mathbb{H}P^5]$, because of $\xi \circ [h_4] = 0$ and of the fact that $\mathbb{H}P^5$ is the mapping cone of the map $h_4 : S^{19} \to \mathbb{H}P^4$. Let $x\dot{+}\gamma$ be the co-action on $x \in [\mathbb{H}P^5, \mathbb{H}P^5]$ by $\gamma \in \pi_{20}(\mathbb{H}P^5)$.

Take two elements $x, y \in Z_\infty(\mathbb{H}P^5)$. Then x has the form $x = 0\dot{+}\gamma = \gamma\circ q_5$ or the form $x = \alpha\dot{+}\gamma$ for some $\gamma \in \pi_{20}(\mathbb{H}P^5)$. For the former case, it is trivial that $y \circ x = 0$. For the latter case, $y \circ x = y \circ \alpha\dot{+}y \circ \gamma = y \circ \alpha$. Therefore, it is enough that we can take α so as to be factored through S^{15} (or S^4).

By [5], $[j_5]\circ(\Sigma\nu')\circ\mu_7 = [j_5]\circ(\Sigma\mu')\circ\eta_{15}$, and $\eta_{15}\circ[q_4]\circ[h_4] = \eta_{15}\circ(4\nu_{16}) = 0$. Therefore, there exists $\alpha' : \mathbb{H}P^5_4 \to S^{15}$ such that $\alpha' \circ [i_4^{4,5}] = \eta_{15}$. Hence $[j_5] \circ (\Sigma\mu') \circ \eta_{15} \circ [q_4] = [j_5] \circ (\Sigma\mu') \circ \alpha' \circ [i_4^{4,5}] \circ [q_4] = [j_5] \circ (\Sigma\mu') \circ \alpha' \circ [q_4^5] \circ [i_4]$. We can put $\alpha = [j_5] \circ (\Sigma\mu') \circ \alpha' \circ [q_4^5]$.

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