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A GENERAL INEQUALITY FOR DOUBLY WARPED PRODUCT SUBMANIFOLDS

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Abstract. In this paper, we consider doubly warped product manifolds and we establish a general inequality for doubly warped products isometrically immersed in arbitrary Riemannian manifolds. Some aplications are derived.

1. Introduction

Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and let $\sigma_1 : N_1 \to (0, \infty)$ and $\sigma_2 : N_2 \to (0, \infty)$ be differentiable functions.

The doubly warped product $N =_{\sigma_2} N_1 \times_{\sigma_1} N_2$ is the product manifold $N_1 \times N_2$ endowed with the metric

$$g = \sigma_2^2 g_1 + \sigma_1^2 g_2.$$

More precisely, if $\pi_1 : N_1 \times N_2 \to N_1$ and $\pi_2 : N_1 \times N_2 \to N_2$ are natural projections, the metric g is defined by

$$g = (\sigma_2 \circ \pi_2)^2 \, \pi_1^* g_1 + (\sigma_1 \circ \pi_1)^2 \, \pi_2^* g_2.$$

The functions σ_1 and σ_2 are called warping functions. If either $\sigma_1 \equiv 1$ or $\sigma_2 \equiv 1$, but not both, then we obtain a warped product. If both $\sigma_1 \equiv 1$ and $\sigma_2 \equiv 1$, then we have a Riemannian product manifold. If neither σ_1 nor σ_2 is constant, then we have a non-trivial doubly warped product.

Let $x :_{\sigma_2} N_1 \times_{\sigma_1} N_2 \to \widetilde{M}$ be an isometric immersion of a doubly warped product ${}_{\sigma_2}N_1 \times_{\sigma_1} N_2$ into a Riemannian manifold $\widetilde{M}$. We denote by h the second fundamental form of x and by $H_i = \frac{1}{n_i} trace h_i$ the partial mean curvatures, where $trace h_i$ is the trace of h restricted to N_i and $n_i = \dim N_i$ $(i = 1, 2)$.

The immersion x is said to be mixed totally geodesic if $h(X, Z) = 0$, for any vector fields X and Z tangent to D_1 and D_2, respectively, where D_i are the distributions obtained from the vectors tangent to N_i (or more precisely, vectors tangent to the horizantal lifts of N_i).

In [3], B. Y. Chen proved the following general optimal result:

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Theorem 1. *Let $x : N_1 \times_f N_2 \to \widetilde{M}(c)$ be an isometric immersion of an n-dimensional warped product $N_1 \times_f N_2$ into an m-dimensional Riemannian manifold $\widetilde{M}(c)$ of constant sectional curvature c. Then:*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}||H||^2 + n_1 c, \tag{1.1}$$

where $n_i = \dim N_i$, $i = 1, 2$, and Δ is the Laplacian operator of N_1. Moreover, the equality case of (1.1) *holds if and only if x is a mixed totally geodesic immersion and $n_1 H_1 = n_2 H_2$, where H_i, $i = 1, 2$, are the partial mean curvature vectors.*

Later, B. Y. Chen and F. Dillen extended this inequality to multiply warped product manifolds in arbitrary Riemannian manifolds (see[4]).The purpose of this article is to extend inequality (1.1) for doubly warped product submanifolds into arbitrary Riemannian manifolds.

2. Preliminaries

In this section, we recall some definitions and basic formulas which we will use later.

Let N be a Riemannian n-manifold isometrically immersed in a Riemannian m-manifold $\widetilde{M}^m$.

We choose a local field of orthonormal frame $e_1, ..., e_n, e_{n+1}, ..., e_m$ in $\widetilde{M}^m$ such that, restricted to N, the vectors $e_1, ..., e_n$ are tangent to N and $e_{n+1}, ..., e_m$ are normal to N.

Let $K(e_i \wedge e_j)$, $1 \leq i < j \leq n$, denote the sectional curvature of the plane section spanned by e_i and e_j. Then the scalar curvature of N is given by

$$\tau = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j). \tag{2.1}$$

Let L be a subspace of T_pN of dimension $r \geq 2$ and $\{e_1, ..., e_r\}$ an orthonormal basis of L. The scalar curvature $\tau(L)$ of the r-plane section L is defined by

$$\tau(L) = \sum_{1 \leq \alpha < \beta \leq r} K(e_\alpha \wedge e_\beta). \tag{2.2}$$

Let h be the second fundamental form and R the Riemann curvature tensor of N.

Then the equation of Gauss is given by

$$\begin{aligned} &\widetilde{R}(X, Y, Z, W) \\ &\quad = R(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)) \end{aligned} \tag{2.3}$$

for any vectors X, Y, Z, W tangent to N.

The mean curvature vector H is defined by

$$H = \frac{1}{n} traceh = \frac{1}{n} \sum_{i=1}^{n} h\left(e_i, e_i\right). \tag{2.4}$$

As is known, M is said to be minimal if H vanishes identically.

Also, we set

$$h_{ij}^{r} = g\left(h\left(e_i, e_j\right), e_r\right),\ i, j \in \{1, ..., n\},\ r \in \{n+1, ..., m\} \tag{2.5}$$

the coefficients of the second fundamental form h with respect to $e_1, ..., e_n$, $e_{n+1}, ..., e_m$, and

$$||h||^2 = \sum_{i,j=1}^{n} g\left(h\left(e_i, e_j\right), h\left(e_i, e_j\right)\right). \tag{2.6}$$

Let M be a Riemannian p-manifold and $\{e_1, ..., e_p\}$ be an orthonormal basis of M. For a differentiable function f on M, the Laplacian Δf of f is defined by

$$\Delta f = \sum_{j=1}^{p} \{\left(\nabla_{e_j} e_j\right) f - e_j e_j f\}. \tag{2.7}$$

We recall the following general algebraic lemma of Chen for later use.

Lemma 2. [3]. *Let $n \geq 2$ and $a_1, a_2, ..., a_n, b$ real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right). \tag{2.8}$$

Then $2a_1a_2 \geq b$, with equality holding if and only if

$$a_1 + a_2 = a_3 = ... = a_n.$$

3. Doubly warped product submanifolds in arbitrary Riemannian manifolds

Theorem 3. *Let x be an isometric immersion of an n-dimensional doubly warped product $N = _{\sigma_2} N_1 \times_{\sigma_1} N_2$ into an m-dimensional arbitrary Riemannian manifold $\widetilde{M}^m$. Then:*

$$n_2 \frac{\Delta_1 \sigma_1}{\sigma_1} + n_1 \frac{\Delta_2 \sigma_2}{\sigma_2} \leq \frac{n^2}{4} ||H||^2 + n_1 n_2 \max \widetilde{K}, \tag{3.1}$$

where $n_i = \dim N_i$, $n = n_1 + n_2$, Δ_i is the Laplacian operator of N_i, $i = 1, 2$ and $\max \widetilde{K}(p)$ denotes the maximum of the sectional curvature function of $\widetilde{M}^m$ restricted to 2-plane sections of the tangent space T_pN of N at each *point p in N. Moreover, the equality case of* (3.1) *holds if and only if the following two statements hold*

(1) *x is a mixed totally geodesic immersion satisfying $n_1H_1 = n_2H_2$, where H_i, $i = 1, 2$, are the partial mean curvature vectors of N_i.*

(2) *at each point $p = (p_1, p_2) \in N$, the sectional curvature function $\widetilde{K}$ of $\widetilde{M}^m$ satisfies $\widetilde{K}(u, v) = \max \widetilde{K}(p)$ for each unit vector $u \in T_{p_1}N_1$ and each unit vector $v \in T_{p_2}N_2$.*

Proof. Let $N =_{\sigma_2} N_1 \times_{\sigma_1} N_2$ be a doubly warped product submanifold into an arbitrary Riemannian manifold $\widetilde{M}^m$. Since ${}_{\sigma_2}N_1 \times_{\sigma_1} N_2$ is a doubly warped product, then

$$\left\{ \begin{array}{c} \nabla_X Y = \nabla^1_X Y - \frac{\sigma_2^2}{\sigma_1^2} g_1(X, Y) \nabla^2 (\ln \sigma_2), \\ \nabla_X Z = Z(\ln \sigma_2) X + X(\ln \sigma_1) Z, \end{array} \right. \tag{3.2}$$

for any vector fields X, Z tangent to N_1 and N_2, respectively, where ∇^1 and ∇^2 are the Levi-Civita connections of the Riemannian metrics g_1 and g_2, respectively (see [5], [8]). Here, $\nabla^2(\ln \sigma_2)$ denotes the gradient of $\ln \sigma_2$ with respect to the metric g_2.

If X and Z are unit vector fields, it follows that the sectional curvature $K(X \wedge Z)$ of the plane section spanned by X and Z is given by

$$K(X \wedge Z) = \frac{1}{\sigma_1}\{(\nabla^1_X X)\sigma_1 - X^2\sigma_1\} + \frac{1}{\sigma_2}\{(\nabla^2_Z Z)\sigma_2 - Z^2\sigma_2\}. \tag{3.3}$$

We choose a local orthonormal frame $\{e_1, ..., e_{n_1}, e_{n_1+1}, ..., e_n\}$ such that $e_1, ..., e_{n_1}$ are tangent to N_1, $e_{n_1+1}, ..., e_n$ are tangent to N_2 and e_{n+1} is parallel to the mean curvature vector H.

Then, using (3.3), we get

$$\sum_{\substack{1\le j\le n_1 \\ n_1+1\le s\le n}} K(e_j \wedge e_s) = n_2\frac{\Delta_1\sigma_1}{\sigma_1} + n_1\frac{\Delta_2\sigma_2}{\sigma_2}. \tag{3.4}$$

From the equation of Gauss, we have

$$2\tau(p) = n^2||H||^2(p) - ||h||^2(p) + 2\widetilde{\tau}(T_pN), \ p \in N, \tag{3.5}$$

where $n_i = \dim N_i$, $n = n_1 + n_2$, $||h||^2$ is the squared norm of the second fundamental form h of N in $\widetilde{M}^m$ and $\widetilde{\tau}(T_pN)$ is the scalar curvature of the subspace T_pN in $\widetilde{M}^m$.

We set

$$\delta = 2\tau - \frac{n^2}{2}||H||^2 - 2\widetilde{\tau}(T_pN). \tag{3.6}$$

Then, (3.5) can be written as

$$n^2||H||^2 = 2\left(\delta + ||h||^2\right). \tag{3.7}$$

With respect to the above orthonormal frame, (3.7) takes the following form:

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = 2\left[\delta + \sum_{i=1}^{n}\left(h_{ii}^{n+1}\right)^2 + \sum_{i\neq j}\left(h_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}\left(h_{ij}^{r}\right)^2\right].$$

If we put $a_1 = h_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1}$ and $a_3 = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}$, the above equation becomes

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2[\delta + \sum_{i=1}^{3} a_i^2 + \sum_{1\leq i\neq j\leq n}\left(h_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}\left(h_{ij}^{r}\right)^2$$
$$- \sum_{2\leq j\neq k\leq n_1} h_{jj}^{n+1}h_{kk}^{n+1} - \sum_{n_1+1\leq s\neq t\leq n} h_{ss}^{n+1}h_{tt}^{n+1}].$$

Thus a_1, a_2, a_3 satisfy the Lemma of Chen (for $n = 3$), i.e.,

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left(b + \sum_{i=1}^{3} a_i^2\right),$$

with

$$b = \delta + \sum_{1\leq i\neq j\leq n}\left(h_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}\left(h_{ij}^{r}\right)^2$$
$$- \sum_{2\leq j\neq k\leq n_1} h_{jj}^{n+1}h_{kk}^{n+1} - \sum_{n_1+1\leq s\neq t\leq n} h_{ss}^{n+1}h_{tt}^{n+1}.$$

Then $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3$.

In the case under consideration, this means

$$\sum_{1\leq j<k\leq n_1} h_{jj}^{n+1}h_{kk}^{n+1} + \sum_{n_1+1\leq s<t\leq n} h_{ss}^{n+1}h_{tt}^{n+1} \geq \tag{3.8}$$
$$\geq \frac{\delta}{2} + \sum_{1\leq\alpha<\beta\leq n}\left(h_{\alpha\beta}^{n+1}\right)^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{\alpha,\beta=1}^{n}\left(h_{\alpha\beta}^{r}\right)^2.$$

Equality holds if and only if

$$\sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}. \tag{3.9}$$

Using again the Gauss equation, we have

$$\begin{aligned}
(3.10)\qquad & n_2\frac{\Delta_1\sigma_1}{\sigma_1}+n_1\frac{\Delta_2\sigma_2}{\sigma_2}\\
&=\tau-\sum_{1\le j<k\le n_1}K\left(e_j\wedge e_k\right)-\sum_{n_1+1\le s<t\le n}K\left(e_s\wedge e_t\right)\\
&=\tau-\widetilde{\tau}\left(D_1\right)-\sum_{r=n+1}^{m}\sum_{1\le j<k\le n_1}\left(h_{jj}^r h_{kk}^r-\left(h_{jk}^r\right)^2\right)-\\
&-\widetilde{\tau}\left(D_2\right)-\sum_{r=n+1}^{m}\sum_{n_1+1\le s<t\le n}\left(h_{ss}^r h_{tt}^r-\left(h_{st}^r\right)^2\right).
\end{aligned}$$

Combining (3.8) and (3.10) and taking account of (3.4), we obtain

$$\begin{aligned}
& n_2\frac{\Delta_1\sigma_1}{\sigma_1}+n_1\frac{\Delta_2\sigma_2}{\sigma_2}\le\tau-\widetilde{\tau}\left(TN\right)+\sum_{1\le s\le n_1}\sum_{n_1+1\le t\le n}\widetilde{K}\left(e_s,e_t\right)-\\
&-\frac{\delta}{2}-\sum_{\substack{1\le j\le n_1\\ n_1+1\le t\le n}}\left(h_{jt}^{n+1}\right)^2-\frac{1}{2}\sum_{r=n+2}^{m}\sum_{\alpha,\beta=1}^{n}\left(h_{\alpha\beta}^r\right)^2+\\
&+\sum_{r=n+2}^{m}\sum_{1\le j<k\le n_1}\left(\left(h_{jk}^r\right)^2-h_{jj}^r h_{kk}^r\right)+\sum_{r=n+2}^{m}\sum_{n_1+1\le s<t\le n}\left(\left(h_{st}^r\right)^2-h_{ss}^r h_{tt}^r\right)=\\
&=\tau-\widetilde{\tau}\left(TN\right)+\sum_{1\le s\le n_1}\sum_{n_1+1\le t\le n}\widetilde{K}\left(e_s,e_t\right)-\frac{\delta}{2}-\sum_{r=n+1}^{m}\sum_{j=1}^{n_1}\sum_{t=n_1+1}^{n}\left(h_{jt}^r\right)^2-\\
&-\frac{1}{2}\sum_{r=n+2}^{m}\left(\sum_{j=1}^{n_1}h_{jj}^r\right)^2-\frac{1}{2}\sum_{r=n+2}^{m}\left(\sum_{t=n_1+1}^{n}h_{tt}^r\right)^2\le
\end{aligned}$$

$$(3.11)\qquad \le\tau-\widetilde{\tau}\left(TN\right)+n_1n_2\max\widetilde{K}-\frac{\delta}{2}=\frac{n^2}{4}||H||^2+n_1n_2\max\widetilde{K},$$

which implies the inequality (3.1).

If the equality sign of (3.1) holds, then all of inequalities in (3.11) become equalities. That means we have statements 1 and 2 as well.The converse statement is straightforward.

Corollary 4. *Let x be an isometric immersion of an n-dimensional doubly warped product $N={}_{\sigma_2}N_1\times_{\sigma_1}N_2$ into a Riemannian m-manifold $R^m\left(c\right)$ of constant curvature c. Then:*

$$(3.12)\qquad n_2\frac{\Delta_1\sigma_1}{\sigma_1}+n_1\frac{\Delta_2\sigma_2}{\sigma_2}\le\frac{n^2}{4}||H||^2+n_1n_2c,$$

where $n_i = \dim N_i$, $n = n_1 + n_2$, Δ_i is the Laplacian operator of N_i, $i = 1, 2$. Moreover, the equality case of (3.12) *holds if and only if x is a mixed totally geodesic immersion satisfying $n_1 H_1 = n_2 H_2$, where H_i, $i = 1, 2$, are the partial mean curvature vectors of N_i.*

4. Doubly warped product submanifolds in generalized Sasakian space forms

Recently, P. Alegre, D. E. Blair and A. Carriazo have introduced the notion of generalized Sasakian space form (see [1]).

If we change the ambient space with generalized Sasakian space forms, applying Theorem 3, we get some interesting results.

Remark 5. *In an earlier paper* [7], *the present author established a general inequality for warped products isometrically immersed in generalized Sasakian space forms.*

A $(2m+1)$-dimensional Riemannian manifold $\left(\widetilde{M}, g\right)$ is said to be an almost contact metric manifold if there exist on $\widetilde{M}$ a $(1,1)$ tensor field ϕ, a vector field ξ (called the structure vector field) and a 1-form η such that $\eta(\xi) = 1$, $\phi^2(X) = -X + \eta(X)\xi$ and $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$, for any vector fields X, Y on $\widetilde{M}$. In particular, on an almost contact metric manifold we also have $\phi\xi = 0$ and $\eta \circ \phi = 0$.

We denote an almost contact metric manifold by $\left(\widetilde{M}, \phi, \xi, \eta, g\right)$.

A generalized Sasakian space form is an almost contact metric manifold $\left(\widetilde{M}, \phi, \xi, \eta, g\right)$ whose curvature tensor is given by (see [1])

$$
\begin{aligned}
(4.1)\quad \widetilde{R}(X,Y)Z \;=\; & f_1\{g(Y,Z)X - g(X,Z)Y\} + \\
& + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + \\
& + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \\
& + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}
\end{aligned}
$$

where f_1, f_2, f_3 are differential functions on $\widetilde{M}$. In such a case, we will write $\widetilde{M}(f_1, f_2, f_3)$.

Theorem 6. *Let x be a isometric immersion of an n-dimensional doubly warped product ${}_{\sigma_2}N_1 \times_{\sigma_1} N_2$ into an $(2m+1)$-dimensional generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$, such that ${}_{\sigma_2}N_1 \times_{\sigma_1} N_2$ is an anti-invariant submanifold normal to ξ. Then:*

$$(4.2)\qquad n_2 \frac{\Delta_1 \sigma_1}{\sigma_1} + n_1 \frac{\Delta_2 \sigma_2}{\sigma_2} \le \frac{n^2}{4} ||H||^2 + n_1 n_2 f_1,$$

where $n_i = \dim N_i$ and Δ_i is the Laplacian operator of N_i, $i = 1, 2$. Moreover, the equality case of (4.2) holds if and only if x is a mixed totally geodesic immersion and $n_1 H_1 = n_2 H_2$, where H_i, $i = 1, 2$, are the partial mean curvature vectors.

Remark 7. *If either $\sigma_1 \equiv 1$ or $\sigma_2 \equiv 1$, then the inequality (4.2) is exactly the inequality from [7] for warped products.*

As applications, we derive certain obstructions to the existence of minimal anti-invariant doubly warped product submanifolds normal to ξ in generalized Sasakian space forms.

Corollary 8. *Let ${}_{\sigma_2}N_1 \times_{\sigma_1} N_2$ be a doubly warped product into a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$, such that ${}_{\sigma_2}N_1 \times_{\sigma_1} N_2$ is an anti-invariant submanifold normal to ξ. If the warping functions σ_1 and σ_2 are harmonic, then ${}_{\sigma_2}N_1 \times_{\sigma_1} N_2$ admits no minimal immersion into a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ with $f_1 < 0$.*

Proof. Assume σ_1 is a harmonic function on N_1, σ_2 is a harmonic function on N_2 and ${}_{\sigma_2}N_1 \times_{\sigma_1} N_2$ is an anti-invariant submanifold normal to ξ which admits a minimal immersion into a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$.

Then, the inequality (4.2) becomes $f_1 \geq 0$.

Corollary 9. *If the warping functions σ_1 and σ_2 of an anti-invariant doubly warped product submanifold ${}_{\sigma_2}N_1 \times_{\sigma_1} N_2$ normal to ξ in a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ are eigenfunctions of the Laplacian on N_1 and N_2, respectively, with corresponding eigenvalues $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively, then ${}_{\sigma_2}N_1 \times_{\sigma_1} N_2$ admits no minimal immersion in a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ with $f_1 \leq 0$.*

Corollary 10. *Let ${}_{\sigma_2}N_1 \times_{\sigma_1} N_2$ be an anti-invariant doubly warped product submanifold normal to ξ in a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$. If one of the warping functions is harmonic and the other one is an eigenfunction of the Laplacian with corresponding eigenvalue $\lambda > 0$, then ${}_{\sigma_2}N_1 \times_{\sigma_1} N_2$ admits no minimal immersion into a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ with $f_1 \leq 0$.*

Next we will derive corresponding results for doubly warped product submanifolds in Sasakian space forms.

A Sasakian space form $\widetilde{M}(c)$ can be viewed as a generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ with $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$.

We want to mention that a submanifold N normal to ξ in a Sasakian space form $\widetilde{M}(c)$ is anti-invariant, i.e., ϕ maps any tangent space of N into the normal space, that is $\phi(T_pN) \subset T_p^{\perp}N$, for every $p \in N$.

Such a manifold is said to be a C-totally real submanifold.

Corollary 11. *Let x be a C-totally real isometric immersion of an n-dimensional doubly warped product ${}_{\sigma_2}N_1 \times_{\sigma_1} N_2$ into an $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$. Then:*

$$n_2 \frac{\Delta_1 \sigma_1}{\sigma_1} + n_1 \frac{\Delta_2 \sigma_2}{\sigma_2} \leq \frac{n^2}{4} ||H||^2 + n_1 n_2 \frac{c+3}{4}, \tag{4.3}$$

where $n_i = \dim N_i$ and Δ_i is the Laplacian operator of N_i, $i = 1, 2$. Moreover, the equality case of (4.3) holds if and only if x is a mixed totally geodesic immersion and $n_1 H_1 = n_2 H_2$, where H_i, $i = 1, 2$, are the partial mean curvature vectors.

Remark 12. *If either $\sigma_1 \equiv 1$ or $\sigma_2 \equiv 1$, then the inequality (4.3) is exactly the inequality from [6] for warped products.*

By using the above corollary (Corollary 11), we can obtain some important consequences:

Corollary 13. *Let ${}_{\sigma_2}N_1 \times_{\sigma_1} N_2$ be a doubly warped product whose warping functions are harmonic. Then ${}_{\sigma_2}N_1 \times_{\sigma_1} N_2$ admits no minimal C-totally real immersion into a Sasakian space form $\widetilde{M}(c)$ with $c < -3$.*

Proof. Assume σ_1 is a harmonic function on N_1, σ_2 is a harmonic function on N_2 and ${}_{\sigma_2}N_1 \times_{\sigma_1} N_2$ admits a minimal C-totally real immersion in a Sasakian space form $\widetilde{M}(c)$. Then, the inequality (4.3) becomes $c \geq -3$.

Corollary 14. *If the warping functions σ_1 and σ_2 of a doubly warped product ${}_{\sigma_2}N_1 \times_{\sigma_1} N_2$ are eigenfunctions of the Laplacian on N_1 and N_2, respectively, with corresponding eigenvalues $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively, then ${}_{\sigma_2}N_1 \times_{\sigma_1} N_2$ admits no minimal C-totally real immersion in a Sasakian space form $\widetilde{M}(c)$ with $c \leq -3$.*

Corollary 15. *Let ${}_{\sigma_2}N_1 \times_{\sigma_1} N_2$ be a doubly warped product . If one of the warping functions is harmonic and the other one is an eigenfunction of the Laplacian with corresponding eigenvalue $\lambda > 0$, then ${}_{\sigma_2}N_1 \times_{\sigma_1} N_2$ admits no minimal C-totally real immersion into a Sasakian space form $\widetilde{M}(c)$ with $c \leq -3$.*

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